UNIVERSITY OF PENNSYLVANIA
The Wharton School

FNCE 100 A. Craig MacKinlay

PROBLEM SET #3

Fall Term 2005

DIVERSIFICATION, RISK AND RETURN

1. We have three securities with the following possible payoffs.

State	Probability of Outcome	Return on Security #1	Return on Security #2	Return on Security #3
1	.10	.25	.25	.10
2	.40	.20	.15	.15
3	.40	.15	.20	.20
4	.10	.10	.10	.25

   (a) What is the expected return and standard deviation on each security?

   (b) What is $\text{Cov}(R_1, R_2)$, $\text{Cov}(R_1, R_3)$, and $\text{Cov}(R_2, R_3)$?
   What is $\text{Corr}(R_1, R_2)$, $\text{Corr}(R_1, R_3)$, and $\text{Corr}(R_2, R_3)$?
   ($\text{Cov}(\cdot, \cdot)$ denotes covariance and $\text{Corr}(\cdot, \cdot)$ denotes correlation).

   (c) What is the expected return, $E(R_p)$, and standard deviation, $\sigma(R_p)$, of a portfolio which has half of its funds invested in Security #1 and half in Security #2?

   (d) What is $E(R_p)$ and $\sigma(R_p)$ of a portfolio which has half of its funds invested in Security #1 and half in Security #3?

   (e) What is $E(R_p)$ and $\sigma(R_p)$ of a portfolio which has half of its funds in Security #2 and half in #3?

   (f) Compare your answers in Parts (a), (c), (d), and (e). Comment on the effects of diversification.

2. Assume there are n securities, each having:

$$\begin{aligned}
E(R_i) &= .01 & i &= 1, 2, \ldots, n \\
\sigma^2(R_i) &= .01 & i &= 1, 2, \ldots, n \\
\text{cov}(R_i, R_j) &= .005 & i &= 1, 2, \ldots, n; \\
& & j &= 1, 2, \ldots, n; \quad j \neq i
\end{aligned}$$

(a) What is the expected return and variance of an equally weighted portfolio containing all n securities?

(i.e., $x_i = 1/n,\ i = 1, \ldots, n$).

(b) What value will the variance approach as n gets larger?

(c) What characteristics of securities are most important in the determination of the variance of a "well diversified" portfolio?

3. A risky security cannot have an expected return that is less than the riskfree rate (R_F) because no risk-averse investor would be willing to hold this asset, in equilibrium. (True, False & explain).

4. "The risk of a portfolio is the variance of its return; however, the variance of the return on an individual asset is not an appropriate measure of its risk." Discuss.

5. A neighbor purchased a lottery ticket yesterday but now, owing to an unpredicted crisis, is in desperate need of cash. He offers to sell the ticket to you. You know the payoff and the probability of winning. All of your considerable fortune is invested in a highly diversified portfolio. How would you determine an appropriate price for the ticket?

6. Some relevant data pertaining to three Dow-Jones stocks over the January 1971–December 1975 period are:

		$\sigma(R_i)$	β_i
A:	Aluminum Company of America	.093	.662
B:	Eastman Kodak	.070	.979
C:	Union Carbide Corp.	.085	1.231

The pairwise correlations between the returns of these three securities are:

$$\begin{aligned}\rho_{AB} &= .137\\ \rho_{AC} &= .476\\ \rho_{BC} &= .422.\end{aligned}$$

Using the capital asset pricing model and assuming that $E(R_m) = .010$ per month and $R_F = .002$ per month, calculate the expected return on each stock. Why is $E(R_B) > E(R_A)$ when $\sigma(R_A) > \sigma(R_B)$?

7. Given two random variables z and y,

Probability of state of nature	State of nature	Variable z	Variable y
.2	I	18	0
.2	II	5	−3
.2	III	12	15
.2	IV	4	12
.2	V	6	1

   (a) Calculate the mean and variance for each of these variables, and the covariance and correlation between them.

   (b) Suppose z and y represent the returns from two assets. Calculate the mean and variance for the following portfolios.

% in z	125	100	75	50	25	0	−25
% in y	−25	0	25	50	75	100	125

   (c) Draw a picture of the mean vs. std. dev. of all feasible portfolios. Indicate the efficient frontier if these are the only two possible assets.

   (d) Find the portfolio which has the minimum variance.

8. The following data have been developed for the Donovan Company, the manufacturer of an advanced line of adhesives:

State	Probability	Market return R_m	Return for the firm R_j
1	.1	−.15	−.30
2	.3	.05	.00
3	.4	.15	.20
4	.2	.20	.50

The risk-free rate is 6%. Calculate the following:

(a) The expected market return.

(b) The variance of the market return.

(c) The expected return for the Donovan Company.

(d) The covariance of the return for the Donovan Company with the market return, and the correlation of the two.

(e) Write the equation of the security market line.

(f) What is the required return for the Donovan Company? How does this compare with its expected return?

9. The following data have been developed for the Milliken Company:

Year	Market Return	Company Returns
1978	.27	.25
1977	.12	.05
1976	−.03	−.05
1975	.12	.15
1974	−.03	−.10
1973	.27	.30

The yield to maturity on Treasury bills is .066 and is expected to remain at this point for the foreseeable future. Calculate the following:

(a) the expected market return;

(b) the variance of the market return;

(c) the expected return for the Milliken Company;

(d) the covariance of the return for the Milliken Company with the return on the market.

(e) Write the equation of the security market line.

(f) What is the required return for the Milliken Company?

(g) Discuss how this approach for estimating (a)–(f) differs from the approach in (13). Which is more accurate?

10. What is the beta of an efficient portfolio with $E(R_j) = 20\%$ if $R_f = 5\%$, $E(R_m) = 15\%$, and $\sigma_m = 20\%$? What is its σ_j? What is its correlation with the market?

11. The average variance of the annual returns from a typical stock is about 1500 and its average covariance with other stocks is about 400. Work out what this implies for the standard deviation of returns from: (a) a fully diversified portfolio, (b) a portfolio of 64 stocks, (c) a portfolio of 16 stocks, (d) a portfolio of 4 stocks, (e) a portfolio of one stock. Assume equal-sized holdings of each stock. Plot your results.

12. The expected return on a given mean-variance efficient portfolio is 25 percent. This was calculated under the assumption that the risk-free rate was 5 percent, the expected return on the market portfolio of risky assets was 20 percent, and that the standard deviation of the efficient portfolio was 4 percent. In this environment, what expected rate of return would a security earn if it had a 0.5 correlation with the market and a standard deviation of 2 percent?

13. How does the market beta of a portfolio depend on the beta of the individual securities in the portfolios? The expected return? The standard deviation?

14. William Shakespeare's character Polonius is credited with the adage "neither a borrower, nor a lender be." Under the assumptions of the Sharpe-Lintner version of the CAPM, what would be the composition of Polonius' portfolio?

15. By diversifying an equally-weighted portfolio among a large number of risky assets, it is possible to reduce the standard deviation of the portfolio return to almost zero if the asset returns are uncorrelated. True or False?

16. Under the assumptions of the CAPM, every security will, in general, be off of the Capital Market Line (in $E(r)$, σ space) and on the Security Market Line (in $E(r)$, β space). Do you agree or disagree? Explain.

17. The risk-free rate is 5 percent and the market portfolio has an expected rate of return of 15 percent. The market portfolio has a standard deviation of 10 percent. Portfolio z has a correlation coefficient with the market of –0.1 and a standard deviation of 10 percent. According to the capital asset pricing model, what is the expected rate of return on portfolio z?

18. (a) What determines the shape of the feasible region of portfolios on an $E(R)$, σ graph?
    (b) In what sense is the "efficient frontier" efficient?
    (c) Show on a graph (with any explanation necessary) that the optimal portfolio on the efficient frontier is an expected utility maximizing portfolio.
    (d) What conditions are necessary for a portfolio made up of two assets to have a smaller standard deviation of return than either of the two assets has individually?
    (e) What conditions are necessary for a portfolio made up of two assets to have a larger expected return than either of the two assets has individually?

19. For two risky securities M and J possible rate of returns and their joint probabilities are given below:

Rate of Return		Probability
r_M	r_J	
0.16	0.16	0.10
0.16	0.18	0.06
0.16	0.22	0.04
0.18	0.18	0.12
0.18	0.20	0.36
0.18	0.22	0.12
0.20	0.18	0.02
0.20	0.20	0.04
0.20	0.22	0.04
0.20	0.24	0.10

(a) Determine the probability distribution for r_M and calculate its expected value, variance and standard deviation.

(b) Determine the probability distribution for r_J and calculate its expected value, variance and standard deviation.

(c) Calculate the covariance and correlation coefficient for r_M and r_J.

(d) Draw the two securities in $E(r)$, a space.

(e) Calculate the portfolio weights on M & J for:

   i. The expected value maximizing portfolio.

   ii. The variance minimizing portfolio.

(f) Assuming that M is the market portfolio, calculate the beta coefficient for security J.

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Solutions to PROBLEM SET #3

Fall Term 2005

Diversification, Risk and Return

1. (a)
$$\begin{array}{rclcrcl} E(R_1) & = & .175 & \quad & \sigma(R_1) & = & .0403 \\ E(R_2) & = & .175 & & \sigma(R_2) & = & .0403 \\ E(R_3) & = & .175 & & \sigma(R_3) & = & .0403 \end{array}$$

$$E(R_1) = (.10 \times .25) + (.40 \times .20) + (.40 \times .15) + (.10 \times .10) = .175$$

$$\begin{aligned} \sigma^2(R_1) &= .10 \times (.25 - .175)^2 + .40 \times (.20 - .175)^2 + \\ &\quad .40 \times (.15 - .175)^2 + .10 \times (.10 - .175)^2 \\ &= .001625 \\ \sigma(R_1) &= \sqrt{.001625} = .0403 \end{aligned}$$

Similarly for securities #2 and #3.

(b)
$$\begin{aligned} \text{cov}(R_1, R_2) &= .10 \times (.25 - .175)(.25 - .175) \\ &\quad + .40 \times (.20 - .175)(.15 - .175) \\ &\quad + .40 \times (.15 - .175)(.20 - .175) \\ &\quad + .10 \times (.10 - .175)(.10 - .175) \\ &= .000625 \end{aligned}$$

$$\begin{aligned} \text{corr}(R_1, R_2) &= \text{cov}(R_1, R_2)/\sigma(R_1)\sigma(R_2) \\ &= .000625/(.0403)(.0403) \\ &= .385 \end{aligned}$$

$$\begin{aligned} \text{cov}(R_1, R_3) &= .10 \times (.25 - .175)(.10 - .175) \\ &\quad + .40 \times (.20 - .175)(.15 - .175) \\ &\quad + .40 \times (.15 - .175)(.20 - .175) \\ &\quad + .10 \times (.10 - .175)(.25 - .175) \\ &= -.001625 \end{aligned}$$

$$
\begin{aligned}
\text{corr}(R_1, R_3) &= -1.0 \\
\text{cov}(R_2, R_3) &= -.000625 \\
\text{corr}(R_2, R_3) &= -.385
\end{aligned}
$$

(c)
$$
\begin{aligned}
E(R_p) &= .5E(R_1) + .5E(R_2) = .5(.175 + .175) = .175 \\
\sigma(R_p) &= .10 \times (.25 - .175)^2 + .40(.175 - .175)^2 \\
&\quad + .40 \times (.175 - .175)^2 + .10(.10 - .175)^2 \\
&= .001125
\end{aligned}
$$

or:

$$
\begin{aligned}
\sigma^2(R_p) &= (.5)^2\sigma^2(R_1) + (.5)^2\sigma^2(R_2) + 2(.5)(.5)\text{cov}(R_1R_2) \\
&= .25(.001625) + .25(.001625) + .5(.000625) \\
&= .001125 \\
\sigma(R_p) &= \sqrt{.001125} = .0335
\end{aligned}
$$

(d)
$$
\begin{aligned}
E(R_p) &= .175 \\
\sigma^2(R_p) &= 0 \\
\sigma(R_p) &= 0
\end{aligned}
$$

(e)
$$
\begin{aligned}
E(R_p) &= .175 \\
\sigma^2(R_p) &= .0005 \\
\sigma(R_p) &= .0224
\end{aligned}
$$

(f) Each of the individual securities and each of the portfolios has an expected return equal to .175. However, we can reduce the variance of the return on the investment by combining securities. The effectiveness of diversification is related to the covariance (or correlation) between the assets. If $\text{corr}(R_i, R_j) = 1.0$, then there is no gain from diversification. If $\text{corr}(R_iR_j) = -1.0$, then all risk can be eliminated by choosing the appropriate portfolio weights. The smaller (algebraically) the correlation, the larger the decrease in variance from diversification.

2. (a)
$$
E(R_p) = \sum_{i=1}^{N} \frac{1}{N} E(R_i) = .01 \sum_{i=1}^{N} \frac{1}{N} = .01
$$

$$
\begin{aligned}
\sigma^2(R_p) &= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{N}\frac{1}{N}\text{cov}(R_i, R_j) \\
&= \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma^2(R_i) + \sum_{i=1}^{N}\sum_{j\neq i} \left(\frac{1}{N}\right)\cdot\left(\frac{1}{N}\right)\text{cov}(R_i, R_j) \\
&= \left(\frac{1}{N}\right)^2 \cdot N \cdot (.01) + \left(\frac{1}{N^2}\right) N(N-1)(.005) = \frac{.01}{N} + \left(1 - \frac{1}{N}\right)(.005)
\end{aligned}
$$

(b) $\sigma^2(R_p) \to .005$ as $N \to \infty$
since $\frac{.01}{N} \to 0$ and $\left(1 - \frac{1}{N}\right)(.005) \to .005$

(c) The average *covariance* between securities will be much more important in determining a portfolio's risk than the average variance, when the portfolio is "well diversified."

3. False. A security can have negative risk (i.e., $\beta_i < 0$). Since this security has a negative covariance with the market it is quite effective in reducing the risk of portfolios. For this reason people will be willing to accept an expected return that is less than R_F. The Sharpe-Lintner CAPM gives us:

$$E(R_i) = R_F + [E(R_m) - R_F]\beta_i$$

where: $E(R_m) - R_F > 0$.

Therefore, $\beta_i < O$ implies that,

$$E(R_i) < R_F$$

Even though R_i is uncertain.

4. Given the assumption that asset returns have a joint probability distribution that is multivariate normal, we know that the returns on any single security or portfolio is normally distributed. Since expected value and variance completely describe a normal distribution, we take the expected return on a portfolio as its measure of central tendency, and we take the variance as its measure of risk. Equivalently, the choice between any two portfolios can be based on a comparison of their expected returns and variance. So variance is an appropriate measure of a portfolio's risk.

The risk of an individual asset, *when held in a portfolio*, is not its variance. The *same* security can contribute different amounts of risk to different portfolios. For example, if portfolio #1 is positively correlated with asset i, while portfolio #2 is negatively correlated with asset i, then adding (or increasing the amount invested in) asset i will change the variances of the portfolio by different amounts. The risk of asset i in portfolio P is given by,

$$\mathrm{Cov}(R_i, R_p)$$

The relative risk of asset i in portfolio P is given by

$$\beta_{ip} = \frac{\mathrm{Cov}(R_i, R_p)}{\sigma^2(R_P)} .$$

The variance of asset i is not the appropriate measure of risk *unless* it is the only asset in the portfolio (i.e., portfolio P *is* asset i). In this case, we have

$$\mathrm{Cov}(R_i, R_p) = \mathrm{Cov}(R_i, R_i) = \sigma^2(R_i).$$

5. The outcome of the lottery is not likely to be correlated in any way with any changes in the level of the market as a whole or in the value of your highly diversified portfolio. Since your portfolio is large, the lottery ticket will represent a small increment, so its

marginal impact on risk is relevant. Since it has, in effect, a beta value of 0, it will add an insignificant amount of risk to your portfolio. As long as you can expect a return at least as large as that available from Treasury bills, the ticket will be a good investment for you. You thus need to calculate the price that will make the expected payoff bring a return equal to that from Treasury bills.

6. $E(R_m) = R_F + (E(R_m) - R_F)\beta$

$$\begin{aligned}
\text{Alcoa:}\quad & E(R_A) = .002 + (.010 - .002).662 && = .0073 \\
\text{Kodak:}\quad & E(R_B) = .002 + (.010 - .002).979 && = .0098 \\
\text{Union Carbide:}\quad & E(R_C) = .002 + (.010 - .002)1.231 && = .0118
\end{aligned}$$

In the important sense of contribution to the risk of a portfolio, Kodak is a more risky security ($\beta_B > \beta_A$). If the CAPM is correct, beta risk is the only risk for which the investor is compensated. Therefore, the security with the higher risk (Kodak) must have the higher expected return. Variance of return is not important in determining the expected value of future security return; it is risk that can be eliminated by diversification.

7. (a)

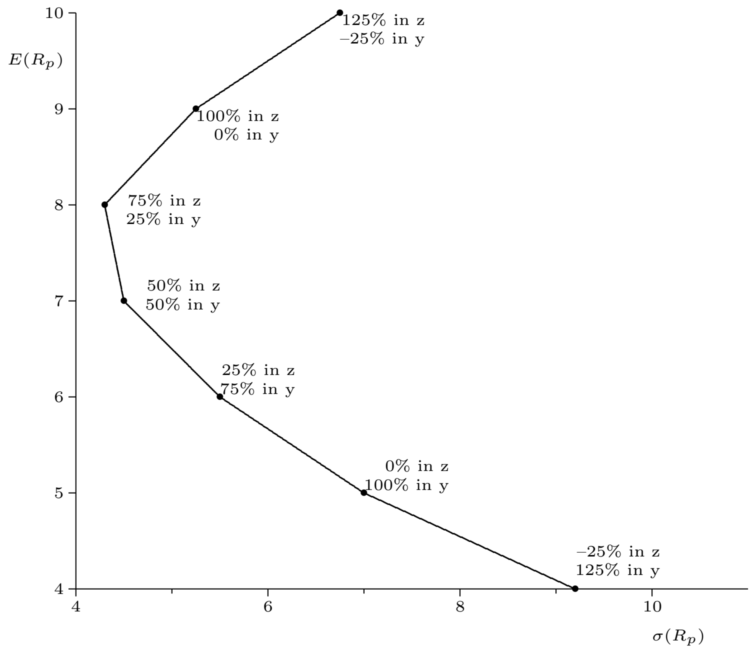
$$\begin{aligned}
E(z) &= (.2)(18) + (.2)(5) + (.2)(12) + (.2)(4) + (.2)(6) &&= 9 \\
E(y) &= (.2)(0) + (.2)(-3) + (.2)(15) + (.2)(12) + (.2)(1) &&= 5 \\
\sigma^2(z) &= (18-9)^2(.2) + (5-9)^2(.2) + (12-9)^2(.2) \\
&\quad + (4-9)^2(.2) + (6-9)^2(.2) &&= 28 \\
\sigma^2(y) &= (0-5)^2(.2) + (-3-5)^2(.2) + (15-5)^2(.2) \\
&\quad + (12-5)^2(.2) + (1-5)^2(.2) &&= 50.8 \\
\text{Cov}(z,y) &= (18-9)(0-5)(.2) + (5-9)(-3-5)(.2) \\
&\quad + (12-9)(15-5)(.2) + (4-9)(12-5)(.2) \\
&\quad + (6-9)(1-5)(.2) &&= -1.2
\end{aligned}$$

$$\rho(z,y) = \frac{\text{Cov}(z,y)}{\sigma(z)\sigma(y)} = \frac{-1.2}{(5.29)(7.13)} = -.032$$

(b)

% in z	125	100	75	50	25	0	–25
% in y	-25	0	25	50	75	100	125
$E(R_p)$	10	9	8	7	6	5	4
$\sigma^2(R_p)$	47.7	28	18.5	19.1	29.9	50.8	81.9
$\sigma(R_p)$	6.9	5.3	4.3	4.4	5.5	7.1	9.1

(c)



(d) 75% in z
25% in y

8. (a) $E(R_M) = -.15(.1) + (.05)(.3) + (.15)(.4) + (.2)(.2) = .1$

(b) $$\sigma^2_M = (-.15 - .1)^2(.1) + (.05 - .1)^2(.3) + (.15 - .1)^2(.4) + (.2 - .1)^2(.2) = .01$$

(c) $$E(R_D) = (-.3)(.1) + (.2)(.4) + (.5)(.2) = .15$$
$$\sigma^2_{R_D} = (-.30 - .15)^2(.1) + (.0 - .15)^2(.3) + (.20 - .15)^2(.4) + (.50 - .15)^2(.2) = .0525$$

(d) $$\sigma_{D,M} = (-.15 - .10)(-.3 - .15)(.1) + (.05 - .1)(0 - .15)(.3) + (.15 - .1)(.2 - .15)(.4) + (.2 - .1)(.5 - .15)(.2) = .0215$$

$$\rho_{D,M} = \frac{\sigma_{D,M}}{\sigma_D \cdot \sigma_M} = \frac{.0215}{\sqrt{.0525}\sqrt{.61}} = .938$$

(e) $E(R_D) = R_F + [E(R_M) - R_F]\beta_D \qquad \beta_D = \text{cov}(D, M)/\sigma^2_M = \frac{.0215}{.01} = 2.15$

(f) According to CAPM, $E(R_D)$ should be:

$$E(R_D) = .06 + [.1 - .06](.0215/.01) = .146. \qquad .146 < .15$$

This means that the stock is underpriced (offers excess return) relative to the CAPM.

9. In this problem you are asked to calculate *estimates* of the mean, variance, etc., given data on past outcomes. Denote estimates by putting a ˆ over a variable. Let R_{Mt} denote the *realization* of R_M in period t (similarly for R_{Dt}).

(a) $\widehat{\overline{R}}_M = \left[\sum_{t=1}^{T} R_{Mt}\right] \Big/ T = (.27 + .12 - .03 + .12 - .03 + .27)/6 = .12$

(b)
$$\begin{aligned} \hat{\sigma}^2_{R_M} &= \left[\sum_{t=1}^{T}(R_{Mt} - \widehat{\overline{R}}_M)^2\right] \Big/ (T-1) \\ &= [(-27 - .12)^2 + (.12 - .12)^2 + (-.03 - .12)^2 + (12 - .12)^2 \\ &\quad +(-.03 - 12)^2 + (27 - .12)^2]/5 = .09/5 = .018 \end{aligned}$$

(c) $\widehat{\overline{R}}_K = (.25 + .05 - .05 + .15 - .10 + .30)/6 = .10$

(d) $\widehat{\text{cov}}(R_M, R_K) = \left[\sum_{t=1}^{T}(R_{Mt} - \widehat{\overline{R}}_M)(R_{Kt} - \widehat{\overline{R}}_K)\right] \Big/ (T-1) = .105/5 = .021$

$$\hat{\beta}_K = \frac{\widehat{\text{cov}}(R_M, R_K)}{\widehat{\text{var}}(R_M)} = \frac{.021}{.018} = 1.17$$

(e) $E(R_K) = R_F + \beta_K \cdot E(R_M - R_F)$

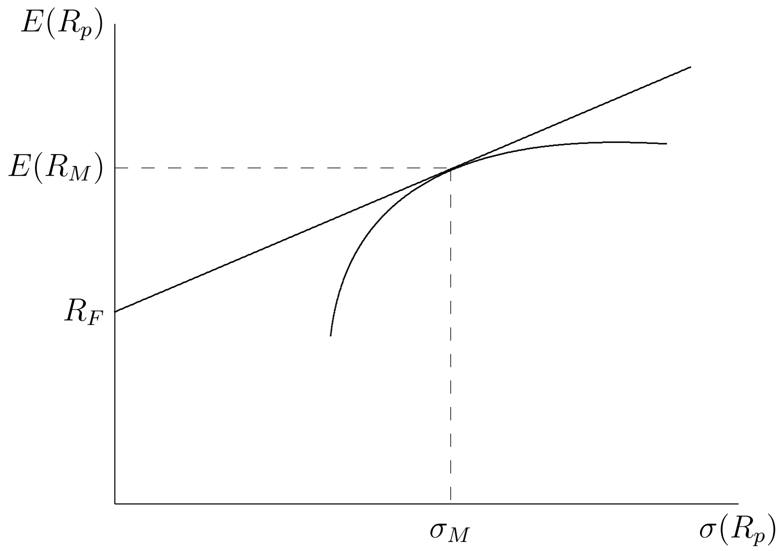
(f) Plugging in our estimates for $E(R_M)$ and β_K we get an estimate of the required return on Milliken $= .066 + 1.17[.12 - .066] = 12.9\%$

(g) In this problem, all of the relevant numbers have to be *estimated*, based on a time series of stock returns. In (13) we could calculate the *true* means, variances, covariance, and beta because we were given the underlying joint probability distribution.

10. When there is a riskless asset, an efficient portfolio lies on the CML. Let R_j be the return on an *efficient* portfolio. Then

$$E(R_j) = R_F + \sigma_{R_j}\left[\frac{E(R_M) - R_F}{\sigma_M}\right]$$

So $\longrightarrow .20 = .05 + \sigma_{R_j}\left[\frac{.15-.05}{.20}\right] \longrightarrow \sigma_{R_j} = .30$



We can use the SML to determine the β of the portfolio.

$$\begin{aligned} E[R_j] &= R_F + \beta_j[E(R_M) - R_F)] \\ .20 &= .05 + \beta_j[.15 - .05] \qquad \longrightarrow \beta_j = 1.5 \end{aligned}$$

Also, the correlation of the efficient portfolio (with the market) must equal 1, since the efficient portfolio is composed of some fraction of the market and some fraction of the riskless asset.

To verify this:

$$\begin{aligned} \rho(R_j, R_M) &= \frac{\text{cov}(R_j, R_M)}{\sigma_{R_j}\sigma_{R_M}} = \frac{\text{cov}(R_j, R_M)}{\sigma^2_{R_M}} \cdot \frac{\sigma_{R_M}}{\sigma_{R_j}} \\ &= \beta_j \cdot \frac{\sigma_{R_M}}{\sigma_{R_j}} = 1.5 \cdot \left(\frac{.20}{.30}\right) = 1 \end{aligned}$$

11. We showed in class that:

$$\sigma^2_p = \frac{1}{N}\overline{\text{var}} + \left(1 - \frac{1}{N}\right)\overline{\text{covar}}; \qquad \overline{\text{var}} = 1500; \quad \overline{\text{covar}} = 400$$

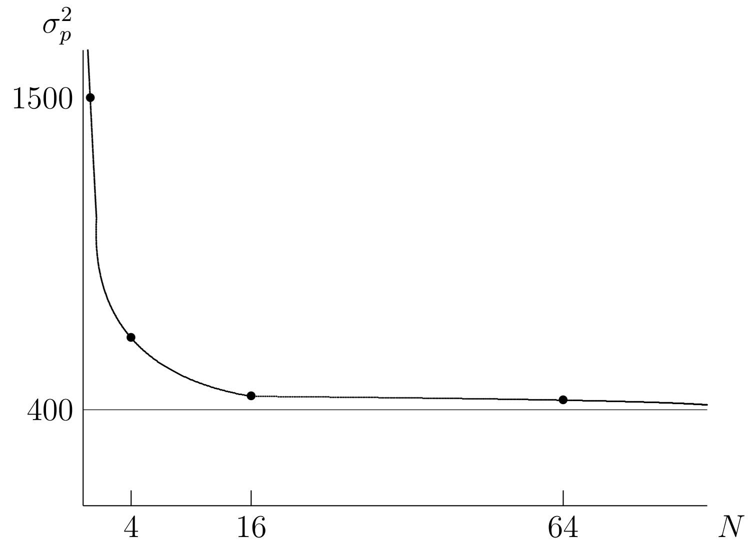
(a) $\sigma^2_p = \overline{\text{covar}} = 400$

(b) $\sigma^2_p(64) = \frac{1}{64}(1500) + \left(1 - \frac{1}{64}\right) 400 = 417.19$

(c) $\sigma^2_p(16) = \frac{1}{16}(1500) + \left(1 - \frac{1}{16}\right) 400 = 468.75$

(d) $\sigma^2_p(4) = \frac{1}{4}(1500) + \left(1 - \frac{1}{4}\right) 400 = 675$

(e) $\sigma^2_p(1) = \frac{1}{1}(1500) + \left(1 - \frac{1}{1}\right) 400 = 1500$



12. C.M.L. $$.25 = .05 + \frac{[.20 - .05]}{\sigma_M}.04 \longrightarrow \sigma_M = .03$$

    S.M.L. $$E(R_i) = .05 + \frac{[.20 - .05](.5)(.02)(.03)}{(.03^2)} = 10\%$$

13. The beta of a portfolio with respect to the market is a weighted average of the individual security betas with respect to the market; the weights correspond to the proportion of the particular security included in the portfolio:

$$\beta_p = \sum_{i=1}^{n} x_i \beta_i$$

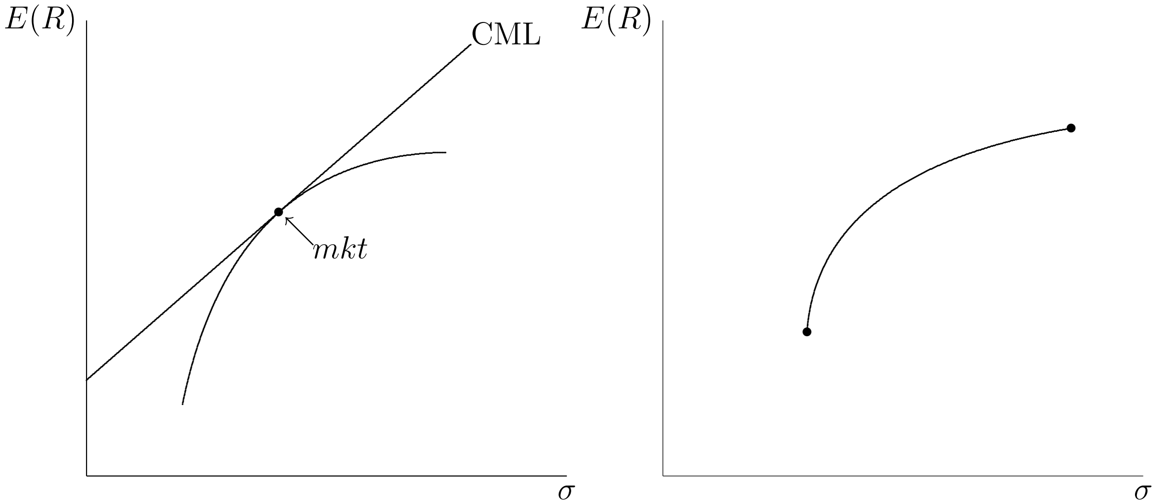
The same is true for expected return:

$$E(R_p) = \sum_{i=1}^{n} x_i E(R_i)$$

The same is *not* true, however, for variance or standard deviation:

$$\begin{aligned} V(R_p) &= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \text{cov}(R_i, R_j) \\ \sigma(R_p) &= \sqrt{V(R_p)} \end{aligned}$$

14. Under the CAPM, if Polonius' optimal portfolio happened to involve neither borrowing nor lending, then he must have chosen the market portfolio.



[Aside: If Polonius would have liked to borrow or lend, but felt compelled by external social forces not to, then he would have had to choose a portfolio somewhere along the parabola that constitutes the efficient frontier with no riskless asset. Where he chose would then depend on his attitude toward risk.]

15. Asset returns uncorrelated:

$$\begin{aligned} \rho_{ij} &= 0 \quad \forall i,j \quad i \neq j \\ \sigma_p^2 &= \overline{\text{covar}} = \overline{\rho\sigma_1\sigma_2} = \bar{0} = 0 \end{aligned}$$

Therefore, this statement is true.

16. Agree. The equation for the C.M.L. captures the risk-return trade-off for "efficient portfolios," i.e., those portfolios which are a combination of risk-free borrowing or lending and investment in the market portfolio. Therefore, in general, individual securities lie below the C.M.L. The implication of the C.M.L. for equilibrium expected returns for individual securities is the S.M.L. All assets in equilibrium lie on S.M.L. or arbitrage with positive profits exist, inconsistent with equilibrium.

17.
$$\begin{aligned} E(r_z) &= r_f + \beta_z[E(r_m) - r_f] \\ \beta_z &= \frac{\sigma_{zm}}{\sigma_m^2} = \frac{\rho_{zm}\sigma_z\sigma_m}{\sigma_m^2} = \frac{(-0.1)(10)(10)}{(10)(10)} = -0.1 \\ E(r_z) &= .05 + (-0.1)[.15 - .05] = .04 \end{aligned}$$

18. (a) The expected returns ($E(R)$) and standard deviations (σ) of the individual assets and the covariances between them. The general leftward curvature results from the less than perfect correlation.

(b) Min risk for a given $E(R)$ *and* Max $E(R)$ for a given risk.

(c) For the mean-variance CAPM to hold, individual's preferences must be based solely on μ and σ, or asset returns must be normally distributed. With either of these two assumptions, indifference curves can be constructed in μ, σ space. Realizing that expected utility increases with increase in μ and decreases with increase in σ, indifference curves moving northwestward represent higher levels of expected utility. Individuals choose that portfolio with a given μ and σ that affords them the greatest expected utility. This will be at a tangency.

(d) The correlation between them is less than +1.0.

(e) The asset with the lowest expected return can be sold short and the proceeds invested in the asset with the higher expected return.

19.

		r_j						
		.16	.18	.20	.22	.24		
	.16	.10	.06	0	.04	0	.20	
r_m	.18	0	.12	.36	.12	0	.60	Probability
	.20	0	.02	.04	.04	.10	.20	
		.10	.20	.40	.20	.10		
				Probability				

(a)
$$\begin{aligned} E(r_m) &= (.16)(.20) + (.18)(.60) + (.20)(.20) &= .18 \\ \sigma^2_{r_m} &= (.16 - .18)^2(.2) + (.18 - .18)^2(.6) \\ &\quad + (.20 - .18)^2(.2) &= .00016; \\ \sigma_{r_m} &= .01265 \end{aligned}$$

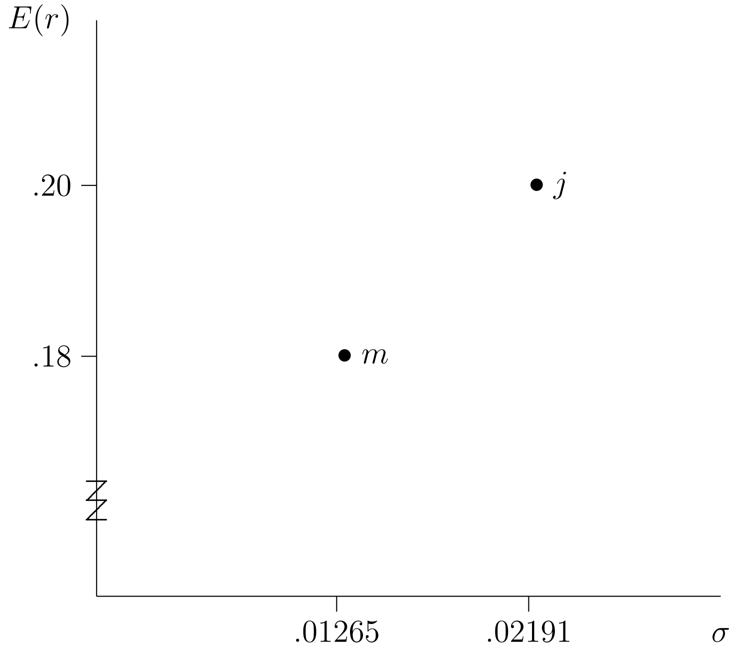
(b)
$$\begin{aligned} E(r_j) &= (.16)(.10) + (.18)(.20) + (.20)(.40) \\ &\quad + (.22)(.20) + (.24)(.10) = .2 \\ \sigma^2_{r_j} &= (.16 - .2)^2(.10) + (.18 - .2)^2(.2) + (.2 - .2)^2(.4) \\ &\quad + (.22 - .2)^2(.2) + (.24 - .2)^2(.1) = .00048 \\ \sigma_{r_j} &= .02191 \end{aligned}$$

(c)

$$
\begin{aligned}
\mathrm{cov}(r_m, r_j) &= (.16-.18)(.16-.2)(.10) && [= .00008\ ] \\
&+ (.16-.18)(.18-.2)(.06) && [= .000024] \\
&+ (.16-.18)(.22-.2)(.04) && [= -.000016] \\
&+ (.18-.18)(.18-.2)(.12) && [= 0\ ] \\
&+ (.18-.18)(.2-.2)(.36) && [= 0\ ] \\
&+ (.18-.18)(.22-.2)(.12) && [= 0\ ] \\
&+ (.20-.18)(.18-.2)(.02) && [= -.000008] \\
&+ (.20-.18)(.2-.2)(.04) && [= 0\ ] \\
&+ (.20-.18)(.22-.2)(.04) && [= .000016] \\
&+ (.20-.18)(.24-.2)(.10) && [= \underline{.00008}\ ] \\
& && .000176
\end{aligned}
$$

$$\rho = \frac{\mathrm{cov}(r_j, r_m)}{\sigma_m \cdot \sigma_j} = \frac{.000176}{(.01265)(.02191)} = .635$$

(d)



(e) i. Consider a portfolio with x in j and $(1-x)$ in m.

$$
\begin{aligned}
E(r_p) &= xE(r_j) + (1-x)E(r_m) = x(.2) + (1-x)(.18) \\
&= .18 + x(.20 - .18)
\end{aligned}
$$

If short sales are allowed, we can get $E(r_p)$ arbitrarily high by going long in j and short in m, i.e., by making x arbitrarily large (> 1).
If no short sales are allowed (i.e., $O < x < 1)$), choosing $x = 1$ will maximize the expected return of the portfolio.

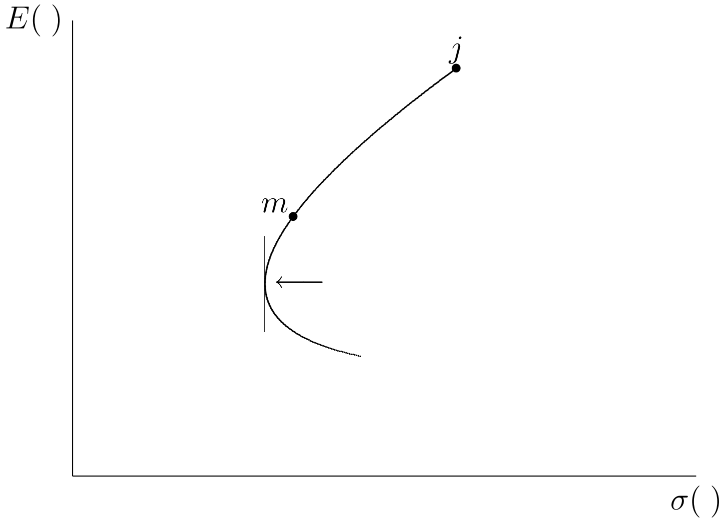
ii. $\mathrm{Var}(r_p) = x^2\mathrm{Var}(r_j) + (1-x)^2\mathrm{Var}(r_m) + 2(x)(1-x)\mathrm{cov}(r_j, r_m)$

If we choose x to minimize this (set $\frac{d\,\mathrm{Var}(r_p)}{dx} = 0$), we get

$$\begin{aligned} x* &= \frac{\mathrm{Var}(r_m) - \mathrm{Cov}(r_j, r_m)}{\mathrm{Var}(r_j) + \mathrm{Var}(r_m) - 2\mathrm{cov}(r_j, r_m)} \\ &= \frac{.00016 - .000176}{.00048 + .00016 - (.000176) \cdot 2} = -.06 \end{aligned}$$

So the weights are -.06 in j and 1.06 in m.

Therefore, the feasible set looks something like:



(f) $\beta_j = \dfrac{\mathrm{cov}(r_j, r_m)}{\sigma^2_{r_m}} = \dfrac{.000176}{.00016} = 1.1$