Is Credit Event Risk Priced? Modeling Contagion via the Updating of Beliefs

Pierre Collin-Dufresne\textsuperscript{2} Robert S. Goldstein\textsuperscript{3} Jean Helwege\textsuperscript{4}

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\textsuperscript{2}Associate Professor at the Graduate School of Industrial Administration, Carnegie Mellon University, GSIA 315A, 5000 Forbes ave, Pittsburgh PA 15213, dufresne@andrew.cmu.edu

\textsuperscript{3}Associate Professor at the Olin School of Business, Washington University in St. Louis, Campus Box 1133, One Brookings Drive, St. Louis, MO 63130, goldstein@olin.wustl.edu

\textsuperscript{4}Assistant Professor at the Fisher School of Business, Ohio State University, Columbus, OH 43210, helwege@cob.ohio-state.edu
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Abstract
We propose a reduced-form model where jumps-to-default are priced because they generate a market-wide jump in credit spreads. While this framework is consistent with a counterparty risk interpretation (e.g., Jarrow and Yu (2001)), it is most naturally interpreted as an updating of beliefs due to an unexpected event. Simple analytic solutions are obtained for the prices of risky debt regardless of the number of firms that share in the contagious response. As a special case, we show that the contagious response can be induced via a liquidity-shock, with no impact on actual default intensities. Empirically, we find that credit events of large firms generate a market wide increase in credit spreads and a significant ‘flight-to-quality’ response in the Treasury market. A calibration argument suggests that the premium associated with jump-to-default risk for a typical investment grade firm has an upper bound of a few basis points per year, but that the risk premium for contagion-risk may be considerably larger.
1 Introduction

It is well documented that structural models of default which are calibrated to historical default rates and restricted to ‘reasonable’ risk-premia specifications generate counterfactually low yield spreads for investment-grade debt, especially for debt of short maturity. (See, e.g., Huang and Huang (2002), Eom, Helwege and Huang (2003)). Often this discrepancy between theoretical and observed yield spreads has been attributed to considerations beyond credit-risk, such as taxes and liquidity. (See, e.g., Elton et al (2002)). However, the reduced form framework, which models default as an unpredictable stopping time, can potentially explain this discrepancy in terms of compensation for credit-risk if in fact jumps-to-default of individual firms are priced. For this to be the case, however, there must be a market-wide response (i.e., a jump in the pricing kernel) at these jump-events. If not, then as emphasized by Jarrow, Lando and Yu (2000), such jump-risk would be conditionally diversifiable, and therefore would not command a risk premium.

One explanation for a market-wide response to a credit event of an individual firm is that of counterparty-risk. This occurs when the default of one firm causes financial distress on entities that it had close business ties with. This distress in turn is transmitted to a second layer of firms through a domino (or contagion) effect. As a recent example, the concern that the default of LTCM would lead to defaults of other major investment banks provided incentive for the Federal Reserve to intervene. Jarrow and Yu (JY 2001) investigate the pricing of risky debt in the presence of counterparty risk. Unfortunately, their framework is not very tractable if one wishes to model contagion symmetrically among many firms.\(^1\) Furthermore, it is not obvious whether such a mechanism can explain contagion across different industries (except possibly for those examples which include the banking system).

A second mechanism by which a credit event of one firm can trigger a market-wide response is through Bayesian updating of the ‘perception’ of risk. A recent example includes Enron, where inaccurate accounting data of one firm led to concern about the accounting numbers of other firms (e.g., GE, Tyco), even though there were no business ties between these companies. Another example includes the leveraged buyout of RJR, where market participants assumed beforehand that some firms were simply too large to be a target of a leveraged buyout. These beliefs were revised dramatically after the LBO announcement, in turn generating a substantial market-wide increase in credit spreads. (See, e.g., Crabbe (1991)).

Below, we propose a general reduced-form framework where an unexpected default of an

\(^1\)Yu (2003) presents a numerical simulation approach which may prove useful to solve such models.
individual firm leads to a market-wide increase in credit spreads. As such, jump-to-default risk is not conditionally diversifiable, and hence commands a risk premium. In contrast to JY, our framework remains tractable regardless of the number of firms that share in the contagious response. Indeed, even though our model falls outside of the ‘Cox-process’ framework it provides tractable solutions for corporate bond prices even when the number of firms that share in the cross-contagion is large. While this framework is consistent with a counterparty-risk interpretation, it is most naturally interpreted as an updating of beliefs due to an unexpected event. Indeed, as a worked-out example, we extend the framework of Duffie and Lando (2000) to multiple firms that share a common ‘accounting quality’. Since this ‘accounting quality’ is unknown by market participants, an unexpected default of one firm will cause market participants to update their beliefs about the accounting quality of all other firms, in turn triggering a market-wide jump in spreads.

While most of the focus in this paper is on contagion due to the credit event of one firm changing the perceived risk of other firms, as a special case we demonstrate that our model can capture a pure liquidity-induced contagion in that, even if a default of some firm-i has (almost) no impact on the perceived default intensities of other firms, the default-event can still generate a sizable jump in credit spreads. This occurs because the credit event also causes a change in risk premia. Such an example might be consistent with the Russian default leading to the demise of LTCM, in turn leading to a liquidity crunch in the corporate bond markets well beyond any perceived increase in risk in these bonds.

We then empirically estimate the size of the risk premium associated with credit-event risk. We note that the feature that uniquely identifies whether or not a jump-risk premium exists is the time-series implication that, at the jump-date, there is a market-wide response. As such, we take an ‘event study’ approach by identifying those months where a credit event of an investment grade firm has occurred, and then investigating whether there are significant differences to aggregate returns for credit event months and non-event months. We find that credit events lead to market-wide increases in credit spreads, and, consistent with a general equilibrium framework investigated below, to ‘flights to quality’ in that the risk-free rate jumps downward during these events. Further, the equity return experiences a negative shock in these months. Consistent with intuition, we find that security prices are more affected by credit events of large firms than by credit events of small firms.

Of course, other factors besides credit events drive changes in credit spreads, Treasury yields, and stock returns. Moreover, other events also generate flights-to-quality (e.g., the market crash of 1987, the Russian default of 1998). We demonstrate that, even after controlling
for these other factors, credit events of large firms remains a statistically significant factor with economically significant factor loadings.

Given that we find credit events command a risk premium, the expected returns on corporate bonds should have two additional components over and above those traditionally found in structural models (or doubly stochastic reduced-form models) of default, namely, a jump-to-default component and a contagion-risk component. Both theoretically and empirically, we find that the second of these two components generates a larger risk premium. Intuitively, this occurs because in both cases the risk premium is a product of three factors: (i) the likelihood of a credit event occurring, (ii) the percentage of bond value lost at the event, and (iii) the jump in the pricing kernel at the credit event. Although the percentage of firm-i’s bond value lost due to the default of firm-i is naturally much larger than the percentage of firm-i’s bond value lost due to default of some other firm-k, the relative likelihood that a given bond suffers a contagious event versus a credit event increases linearly with the number of firms that share in the contagion. That is, for example, in an economy with 1000 investment-grade firms, it is 999-times more likely for some firm k(≠ i) to default, leading to the credit spread of firm-i to jump via a contagious response, than it is for a particular firm-i to default. Empirically, we find that the market-wide ‘contagious jump’ in credit spreads due to a credit event is of sufficient magnitude so that the product of factors (i) and (ii) generates a significantly higher risk premium for the contagion-risk than for the jump-to-default risk. Indeed, we find that the size of the risk premium associated with jump-to-default has an upper bound of a few basis points per year, implying that such a risk premium cannot explain the ‘credit spread puzzle’ alluded to above.

In a recent related paper, Driessen (2002) estimates the size of the risk premium associated with jump-to-default risk to be between 5 and 30 basis points per year. In contrast to our time-series approach, Driessen (2002) obtains a ‘cross-sectional’ estimate for the jump-to-default-premium by estimating all other sources of risk-premia in his model, and then defining the residual as a premium for jump-to-default risk. Note, however, that his approach does not account for any type of market-wide response in credit spreads at default events, and thus provides no justification for the existence of a jump-to-default risk premium. Admittedly, Driessen’s approach can be justified if it is assumed that default events command a risk premium because aggregate equity prices, and not credit spreads, jump at credit events. However, such a scenario is in conflict with our empirical findings that there is a market-wide impact on credit spreads due to credit events of individual firms. As such, it is conceivable that his parameter estimates for jump-to-default risk are biased due to model misspecifica-
tion. Furthermore, if in fact there is a contagious response in the corporate bond markets to jumps-to-default of individual firms, the framework used in Driessen (2002) will underestimate the effect that a jump-to-default has on a corporate bond portfolio.

Many other papers examine correlated default risk. Duffie and Singleton (2000) present various simulation techniques for estimating correlation risk within reduced form models where default correlation is the result of correlated intensities. Das, Freed, Geng and Kapadia (2002) provide an empirical investigation of this type of intensity-based default correlation. Davis and Lo (2000) study a static model of ‘infectious’ default, which shares some of the notion of contagion present in our framework. However, their model is a purely static one-period model in that the default of one firm may trigger the default of other firms, but there is no dynamic updating of default probabilities. Zhou (2001) and Cathcart and El-Jahel (2002) consider structural models that allow for jumps in firm value. Their models are not very tractable for more than two firms, however. Giesecke (2001) is closest to our paper in that investors learn over time about the probability of default of one firm from the defaults of other firms. A major difference is that, in his model, firms can only default if they are currently at an all-time low price. In other words, there does not exist an intensity for the default arrival time unless firm value is at an all-time low. This feature contrasts considerably with our framework, where credit events can occur at any time, and more closely captures such cases as Enron and RJR. Our paper is also related to recent work by Schönbucher and Schubert (2001) who use a Copula approach in a reduced-form framework to model correlated default-risk. Since the Copula approach is inherently static, they rely on a clever use of a ‘doubly stochastic’ Cox-process construction.

This paper is also closely related to the learning and contagion literature. In contrast to existing learning models (e.g., Detemple (1986), Feldman (1989), David (1997) and Veronesi (2000)) that use results on filtering theory for diffusions (see, e.g., Lipster and Shiryaev (1974)), we derive results for continuous time updating based on information revealed through point processes. Casual observation suggests that there have been many incidences where ‘learning’ (e.g., Enron, Worldcom, etc.) is more suitably modeled as a jump event rather than as a diffusion.

The information-based mechanism for contagion modeled in this paper is similar to that proposed by King and Wadhwani (1990), and Kodres and Pritsker (2002), who investigate contagion across international financial markets. There is also a large empirical literature that

\footnote{By not modeling contagion-risk, Driessen's methodology combines both diffusion-risk and contagion-risk into a single component.}
studies contagion in equity markets (Lang and Stulz (1992)) and in international finance (see e.g., Bae, Karolyi and Stulz (2000)).

The rest of the paper is as follows. Section 2 discusses various potential justifications for jump risk premia. In Section 3 we motivate the dynamics of our model using a simple framework. We then generalize the dynamics and show that it leads to tractable solutions for the pricing of defaultable bonds. We show that if the bond market is mean-variance efficient, then the jump-to-default risk premium has an upper bound of a few basis points per year, and is dominated by the contagion risk premium. Furthermore, we show that our framework can generate a contagious response even if the perceived default probabilities are unaffected by default events. In Section 4 we investigate empirically what effect jump-events have had on market indices. We conclude in Section 5. In Appendix A we generalize the framework of Duffie and Lando (2001) to show how a structural model of default can capture the notion of contagion. In Appendix B we provide a simple general equilibrium framework which generates a functional form for the jump-to-default premium, and predicts ‘flight-to-quality’ at jump events. Many of the proofs are relegated to Appendix C.

2 Reduced Form Models, Credit Risk Premia and Default Event Correlation

Reduced form models of default\(^3\) assume that default is triggered by the jump of an unpredictable point process \(1_{\{\tau < t\}}\), whose historical intensity \(\lambda^P\) and risk neutral intensity \(\lambda^Q\) satisfy

\[
\begin{align*}
E^P_t \left[ d1_{\{\tau < t\}} \right] &= \lambda^P_t 1_{\{\tau > t\}} \, dt \\
E^Q_t \left[ d1_{\{\tau < t\}} \right] &= \lambda^Q_t 1_{\{\tau > t\}} \, dt.
\end{align*}
\]

If in fact there is a jump in the pricing kernel (i.e., loosely speaking, a market-wide response at the default date \(\tau\)) then the source of risk \(d1_{\{\tau \leq t\}}\) will appear in the dynamics of the pricing kernel, implying that it is priced, and thus \(\lambda^Q_t\) will not equal \(\lambda^P_t\).\(^4\)

The intensity \(\lambda^P\) (and hence, in general, also \(\lambda^Q\)) itself is typically stochastic, capturing the notion that the likelihood of default changes over time. In particular, its dynamics can change randomly due to both Brownian motions \(\zeta\) and jumps \(q\):

\[
d\lambda^P_t = \mu^P_t \, dt + \sigma^P_t \, d\zeta^P_t + \Gamma^P_t \, dq_t.
\]

\(^3\)See, for example, Jarrow, Lando and Turnbull (1995), Madan and Unal (1998), Duffie and Singleton (1999).

\(^4\)This is the well-known result of the change of measure, i.e., Girsanov’s theorem for point processes. If the Radon-Nykodim derivative has a common jump with the point process then its intensity may be modified under the new measure. An example of this is provided in equation (33) and Lemma 3 below.
If changes in $\lambda^P$ are correlated with changes in the pricing kernel, then at least one of the sources of risk $dz$ and $dq$ is priced, and the dynamics for $\lambda^P$ will differ under the historical and risk-neutral measures.

As equations (1)-(3) suggest, within a reduced-form framework, risk premia can show up in two different manners. First, risk premia can be due to sources of risk that drive the dynamics of the intensity $(dz^P, dq)$. Elton et al. (2001) and Driessen (2002) provide convincing evidence that such risk premia exist. Second, the jump-to-default random variable $d\mathbf{1}_{\{\tau \leq t\}}$ can command a risk premium itself, in which case $\lambda^P \neq \lambda^Q$. We emphasize that these risk premia are empirically distinguishable: in time series, for example, $d\mathbf{1}_{\{\tau \leq t\}}$ is priced only if there is a market-wide response at the default event. In cross-section, abstracting from taxes and liquidity, only jump-to-default risk can generate credit spreads that are higher than expected loss rates at the very short end of the yield curve.

Jarrow, Lando and Yu (JLY 2001) discuss the conditions for which no systematic jump-to-default-risk exists. Essentially, their results show that if the following two conditions are satisfied:

(i) Conditional on the state variables driving intensities, default events are independent.

(ii) A large number of bonds are available for trading,

then jump-risk is conditionally diversifiable, and therefore should not command a risk-premium. We note that this does not imply that default events are (unconditionally) independent. Indeed, the default intensities may be correlated (i.e., $d\lambda_i(t) d\lambda_k(t) \neq 0 \; \forall \; i, k \in [1, N]$).

There are at least two different scenarios where we can expect jump-risk to be priced. First, there can be systemic risk in the sense that firms default at the same time (i.e., $d\mathbf{1}_{\{\tau_i \leq t\}} d\mathbf{1}_{\{\tau_k \leq t\}} \neq 0 \; \forall \; i, k \in [1, N]$). Intuitively, if the number of firms $N$ is large enough that a non-negligible part of the economy defaults at the same date, then such a risk would command a risk-premium. However, there is little empirical support for such a notion. (Of course, there is always the concern of a ‘Peso-problem’).

Second, there can be contagion-risk in the sense that the default of one firm can trigger an increase in the risk (i.e., an increase in the intensity) of default of other firms. Mathematically, we can write this as $d\mathbf{1}_{\{\tau_i \leq t\}} d\lambda_k(t) \neq 0 \; \forall \; i, k \in [1, N]$. One example of such a situation is the

\[ = \mu^Q \ dt + \sigma^Q \ dz^Q_t + \Gamma^Q \ dq_t. \tag{3} \]
(N = 2) counterparty-risk model proposed by Jarrow and Yu (JY 2001):

\[ \lambda_i(t) = a_{i1} + a_{i2} \mathbf{1}_{\{ \tau_2 \leq t \}} \]  
\[ \lambda_j(t) = a_{j1} \mathbf{1}_{\{ \tau_1 \leq t \}} + a_{j2}. \]  

While analytic solutions for survival probabilities and bond prices exist for these types of models, they quickly become intractable as \( N \) becomes large, even for the simplest case where the coefficients \( \{ a_{ik} \} \) are deterministic constants.\(^6\)

The most natural economic motivation for the JY model of cross-dependent contagion is that firms have economic ties that render each firm vulnerable to the default of the other. While such mechanism is perhaps appropriate for firms with economic ties such as supplier-producer relations, it would not appear able to explain common jumps in spreads across bonds of various industries, such as what happened after the RJR and Enron credit events. To provide a mechanism for that type of an example, here we model contagion through Bayesian updating of beliefs. We first motivate the dynamics of the model using a simple example driven by a single state variable. Then, we generalize the framework to multiple state variables.

3 A simple model of contagion-risk based on updating of beliefs

Consider \( N \) firms indexed by \( i \in (1, N) \) with default intensities that are equal to \( \lambda_i^H \) if the economy is in state \( H \), or \( \lambda_i^L \) if the economy is in state \( L \). Intuitively, we can think of \( H \) as the risky fundamental state, since we specify \( \lambda_i^H > \lambda_i^L \). Investors do not know whether the economy is in state \( H \) or \( L \), but form a prior \( p^H(t) = \Pr(H | F_t) \), where \( F_t \) represents all information investors have available at date-\( t \). Thus, from the point of view of investors, the \( F_t \)-default intensity is

\[ \overline{\lambda}_i(t) = p^H(t) \lambda_i^H + (1 - p^H(t)) \lambda_i^L, \]  

where the default intensity is defined implicitly through

\[ E_t \left[ \mathbf{1}_{\{ \tau \leq t \}} \right] = \overline{\lambda}_i(t) \mathbf{1}_{\{ \tau > t \}} dt. \]  

Investors continuously update their estimate of \( p^H(t) \) conditional upon whether or not they observe a default event during the interval \( dt \). Here we provide a heuristic derivation for the updating process. In Lemmas 1 and 2 below we provide a more rigorous proof.

Since defaults are triggered by point processes, investors observe, at most, one event per unit time. Define \( \mathbf{d}_{1} \equiv \mathbf{d}_{\{ \tau < t \}} \) as the vector of jump events. Consider first the case

\(^6\)The solutions may be computed using results from Collin-Dufresne, Goldstein and Hugonnier (2003). The cases \( N = 2 \) and \( N = 3 \) are available upon request.
where no default is observed in a period $dt$. Applying Bayes’ rule and keeping terms only to $O(dt)$, we obtain:

$$\Pr[H, d1_i = 0 | \mathcal{F}_t] = \Pr[d1_i = 0 | H, \mathcal{F}_t] \times \Pr[H | \mathcal{F}_t]$$

$$= \left( \prod_{i=1}^{N} e^{-\lambda_i^H \mathbf{1}_{\{\tau_i > t\}} dt} \right) \times p^H(t)$$

$$\overset{O(dt)}{=} p^H(t) \left( 1 - \sum_{i=1}^{N} \lambda_i^H \mathbf{1}_{\{\tau_i > t\}} dt \right).$$

(8)

Analogously, we find

$$\Pr[L, d1_i = 0 | \mathcal{F}_t] \overset{O(dt)}{=} \left( 1 - p^H(t) \right) \left( 1 - \sum_{i=1}^{N} \lambda_i^L \mathbf{1}_{\{\tau_i > t\}} dt \right).$$

(9)

Again using Bayes’ rule, we find that the process for $p^H(t)$, conditional upon no firm defaulting during the interval $(t, t + dt)$ evolves via

$$p^H(t + dt) \big|_{d1_i = 0} = \Pr[H | \mathcal{F}_t, d1_i = 0]$$

$$= \frac{\Pr[H, d1_i = 0 | \mathcal{F}_t]}{\Pr[d1_i = 0 | \mathcal{F}_t]}$$

$$= \frac{\Pr[H, d1_i = 0 | \mathcal{F}_t]}{\Pr[H, d1_i = 0 | \mathcal{F}_t] + \Pr[L, d1_i = 0 | \mathcal{F}_t]}$$

$$= \frac{p^H(t) \left( 1 - \sum_{i=1}^{N} \lambda_i^H \mathbf{1}_{\{\tau_i > t\}} dt \right)}{p^H(t) \left( 1 - \sum_{i=1}^{N} \lambda_i^H \mathbf{1}_{\{\tau_i > t\}} dt \right) + (1 - p^H(t)) \left( 1 - \sum_{i=1}^{N} \lambda_i^L \mathbf{1}_{\{\tau_i > t\}} dt \right)}$$

$$\overset{O(dt)}{=} p^H(t) - p^H(t) \left( 1 - p^H(t) \right) \frac{\sum_{i=1}^{N} (\lambda_i^H - \lambda_i^L) \mathbf{1}_{\{\tau_i > t\}} dt}{\sum_{i=1}^{N} (\lambda_i^H - \lambda_i^L) \mathbf{1}_{\{\tau_i > t\}} dt}.\$$

(10)

Hence, if there are no jumps during the interval $dt$, then the Bayesian updating follows the process:

$$dp^H(t) \big|_{d1_i = 0} = -p^H(t) \left( 1 - p^H(t) \right) \sum_{i=1}^{N} (\lambda_i^H - \lambda_i^L) \mathbf{1}_{\{\tau_i > t\}} dt.\$$

(11)

In contrast, if one firm (e.g., firm $i$) defaults during the interval $dt$, then, applying Bayes rule we obtain:

$$\Pr[H, d1_{\{\tau_i < t\}} = 1 | \mathcal{F}_t] \equiv \Pr[d1_{\{\tau_i < t\}} = 1 | H, \mathcal{F}_t] \times \Pr[H | \mathcal{F}_t]$$

$$= p^H(t) \lambda_i^H dt.\$$

(12)

Similarly,

$$\Pr[L, d1_{\{\tau_i < t\}} = 1 | \mathcal{F}_t] = (1 - p^H(t)) \lambda_i^L dt.\$$

(13)
Hence, it follows that, conditional on firm-$i$ defaulting during the interval $(t, t + dt)$, the process for $p^H(t)$ evolves via

$$
p^H(t + dt)ig|_{d1_{\{\tau < t\}}} = 1 = \Pr \left[ H \left| \mathcal{F}_t \right., \, d1_{\{\tau < t\}} = 1 \right] = \frac{\Pr \left[ H, \, d1_{\{\tau < t\}} = 1 \left| \mathcal{F}_t \right. \right]}{\Pr \left[ d1_{\{\tau < t\}} = 1 \left| \mathcal{F}_t \right. \right]} = \frac{\Pr \left[ H, \, d1_{\{\tau < t\}} = 1 \left| \mathcal{F}_t \right. \right]}{\Pr \left[ H, \, d1_{\{\tau < t\}} = 1 \left| \mathcal{F}_t \right. \right] + \Pr \left[ I, \, d1_{\{\tau < t\}} = 1 \left| \mathcal{F}_t \right. \right]} = \frac{p^H(t) \lambda_i^H}{p^H(t) \lambda_i^H + (1 - p^H(t)) \lambda_i^L} = \frac{p^H(t) \lambda_i^H}{\lambda_i(t)}, \tag{14}
$$

where we have used equation (6) in the last line. Therefore, we can write

$$
dp^H(t)\big|_{d1_{\{\tau < t\}}} = 1 = p^H(t) \left( \frac{\lambda_i^H}{\lambda_i(t)} - 1 \right) d1_{\{\tau < t\}} = p^H(t) \left( 1 - p^H(t) \right) \frac{\lambda_i^H - \lambda_i^L}{\lambda_i(t)} d1_{\{\tau < t\}}. \tag{15}
$$

Combining equations (11) and (15) we obtain the updating process for $p^H(t)$:

$$
dp^H(t) = p^H(t) \left( 1 - p^H(t) \right) \sum_{i=1}^{N} \left( \frac{\lambda_i^H}{\lambda_i(t)} - 1 \right) \left( d1_{\{\tau < t\}} - \lambda_i(t) \right) d1_{\{\tau > t\}} = p^H(t) \sum_{i=1}^{N} \left[ \left( \frac{\lambda_i^H}{\lambda_i(t)} - 1 \right) \left( d1_{\{\tau < t\}} - \lambda_i(t) \right) \right], \tag{16}
$$

$$
\text{(The form of Equation (17) is more useful than the form of Equation (16) when generalizing dynamics to multiple states of nature, as we demonstrate below). Note that this process has many intuitive properties. First, if the prior $p^H$ is either 0 or 1, then there is no updating. That is, in an economy where the agents know for sure the intensity of the firms, then there is no learning to be done. Second, when no default is observed over an interval $dt$, investors revise downward the probability of being in the high-default state (i.e., $p^H(t)$ drifts downward). Conversely, when a default is observed, they revise upward the probability that the economy is in the high default-state (i.e., $p^H(t)$ jumps upward). Finally, note that $p^H(t) \equiv E \left[ H \left| \mathcal{F}_t \right. \right]$ is a martingale in that $E_t \left[ dp^H(t) \right] = 0$, as can be seen from equations (7) and (17), .}
The model specified by equations (6) and (16) is reminiscent of the counterparty risk example of JY given in equations (4)-(5) above in that the intensity of default $\lambda_i(t)$ for firm-$i$ increases when some other firm-$k$ defaults. In contrast to JY, however, contagion is explicitly modeled as a result of the updating of beliefs. Besides providing a second mechanism for generating contagion (which seems to be consistent with the Enron and RJR LBO events), the advantage of this framework is that it remains tractable even when the number of firms $N$ that share in the contagion is large. Indeed, from Bayes’ rule, we find that the survival probability for any firm-$i$ is given by:

$$E[1_{\{\tau_i > T\}} | F_t] = p^H(t) E[1_{\{\tau_i > T\}} | F_t, H] + (1 - p^H(t)) E[1_{\{\tau_i > T\}} | F_t, L]$$

$$= p^H(t) e^{-\lambda^H_i(T-t)} + (1 - p^H(t)) e^{-\lambda^L_i(T-t)}. \quad (18)$$

Note that the survival probabilities are not equal to

$$E \left[ 1_{\{\tau_i > T\}} \left| F_t \right. \right] \neq E \left[ e^{-\int_t^T \lambda_i(s) \, ds} \left| F_t \right. \right] = E \left[ e^{-\int_t^T \left( p^H(s) \lambda^H_i + (1-p^H(s)) \lambda^L_i \right) \, ds} \left| F_t \right. \right]. \quad \quad (19)$$

This result follows from the fact that the intensity of default for firm-$i$ jumps at the same date that the default occurs. That is, our model falls outside of the ‘no-jump’ framework of Duffie, Schroeder and Skiadis (DSS 1996), Lando (1998), and Duffie and Singleton (DS 1999).

Not only does our framework tractably account for a large number of firms sharing in the contagion, but, as we demonstrate below, our framework remains tractable even if (i) the intensities $\{\lambda^H_i(s), \lambda^L_i(s)\}$ follow affine or squared-Gaussian stochastic processes, and (ii) there are more than two ‘risk-states’. In particular, survival probabilities and the prices of typical defaultable claims, such as risky bonds and credit derivatives, can be obtained in closed-form at little additional cost over and above the traditional framework that ignores contagion.

### 3.1 A reduced-form model of contagion risk and changing risk-perception

Consider a reduced form model of default for $N$ firms. We assume that the default time for each firm $i \in 1,\ldots,N$ is modeled by an unpredictable stopping time $\tau_i$ with physical measure intensity $\overline{\lambda}_i(t)$ on some standard filtered probability space $(\Omega, \{ F_t \}_{t \leq T}, \mathcal{F}, P)$. It follows from the martingale characterization of intensity (see, e.g., Theorem 9 of Brémaud (1981)) that the indicator of the default event satisfies

$$E_t \left[ d1_{\{\tau_i \leq t\}} \right] \equiv \overline{\lambda}_i(t) 1_{\{\tau_i > t\}} \, dt. \quad \text{In other words,}$$

The affine restriction on $\lambda^H_i(s), \lambda^L_i(s)$ is only necessary to obtain closed form solutions for prices or survival probabilities. The filtering equation for the prior $\overline{p}^H(t)$ holds for arbitrary (positive integrable) conditional intensity processes.

Unless otherwise indicated we shall denote $E_t[X] := E[X \mid F_t]$. 


the process $M_i(t)$ with dynamics
\[
dM_i(t) \equiv \left( d1_{\{\tau_i \leq t\}} - \overline{\lambda}_i(t)1_{\{\tau_i > t\}} \right) dt
\]
(20)
is a P-martingale (assuming that the intensity process $\overline{\lambda}_i(t)$ is sufficiently well-behaved).

We specify the intensity $\overline{\lambda}_i(t)$ as a function of the state variables $\{p^j(t)\}$:
\[
\overline{\lambda}_i(t) = \sum_{j=1}^{J} \lambda_{ij}(t) p^j(t),
\]
(21)
where, for each firm-i, the $\lambda_{ij}(t)$ are positive integrable $\mathcal{F}$-adapted processes that can be ordered in the sense that\(^9\)
\[
0 < \lambda_{i1}(t) < \lambda_{i2}(t) < \ldots < \lambda_{ij}(t).
\]
(22)

We initialize the values of the “perception of risk” state variables $p^j$ to be positive $(p^j(t_0) > 0)$ and to sum to unity: $\sum_{j=1}^{J} p^j(t_0) = 1$. Further, we specify the dynamics of the $p^j$ to be (compare with equation (17)):
\[
\frac{dp^j(t)}{p^j(t^-)} = \sum_{i=1}^{N} \left[ \left( \frac{\lambda_{ij}(t^-)}{\overline{\lambda}_i(t^-)} - 1 \right) \left( d1_{\{\tau_i \leq t\}} - \overline{\lambda}_i(t)1_{\{\tau_i > t\}} \right) dt \right]
\]
(23)
\[
= \sum_{i=1}^{N} \alpha_{ij}(t^-) dM_i(t).
\]
(24)

While we can interpret the dynamics of the state vector $p^j$ as generating a mechanism for counter-party risk contagion similar in spirit to JY, the most natural interpretation for the $p^j$ is that they are probabilities. In fact, the next two lemmas show that the $p^j(t)$ can be interpreted as the probabilities, conditional on the information $\mathcal{F}(t)$, that the economy is in state-$j$, and hence, that firm-i has a default intensity of $\lambda_{ij}(t)$.

**Lemma 1** The $p^j$ satisfy $\left( 0 < p^j(t) < 1 \right)$ and $\sum_{j=1}^{J} p^j(t) = 1$ a.s. $\forall (t, \omega)$.

**Proof:** See Appendix C. □

For what follows, it is convenient to define $M_{ij}$ as:
\[
dM_{ij}(t) = d \left( p^j(t) 1_{\{\tau_i \leq t\}} \right) - \lambda_{ij}(t) p^j(t) 1_{\{\tau_i > t\}} dt.
\]
(25)

We claim:

\(^9\)We note that this restriction is not essential, but is consistent with a market-wide jump in yield spreads in response to a credit event, which is our focus here.
Lemma 2 $M_{ij}(t)$ is a $P$-martingale.

Proof: See Appendix C. 

Note the similarities in the structure of the dynamics between $dM_i(t)$ in equation (20) and $dM_{ij}(t)$ in equation (25). Indeed, we interpret Lemma 2 to imply that $p^j(t)$ can be interpreted as the conditional probability of being in state $j$ in the sense that, when multiplied by $p^j$, the jump-to-default indicator function for firm $i$ has associated with it an intensity $\lambda_{ij}$. Within this context, equation (24) implies that at the default date of some firm-$k$, the market updates their beliefs so that the probabilities $p^j(t)$ of those states-$j$ with $(\lambda_{kj}(t) > \bar{\lambda}_k(t))$ are revised upward after a default.\footnote{Naturally, the probabilities of those states of nature associated with lower intensities of default are revised downward, since the probabilities must always sum to unity.} Note that this framework captures the idea of contagion in that, if firm-$k$ defaults, then the intensity process for firm-$i$ ($\neq k$) jumps by $\Delta \bar{\lambda}_i(t) = \sum_{j=1}^{J} \lambda_{ij}(\tau^{-}_k) \Delta p^j(t) = \sum_{j=1}^{J} \lambda_{ij}(\tau^{-}_k) \alpha_{ij}(\tau^{-}_k) p^j(t^{-})$, where $\alpha_{ij}(\tau^{-}_k)$ is defined in equation (24).

Note that the dynamics of the 'conditional intensities' $\lambda_{ij}(t)$ can be stochastic processes. For example, a very tractable framework obtains if the $\lambda_{ij}(t)$ are specified as

$$\lambda_{ij}(t) = a_{ij} + (b_{ij})^\top X(t),$$

where the $a_{ij}$ are positive constants, $X(t)$ is the state vector specified to have square-root affine dynamics (Duffie and Kan (1996)), and the $b_{ij}$ are constant positive vectors. Such a specification guarantees that the $\{\lambda_{ij}\}$ maintain their ordering as in equation (22) so long as $(a_{i1}, b_{i1}) < (a_{i2}, b_{i2}) < \ldots < (a_{ij}, b_{ij})$. With this particular application in mind, we make the following simplifying assumption:\footnote{This assumption can be relaxed. For example, all results go through if instead of assumption (A1) we assume that the expectation $E_t \left[ e^{-\int_t^T \lambda_{ij}(s) \, ds} \right]$ does not experience jumps at default times, i.e., $\Delta E_t(\tau^{-}_i) = 0 \ \forall k, i = 1, \ldots, N$ and $j = 1, \ldots, J$. (Note that this is automatically satisfied under A1.) In order to focus on the more important issues of this paper, A1 suffices.}

Assumption (A1) The 'conditional intensities' $\lambda_{ij}(t)$ $\forall i = 1, \ldots, N$ and $j = 1, \ldots, J$ are progressively measurable with respect to the natural filtration $\mathcal{F}^z(t) \subset \mathcal{F}(t)$ generated by a vector of $(P, \mathcal{F})$-Brownian motions $z(t) = [z_1, \ldots, z_d]$.

With this assumption we obtain the following result:

Proposition 1 The historical probability of survival for counterparty-$i$ up to maturity $T$ is given by:

$$E_t \left[ \mathbf{1}_{\{\tau_i > T\}} \right] = \sum_{j=1}^{J} p^j(t) E_t \left[ e^{-\int_t^T \lambda_{ij}(s) \, ds} \right] \mathbf{1}_{\{\tau_i > T\}}.$$
In particular, if the \( \lambda_{ij}(t) \) follow an affine process, then the probability of bankruptcy is a weighted average of exponential affine functions.

**Proof:** See Appendix C. \( \square \)

The intuition for this result is the following. Using the insight of Lemmas 1 and 2 that the \( \{p^j\} \) can be interpreted as conditional probabilities, we can apply Bayes' rule to write the left hand side of equation (27) as

\[
E_t\left[1_{\{\tau_i > \tau\}}\right] = \sum_{j=1}^{J} p^j(t) E\left[1_{\{\tau_i > \tau\}} \mid \mathcal{F}_t, \text{state} = j\right] 1_{\{\tau_i > t\}}. \tag{28}
\]

Now, *conditional* upon being in state-\( j \), assumption A1 implies that we are in a doubly-stochastic framework, in turn implying from DSS and DS that the right hand side of equation (28) can be written as the right hand side of equation (27). The proof in the appendix verifies that this intuitive result is in fact correct.\(^{12}\)

We emphasize, however, that at date-\( t \) *we do not actually know* which state-\( j \) the economy is in. Thus, when we average over our prior beliefs, the no-jump condition is not satisfied. In particular, note that the probability \( p^i \), and hence the intensity \( \lambda_i \), experiences a jump at the default time \( \tau_i \) in this framework. Because of this feature, the standard approach of discounting at the risk-adjusted rate does not directly apply in this framework, in contrast to most reduced-form model specifications. This example illustrates the importance of the ‘no-jump’ assumption discussed in DSS and DS. Indeed, in contrast to standard reduced-form models, Proposition 1 implies that:

**Corollary 1** The probability of no default prior to \( T \) defined by \( E_t[1_{\{\tau_i > \tau\}}] \) is *not* equal to \( E_t[e^{-\int_0^t \lambda_i(s) ds} 1_{\{\tau_i > t\}}] \).

Although the model presented above is a reduced-form model, it is easily reconciled with the structural framework following the intuition of Duffie and Lando (DL 2000). DL show that, in contrast to a standard (i.e., Merton (1974)) structural model where the default time is typically predictable, if the underlying firm value is imperfectly observed by investors, then, from the point of view of investors, the default time becomes inaccessible, i.e., default arrives as a surprise event. Our framework can be interpreted as an extension of DL’s model to multiple firms that share a common (but unknown) accounting accuracy. If the economy comprises

\[^{12}\text{We note that assumption (A1) may be relaxed to allow for jumps in the conditional intensities following the general approach of Collin-Dufresne, Goldstein and Hugonnier (2003).}\]
several firms whose asset value is imperfectly known to investors, but if the (accounting) signal quality is correlated across firms, then when one firm defaults this triggers an updating of beliefs by investors about the shared information quality and hence will affect the perceived likelihood of default of all the firms. We show in appendix A that an extension of DL’s model to multiple firm gives rise to a special case of our general reduced-form framework presented above.

Up to this point, the state variable dynamics have been specified under the historical measure. In the following section we address the issue of pricing defaultable securities in the presence of contagion risk and systematic jump risk. We do this by introducing a pricing kernel, which allows us to identify risk-neutral dynamics.

3.2 Pricing defaultable securities in the presence of systematic Jump and Contagion-risk.

If the number of firms $N$ that are affected by the default of another firm is sufficiently large so as to be non-diversifiable, then such default risk will be priced. For example, in appendix B we investigate a general equilibrium model where, even though each individual firm is modeled as ‘small’ in the sense that it does not affect current aggregate production, the information conveyed by the default itself is sufficient for it to generate a risk-premium. Motivated by this example, we consider the pricing of defaultable securities based on a pricing kernel whose dynamics are sufficiently flexible to generate jump and contagion risk-premia. In particular, we specify the pricing kernel as:

$$\Lambda(t) = e^{-\int_0^tr_sds} \xi(t) \bar{\xi}(t),$$

where we have defined

$$\bar{\xi}(t) = \sum_{j=1}^J \bar{\xi}^j(t).$$

The dynamics of the state variables $\xi^c(t)$ and $\xi^j(t)$ are specified as

$$\frac{d\xi^c(t)}{\xi^c(t)} = -\theta^T d\zeta(t),$$

$$\frac{d\xi^j(t)}{\xi^j(t)} = \sum_{i=1}^N \left( \gamma_{ij}^1 d\mathbf{1}_{\{\gamma_i \leq t\}} - \lambda_{ij}^1 (t) \tilde{\gamma}_{ij} \mathbf{1}_{\{\gamma_i > t\}} dt \right),$$

with initial conditions $\xi^c(0) = \xi^j(0) = 1 \forall j$. For simplicity, we make the following technical assumptions about the various market prices of risk (these assumptions are useful for applying
standard results on changes of measure, such as Girsanov’s theorem).

**Assumption (A2)** The vector of market prices of Brownian motion risk \( \theta_t \) is progressively measurable with respect to \( \mathcal{F}^z(t) \) and satisfies the Novikov condition. Further, the market prices of jump risk \( \gamma_{ij} \) are i.i.d. \( \mathcal{F}(\tau) \)-measurable random variables with density \( f_{ij}(\gamma) \), mean \( \hat{\gamma}_{ij} = \int \gamma_{ij} f_{ij}(\gamma) \, d\gamma_{ij} \) and finite variance (and which have the same support).

The structure of the pricing kernel is intuitive and analytically convenient: \(^{13}\) \( \xi^c \) is the component of the state price density which captures diffusion risk, and each \( \xi^j \) captures the ‘cost’ of event risk conditional on being in state \( j \). Let us define \( \xi(t) \equiv e^{c_0^t r(s) \, ds} \Lambda(t) = \xi^c(t) \xi(t) \). From Itô’s lemma, its dynamics are given by:

\[
\frac{d\xi(t)}{\xi(t^{-})} = -\theta(t)^\top dz(t) + \sum_{i=1}^{N} \left( \Gamma_{\xi,i} d1_{\{\tau_i \leq t\}} - \lambda_i(t) \hat{\Gamma}_{\xi,i} 1_{\{\tau_i > t\}} dt \right),
\]

where \( \Gamma_{\xi,i} \) specifies the size of the jump in \( \xi(t) \), and is defined via:

\[
1 + \Gamma_{\xi,i} = \left( \frac{1}{\lambda_i(t^{-}) \xi(t^{-})} \right) \sum_{j=1}^{J} (1 + \gamma_{ij}) \lambda_{ij}(t-) p^j(t-) \xi^j(t^{-}).
\]

Note that \( \Gamma_{\xi,i} \) is basically a weighted average of the random variables \( \{\gamma_{ij}\} \). In fact, \( \Gamma_{\xi,i} = \gamma_{ij} \) only if the state is known with certainty (i.e., \( p^j = 1 \) for some \( j \)) or if all states are identical.

Whereas equation (32) implies that the individual \( \{\xi^j\} \) are not \((P, \mathcal{F})\) martingales, equation (33) shows that \( \xi \) is a positive \((P, \mathcal{F})\)-martingale. It thus defines a martingale measure \( Q \) equivalent to \( P \) through the conditional likelihood ratio \( \xi(t) \equiv E^P_1 \left[ \frac{dQ}{dP} \right] \) and under which discounted asset prices are martingales.\(^{14}\)

The parameter \( \hat{\Gamma}_{\xi,k} \) represents the market price of jump risk for firm \( k \). Indeed, if \( \hat{\Gamma}_{\xi,k} = 0 \), that is, if the pricing kernel is not affected by individual firm \( k \)’s default on average, then the risk-neutral (instantaneous) probability of default is unchanged, as implied in the following lemma:

**Lemma 3** The risk-neutral default intensity for firm \( k \) is given by \( \lambda^Q_k(t) = (1 + \hat{\Gamma}_{\xi,k}) \lambda_k(t) \).

---

\(^{13}\)We note that the pricing kernel dynamics specified in equations (29)-(32) can be obtained endogenously within a general equilibrium exchange economy similar to Lucas (1978). Indeed, assume there exists a representative agent with constant relative risk aversion preferences and specify the dynamics of aggregate output \( \delta \) (which equals consumption in equilibrium) such that \( \delta(t)^{-} = \Lambda(t) \), where \( \gamma \) is the CRRA coefficient.

\(^{14}\)It is well-known that if markets are incomplete then there can be multiple equivalent measures. Here we simply assume that \( \Lambda \) is an empirical pricing kernel which prices existing assets. See, e.g., Cochrane (2001) for a discussion.
Proof:

\[ \lambda^Q_k(t) \mathbf{1}_{\{\tau_k > t\}} \, dt \equiv \mathbb{E}^Q_t[d\mathbf{1}_{\{\tau_k \leq t\}}] \\
= \mathbb{E}^P_t \left[ \frac{d }{d \xi(t) \mathbf{1}_{\{\tau_k \leq t\}}} \right] \\
= \mathbb{E}^P_t \left[ d\mathbf{1}_{\{\tau_k \leq t\}} + \mathbf{1}_{\{\tau_k \leq t\}} \mathbb{E}^P_t \left[ \frac{d \xi(t) \mathbf{1}_{\{\tau_k \leq t\}}}{\xi(t)} \Delta \xi(t) \mathbf{1}_{\{\tau_k \leq t\}} \right] \\
= \left( 1 + \hat{\Gamma}_{\xi,k}(t) \right) \lambda_k(t) \mathbf{1}_{\{\tau_k > t\}} \, dt, \quad (35) \]

where we use the fact that \( \lambda_k(t) \) is the \( P \)-intensity of \( \tau_k \), and that \( \xi \) is a \( P \)-martingale. \( \square \)

Lemma 3 emphasizes that, because each firm’s individual default event is priced (i.e. affects the pricing kernel via equation (32)), the risk-neutral default intensity is different than the physical measure intensity. Assuming that the average jump in the pricing kernel on a default event date is positive, i.e., \( \hat{\Gamma}_{\xi,k} > 0 \), the risk-neutral default intensity is higher than the physical measure intensity \( \lambda^Q > \lambda^P \). As noted by Jarrow, Lando and Yu (2001) and Driessen (2002), the possibility that \( \lambda^Q > 1 \) provides a potential explanation for why credit spreads are higher than observed expected loss rates at the short end of the credit spread term structure.

In contrast to these papers, our model identifies two sources of risk-premia associated with credit spread jumps for a given firm-\( i \), namely: those associated directly with credit-events of firm-\( i \), and those associated with a contagious response by firm-\( i \) due to the credit event of some other firm-\( k \). To prove this, we first show that the conditional probability of being in state \( j \) is risk-adjusted when changing measures. Indeed, let us define

\[ q^j(t) \equiv p^j(t) \frac{\xi^j(t)}{\xi(t)} \quad j = 1, \ldots, J, \quad (36) \]

and

\[ \lambda^Q_{ij}(t) \equiv \lambda_{ij}(t) \left( 1 + \hat{\gamma}_{ij} \right). \quad (37) \]

From their definition, the \( \{q^j(t)\} \) are positive. Further, from equation (30), the \( \{q^j(t)\} \) sum to unity. The following lemma shows that, analogous to equation (25) and Lemma 2, \( q^j(t) \) can be interpreted as the risk-neutral probability of being in state \( j \), whereas \( \lambda^Q_{ij}(t) \) can be interpreted as the risk-neutral intensity for firm-\( i \) conditional on being in state-\( j \). Indeed, define the process \( L_{ij} \) by

\[ dL_{ij}(t) = d \left( q^j(t) \mathbf{1}_{\{\tau_i \leq t\}} \right) - \lambda^Q_{ij}(t) q^j(t) \mathbf{1}_{\{\tau_i > t\}} \, dt, \quad (38) \]

with initial condition \( L_{ij}(0) = 0 \). We claim:
**Lemma 4** \( L_{ij}(t) \) is a \( Q \)-martingale.

**Proof:** See Appendix C. \( \Box \)

Equations (36) and (37) allow us to write the risk-neutral probability of survival in an intuitive form:

**Proposition 2** In the presence of priced credit event risk and contagion risk, the risk-neutral survival probability of firm \( k \) is given by:

\[
E^Q_t \left[ \mathbf{1}_{\{\tau_{k} > \tau\}} \right] = \mathbf{1}_{\{\tau_{k} > t\}} \sum_{j=1}^{J} q^j(t) E^Q_t \left[ e^{- \int_t^T \lambda^Q_{kj}(s) ds} \right],
\]

where, under \( Q \), the process \( z^Q(t) = z^P(t) + \int_0^t \theta_s ds \) is a standard Brownian motion.

**Proof:** See Appendix C. \( \Box \)

We note that the right-hand side expectation in Proposition 2 can be readily calculated. In particular, if the choice of the Brownian risk-premia \( \theta_t \) is essentially affine in the sense that the dynamics of the state vector \( X \) remains affine under the \( Q \)-measure (Duffee (2002)) then the risk-neutral expectation will be equal to a weighted average of exponential affine conditional survival probabilities each corresponding to the survival probability of a firm with default intensity \( \lambda^Q_{kj}(t) \).

A generalization of Proposition 2 provides a pricing formula for a defaultable claim. To focus on the important issues, here we assume:

**Assumption (A3)** The interest rate process \( r_t \) is positive and progressively measurable with respect to \( \mathcal{F}^z(t) \).

This additional assumption implies that the interest rate is not affected by the jumps in individual firm defaults.\(^{15}\) In that case we find:

**Proposition 3** In the presence of priced credit event risk and contagion risk, the price of a risky defaultable claim issued by firm \( k \) which pays \( \$X \) conditional on no-default, and zero otherwise, is given by:

\[
B_k(t) \equiv E^Q_t \left[ e^{- \int_t^T r(s) ds} \mathbf{1}_{\{\tau_{k} > \tau\}} \right] \]

\[
= \mathbf{1}_{\{\tau_{k} > t\}} \sum_{j=1}^{J} q^j(t) E^Q_t \left[ e^{- \int_t^T (r(s)+\lambda^Q_{kj}(s)) ds} \mathbf{1}_{\{\tau_{k} > \tau\}} \right].
\]

\(^{15}\)This assumption can be relaxed using the results in Collin-Dufresne, Goldstein and Hugonnier (2003). Such a generalization may prove important, since empirically we find flights-to-quality associated with credit events.
**Proof:** Similar to that of Proposition 2 and thus omitted. \(\square\)

Proposition 3 allows us to derive the implied risk premium on a claim subject to default risk and contagion risk. Indeed, it is convenient to define \(B_{\omega j}(t) \equiv E_t^Q \left[ e^{-\int_0^t (r(s) + \lambda^{Q}_{\omega j}(s)) \, ds} \right] \).

Then, applying Itô’s lemma to equation (40) and using equations (36), (23), (32) and (33), we obtain:

\[
\frac{d B_{\omega j} (t)}{B_{\omega j}(t^-)} = \mu_{\omega j}(t) \, dt + \sigma_{\omega j}(t) \, dz_t - \sum_{i \neq k} \Gamma_{k,i} \, \bar{X}_t(t \wedge \gamma_k) - \Gamma_k \, d 1_{\{t \leq \gamma_k\}},
\]

where

\[
\Gamma_k = 1
\]

\[
\Gamma_{k,i} = 1 - \left( \sum_{j=1}^{J} \frac{q^j(t^-) \lambda_j(t^-) (1 + \gamma_{ij}) B_{\omega j}(t^-)}{B_k(t^-) \bar{X}_t(t)} \right).
\]

The relation \(\Gamma_k = 1\) captures the fact that at the jump date the value of the risky bond drops to zero. The diffusion and drift terms in equation (41) can also be computed for specific (e.g., affine) dynamics of \(X\), but for the purpose of identifying the contribution of contagion risk for credit risk-premia, the above representation suffices. Indeed, we find:

**Proposition 4** For dates \((t < \tau_k)\), the instantaneous premium for credit risk is given by:

\[
\frac{1}{dt} \mathbf{E}_t^P \left[ \frac{dB_{\omega j}(t)}{B_k(t)} \right] - r_t = \sigma_{\omega j}(t) \bar{\theta}(t) + \sum_{i \neq k} \bar{X}_t(t \wedge \gamma_k) \Gamma_k \bar{1}_{\{t \leq t \wedge \gamma_k\}} + \bar{X}_t(t) \Gamma_k \Gamma_{\xi,k}.
\]

**Proof:** By definition of the pricing kernel we have

\[
\mathbb{E}_t^P \left[ \frac{d \left( \xi(t) B_{\omega j}(t) \right)}{\xi(t^-) B_k(t^-)} \right] = r(t) \, dt.
\]

Applying Itô’s lemma and using equations (33) and (41) we find:

\[
\mathbb{E}_t^P \left[ \frac{d \left( \xi(t^-) B_{\omega j}(t^-) \right)}{\xi(t^-) B_k(t^-)} \right] = \mathbb{E}_t^P \left[ \frac{d \xi(t^-)}{\xi(t^-)} + \frac{dB_{\omega j}(t^-)}{B_k(t^-)} + \frac{\Delta B_{\omega j}(t^-) \Delta \xi(t^-)}{B_k(t^-) \xi(t^-)} \right]
\]

\[
= \mathbb{E}_t^P \left[ \frac{dB_{\omega j}(t^-)}{B_k(t^-)} - \sigma_{\omega j}(t) \bar{\theta}(t) \, dt - \sum_{i=1}^{N} \Gamma_{\xi,\omega j} \Gamma_{k,i} \bar{1}_{\{t \leq \gamma_k\}} - \Gamma_{\xi,\omega j} \Gamma_k \bar{1}_{\{t \leq \gamma_k\}} \right].
\]

The result then follows from the fact that \(\bar{X}_t(t)\) is the \(P\)-intensity of \(\tau_i\).

\(\square\)

Proposition 4 implies that in the presence of systematic jump risk and contagion there are two additional sources of risk-premia that can affect the instantaneous credit spread over and
above continuous co-variation risk. First, if firm $k$’s default event affects the pricing kernel (i.e., if $\hat{\Gamma}_{\xi,k} = 0$), then the expected common jump in the firm’s price and the pricing kernel will contribute to the spread $(\widehat{y}_{\beta}(t) \Gamma_{\beta} \hat{\Gamma}_{\xi,k})$. Second, if there is contagion in the sense that firm-$i$’s default affects the intensity of firm-$k$’s default ($\Gamma_{k,i} = 0$), and if firm $i$’s default event is systematic ($\hat{\Gamma}_{\xi,i} = 0$), then the sum of all these interaction will also contribute to the spread of firm $k$ through the term $\sum_{i \neq k} \hat{x}_{i}(t) \mathbb{E}^Q [\Gamma_{k,i} \Gamma_{\xi,i} \mathbb{1}_{\{\gamma_i > t\}}]$.

### 3.3 Estimating jump premia when the corporate bond market is (instantaneously) mean-variance efficient

One disadvantage of exogenously specifying the pricing kernel is that we do not obtain an endogenous estimate for the size of the pricing kernel jump at default events. Intuitively, if a jump-to-default is priced because the contagious response makes it non-diversifiable, then the size of the jump of the pricing kernel $\hat{\Gamma}_{\xi,k}$ should be intimately linked to the size of the contagious jump $\hat{\Gamma}_{i,k}$. Indeed, the implications of JLY stated on page six suggest that $\hat{\Gamma}_{\xi,k} \rightarrow 0$ as $\hat{\Gamma}_{i,k} \rightarrow 0$ for sufficiently large $N$. To illustrate this link we focus on a special case where the aggregate corporate bond market return is instantaneously mean-variance efficient, implying that it can be used as a proxy for the pricing kernel.

In order to focus on the relevant issue here, we assume that the magnitude of contagion jumps are constant and symmetric across all firms $\Gamma_{\xi,k} = \Gamma_{\text{cont}}$, as are all jumps-to-default $\Gamma_{k} = \Gamma_{\text{de,j}}$. Further, we assume that all bonds have the same price initially: $B_k(t) = B$, with the same volatility $\sigma_k = \sigma$. We emphasize that these simplifying assumptions do not generate the results below. However, these assumptions do restrict the applicability of our conclusions to only the ‘typical’ corporate bond in the market. In particular, they need not apply to the largest firms in the economy.

Under these assumptions, the current value of the aggregate corporate bond market is

$$M \equiv \sum_{k=1}^{N} B_k(t) = NB$$

Furthermore, from equation (41) we have

$$dM \equiv \sum_{k=1}^{N} dB_k(t)$$

\[16\text{The condition } \hat{\Gamma}_{i,k} \rightarrow 0 \text{ implies that the jump-to-default risk is diversifiable, and the ‘sufficiently large } N \text{’ condition guarantees that any one firm is a negligible part of the economy.}\]
\[ dM \quad = \quad \frac{1}{N} \sum_{k=1}^{N} \left[ \sigma \, dz - \Gamma_{cont} \sum_{i \neq k}^{N} \, d1_{\{\tau_i < t\}} - \Gamma_{def} \, d1_{\{\tau_k < t\}} \right] \]

Combining equations (47)-(48) we find

\[ \frac{dM}{M} \quad = \quad \frac{1}{N} \sum_{k=1}^{N} \left[ \sigma \, dz - \Gamma_{cont} \sum_{i \neq k}^{N} \, d1_{\{\tau_i < t\}} - \Gamma_{def} \, d1_{\{\tau_k < t\}} \right] \]

\[ = \quad \sigma \, dz - \frac{1}{N} \left( (N-1) \Gamma_{cont} + \Gamma_{def} \right) \sum_{i=1}^{N} d1_{\{\tau_i < t\}}, \quad (49) \]

where we have used the identity \( \sum_{k=1}^{N} \sum_{i \neq k}^{N} = \sum_{i=1}^{N} \sum_{k \neq i}^{N} \).

The assumption that the corporate bond market is mean-variance efficient for pricing corporate debt implies that, for some constant \( \beta \) we can write expected excess returns as

\[ \frac{1}{dt} \mathbb{E}^p \left[ \frac{dB_j(t)}{B_j(t)} \right] - r(t) \quad = \quad \frac{1}{dt} \mathbb{E}^p \left[ \beta \, \frac{dM(t)}{M(t)} \frac{dB_j(t)}{B_j(t)} \right] \quad = \quad \theta_{def} + \theta_{cont} + \theta_{def}, \quad (50) \]

where the diffusion, contagion, and default components are

\[ \theta_{def} \quad = \quad \beta \sigma^2 \quad (51) \]

\[ \theta_{cont} \quad = \quad \beta \frac{\lambda}{N} \left( (N-1) \hat{\Gamma}_{cont} + \hat{\Gamma}_{def} \right) (N-1) \hat{\Gamma}_{cont} \quad (52) \]

\[ \theta_{cont} \quad = \quad \beta \frac{\lambda}{N} \left( (N-1) \hat{\Gamma}_{cont} + \hat{\Gamma}_{def} \right) \hat{\Gamma}_{def}. \quad (53) \]

As predicted above, for sufficiently large \( N \), the jump-to-default risk premia is linear in the size of the contagion jump \( \hat{\Gamma}_{cont}^\lambda \):\(^{17} \)

\[ \theta_{def} \bigg|_{N \to \infty} \quad \sim \quad \beta \lambda \hat{\Gamma}_{cont}^\lambda \hat{\Gamma}_{def}, \quad (54) \]

implying that jump-to-default risk is priced only because of contagion-risk. Furthermore, note that, while the jump-to-default premium is independent of the number \( N \) of firms that share in the contagion, the contagion premium increases linearly with \( N \):

\[ \theta_{cont} \bigg|_{N \to \infty} \quad \sim \quad \beta \lambda N \hat{\Gamma}_{cont}^2. \quad (55) \]

Combining these last two results, we find that the ratio of the default risk premia to the contagion risk premia is

\[ \frac{\theta_{def}}{\theta_{cont}} \bigg|_{N \to \infty} \quad \sim \quad \frac{\hat{\Gamma}_{def}}{N \hat{\Gamma}_{cont}}, \quad (56) \]

\(^{17}\)Indeed, from equation (49), we see that under the assumption of mean-variance efficiency \( \hat{\Gamma}_{\xi t} = \hat{\Gamma}_{cont} \).
suggesting that contagion risk might be substantially larger than jump-to-default risk. In fact, below we empirically estimate the ratio \( \frac{\Gamma_{de}}{\Gamma_{cont}} \) to be about twenty. Hence, with \( 10^3 \) firms, we estimate contagion risk premia to be approximately 50-times larger than jump-to-default risk premia. Admittedly, because actual jump-to-default events are so rare, our empirical procedure uses large credit spread jumps rather than actual jumps-to-default. Therefore, one might argue that our empirically obtained ratio \( \frac{\Gamma_{jump}}{\Gamma_{cont}} \) is not a good proxy for \( \frac{\Gamma_{de}}{\Gamma_{cont}} \). Therefore, before introducing our empirical findings, here we provide a theoretical argument for why we feel the jump-to-default risk premium for the typical firm is small.

In order to obtain a theoretical estimate, we first note that the parameter \( \beta \) is not a free parameter, but rather can be determined from the expected return of the market portfolio. Indeed, by pre-multiplying equation (50) by \( \frac{1}{N} \sum_{k=1}^{N} \) we find

\[
\frac{1}{dt} \mathbb{E}^{P} \left[ \frac{dM(t)}{M(t)} \right] - r(t) = \frac{1}{dt} \mathbb{E}^{P} \left[ \beta \left( \frac{dM(t)}{M(t)} \right)^2 \right]
\]

\[
= \beta \left[ \sigma^2 + \frac{\lambda}{N} \left( (N-1) \Gamma_{cont} + \Gamma_{diff} \right)^2 \right].
\] (57)

Since all bonds are initially equivalent, all bonds, and hence the bond market portfolio, have the same expected return. Thus, defining \( \mu \equiv \frac{1}{dt} \mathbb{E}^{P} \left[ \frac{dM(t)}{M(t)} \right] = \frac{1}{dt} \mathbb{E}^{P} \left[ dB_{i}(t) \right] \), we find

\[
\beta = \frac{\mu - r}{\sigma^2 + \frac{\lambda}{N} \left( (N-1) \Gamma_{cont} + \Gamma_{diff} \right)^2}.
\] (58)

That is, \( \beta \) is the ratio of the excess return to the variance of the bond market portfolio.

Using weekly historical data from 1962-2003, we estimate annual excess returns for investment-grade corporate debt to be 0.03, and the volatility to be 0.08. We note that most estimates of volatility are lower because they include returns of callable corporate debt, which as emphasized by Duffee (98) will create a large ‘spurious’ negative correlation between Treasury rates and credit spreads. Although Duffee (98) finds there still is some negative relationship between Treasury rates and credit spreads of non-callable debt, it is not too large. Hence, we take the volatility of medium-term Treasuries as a better proxy than the volatility of observed aggregate corporate bonds themselves. Finally, we estimate the Sharpe ratio to be approximately 0.3. Note that raising the Sharpe ratio estimate will lower the implied jump-to-default risk premium since, for a given excess return, raising the Sharpe ratio implies a lower total volatility. Note that total volatility is a combination of diffusion volatility and jump-induced volatility. Thus, since we have estimated the diffusion volatility to be 0.08, raising the Sharpe ratio only serves to decrease the size of the jump-to-default parameter \( \Gamma_{de} \). Finally, note that
\[ \Gamma_{def} \] has an upper bound of one due to limited liability. Historically, recovery rates suggest that a better estimate for \( \Gamma_{def} \) might be closer to 0.5. Here, we use larger estimates to obtain an upper bound.

In Table (1), we estimate the implied estimates for the three risk premia, and the ratio \( \frac{\lambda^Q}{\lambda^P} = 1 + \frac{\beta}{N} [(N - 1) \Gamma_{cont} + \Gamma_{def}] \) for several cases of input parameters. We see that in all cases, the jump-to-default risk premium is small in absolute terms and small relative to the contagion-risk premium.

The first scenario, our base-case, is our best estimates for the parameters. Note that we estimate the probability of an investment-grade firm jumping to default to be \( 10^{-4} \), implying that, for \( 10^3 \) investment grade firms, such jumps-to-default are expected to occur about once every ten years. Note that this number is much smaller than the one-year default probabilities for investment-grade debt. As we discuss in greater detail below, we are aware of only four companies since the Great Depression that have defaulted on public debt carrying an investment-grade status from Moody’s. As such, we feel that the one-year default rate is a significant over-exaggeration of the jump-to-default intensity, as most investment grade firms are downgraded to non-investment grade before defaulting (recall that in a reduced-form model the intensity truly is the probability of a jump to default). Still, as shown in scenario-3, even when we set \( \lambda = 10^{-2} \), we still only get jump-to-default risk premium of 3.5 basis points per year.

We conclude from Table 1 that an upper bound for the risk premium on jump-to-default risk is a few basis points per year. However, the risk premium for contagion risk can be significantly larger. We emphasize, however, that our calibration is silent about the jump-to-default risk premium for the largest and safest firms. Intuitively, we suspect that for these

<table>
<thead>
<tr>
<th>Input Parameters</th>
<th>Implied Parameters</th>
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<tr>
<td>Scenario</td>
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</tr>
<tr>
<td>------------------</td>
<td>-------------------</td>
</tr>
<tr>
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<td>0.08</td>
</tr>
<tr>
<td>2</td>
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<td>6</td>
<td>0.07</td>
</tr>
<tr>
<td>7</td>
<td>0.07</td>
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Table 1: Implied Jump-to-default risk premia for select inputs when corporate bond market is mean-variance efficient.
firms the default risk premium is substantially larger.

3.4 Special Case: ‘Liquidity’-Induced Contagion

There seems to be evidence that credit spreads can vary considerably without a perceptible change in default probabilities. (See, for example, Duffie (2003)). One explanation for this apparent time-varying risk premium is a time-varying liquidity premium. A recent example includes the demise of LTCM, where the Russian default caused significant damage to a major player in the corporate bond market. In turn, LTCM was forced to unwind many other positions in corporate bonds, in turn creating a market-wide liquidity crunch. In this section, we demonstrate that our framework can capture significant changes in credit spreads without any change in default probabilities.

Consider the extreme case where \( \lambda_{ij} = \lambda_i \forall (i, j) \). Note from equations (21) and (23) that

\[
\bar{\lambda}_i(t) = \lambda_i \sum_{j=1}^{J} p_j(t) = \lambda_i
\]

(59)

\[
\frac{dp_j(t)}{p(t^-)} = 0.
\]

(60)

Hence, in this framework the \( \{p_j(t)\} \) are constants, equal to their initially specified values. Further, the intensity of default \( \bar{\lambda}_i(t) = \lambda_i \) is constant over time. Yet, as we demonstrate below, jumps-to-default still cause a contagious response in spite of the constant intensity.

Note that in this special case, the individual \( \{\xi^j(t)\} \) are each martingales, as can be seen from equation (32). More importantly, at the jump-event of firm-i, we find that the individual \( \xi^j(t) \) and the \( \bar{\xi} \) jump according to:

\[
d\xi^j(t)\bigg|_{t \in (t^+, t^+ + dt)} = \xi^j(t) \gamma_{ij}
\]

(61)

\[
d\bar{\xi}(t)\bigg|_{t \in (t^+, t^+ + dt)} = \sum_{j=1}^{J} p_j(t) \xi^j(t) \gamma_{ij}.
\]

(62)

Hence, if we specify the \( \gamma_{ij} \) to be increasing in \( j \), we see from equation (36) that those \( q^j \) for higher values of \( j \) will jump upward at default events. Further, from equation (37), these states-\( j \) are associated with higher risk-neutral default intensities. Hence, even though the actual intensity of default for some firm-\( k \) is unaffected by the default of firm-i, its bond price given in equation (40) will experience a jump at these dates. Of course, strictly speaking, if the state are indistinguishable then our framework is indistinguishable from a Cox process framework and jumps should be conditionally diversifiable in the sense of Jarrow Lando Yu.
(2002). The point is to note that if the states are very similar under the \( P \)-measure (i.e., \( \lambda_{ij} \approx \lambda_i \forall (i, j) \)) then updating will barely affect the \( P \)-measure default intensity. And yet, the updating mechanism could trigger large variation in risk-neutral default intensities and hence prices, if the various states command different risk-premia.\(^{18}\) We illustrate this numerically below.

### 3.5 A numerical illustration

To illustrate the implications of the framework, we investigate the following example. Consider a portfolio of 100 risky 30-year zero-coupon bonds. Suppose there are two possible states of nature (i.e., \( J = 2 \)) corresponding to a high intensity of default (\( \lambda^H \)) and low intensity of default (\( \lambda^L \)). For simplicity, we make the following assumptions:

- Both (\( \lambda^H = 10^{-3} \)) and (\( \lambda^L = 10^{-4} \)) are constants.
- The risk free rate (\( r = 0.06 \)) is constant.
- There are a total of \( N = 1000 \) firms that are linked by contagion risk (i.e., they share a common intensity, either \( \lambda^H \) or \( \lambda^L \), which is unknown). This number is constant over time.
- The initial estimate that the economy is in the high-intensity state is \( p^H(0) = 0.5 \).
- The recovery rate is zero in the event of default.
- The market price of jump risk is constant \( \gamma^H = \gamma^L = 0.25 \).

The size of the intensity is consistent with estimates (presented below) of the intensity of a large credit event for an investment grade firm. The assumption that the number of firms that share in the contagion is constant over time can be justified by assuming that at default dates either new firms enter the economy, or that the defaulted firm is reorganized. This assumption can easily be relaxed, but it avoids the mechanical impact of updating on a shrinking cross-section of firms as some firms default. The market price of risk is quite high, but as we demonstrate below, a much larger number would be necessary to explain the size of credit spreads using systematic jump risk premia alone.

\(^{18}\) Of course, economically the question is why such similar states in terms of physical measure default intensity should carry different risk-premia. One possibility is that, if enough firms share in the contagion, then a very small difference in each firm’s likelihood of default may be lot riskier from an aggregate point of view.
Continuous updating of prior if no default. In particular, as is apparent from the graph, the distribution is similar to a mean-preserving spread in that it takes weight from the center of the distribution when does the competition-free model. In general, the effect of the competition model distribution then does more weight into the tails of the graphs 2 demonstrate that the competition-free model puts more weight into the tails of the graphs. For this example we find $\approx 0.0003$. 

$\frac{dp_H}{dt} = p_H(t^{-1}; p) - \frac{1}{N} \sum_{i=1}^{N} \alpha \frac{(p_H(t^{-1}; p) - 1)}{\gamma} dp$
The model puts a much higher weight on multiple borrowers defaulting (similar to the correlations with contagion).

Figure 2: Ten-year probability of bankruptcy assuming constant default intensity vs. model with contagion.
Figure 3: Simulated path of an economy with contagion. The left hand graph shows the path of the p and q-measure conditional probability of the high-intensity state. The middle graph shows the time series of the P and Q-measure 30-year survival probabilities for the average bond in the portfolio. We also show as a benchmark the P and Q-measure survival probabilities corresponding to an iid world. Finally, the right-hand side graph shows the time series of the value of the portfolio. The benchmark is the value as it would be in an iid world.

Finally, in figure 5 we illustrate the case discussed in section 3.4 above, where the physical measure intensities are almost the same across states ($\lambda^H = 0.001$ and $\lambda^L = 0.0011$). In that case there is almost no change in the perceived default probabilities across firms when a default occurs. But, if the risk-premia associated with each state are important and differ in magnitude (here $\gamma^H = 0.5$ and $\gamma^L = 0$) then there can be substantial time variation in the risk-neutral default intensities as illustrated by the first panel of figure 5. As a result, there will be substantial variation in the risk-neutral survival probabilities and hence prices of defaultable securities relative to physical measure survival probabilities (i.e., expected cash-flows), as shown in the second and third panel of figure 5.

4 Empirical Analysis

We noted above that a jump-to-default can be priced in at least two situations: if multiple firms can default simultaneously, or if the default of one firm increases the intensity of default of many other firms. In addition, a jump event that generates an increase in intensity across many firms, but does not generate any defaults, can also be priced. Note, however, that such a jump process can be empirically distinguished from the first two cases since it does not generate a credit spread above the expected loss rate for short maturities (i.e., $\lambda^P = \lambda^Q$ in this case).

In practice, very few investment-grade firms actually ‘jump’ to default. Indeed, since 1937, we are aware of only four firms that have defaulted on a bond which had an investment-
grade rating from Moody’s.\footnote{This list includes Pacific Gas & Electric (2001), Southern California Edison (2001), Philadelphia, Washington and Baltimore Railroad (1970: due to the Penn Central default), and Johns Manville (1982). Note that both Enron and Columbia Gas were downgraded before they defaulted. This figure was obtained from conversations with Richard Cantor of Moody’s, who also noted that since the Depression era two other firms defaulted on bank loans, but not the publicly issued bonds that carried the investment-grade status.} This statistic suggests that, unless there is an enormous risk-premium on jump-to-default risk, either the default intensities attributed to investment-grade debt greatly exaggerate the likelihood of default occurring over the next ‘instant’, or there are other factors (e.g., liquidity, taxes) embedded in the implied intensity.

In the literature, it has been common to associate the default intensity of investment-grade debt with the fraction of firms that have defaulted within one year of holding investment-grade status, even if that firm’s ratings were lowered over that year. Clearly, such an approach greatly exaggerates the actual jump-to-default intensity. Even so, this statistic is very low, averaging about 0.17% over the past century, and considerably smaller if the Depression era is excluded.\footnote{See Hamilton, Varma, Ou and Cantor (2003).}

To the extent that investment-grade firms default, they more often ‘limp’ to default, experiencing several spread increases over several years before finally defaulting (e.g., Western Union). In such situations, we would not expect a market-wide response at the default-event, because it was not so unexpected.\footnote{Lang and Stulz (1992) find only a 1% negative stock market response for competitors in the same industry when a firm files for bankruptcy.} On the other hand, many corporate bonds experience a large jump in their yield spreads without ever defaulting (e.g., the RJR LBO). Intuitively,
Figure 5: Simulated path of an economy with contagion where $\lambda^H = 0.001$ and $\lambda^L = 0.0011$, $\gamma^H = 0.5$ and $\gamma^L = 0$. The left hand graph shows the path of the p and q-measure default intensities. The middle graph shows the time series of the P and Q-measure 30-year survival probabilities for the average bond in the portfolio. We also show as a benchmark the P and Q-measure survival probabilities corresponding to an iid world. Finally, the right-hand side graph shows the time series of the value of the portfolio. The benchmark is the value as it would be in an iid world.

we would expect that if any jumps are to have a market-wide impact, it would be those that were surprises to investors, and hence accompanied by a large jump in the yield spread of an individual firm.

As mentioned previously, while the existence of a jump-risk premium is only one of many potential explanations for the cross-sectional observation that yield spreads on investment-grade bonds are ‘too high’, the feature that uniquely identifies whether or not a jump-risk premium exists is the time-series implication that, at the jump-date, there is a market-wide response. As such, in this section we investigate empirically the impact that credit events have had on aggregate portfolios. In particular, if these credit events are priced, then we expect their occurrence to be associated with increases in aggregate credit spreads, and possibly also negative excess returns for equity indices. Furthermore, consistent with the implications of the general equilibrium framework in Appendix B, we investigate whether such jumps are associated with ‘flights-to-quality’, i.e., downward movements in risk-free rates.\footnote{Note that we do not investigate incidences of major spread decreases, since the effect of these jumps on the ‘market portfolio’ is not expected to be symmetric.}

Regardless of whether the contagion is due to ‘counterparty risk’ or to ‘updating-of-beliefs’, one would expect that jumps in the yield spreads of larger, ‘safer’ firms would produce a greater impact on the market portfolio than would ‘riskier’ firms. As such, we limit our empirical investigation to investment-grade bonds.
4.1 Data

In order to gather a sufficiently large number of credit events in the investment-grade market, we use the Warga Fixed Income Database (FID), which contains trader quotes provided by Lehman Brothers. This database extends over nearly 25 years and contains reasonably good quality bond price data. The bond prices are month-end quotes and these data constitute the basis for the calculation of net asset values by mutual funds and other money managers.

We obtain spreads on bonds from the FID, which contains month-end bond price for the period January 1973-March 1998. The FID also contains the history of Lehman Brothers' corporate bond index and Treasury bond index over the January 1973-October 1997 period. In order to examine the effect on the bond market as a whole, we restrict our analysis to the months for which the corporate bond index is available. Corporate bond spreads are calculated as the difference between the bond's yield to maturity (YTM) and the interpolated YTM on a Treasury bond with a similar maturity. We obtain the interpolated YTM by using Nelson-Siegel (1987) estimates of the yield curve from the Federal Reserve's Constant Maturity Treasury (CMT) daily series. The CMT series is essentially a database of yield estimates for the on-the-run Treasuries, but occasionally the CMT will continue to estimate a yield when the bonds are no longer auctioned. We only use yields from the CMT series in time periods when the bond is still being auctioned.\textsuperscript{24}

The corporate bond yield spread is the difference between the bond's yield and the estimated yield on the interpolated Treasury bond with the nearest maturity. Rather than estimates, we use actual CMT yields on corporate bonds with the same maturity as the on-the-run bonds.

Corporate bond prices are known to be inaccurate (see Warga (1991) and Warga and Welch (1993)). Few databases report transactions, and none of those contain a lengthy time-series of dealer market transactions. The Lehman Brothers database (the FID) is not a transaction database, but rather a database of quotes on individual bonds supplied at month-end. Unlike many other quote sources, the FID distinguishes between matrix prices and trader quotes. The latter are prices quoted by the Lehman employees who trade the bond, whereas the matrix price is inserted when a trader has no quote. We only use yields that are based on trader quotes, deleting matrix prices from the analysis.

We consider spreads on all corporate bonds of investment-grade in the FID as long as they

\textsuperscript{24}To estimate the yield curve using the Nelson-Siegel method, one must find the estimate based on the rate of decay of the regressors, \( \tau \), that produces the least squared error. We use 400 values of \( \tau \) to obtain the most efficient estimate of the yield curve for each day in the sample. Once we find the best \( \tau \), we take 30 data points from the estimated yield curve, one for each possible yearly maturity.
are not private placements, medium term notes, or Euro-bonds. By corporate bonds, we mean bonds issued by US firms that are not government-sponsored enterprises or supranational organizations, and which are not mortgage-backed or other asset-backed securities. Bonds convertible into preferred stock are also excluded.

We define a credit event as a major change in a bond’s credit spread from one month to the next. Among the set of bonds that experience such shocks, we exclude bonds with less than two years until maturity. We also exclude those bonds which have already fallen below a flat price of $80 (which would be the second piece of bad news about the firm). Furthermore, we exclude bonds where the post-credit-shock price is above $95. The purpose of this exclusion is to avoid identifying a coding-error as a credit-event.

The FID does not describe a bond as either floating or fixed in the same way that the Securities Data Corporation (SDC) does in its database. The only indication in the FID that a bond has a floating rate coupon is that the coupon variable changes from one month to the next. If a researcher uses the stated maturity on a floating rate bond to determine the appropriate yield on the riskfree rate, errors in calculated spreads are likely. As such, we eliminate floating rate bonds from the analysis. We do this using three methods. First, we match the bonds in FID by cusip with the SDC bond issuance database. This method does not work if the bond was issued before the first issue date in the SDC database (late 1970s). Second, we eliminate bonds in the FID if the standard deviation of the coupon calculated over the months it appears in the sample is not zero. Finally, we cross-check our bonds with Moody’s bond record to make sure we have eliminated all floating rate bonds.

With these qualifications, we obtain 52,828 spread-widenings. Table 1 shows the distribution of reported spread increases on corporate bonds in the FID over the sample period. The vast majority of the increased spreads are quite small. Indeed, less than 3% involve credit spread jumps of more than 50 bp.

We are looking for events that are truly rare, yet we would like a database that does not cause small sample problems. Thus, we consider all spread widenings of 200 bp or more, which includes 158 bonds.

One concern is that out of the hundreds of thousands of observations in the FID, some of

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25We do this for two reasons. First, the trading of short maturity bonds is rather illiquid. Hence, it is conceivable that a large drop in credit spreads could occur without a credit-event. Second, a large jump in the yield spread does not necessarily imply a large negative return when maturities are sufficiently low.

26We recognize the possibility that a bond can be selling at a sizeable discount (e.g., below $80) simply because yields have risen substantially since the time the bond was issued. Analogously, there is the possibility that a bond was selling at a sizeable premium because yields have fallen since the time the bond was issued, so that even if this bond suffers a credit event, its price might remain above $95. We emphasize, however, that these type of ‘errors’ would only reduce the likelihood of us finding the significant results reported below.
the credit events (i.e., spread increases of 200bp or more) are merely due to data entry errors. In an attempt to eliminate such errors, we investigate each of our sample's spread changes using Lexis-Nexus and Standard and Poor's Creditweek. If we see evidence that the bonds could have lost considerable value, we include it in our analysis. Evidence consistent with a major bond price movement includes: news of a bond rating downgrade, dividend cut, major losses or other negative information in an earnings announcement, depressed stock prices, a major lawsuit or accident that could cause insolvency, a subsequent default or bankruptcy, or a leverage-increasing merger such as an LBO. If we could not find evidence of a credit shock to the bond, we also checked the bond price recorded in Moody's Bond Record to see if a sale price was recorded at a level close to our bond price. If the price was not similar and there was no news to indicate a problem with the firm, we assumed it was a typo in the FID.\textsuperscript{27} We have determined that 112 of the 158 bonds suffered wider spreads as a result of a credit event. These bonds belonged to 40 firms, two of which suffered two episodes of credit risk (Chrysler and RJR Nabisco). The large number of bonds relative to the number of firms reflects the fact that many of the firms had numerous bonds outstanding, several of which had spreads widen by over 200 bp.

The various credit events are listed in Table 2. The largest number of bonds affected by a credit shock can be classified as being the result of economic hardship. Many of these bonds belong to Chrysler, which suffered in the 1970's and again in 1990. Often the economic hardship events involve news indicating that a company is having difficulty producing operating profits, and some of these occur during recession years. Another common event involves an industrial firm that is the target of a leveraged buyout or similar event (e.g., RJR Nabisco or Marriott). The third largest category is news that a bank has lent funds that it is unlikely to recover. Many of these banks bet badly on real estate loans in the 1990-1991 recession, leading to a number of separate companies having trouble in the same months in the Fall of 1990. A few banks appear in the early 1980s because of loans to Latin America.

Some of the corporate bonds in our sample lost substantial value in one month (i.e., they meet our definition of a credit shock), and then went on to lose additional value in ensuing months. Because it is conceivable that such bonds were no longer considered ‘investment-grade’ in the minds of the marketplace after the first event (even if the ratings firms had not yet downgraded them to non-investment grade status), we choose to use only the first credit shock in a series of episodes in the analysis. By a series of episodes, we mean more than one credit shock for a bond over the course of a year. By using only the first episode of a firm,

\textsuperscript{27}Again, any error here only reduces the likelihood of finding the significant results we report below.
we identify 25 months over the sample period in which a credit event first occurs. That leaves 273 months in which no credit event occurred.

Consistent with either a contagion via counter-party risk hypothesis or a contagion via updated-expectations hypothesis, we investigate whether the size of the firm affects the impact that a credit event has on the stock and bond indices. We measure size by two methods. The first method aggregates the amount of debt outstanding in the FID for the issuer’s six-digit cusip. Note that this measure of ‘size’ reflects the firm’s presence in the corporate bond market. Such a proxy for size could be important if the supply of bonds affects prices. We note that this measure can be noisy because six-digit cusips do not always capture all of the bond issues of a firm, either because bonds are issued by subsidiaries, or because the parent issued bonds under old cusips (that differ as a result of mergers or reorganizations).

The second proxy we use for measuring issuer size is the total assets that the firm has in the year in which the credit event occurs. To identify asset size, we looked for the firm using the first six digits of the bond’s cusip in Compustat. If that search failed, we searched for the firm’s ticker using the look-up window in Compustat. If we could not find a ticker by that method, we then looked for its ticker in Standard & Poor’s Stock Guide in the month in which the credit event occurred. Although this often required a crosscheck with the matched CRSP/Compustat file’s company number, by this method we were able to obtain asset sizes on Compustat for all but one firm. For this remaining firm, which was private, we found assets reported in Moody’s Transportation Manual.

Once the true credit events had been identified, we then examined how these events affected returns of aggregate portfolios. In particular, we investigated the impact of credit events on i) the Lehman corporate bond index, ii) the CRSP value-weighted stock index (including all NYSE, AMEX and Nasdaq firms), and iii) the Lehman Treasury index. Our first methodology basically followed an event study approach. In particular, we identified the average returns (or excess returns) for these three markets during months when a credit event occurred and compared these results to those months when no events occurred. We then performed t-tests analysis to determine whether the credit-event month returns were significantly different than the non-event months.

Of course, aggregate returns are affected by many factors besides credit events. Furthermore, flights-to-quality are caused by many factors besides credit events (e.g., the Russian default of 1998; the stock market crash of 1987). Therefore we also investigate via regression analysis the marginal impact of credit events on (excess) returns accounting for several other factors that have been previously identified to have statistical significance. For example, we
include those macroeconomic factors identified by Fleming and Remolona (1997, 1999) when performing regression analysis on Treasury returns. Furthermore, we also control for flights-to-quality due to other events besides credit-events. We investigate three proxies: First, following Longstaff (2003) we use money fund flows. Second, we use the difference between the on-the-run 30-year Treasury bond’s yield and the yield of the most recently issued off-the-run 30 year bond, as suggested by, for example, Collin-Dufresne, Goldstein and Martin (2001). Third, we use an indicator variable based on a search of the financial press, where the indicator variable is one if the financial press mentions a flight to quality in the Treasury market.

4.2 Results

We compare average returns in the Treasury, corporate bond and stock markets in the months in which credit events occur to the average returns during those months when no event occurred. We then determined the t-statistics for the difference between the mean returns for months where a credit event did and did not occur. Since some of the changes in the corporate bond index reflect changes in Treasury rates (this is especially true of the late 1970s when Treasury market volatility was extremely high), we focus on returns to the corporate bond index in excess of Treasury returns. The Treasury returns are aggregate returns on the market captured by the Lehman Brothers aggregate Treasury market index.

Our events are quite rare and usually involve only a single firm in any given month. Only seven months include credit shocks to more than a single firm. Moreover, only three of the months include credit shocks to more than a dozen bonds. The month with the largest number of affected bonds is September 1990, during which eleven firms and 27 bonds experienced a spread widening of 200 bp or more. In estimating the effect of the credit shocks on the bond market as a whole, a major concern is whether these 27 bonds (fewer bonds in other months) constitute such a large fraction of the corporate bond index that there own price movements drive the returns on the bond index. This is not the case, as the Lehman corporate bond index is typically based on thousands of bonds. For example, in September 1990, 3811 corporate bonds were included in the index. Furthermore, we find nearly identical results (and hence, do not report them here) for the point estimates if we include only those months where a single firm suffered a credit shock.

The results of the t-tests are reported in Tables 3 and 4. Table 3 reports the bond returns according to size, where size is measured as the amount of debt outstanding in the bond market. Table 4 reports bond returns according to size measured by total assets.

Table 3a shows that in the months in which a credit shock occurred the average excess
return on the corporate bond index is negative and significantly less than the return in the other 273 months. In particular, the excess return on the corporate bond index was -0.33% for months with a credit event, and +0.06% for months without a credit event. The t-stat for the difference in means is 1.74. As expected, the difference in excess returns is mainly driven by the largest bond issuers in the sample. Indeed, the mean excess return for the corporate bond index was -1.05%, versus +0.07% during months with no credit event if the bad news in the corporate bond market was associated with a large investment-grade firm. It is worth noting that, according to the results in Table 3b, about one-half of the -1.05% excess return is due to a 0.50% drop (1.22% vs. 0.73%) in Treasury returns, indicating a flight-to-quality during months where a large firm suffered a credit shock. Interestingly, unlike the corporate results shown in 3a, the Treasury returns seem to be affected just as strongly whether a small or a large firm suffers a credit shock. Finally, as shown in Table 3c, returns to the CRSP index appear to be mostly unaffected by these credit shocks. The smaller impact of credit events on equities is not too surprising, as some of the credit events stem from leveraged buyouts and recapitalizations that are positive news for the shareholders involved and other potential takeover targets. This non-result, however, is somewhat misleading. Indeed, below we report that in fact credit events do have a statistically significant impact on stock returns via regression analysis when other variables are controlled for.

As shown in Table 4, similar results are obtained when size is measured via assets rather than amount of debt outstanding. Again, the corporate bond index is mostly affected only when large firms suffer a credit shock. However, now the Treasury returns do appear to be more affected when a large firm rather than a small firm suffers a shock. Again, the stock index is mostly unaffected by such shocks.

Maybe the most surprising result from Tables 3 and 4 is the magnitude of Treasury return during credit event months. One concern is that the implied causality is wrong: maybe those months where Treasury returns are very good are due the Federal Reserve easing, which occurs when the economy is weak. If so, then a large number of credit spread increases would randomly result in at least one corporate bond suffering a 200bp spread widening. If this is the case, then good returns in the Treasury market may not represent a flight to quality but reverse causality. To test this potential explanation for our results, we investigated the 273 months that were not associated with a credit event and identified those 25 months with the highest returns in the Treasury market. We found that the average excess return for the corporate bond market during these 25 months was actually significantly positive. This means that good Treasury returns are not automatically associated with poor corporate bond returns and that
it is highly unlikely that Treasury returns are causing us to find spread widenings without a true credit event.

4.3 Regression Analysis

The t-tests in Tables 3 and 4 are based on the implicit assumption that only credit shocks affect monthly asset returns, as no other factors are taken into account. In Table 5 we report the results of regression analysis of the three portfolios, controlling for other effects in these markets. In each set of regressions, we include factors that represent changes in the state of the economy. The macroeconomic variables are suggested by Fleming and Remolona (1997, 1999) who investigate intraday Treasury market moves. In the corporate bond and stock market regressions, we also include the slope of the term structure, as Estrella and Hardouvelis (1991) show that it predicts recessions. To control for the effect of changes in real rates, we also include the current month’s change in the actual Federal Funds rate or alternatively the Federal Funds rate relative to inflation (the Taylor rule variable: FF-inflation -2 (see Taylor (1993))).

For the Treasury return regression, we prefer not to include the slope of the Treasury curve as an explanatory variable, as it may cause a spurious relationship in the estimation. In addition to macroeconomic control variables, we include our measures of flight to quality in the Treasury market. These include indicator variables for instances of flight to quality that are not related to the credit events we identify, as well as changes in retail and institutional money market mutual fund flows.

Longstaff’s analysis of flight to quality/flight to liquidity involves analysis of Resolution Trust Corporation (RTC) bonds, which are less liquid than Treasuries but have identical credit risk. RTC bonds were not issued before the late 1980s, at which time money market mutual funds were a well established product in the U.S. However, these mutual funds were in their infancy at the start of our sample period. Even detrending this series to obtain innovations in fund flows does not effectively deal with the immaturity of this product, because the rate of growth beyond the trend is quite high initially. Nevertheless, for comparison sake we include both institutional and retail money fund flow innovations as a measure of the flight to quality.

Our indicator variable for flight to quality episodes is based on information culled from the financial press. For the part of the sample from June 1979 on, we set the indicator to one whenever the Wall Street Journal (WSJ) uses the phrase “flight to quality” to explain why prices of Treasuries have risen in that calendar month. If the WSJ uses the phrase to describe a long term trend (say 3 months) in Treasury prices, we do not set the indicator to one. For the period from 1973 to May 1979, the WSJ index is not available in electronic form and we
cannot search on the phrase "flight to quality." For that part of the sample period we search for references to flight to quality in three sources. First, we read the weekly write-up of the bond market in Moody’s Bond Survey for mention of a flight to quality phenomenon. We draw confidence from this method by the fact that the write-up for October 26, 1987 refers to the impact on Treasuries of the stock market crash as "a spectacular flight to quality." A second source for the early time period is the market commentary of Aubrey Lanston (various issues, 1973-1979), which specializes in government bond trading. Lastly, we search Lexis-Nexus for references to flight to quality over these years. If any of the sources mention a flight to quality in Treasuries comparable to those used for the WSJ, we set the indicator to one for that month.\textsuperscript{28}

We obtain closing VIX implied volatilities from the CBOE website (data begins in 1986) and on-the-run/off-the-run Treasury spreads from Datastream (data starts in 1980). Corporate bond upgrade to downgrade ratios are obtained from Moody’s Investors Service, as are monthly default rates.

Table 5 shows estimations of excess returns in the corporate bond market. The results are similar to those found in the t-tests, in that the credit shocks lead to lower monthly bond returns and are greater if the credit shock belongs to a larger firm. These regressions are corrected for heteroskedasticity and autocorrelation by the Newey-West method (using five lags). In the first column of Table 5, the estimated model incorporates an indicator for all credit shocks, regardless of the size of the firm, and includes standard measures of the macroeconomy. Corporate returns are significantly lower, by 43 bp, in months in which a credit event occurs in this specification. Meanwhile, the signs of the control variables are largely as expected: a steep yield curve implies a strong economy and thus a low chance of default, while Fed tightening indicates that the peak in the current business cycle has arrived (with defaults increasing soon). The trade deficit situation is also very significant, which may reflect the fact that the corporate bond market includes more exporters than the economy in general. Defaults and rating changes are not significant, regardless of the lead/lag relationship used in the regression (results not reported).

Several of the macro variables are insignificant, and are dropped for the sake of parsimony.\textsuperscript{28}We also estimated the effect on Treasury returns of several variations on this indicator variable (results not reported). First, we created an indicator variable that is only set to one when the flight to quality episode is not reversed within the calendar month. The reference to the reversal of the flight to quality is obtained by the same search of the financial press. And then we created a third indicator variable that allows the flight to quality indicator to be set to one even if there is a reversal as long as there is evidence of severity of the flight to quality impact (more than one reference to flight to quality in the financial press for that month). These variations on the flight to quality indicator variable result in similar coefficient estimates.
in the remaining specifications. Likewise the flight to quality indicators are not important for the corporate bond market. This likely reflects the relative safety of bonds in the various episodes. For example, when Russia defaulted, high grade corporate bonds became appealingly safe relative to emerging markets debt. But in the October 1987 stock market crash, corporate bonds were considered quite risky and the flight to quality was into Treasuries.

In the last two columns of Table 5 we report regressions of excess bond returns where the credit event months are split into those with large firms and those with small firm shocks. As in the t-tests, the impact is restricted to months when large firms experience credit shocks. The last column of Table 5 differs by inclusion of the VIX volatility measure, which is only available after 1986. Although the credit shocks are still significantly negative for large firm months in this specification, the October 1987 crash is not. Moreover, the default rate and ratio of upgrades to downgrades now come in with the correct signs and sharply stronger significance levels. This owes to the fact that defaults and rating changes are more common for speculative-grade firms, and the junk bond market is not established until the mid-1980s.

Table 6 shows the results of regressions explaining the monthly variation in Treasury index returns. In each of the specifications, the corporate bond market shock has a positive impact on Treasury returns, although the impact is not statistically significant in the specification shown in the last column. The coefficient on the flight to quality indicator variable, shown in the first column, suggests the flight to quality phenomenon is quite broad and usually has a large impact than that arising from our corporate credit shocks. While the credit shock indicator has a coefficient of 54 bp, the other flight to quality episodes have an average impact of more than 70 bp. And, this does not include the October 1987 stock market crash, which is estimated to have an impact nearly four times larger. The models in the second through fourth columns indicates that the impact of the credit shocks is mainly in months in which large firms suffer negative credit events. Our alternative measures of flight to quality episodes, the institutional and retail money fund flow variables, have contradictory results. While the institutional fund flow variables carries the correct sign, the effect is not statistically significant. The retail fund flow variable is statistically significant, but it suggests that money outflows improve the returns in the Treasury market. Moreover, when this variable is included, the credit event indicator variable is no longer significant. If we restrict the time period (results not shown), the impact of the retail money fund flows is only significant earlier in the sample period, likely reflecting the surge in retail money funds that occurred in the face of high inflation and federal restrictions on interest rates paid on retail deposits.

Table 7 shows regressions analyzing the effect of our credit shock months on excess stock
returns (the CRSP firms’ value weighted index return less the Treasury return for the same month). The first column shows a 52 bp lower return on stocks in months in which these credit events occur, but the level of significance is just outside conventional levels. Note that the macroeconomic control variables are quite different in this estimation than in the models of bond returns. The slope of the Treasury yield curve is not significant in any of the specifications and increases in the Federal Funds rate has a positive impact (and significantly so in several of the models shown). Of particular interest is the role of consumer confidence, which does not have a significant impact on bond returns. For stocks, higher consumer confidence always leads to higher returns. This variable impinges on the significance of the credit events, suggesting that confidence drops when these events occur. In the second column of Table 6, we drop the consumer confidence variable to better understand the impact of the credit event months. While this lowers the adjusted R-square, it prevents us from estimating an equation with two variables that capture the impact of the same events. The credit event indicator variable is significant at the 5% level once the change in consumer confidence is left out of the regression. In the third column of the table, we once again see that the significance of the credit shocks stems from credit shocks of large companies. The last column shows the effect of the flight to quality indicator variable versus money fund flows. If instead we measure flight to quality with retail fund flows (not shown) the results are insignificantly positive.

In summary, we find that credit shocks lead to significant losses on a corporate bond portfolio, especially if the shocks are associated with large investment grade firms. While these shocks do not lead to dramatic flights to quality (the stock market crash of 1987 is clearly outsized in comparison), the evidence is consistent with credit risk contagion being an important factor in the Treasury market. Lastly, we find that these events are largely negative news for the stock market, despite the fact that some of the events are related to shareholder-enhancing takeover events.

4.4 Systematic jump event risk vs. contagion risk premia: a simple calibration

As shown in Proposition 4, for an economy with contagion and systematic jump risk, the risk-premium for any risky security with price $B_k$ is given by:

$$\frac{1}{dt}E^P \left[ \frac{dB_k(t)}{B_k(t)} \right] - r(t) = \sigma_b(t)^\top \theta(t) + \sum_{i \neq k} \overline{\lambda}_i(t) E^P \left[ \Gamma_{k,i} \Gamma_{\xi,i} \right] 1_{\{\tau_i > t\}} + \overline{\lambda}_k(t) \Gamma_k \hat{r}_{\xi,k}.$$  (64)

Since we are interested in credit spreads, it is more convenient to express equation (64) as excess returns over treasury returns of the same maturity. While Treasury prices are not
subject to default risk, they may experience jumps on the credit event dates as a result of a flight to quality (i.e., a jump in the risk-free rate). This implies that the term premium on Treasury securities \( P^T(t) \) with maturities \( T > t \) can be written in a form similar to that of equation (64). Hence, we obtain:

\[
\frac{1}{dt} \mathbb{E}_t \left[ \frac{dB_t(t)}{B_t(t)} - \frac{dP^T(t)}{P^T(t)} \right] = (\sigma_k - \sigma_P) \theta + \sum_{i \neq k} \lambda_i(t) \mathbb{E}^{P} \left[ \left( \Gamma_{k,i} - \Gamma_{P,i} \right) \Gamma_{\xi,i} \right] 1_{\{\tau_i > t\}} + \lambda_k(t) \mathbb{E}^{P} \left[ \left( \Gamma_k - \Gamma_{P,k} \right) \Gamma_{\xi,k} \right].
\]

From equation (33), the instantaneous volatility of the pricing kernel is

\[
\frac{1}{dt} \text{Var} \left[ \frac{d\Delta(t)}{\Delta(t)} \right] = \theta^2 + \sum_{i=1}^{N} \lambda_i(t) \mathbb{E}^{P} \left[ \Gamma_i \right]^2.
\]

This statistic is useful because, following the duality analysis of Hansen-Jaganathan (1991) bounds, it represents an upper bound to the highest Sharpe ratio attainable in an economy. This implies that to explain the historical (market portfolio) equity Sharpe ratio of \((\frac{1}{\sigma_{\Delta P}}) = (0.09-0.01) \approx 0.5\), the variance of the pricing kernel needs to be \( \geq (0.5)^2 = 0.25 \). While some have argued that the corporate bond market is somewhat segmented from the equity market, it seems reasonable to assume that the total variance in the pricing kernel attributable solely to credit-spread jump risk is bounded above by 0.25. In other words we shall assume that:29

\[
V_\Delta \equiv \sum_{i=1}^{N} \lambda_i(t) \mathbb{E}^{P} \left[ \Gamma_i \right]^2 = 0.25.
\]

There are many different ways to estimate \( \lambda^P \) for investment grade debt. One estimate would be to use Moody’s one-year default rate for investment grade debt. This would generate an order of magnitude of \( \lambda^P \approx 10^{-2} \). However, we argue that this number is too high. Indeed, most debt that goes from investment grade to default within a year does so by first dropping from investment grade to speculative grade before defaulting. As such, the above estimate would include many firms which do not exhibit a ‘surprise’ jump-to-default.

Table 1 implies that if a jump-to-default is associated with a credit spread jump of over 300bp during a single month, then an appropriate estimate for the intensity of a credit event is on the order of \( \lambda^P \in (10^{-3}, 10^{-4}) \). Hence, with \( N \approx 10^3 \) firms, we would expect to see a true jump-to-default of an investment grade firm about once every one-to-ten years. Below, we will use as an upper bound \( \lambda^P \approx 10^{-3} \).

\[29\text{This leads to the component of volatility of the pricing kernel due solely to ‘corporate bond event risk’ of about 50% (enough to explain the equity premium only with bond price jump risk!).}\]
Finally we need to calibrate the loss upon default of a typical firm, say $k$, ($\Gamma_k$) and the contagion jump in firm $k$ upon default of firm $j$ ($\Gamma_{kj}$). We set $\Gamma_k = 50\%$, which is approximately equal to the historical average recovery rate reported by Moody's. Finally, we set $\Gamma_{kj} = 0.39\%$ as implied in Tables 3A and 4A. (Tables 3A and 4A show that the excess returns during the 25 months where a credit event took place were 0.39% on average.)

The table below estimates the size of the credit premium due to jump-to-default risk and to contagion risk. We find that approximately 2.5 bp is attributable to jump-to-default risk premium, whereas as much as 19.5 bp can be attributed to contagion risk. Given the rather 'generous' assumption about the volatility of the pricing kernel, these should be considered as upper bounds.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$V_A$</th>
<th>$\lambda^p$</th>
<th>$\Gamma_{k,i}$</th>
<th>$\Gamma_{k,i}$</th>
<th>$\Gamma_k$</th>
<th>Jump to default premium</th>
<th>Contagion premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>0.5</td>
<td>0.001</td>
<td>0.5</td>
<td>0.0039</td>
<td>0.5</td>
<td>0.00025</td>
<td>0.00195</td>
</tr>
</tbody>
</table>

The intuition for our finding that the amount of credit spread due to jump-to-default cannot be too large is that the volatility of the pricing kernel increases linearly in the number of firms $N$ whose debt commands jump-to-default premia. Thus in order to keep a reasonable overall volatility of the pricing kernel the common jumps of individual firms and pricing kernel cannot be large. In contrast, allowing for contagion risk increases spreads substantially without raising the volatility of the pricing kernel.

5 Conclusion

We have introduced a framework that provides a tractable solution to the contagion problem. Although consistent with the counterparty-risk hypothesis, our framework is most naturally interpreted as an updating of beliefs hypothesis, where the upward jump in the yield spread of one firm increases the perception of risk in the risky bonds of other firms. We show that our reduced-form model can be reconciled with a simple structural model along the lines of Duffie and Lando (2001), where the default of one firm affects the market's perception of the quality of accounting numbers of other firms. If contagion risk is to be priced (non-diversifiable) one expects to observe a market wide response on the day an event occurs. We study empirically the response of return on corporate bond, Treasury and equity indices to 'surprise' credit events incurred by individual firms. Consistent with intuition, we find that credit events of large firms have a more significant effect on the market than credit events of small firms. Furthermore, these events lead to market-wide increases in credit spreads, and downward jumps in risk-free rates. The positive impact on Treasury returns of credit events is substantial and seems
robust. It is consistent with the notion of ‘flight to quality,’ and indeed with the predictions of a simple general equilibrium framework. Using our results we try to estimate the respective impact of systematic jump risk and contagion risk for credit spreads. A crude calibration (and an intuitive argument) suggests that systematic jump risk alone cannot explain a significant portion of the observed credit spreads (unless the volatility of the pricing kernel exceeds many times the upper bound usually accepted in the literature). However, contagion risk can account for a substantially larger component of the spread.

Given the low historical default rate (as pointed out by, e.g., Huang and Huang (2002)) contagion-risk may have to be interpreted as ‘liquidity’ risk rather than true updating of future default-risk. That is, it is likely that the market wide jump in credit spreads after a credit event overshoots the increased probability of default. This is consistent with our model if states with similar physical measure conditional default intensities carry different risk-premia. We leave it open for future research as to why risk-premia should vary so much across apparently similar states.

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30 In this case as illustrated in the third example figure 5 in section 3.5, we may see very little variation in physical measure default intensities and yet a lot of variation in risk-neutral measure intensities and prices. This seems to be consistent with recent results by Duffie et al. (2003).
References


<table>
<thead>
<tr>
<th>Number of Average Excess Average</th>
<th>observations</th>
<th>Percentage</th>
<th>Return</th>
<th>Return</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Over 10 percentage points</td>
<td>7</td>
<td>0.01%</td>
<td>-31.33%</td>
<td>-31.37%</td>
<td>2.36%</td>
</tr>
<tr>
<td>5 to 10 percentage points</td>
<td>6</td>
<td>0.01%</td>
<td>-17.43%</td>
<td>-16.79%</td>
<td>3.17%</td>
</tr>
<tr>
<td>3 to 5 percentage points</td>
<td>32</td>
<td>0.06%</td>
<td>-12.40%</td>
<td>-12.91%</td>
<td>4.26%</td>
</tr>
<tr>
<td>2 to 3 percentage points</td>
<td>113</td>
<td>0.21%</td>
<td>-7.74%</td>
<td>-8.93%</td>
<td>4.57%</td>
</tr>
<tr>
<td>1.5 to 2 percentage points</td>
<td>146</td>
<td>0.28%</td>
<td>-5.42%</td>
<td>-6.69%</td>
<td>4.76%</td>
</tr>
<tr>
<td>1.25 to 1.5 percentage points</td>
<td>131</td>
<td>0.25%</td>
<td>-4.08%</td>
<td>-5.50%</td>
<td>5.10%</td>
</tr>
<tr>
<td>1 to 1.25 percentage points</td>
<td>273</td>
<td>0.52%</td>
<td>-2.85%</td>
<td>-4.08%</td>
<td>4.87%</td>
</tr>
<tr>
<td>0.75 to 1 percentage points</td>
<td>572</td>
<td>1.08%</td>
<td>-2.00%</td>
<td>-3.63%</td>
<td>5.70%</td>
</tr>
<tr>
<td>0.5 to 0.75 percentage points</td>
<td>1919</td>
<td>3.63%</td>
<td>-1.08%</td>
<td>-2.84%</td>
<td>6.30%</td>
</tr>
<tr>
<td>0.25 to 0.5 percentage points</td>
<td>5776</td>
<td>10.93%</td>
<td>-0.54%</td>
<td>-1.18%</td>
<td>6.39%</td>
</tr>
<tr>
<td>Less than 0.25 percentage points</td>
<td>43853</td>
<td>83.01%</td>
<td>0.08%</td>
<td>-0.23%</td>
<td>7.71%</td>
</tr>
</tbody>
</table>

Table 2: Distribution of Spread increases

<table>
<thead>
<tr>
<th>Event Type</th>
<th>Cumulative Frequency</th>
<th>Cumulative Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accidents (e.g., toxic train crash, nuclear power plant accident)</td>
<td>5</td>
<td>4.46%</td>
</tr>
<tr>
<td>Economic hardship of firm (e.g., declining revenue, higher costs)</td>
<td>36</td>
<td>32.14%</td>
</tr>
<tr>
<td>Major lawsuit against firm (e.g., asbestos, tobacco)</td>
<td>5</td>
<td>4.46%</td>
</tr>
<tr>
<td>LBO or other leverage increasing event</td>
<td>34</td>
<td>31.25%</td>
</tr>
<tr>
<td>Liquidity and lack of access to new funds</td>
<td>1</td>
<td>0.89%</td>
</tr>
<tr>
<td>Nonperforming bank loans or leases</td>
<td>30</td>
<td>26.79%</td>
</tr>
</tbody>
</table>

Table 3: Description of Event Types
### Table 3a: Corporate Bond Index Returns

<table>
<thead>
<tr>
<th>Event</th>
<th>Number of months</th>
<th>Mean excess return in months</th>
<th>Mean excess return in months when an event occurs</th>
<th>Mean excess return in other months</th>
<th>Difference in mean returns</th>
<th>t-test statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>All events</td>
<td>25</td>
<td>-0.33</td>
<td>0.06</td>
<td>0.39</td>
<td>1.74</td>
<td>0.084</td>
<td></td>
</tr>
<tr>
<td>Largest firms</td>
<td>11</td>
<td>-1.05</td>
<td>0.07</td>
<td>1.12</td>
<td>3.42</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Smallest firms</td>
<td>14</td>
<td>0.24</td>
<td>0.02</td>
<td>-0.22</td>
<td>-1.50</td>
<td>0.149</td>
<td></td>
</tr>
</tbody>
</table>

### Table 3b: Treasury Bond Index Returns

<table>
<thead>
<tr>
<th>Event</th>
<th>Number of months</th>
<th>Mean return in months</th>
<th>Mean return when an event occurs</th>
<th>Mean return in other months</th>
<th>Difference in mean returns</th>
<th>t-test statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>All events</td>
<td>25</td>
<td>1.28</td>
<td>0.70</td>
<td>-0.59</td>
<td>-1.70</td>
<td>0.090</td>
<td></td>
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<tr>
<td>Largest firms</td>
<td>11</td>
<td>1.22</td>
<td>0.73</td>
<td>-0.49</td>
<td>-0.96</td>
<td>0.336</td>
<td></td>
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<tr>
<td>Smallest firms</td>
<td>14</td>
<td>1.33</td>
<td>0.72</td>
<td>-0.62</td>
<td>-1.37</td>
<td>0.173</td>
<td></td>
</tr>
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</table>

### Table 3c: Stock market (NYSE) Returns

<table>
<thead>
<tr>
<th>Event</th>
<th>Number of months</th>
<th>Mean return in months</th>
<th>Mean return when an event occurs</th>
<th>Mean return in other months</th>
<th>Difference in mean returns</th>
<th>t-test statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>All events</td>
<td>25</td>
<td>0.80</td>
<td>1.13</td>
<td>0.33</td>
<td>0.36</td>
<td>0.721</td>
<td></td>
</tr>
<tr>
<td>Largest firms</td>
<td>11</td>
<td>0.05</td>
<td>1.14</td>
<td>-1.09</td>
<td>0.57</td>
<td>0.578</td>
<td></td>
</tr>
<tr>
<td>Smallest firms</td>
<td>14</td>
<td>1.38</td>
<td>1.09</td>
<td>-0.30</td>
<td>-0.25</td>
<td>0.807</td>
<td></td>
</tr>
</tbody>
</table>

### Table 4: Effects of Credit Events on Corporate Bond, Treasury and Stock Indices Size of firm measured by bonds outstanding Monthly returns from January 1973 to October 1997 (298 months)
Table 4a: Corporate Bond Index Returns

<table>
<thead>
<tr>
<th></th>
<th>Number of months</th>
<th>Mean excess return in months</th>
<th>Mean excess return in which event occurs</th>
<th>Difference in mean returns</th>
<th>t-test statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>All events</td>
<td>25</td>
<td>-0.33</td>
<td>0.06</td>
<td>0.39</td>
<td>1.74</td>
<td>0.084</td>
</tr>
<tr>
<td>Largest firms</td>
<td>13</td>
<td>-0.53</td>
<td>0.06</td>
<td>0.58</td>
<td>1.91</td>
<td>0.057</td>
</tr>
<tr>
<td>Smallest firms</td>
<td>12</td>
<td>-0.11</td>
<td>0.04</td>
<td>0.15</td>
<td>0.47</td>
<td>0.641</td>
</tr>
</tbody>
</table>

Table 4b: Treasury Bond Index Returns

<table>
<thead>
<tr>
<th></th>
<th>Number of months</th>
<th>Mean return in months</th>
<th>Mean return when an event occurs</th>
<th>Difference in mean returns</th>
<th>t-test statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>All events</td>
<td>25</td>
<td>1.28</td>
<td>0.30</td>
<td>-0.99</td>
<td>-1.70</td>
<td>0.090</td>
</tr>
<tr>
<td>Largest firms</td>
<td>13</td>
<td>1.55</td>
<td>0.71</td>
<td>-0.84</td>
<td>-3.05</td>
<td>0.008</td>
</tr>
<tr>
<td>Smallest firms</td>
<td>12</td>
<td>0.30</td>
<td>0.76</td>
<td>-0.46</td>
<td>0.72</td>
<td>0.471</td>
</tr>
</tbody>
</table>

Table 4c: Stock market (NYSE) Returns

<table>
<thead>
<tr>
<th></th>
<th>Number of months</th>
<th>Mean return in months</th>
<th>Mean return when an event occurs</th>
<th>Difference in mean returns</th>
<th>t-test statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>All events</td>
<td>25</td>
<td>0.80</td>
<td>1.13</td>
<td>0.33</td>
<td>0.36</td>
<td>0.721</td>
</tr>
<tr>
<td>Largest firms</td>
<td>13</td>
<td>0.24</td>
<td>1.14</td>
<td>0.90</td>
<td>0.72</td>
<td>0.474</td>
</tr>
<tr>
<td>Smallest firms</td>
<td>12</td>
<td>1.40</td>
<td>1.09</td>
<td>-0.31</td>
<td>-0.24</td>
<td>0.811</td>
</tr>
</tbody>
</table>

Table 5: Effects of Credit Events on Corporate Bond, Treasury and Stock Indices Size of firm measured by total assets. Monthly returns from January 1973 to October 1997 (298 months)
<table>
<thead>
<tr>
<th>Table 5</th>
<th>Corporate Bond Excess Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>constant</td>
<td>-0.220</td>
</tr>
<tr>
<td></td>
<td>(-1.884)</td>
</tr>
<tr>
<td>Credit event month</td>
<td>-0.430</td>
</tr>
<tr>
<td></td>
<td>(-2.140)</td>
</tr>
<tr>
<td>Big firm credit event month</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(-3.458)</td>
</tr>
<tr>
<td>Small firm credit event month</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(-0.664)</td>
</tr>
<tr>
<td>Change in Fed Funds</td>
<td>-0.222</td>
</tr>
<tr>
<td></td>
<td>(-1.627)</td>
</tr>
<tr>
<td>Change in Taylor rule measure</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(-2.398)</td>
</tr>
<tr>
<td>Change in defaults (6 mo. lead)</td>
<td>14.447</td>
</tr>
<tr>
<td></td>
<td>(0.901)</td>
</tr>
<tr>
<td>Change in upgrades/downgrades</td>
<td>0.020</td>
</tr>
<tr>
<td>(6 months ahead)</td>
<td>(0.074)</td>
</tr>
<tr>
<td>Slope of the term structure</td>
<td>0.352</td>
</tr>
<tr>
<td></td>
<td>(3.255)</td>
</tr>
<tr>
<td>Change in consumer confidence</td>
<td>-0.788</td>
</tr>
<tr>
<td></td>
<td>(-0.798)</td>
</tr>
<tr>
<td>Change in current account</td>
<td>0.170</td>
</tr>
<tr>
<td></td>
<td>(2.927)</td>
</tr>
<tr>
<td>Change in payroll employment</td>
<td>0.0003</td>
</tr>
<tr>
<td></td>
<td>(0.767)</td>
</tr>
<tr>
<td>Change in industrial production</td>
<td>-0.0006</td>
</tr>
<tr>
<td></td>
<td>(-0.006)</td>
</tr>
<tr>
<td>Indicator for 87 stock crash</td>
<td>-1.321</td>
</tr>
<tr>
<td></td>
<td>(-5.739)</td>
</tr>
<tr>
<td>Flight to quality indicator</td>
<td>0.283</td>
</tr>
<tr>
<td></td>
<td>(1.548)</td>
</tr>
<tr>
<td>Shock to retail money funds</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(-1.552)</td>
</tr>
<tr>
<td>Shock to inst. money funds</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-</td>
</tr>
<tr>
<td>Change in VIX</td>
<td>-</td>
</tr>
<tr>
<td>(post 1986 only)</td>
<td>-</td>
</tr>
<tr>
<td>Adjusted R²</td>
<td>0.124</td>
</tr>
</tbody>
</table>

Table 6: (Regression of excess returns of corporate bonds, and Treasury returns. t-values are in parentheses)
| Table 6: (Regression of Treasury bond returns. t-values are in parentheses) |
|-------------------------|-----------------|-----------------|-----------------|-----------------|
|                          | (1)             | (2)             | (3)             | (4)             |
| constant                 | 1.189           | 1.187           | 1.383           | 1.466           |
| Credit event month       | (5.197)         | (5.179)         | (5.707)         | (6.188)         |
|                         | 0.540           | -               | -               | -               |
| Big firm credit event month | -              | 0.666           | 0.569           | 0.445           |
|                         | (2.082)         | (1.741)         | (1.330)         | (0.760)         |
| Small firm credit event month | -              | 0.415           | 0.313           | 0.213           |
|                         | (0.777)         | (0.580)         | (0.399)         | (0.760)         |
| Change in Fed Funds      | -0.318          | -0.320          | -0.239          | -0.341          |
|                         | (-1.635)        | (-1.644)        | (-1.221)        | (-1.747)        |
| Change in consumer price index | -1.141        | -1.147          | -1.350          | -1.297          |
|                         | (-2.754)        | (-2.752)        | (-3.289)        | (-3.389)        |
| Change in consumer confidence | -4.941        | -4.930          | -5.816          | -6.185          |
|                         | (-2.625)        | (-2.610)        | (-2.881)        | (-3.085)        |
| Change in current account| 0.120           | 0.115           | 0.061           | 0.074           |
|                         | (1.246)         | (1.164)         | (0.612)         | (0.760)         |
| Change in payroll employment | -0.0007        | -0.0007         | -0.0009         | -0.0009         |
|                         | (1.329)         | (-1.273)        | (-1.507)        | (-1.447)        |
| Indicator for 87 stock crash | 2.818          | 2.812           | 3.497           | 3.377           |
| Flight to quality indicator | 0.719          | 0.718           | -               | -               |
|                         | (2.433)         | (2.425)         |                 |                 |
| Shock to inst. money funds | -              | -               | 0.478           | -               |
|                         |                 |                 | (1.469)         |                 |
| Shock to retail money funds | -              | -               | -0.700          | -0.830          |
| Adjusted R²              | 0.158           | 0.155           | 0.150           | 0.157           |

Table 7: (Regression of Treasury bond returns. t-values are in parentheses)
<table>
<thead>
<tr>
<th>Table 7</th>
<th>Stock Market Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>constant</td>
<td>-1.094</td>
</tr>
<tr>
<td>Credit event month</td>
<td>(-4.277)</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Big firm credit event month</td>
<td>-0.519</td>
</tr>
<tr>
<td>(1.613)</td>
<td>(1.987)</td>
</tr>
<tr>
<td>Small firm credit event month</td>
<td>-0.499</td>
</tr>
<tr>
<td>Change in Fed Funds</td>
<td>0.285</td>
</tr>
<tr>
<td>(1.368)</td>
<td>(1.782)</td>
</tr>
<tr>
<td>Change in consumer price index</td>
<td>1.021</td>
</tr>
<tr>
<td>(2.200)</td>
<td>(1.305)</td>
</tr>
<tr>
<td>Change in consumer confidence</td>
<td>5.011</td>
</tr>
<tr>
<td>(2.660)</td>
<td></td>
</tr>
<tr>
<td>Change in current account</td>
<td>-0.129</td>
</tr>
<tr>
<td>(1.275)</td>
<td>(1.784)</td>
</tr>
<tr>
<td>Change in payroll employment</td>
<td>0.0008</td>
</tr>
<tr>
<td>(1.330)</td>
<td>(1.287)</td>
</tr>
<tr>
<td>Slope of term structure</td>
<td>-0.071</td>
</tr>
<tr>
<td>(0.575)</td>
<td>(-0.490)</td>
</tr>
<tr>
<td>Indicator for 87 stock crash</td>
<td>-3.027</td>
</tr>
<tr>
<td>(9.237)</td>
<td>(-8.730)</td>
</tr>
<tr>
<td>Flight to quality indicator</td>
<td>-0.702</td>
</tr>
<tr>
<td>(2.345)</td>
<td>(-2.456)</td>
</tr>
<tr>
<td>Shock to inst. money funds</td>
<td>-</td>
</tr>
<tr>
<td>(1.916)</td>
<td></td>
</tr>
<tr>
<td>Adjusted R²</td>
<td>0.158</td>
</tr>
</tbody>
</table>

Table 8: (Regression of Excess Stock Market Returns. t-values are in parentheses)
A Structural model of contagion

While reduced-form models have the advantage of tractability, structural models of default provide more economic insight and predictive power (e.g., predicting the change in a bond price for a given change in leverage). Interestingly, Duffie and Lando (2001) demonstrate in a structural framework that default becomes a jump process once realistic imperfections (such as imperfectly observed market values) are incorporated. Here, we extend the framework of Duffie and Lando to incorporate contagion and show that it corresponds to a special case of our general reduced-form model.

Following the standard analysis of Merton (1974), Leland (1994), and Duffie and Lando (2001), we assume that each firm in the economy has total asset value given by $X_i(t)$ which follows a geometric Brownian motion:

$$\frac{dX_i(t)}{X_i(t)} = r \, dt + \sigma_i \, dz_i(t).$$  \hspace{1cm} (A.68)

Here, the drift $r$ and volatility $\sigma_i$ are constant, and $dz_i(t)$ is a standard Brownian motion. For simplicity, and to emphasize our point that firms do not need to have economic ties for their spreads to have common jumps, we assume that Brownian motions are uncorrelated across firms: $dz_i(t) \, dz_j(t) = dt \, 1_{i \neq j}$. For simplicity, we assume each firm has previously issued a perpetuity which pays a constant coupon rate. Further, each firm will default on its debt payments when the firm value reaches a known threshold level $X_i^B$.31

Following Duffie and Lando (2001), we assume that investors do not observe the actual current firm value. In such a situation, DL have demonstrated that the default event becomes unpredictable. That is, from the investor’s standpoint, default is effectively a jump process. A similar result obtains in our framework even though we assume a somewhat simpler information mechanism than DL. In particular, we assume that investors observe a signal which corresponds to some lagged firm value $x_i(t) = X_i(t - \ell_i)$, where the lag $\ell$ is not known perfectly. For simplicity, we assume that for each firm, $\ell_i$ can take on only one of two values, $\ell_i^H$ or $\ell_i^L$, where $\ell_i^H > \ell_i^L$. Further, we assume that the ‘accounting quality’ of all firms is perfectly correlated in that all firms are either in the high-delay state or the low-delay state. This captures the

31Some empirical support in favor of the importance of information accuracy for spreads is given in Yu (2003).

32This assumption implies that upon default of firm-i, there is no updating on the current values of other firms conditional upon whether the economy is in state-H or state-L. Rather, it only affects the belief whether the economy is in state-H or state-L. Consistent with the CAPM, given that the drift is the risk-free rate $r$, we assume that innovations of this firm are purely idiosyncratic.

33Following Leland (1994) and Duffie and Lando (2001), it is straightforward to determine the threshold boundary and optimal coupon endogenously. However, in order to focus on the relevant issues, after the issuance-date, we can think of the perpetuity payments $C$ and the default threshold level $X_i^B$ as just some exogenously specified constants.
idea that all firms are using similar information technology (e.g., accounting ‘techniques’). The longer the delay, the less is known about how close the current cash flows are to the default boundary. We note that within this framework, the most recently observed data point is a ‘sufficient statistic’ to price the risky debt. In particular, there is no need to Bayesian update on past values.

At date-$t$, the prior belief about this probability is defined as:

$$p^H(t) \equiv \pi \left( \ell_i = \ell^H \forall i = (1, N) \right) | \mathcal{F}_t . \tag{A.69}$$

Define $y_i(t) \equiv \log \frac{x_i(t)}{X_i}$ and $Y_i(t) \equiv \log \frac{X_i(t)}{X_i}$. Default occurs at the first time that $Y_i(t)$ reaches zero: $\tau_i = \inf\{t : Y_i(t) = 0\}$. Before default occurs, $y_i(t)$ (and $Y_i(t)$ for that matter) follows the process

$$dy_i(t) = \left( r - \frac{\sigma_i^2}{2} \right) dt + \sigma_i\, dz_i(t)$$

$$\equiv m_i \, dt + \sigma_i\, dz_i(t). \tag{A.70}$$

Below, we first derive the default intensity for a single firm assuming that the delay is known. We then solve for the default intensity for the more complicated case when the delay is unknown.

**A.1: Default intensity for a given known information lag.**

Consider a single firm with a known delay $\ell$ (we drop subscripts in this section for simplicity). As in DL (2001), since firm value is imperfectly observed, its default time is unpredictable conditional on the information available to investors. Here we provide a proof of this proposition in a framework that is simpler than the one investigated by DL (2001).

**Proposition 5** If the information lag is known to be $\ell$, then each firm’s default-date is an unpredictable stopping time with a default intensity defined on the set $\tau > t$ by:

$$\lambda(t) = \frac{1}{\sqrt{2\pi \sigma^2 \ell}} \left( \frac{y(t)}{\ell} \right) e^{-\frac{\left( m + \frac{y(t)}{\sigma \sqrt{\ell}} \right)^2}{2\sigma^2 \ell}} N \left( \frac{-m - \frac{y(t)}{\sigma \sqrt{\ell}}}{\sigma \sqrt{\ell}} \right). \tag{A.71}$$

**Proof:** We can apply the general result of Duffie and Lando (2001) which shows that the intensity is given by:

$$\lambda(t) = \frac{1}{2} \sigma^2 F_Y(y(t), \ell), \tag{A.72}$$
where $F(Y_t | Y_0, t)$ is the density of $Y_t$ conditional on both i) $Y_t$, $s \in (0, t)$ not reaching zero prior to $t$, and ii) the initial value $Y_0$. Here, $Y_t$ is an $(m, \sigma)$ arithmetic Brownian motion. The solution is:

$$F(Y_t | Y_0, t) \equiv \pi(Y_t | Y_0, \tau > t) = \frac{\pi(Y_t, \tau > t | Y_0)}{\pi(\tau > t | Y_0)}.$$  \hfill (A.73)

Both the numerator and denominator can be derived using well-known results for normally distributed variables (See, for example, Borodin and Salminen (1996):)

$$\pi(Y_t, \tau > t | Y_0) \equiv \pi \left( Y_t, \min_{s \in (0,t)} Y_s > 0 \bigg| Y_0 \right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} \left[ e^{-\frac{(Y_t-Y_0-mt)^2}{2\sigma^2t}} - e^{-\frac{2Y_0m}{\sigma^2} e^{-\frac{(Y_t+Y_0-mt)^2}{2\sigma^2t}}} \right].$$  \hfill (A.74)

$$\pi(\tau > t | Y_0) \equiv \pi \left( \min_{s \in (0,t)} Y_s > 0 \bigg| Y_0 \right)$$

$$= N \left( \frac{Y_0+mt}{\sigma\sqrt{t}} \right) - e^{-\frac{2Y_0m}{\sigma^2}} N \left( \frac{-Y_0+mt}{\sigma\sqrt{t}} \right).$$  \hfill (A.75)

Using these two results, we can explicitly compute the default intensity in equation (A.71) above.

Alternatively, this result can be demonstrated by noting that a default intensity is given by:

$$\lambda(t) \equiv \left( \frac{1}{dt} \right) \pi \left( \tau \in (t, t+dt) \bigg| \tau > t, Y(t-\ell) \right)$$

$$= \left( \frac{1}{dt} \right) \frac{\pi \left( \tau \in (t, t+dt) \bigg| Y(t-\ell) \right)}{\pi \left( \tau > t \bigg| Y(t-\ell) \right)},$$  \hfill (A.76)

where the denominator is obtained from equation (A.75) and the numerator is simply:

$$\left( \frac{1}{dt} \right) \pi \left( \tau \in (t, t+dt) \bigg| Y(t-\ell) \right) = \frac{\partial}{\partial \ell} \pi \left( \min_{s \in (t,t+\ell)} y(s) > 0 \bigg| y(t) \right) dt$$

$$= \frac{1}{\sqrt{2\pi\sigma^2\ell}} \left( \frac{y(t)}{\ell} \right) e^{-\frac{(m+\gamma(t))^2}{2\sigma^2\ell}}.$$  \hfill (A.77)

Hence, as in DL, conditional on the information available to investors, default arrivals are unpredictable, even though the firm value dynamics follows a diffusion process. Therefore, bond prices will take on the characteristics of those in a reduced form model. For example,
even short maturity bonds will command a credit spread. As in standard reduced-form models, the survival probability is given by the expectation $E_\ell[e^{-\int_T^\ell \lambda(s)ds}]$, since there is no jump in this value at the default date. Interestingly, even though equation (A.71) shows that the intensity is a highly non-linear function of the state variable and time, here we demonstrate that the survival probability possesses a simple analytic solution:

**Corollary 2** For a given default lag $\ell$, the survival probability $\pi_\ell (\tau > T|\tau > t, Y(t-\ell))$ conditional that default has not occurred by date-$t$ is given by:

$$
\pi_\ell (\tau > T|\tau > t, Y(t-\ell)) = \frac{E_\ell[e^{-\int_T^\ell \lambda(s)ds}]}{\pi (\tau > T|Y(t-\ell))}, \quad (A.78)
$$

where:

$$
\pi (\tau > u|Y_0) \equiv \pi (\min_{s\in(0,u)} Y_s > 0 | Y_0) = N\left(\frac{Y_0 + mu}{\sigma \sqrt{u}}\right) - e^{\frac{2\lambda_0 mu}{\sigma^2}} N\left(\frac{-Y_0 + mu}{\sigma \sqrt{u}}\right). \quad (A.79)
$$

**Proof:**

The first equality in equation (A.78) follows directly from, for example, Lando (1998), since for a given lag the intensity follows a diffusion process, and hence the no-jump condition is satisfied. The second equality follows by using Bayes’ rule twice:

$$
\pi (\tau > T|\tau > t, Y(t-\ell)) \pi (\tau > t|Y(t-\ell)) = \pi (\tau > T, \tau > t|Y(t-\ell))
$$

$$
= \pi (\tau > t|\tau > T, Y(t-\ell)) \pi (\tau > T|Y(t-\ell))
$$

$$
= \pi (\tau > T|Y(t-\ell)), \quad (A.80)
$$

where the last step follows from the fact that, if $\tau > T$, then necessarily $\tau > t$. Note that both the numerator and denominator of equation (A.78) can be expressed in terms of equation (A.79). \qed

Corollary 2 demonstrates that, conditional on knowing the information lag, default intensities and survival probabilities possess simple analytic solutions. One reason that such a simple solution obtains is that the intensity processes are continuous, since they are functions of continuous state variables. Below, we consider the more difficult case where the intensity of one firm jumps at the default dates of other firms.
A.2: Uncertainty about information quality and the updating of beliefs

We assume investors do not know whether the accounting information quality (i.e., the lag) is high or low. The filtration $\mathcal{F}_t$ representing the information available to agents is generated by i) the observation of the paths of signals $\{y_i(s), i \in (1,N)\}_{s \leq t}$, and ii) the history of defaults $\{1_{\{\tau_i < s\}}; i \in (1,N)\}_{s \leq t}$. We assume agents start with an initial prior about the quality of the information at date 0. Since the information quality is shared across all firms, investors update their priors conditional on observed defaults. More formally, we assume investors update $p^H(t) \equiv \pi \left( \ell = \ell^H \middle| \mathcal{F}_t \right)$ conditional on the observed history of defaults, where we have used the abbreviated notation: $\{\ell = \ell^H\}$ for $\{\ell_i = \ell^H_i \ \forall i\}$. As discussed previously, associated with each jump process $d1_{\{\tau_i < t\}} (i = 1, \ldots, N)$ are two intensity processes $\lambda^H_i(t), \lambda^L_i(t)$ defined on the set $\{\tau_i > t\}$ via

$$
\lambda^H_i(t) = \left( \frac{1}{dt} \right) E_t \left[ d1_{\{\tau_i < t\}} \middle| \mathcal{F}_t, \ell = \ell^H \right] \quad \text{(A.81)}
$$

$$
\lambda^L_i(t) = \left( \frac{1}{dt} \right) E_t \left[ d1_{\{\tau_i < t\}} \middle| \mathcal{F}_t, \ell = \ell^L \right] . \quad \text{(A.82)}
$$

While each firm may have different default intensities (i.e., $\lambda^H_i(t) \neq \lambda^H_j(t)$ for $i \neq j$), they all share in ‘accounting quality’ in that, if $\ell_i = \ell_i^H$, then $\ell_j = \ell_j^H$, and similarly for the low intensity case. This assumption insures that the default of one firm will carry information about the quality of the accounting information for firms that share this risk.\(^34\)

With these assumptions, investors will compute the intensity of firm $i$ defaulting on the set $t < \tau_i$ as:

$$
\overline{\lambda}_i(t) = p^H(t) \lambda^H_i(t) + \left(1 - p^H(t)\right) \lambda^L_i(t) . \quad \text{(A.83)}
$$

Note that this is a special case of our general reduced form setup. In particular, we can apply previous results to find the updating equation of investors priors:

**Proposition 6** Investors update the quality of beliefs conditional on the observed history of defaults according to:

$$
dp^H(t) = p^H(t) \left(1 - p^H(t)\right) \sum_{i=1}^{N} \overline{\lambda}_i(t) \left( \frac{\lambda^H_i(t) - \lambda^L_i(t)}{\overline{\lambda}_i(t)} \right) 1_{\{\tau_i > t\}} \left( d1_{\{\tau_i < t\}} - \overline{\lambda}_i(t) dt \right) . \quad \text{(A.84)}
$$

The interpretation is as before: As long as no jump are observed, agents update their beliefs towards the more optimistic scenario of a short lag (hence the negative drift in the updating

\(^34\)This assumption is made for simplicity, and can be weakened substantially. There only needs to be some non-zero correlation between lags of different firms to justify the Bayesian updating procedure.
process). When a jump occurs, agents will tilt their beliefs towards the longer lag (i.e., poor accounting information quality) scenario.

An implication of this proposition is that the $\mathcal{F}_t$ conditional default intensities $\tilde{\lambda}_i$ of all firms experience common jumps at times of defaults of individual firms. Note that equation (A.84) is a special case of the general reduced form model presented in Section 2. Therefore, all the results derived there apply to this framework. In particular, we derive the survival probabilities:

**Corollary 3** The survival probability with incomplete information is given by:

$$P(\tau_i > T) = p^H(t) E_t[e^{-\int_t^T \tilde{\lambda}_i^H(s) ds}] + (1 - p^H(t)) E_t[e^{-\int_t^T \tilde{\lambda}_i^L(s) ds}]$$

$$= p^H(t) \frac{\pi(\tau_i > T | Y(t - \bar{t}_i^H))}{\pi(\tau_i > t | Y(t - \bar{t}_i^H))} + (1 - p^H(t)) \frac{\pi(\tau_i > T | Y(t - \bar{t}_i^L))}{\pi(\tau_i > t | Y(t - \bar{t}_i^L))},$$

(A.85)

**Proof:** The result follows immediately from Proposition 1 and Corollary 2. $\square$

We have thus demonstrated in both a reduced-form framework and a structural framework how to model contagion due to jumps-to-default. In the next section, we consider a general equilibrium framework that parsimoniously demonstrates how the default of infinitesimal firms (in that aggregate consumption is unaffected) can generate a market-wide response without direct counter-party risk.
B Jump Risk Premia in a Simple Production Economy

Assuming that the perception of information quality is a common factor driving all risky bonds, one would expect this factor to command a systematic risk-premium (assuming bonds are a substantial part of the market portfolio). In this section we propose a simple general equilibrium production economy to identify the structure of the risk-premia that jump risk should carry. In particular, we investigate the production economy of Cox, Ingersoll and Ross (1985) with many identical agents (i.e., there exists a representative agent) who maximize their expected utility of consumption

$$J(t) = \max_{c,n} \mathbb{E} \left[ \int_t^\infty ds \, e^{\delta s} \frac{c^{1-\gamma}}{1-\gamma} |\mathcal{F}_t| \right]$$

(B.86)

by investing in a single production technology which may experience bad shocks. These shocks are modeled as rare events as in Ahn and Thomson (1988). In particular, the return on the risky technology is modeled as

$$d\eta = \mu \, dt + \sigma \, dz_t - \alpha \, dq_1(t), \quad \text{(B.87)}$$

where $q_1$ is a counting process associated with a Poisson process which represents the rare ‘bad’ events. Similar to our partial equilibrium framework, we assume that agents have imperfect information about the probability of such events. In particular, they know that the jump intensity is either $\lambda_1^H$ or $\lambda_1^L$ (with $\lambda_1^H > \lambda_1^L$), and form priors represented by $p^H(t) = \mathbb{E}[\lambda = \lambda^H | \mathcal{F}_t]$. Here, $\mathcal{F}_t$ represents all information available to agents at date-$t$. In addition to observing the return to the technology, we assume investors observe an independent source of information modeled as a point process $q_2$ which has intensity $\lambda_2^H$ or $\lambda_2^L$ (with $\lambda_2^H > \lambda_2^L$). While information arrival is independent of the technology, (i.e., $dq_1, dq_2 = 0$), we assume, as in the partial equilibrium model, that investors know that the vector of point processes $q = \{q_1, q_2\}$ has an intensity vector of either $\lambda^H = \{\lambda_1^H, \lambda_2^H\}$ or $\lambda^L = \{\lambda_1^L, \lambda_2^L\}$. Hence, in a manner similar to the partial equilibrium models investigated above, the process $q_2$ provides information about the actual intensity of the technological risk. This source of information can be interpreted as observed defaults of firms in the economy that are too small to directly affect aggregate production.\footnote{Alternatively we can think of this as modeling investment in a new technology, such as Nuclear energy, whose riskiness is not known perfectly. Agents are performing experiments which allow them to learn about the riskiness of the technology, but they themselves are independent from the return on the technology.} It allows the agents to update their beliefs about the riskiness of the technology, which in turn affects their consumption-investment decisions. Thus, information arrival has real effects, and in turn, the associated belief process commands a systematic risk-premium,
even though it does not directly affect the real return to the production technology. As a consequence, the risk-neutral default intensity will differ from the actual intensity.

As in previous sections, agents update their beliefs about the intensity of jumps by conditioning on information and actual observed jumps in the risky technology. This leads to the following updating of beliefs:

\[
    dp^H(t) = p^H(t) \left(1 - p^H(t)\right) \sum_{i=1}^{2} \frac{\lambda_i^u(t) - \lambda_i^L(t)}{\lambda_i(t)} \left(dp_i(t) - \lambda_i(t) \, dt\right). \tag{B.88}
\]

Here, the \(F_t\)-conditional intensity of an event \(dp_i(t) = 1\) is defined as

\[
    \lambda_i(t) = \lambda_i^u p^H(t) + \lambda_i^L \left(1 - p^H(t)\right). \tag{B.89}
\]

The agents choose to optimally allocate a proportion \(\theta\) of their wealth to production, to consume at rate \(c_t\), and to save the remainder by investing at the risk-free rate \(r\). As customary in these types of models, since agents are identical in equilibrium, net saving is zero, and thus the risk-free rate \(r\) actually represents the shadow borrowing and lending rate. Each agent’s dynamic wealth equation is:

\[
    dW_t = r_t W_t \, dt + \theta_t W_t \left(\frac{d\eta_t}{\eta_t} - r_t \, dt\right) - c_t \, dt. \tag{B.90}
\]

Using the Bellman equation satisfied by the value function we find that it is of the form:

\[
    J(t, W_t, p^H(t)) = e^{-\delta t} J(W_t, p^H(t)), \quad \text{where } J(W_t, p^H(t)) \text{ satisfies:}
\]

\[
    0 = \max_{c, \theta} \left\{ \frac{c^{1-\gamma}}{1-\gamma} - \delta J + J_w \left(r W + \theta W(\mu - r) - c\right) + \frac{1}{2} J_{ww} W^2 \theta^2 \sigma^2 \right. \\
    -J_{p^H} \left(p^H (1 - p^H) \sum_{i=1}^{2} \left[\lambda_i^u - \lambda_i^L\right]\right) + \lambda_i(p^H) \left[ J \left(W(1 - \alpha \theta), p^H \frac{\lambda_i^u}{\lambda_i}\right) - J \left(W, p^H\right)\right] \\
    + \lambda_i(p^H) \left[ J \left(W, p^H \frac{\lambda_i^u}{\lambda_i}\right) - J \left(W, p^H\right)\right]\right\}. \tag{B.91}
\]

The first order conditions (FOC) are:

\[
    0 = c^{\gamma} - (J_w) \tag{B.92}
\]

\[
    0 = W J_w (\mu - r) + W^2 J_{ww} \theta \sigma^2 - \alpha \lambda_i(p^H) W J_i(W(1 - \theta \alpha), p^H \frac{\lambda_i^u}{\lambda_i}), \tag{B.93}
\]

where \(J_i(x, y) = J_x(x, y)\). Since bonds are in zero net supply, it follows that, in equilibrium, \(\theta = 1\). Plugging the FOC’s into the HJB Equation and looking for a solution of the form,

\[
    J(W, p^H) = B(p^H) \frac{W^{1-\gamma}}{1-\gamma}, \tag{B.94}
\]

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we find that the Bellman equation reduces to

\[ p^H (1 - p^H) \sum_{i} (\lambda^H_i - \lambda^L_i) B'(p^H) = \gamma B^{1-\frac{1}{\gamma}} + \left( -\delta + (1 - \gamma)(\mu - \gamma \frac{c^2}{2}) \right) B \]

\[ + \lambda_1(p^H) \left( (1 - \alpha)^{1-\gamma} B(p^H \frac{\lambda^H}{\lambda_1}) - B(p^H) \right) + \lambda_2(p^H) \left( B(p^H \frac{\lambda^H}{\lambda_2}) - B(p^H) \right) , \] (B.95)

while the FOC’s reduce to

\[ c = B(p^H)^{-1/\gamma} W \] (B.96)

\[ r = \mu - \gamma \sigma^2 - \alpha (1 - \alpha)^{-\gamma} \lambda_1(p^H) \frac{B \left( p^H \frac{\lambda^H}{\lambda_1} \right)}{B(p^H)}. \] (B.97)

Note that \( p^H \) has absorbing boundaries at 0 and 1. Intuitively, if the market knows for sure whether the production technology is in a large jump state or small jump state, then there is no updating, and the model reduces back to that of Ahn and Thompson (1988). In these cases, we can solve for \( B(\cdot) \) in closed form:

\[ B(0) = \left( \frac{\delta - (1 - \gamma)(\mu - \gamma \frac{c^2}{2}) + \lambda^L_1 (1 - (1 - \alpha)^{1-\gamma})}{\gamma} \right)^{-\gamma} \] (B.98)

\[ B(1) = \left( \frac{\delta - (1 - \gamma)(\mu - \gamma \frac{c^2}{2}) + \lambda^H_1 (1 - (1 - \alpha)^{1-\gamma})}{\gamma} \right)^{-\gamma}. \] (B.99)

Since the value function of agents is a decreasing function of the probability that shocks occur with high frequency, we have the following:

**Lemma 5** For \( \gamma \neq 1 \), the function \( B(p) \) satisfies:

\[ \begin{align*}
    & B'(p^H) \leq 0 \quad \text{if} \quad 0 < \gamma < 1 \\
    & B'(p^H) \geq 0 \quad \text{if} \quad \gamma > 1.
\end{align*} \]

**Proof:** Intuitively, this lemma follows from the fact that expected utility is a decreasing function of the state variable \( p^H \). That is, \( \frac{\partial J(p^H, W)}{\partial p^H} = B'(p^H) \frac{W}{1-\gamma} \leq 0. \)

As a formal proof, note that the dynamics for \( p^H(t) \) imply that, for two different starting values \( p^H_a(0), p^H_b(0) \), with \( p^H_a(0) > p^H_b(0) \), then, path-by-path, \( p^H_a(t) > p^H_b(t) \) for all future dates \( t \). Hence, by first order stochastic dominance, it follows that:

\[ J(W(0), p^H_a(0)) \equiv \mathbb{E} \left[ \int_t^\infty ds e^{-\delta s} C_s(p^H_a(s))^{1-\gamma} \frac{1}{1-\gamma} \middle| W(0), p^H_a(0) \right] \]

\[ \leq \mathbb{E} \left[ \int_t^\infty ds e^{-\delta s} C_s(p^H_b(s))^{1-\gamma} \frac{1}{1-\gamma} \middle| W(0), p^H_b(0) \right] \] (B.100)
Furthermore, since path-by-path the above strategy is available, it follows that

\[
J(W(0), p_b^H(0)) = \mathbb{E} \left[ \int_t^\infty ds \, e^{-\delta s} \frac{C^*(p_b^H(s))^{1-\gamma}}{1-\gamma} \left| W(0), p_b^H(0) \right. \right. \\
\geq \mathbb{E} \left[ \int_t^\infty ds \, e^{-\delta s} \frac{C^*(p_b^H(s))^{1-\gamma}}{1-\gamma} \left| W(0), p_b^H(0) \right. \right. \\
\tag{B.101}
\]

Equations (B.100) and (B.101) together imply that \( J(W(0), p_b^H(0)) \geq J(W(0), p_\alpha^H(0)) \). □

In equilibrium it is well-known (Cox, Ingersoll and Ross (1985)) that the state price density is

\[
\Pi(t) = \frac{J_w(W_t, p^H(t))}{J_w(W_0, p^H(0))} = e^{-\int_0^t r_s ds \xi_t}, \tag{B.102}
\]

where we have defined the Radon-Nykodim density \( \xi_t = \frac{dQ}{dP} \bigg|_{0 \leq s \leq t} \). Applying Itô’s lemma we find that:

\[
\frac{d\xi_t}{\xi_t} = -\gamma \sigma d\xi_t + \left( (1 - \alpha)^{-\gamma} \frac{B(p^H \lambda_1^H)}{B(p^H)} - 1 \right) dM_1(t) + \left( \frac{B(p^H \lambda_2^H)}{B(p^H)} - 1 \right) dM_2(t), \tag{B.103}
\]

where we have defined the \((P, \mathcal{F}_t)\) martingales \( M_i(t) = q_i(t) - \int_0^t \lambda_i(s) ds \) \( i = 1, 2 \). Thus we find the risk-adjustment for both \( q_1 \) and \( q_2 \) risks.

**Proposition 7** The risk-neutral measure intensity for information risk is given by:

\[
\lambda_2^Q = \frac{B(p^H \lambda_2^H)}{B(p^H)}, \tag{B.104}
\]

The intensity for production jump risk is:

\[
\lambda_1^Q = (1 - \alpha)^{-\gamma} \frac{B(p^H \lambda_1^H)}{B(p^H)}. \tag{B.105}
\]

**Proof:** We apply Itô’s lemma to show that \( \mathbb{E}_t[d(\xi_t M_1(t))] = \xi_t \left( \lambda_1^Q(t) - \lambda_1(t) \right) dt \). The result then follows from the definition of \( M_i \), Girsanov’s theorem for point processes and the martingale characterization of intensity (see Brémaud (1981)). □

Note that, even for the special case \( \lambda_1^H = \lambda_2^H \) and \( \lambda_1^L = \lambda_2^L \), the risk-adjustments for the two point processes \( q_1 \) and \( q_2 \) will be different. This difference is due to the fact that only \( q_1 \) affects equilibrium wealth. Indeed, in the absence of uncertainty about the quality of information (i.e.,
when $p^H = 0$ or $p^H = 1$), we see that $q_2$ risk carries no risk-adjustment, whereas $q_1$ still does. This proposition demonstrates that for an economy populated by agents more risk-averse than the log-investors (i.e., $\gamma > 1$), the risk-neutral intensity is higher than the actual intensity. However, production jump-risk can command a positive jump risk-premium when investors are less risk-averse than log (due to the factor $(1 - \alpha)^{-\gamma} > 1$).

The equilibrium risk-free rate is given in equation (B.97). We see that the risk-free rate experiences shocks in response to arrival of new information $dq_2$ through $dp^H$. Solving the ODE B.95 numerically for the $B$ function, we find that for reasonable parameter values (in fact, all those we tried) the interest rate is a decreasing function of $p(t)$, i.e., $\frac{\partial r(p)}{\partial p} \leq 0$. Further, the numerical results suggest that the interest rate is a convex function of $p$ for investors that are more risk-averse than log.

In this simple production economy, arrival of bad news triggers a response in the risk-free rate similar to a ‘flight-to-quality’ even though the event itself does not affect the return on the production technology.
C Proofs

C.1: Proof of Lemma 1

Using equation (24) and summing over states-\( j \), we find

\[
d\left( \sum_{j=1}^{J} p^j(t) \right) = \sum_{i=1}^{N} \left( 1 - \sum_{j=1}^{J} p^j(t) \right) dM_i(t). \tag{C.106}
\]

Hence, the process \( \left( \sum_{j=1}^{J} p^j(t) \right) \) has an absorbing barrier at \( \left( \sum_{j=1}^{J} p^j(t) \right) = 1 \). Given the initial condition \( \sum_{j=1}^{J} p^j(t_0) = 1 \), we conclude \( \sum_{j=1}^{J} p^j(t) = 1 \) a.s. \( \forall t \geq t_0 \). Furthermore, as is evident from equation (23), \( p^j(t) \) has an absorbing barrier at zero. Hence, \( p^j(t) \) cannot cross zero continuously. Finally, at a jump date, each of the \( p^j(t) \) jump to the value \( p^j(\tau_i) = p^j(\tau_i^-) \lambda_i(\tau_i^-) / \lambda_i(\tau_i^+) \), which is greater than zero since the intensities \( \lambda_i(\tau_i^-) \) are constrained to be positive. Hence, given the initial condition \( p^j(t_0) > 0, \forall j \), we conclude that \( p^j(t) > 0 \) a.s. \( \forall t \geq t_0 \). Finally, since the sum \( \sum_{j=1}^{J} p^j(t) = 1 \) for all dates-\( t \), it necessarily follows that \( p^j(t) < 1 \) a.s. \( \forall t \geq t_0 \).

C.2: Proof of Lemma 2

Using the generalized Ito formula for jump processes and equation (23), we have:

\[
d \left( p^j(t) 1_{\{\tau_i \leq t\}} \right) = 1_{\{\tau_i \leq t\}} dp^j(t) + p^j(t^-) d1_{\{\tau_i \leq t\}} + \Delta p^j(t) 1_{\{\tau_i < t\} \}, \tag{C.107}
\]

where we have used equation (20) in the last step. Thus it follows from its definition in equation (25) that

\[
dM_{ij}(t) = d \left( p^j(t) 1_{\{\tau_i \leq t\}} \right) - \lambda_{ij}(t)p^j(t) 1_{\{\tau_i > t\}} dt
\]

\[
= 1_{\{\tau_i \leq t\}} dp^j(t) + p^j(t^-) \left( \frac{\lambda_i(t^-)}{\lambda_i(t^-)} \right) dM_i(t), \tag{C.108}
\]

which is clearly a P-martingale. □
C.3: Proof of Proposition 1

Note that since for all dates \( t \) we have \( \sum_{j=1}^{J} p^j(t) = 1 \) a.s., it follows that:

\[
E_t \left[ 1_{\{\tau_i > T\}} \right] = \sum_{j=1}^{J} E_t \left[ p^j(T) 1_{\{\tau_i > T\}} \right].
\]  
(C.109)

Comparing equations (27) and (C.109) we see that the proposition is proved if we can demonstrate that

\[
p^j(t) 1_{\{\tau_i > t\}} E_t \left[ e^{- \int_t^T \lambda_{ij}(s) ds} \right] = E_t \left[ p^j(T) 1_{\{\tau_i > T\}} \right].
\]  
(C.110)

It is convenient to define \( p^j_i(s) \equiv p^j(s) 1_{\{\tau_i > s\}} \), and \( I_i^j(t) = E_d [ e^{- \int_0^T \lambda_{ij}(s) ds} ] \). Hence, it is sufficient to prove that \( p^j_i(t) I_i^j(t) = E_d [ e^{- \int_0^T \lambda_{ij}(s) ds} ] \) is a P-martingale. It therefore follows that:

\[
d I_i^j(t) - \lambda_{ij}(t) I_i^j(t) dt = dm_t
\]  
(C.111)

for some P-martingale \( m(t) \). Further, from equation (25) we have

\[
d p^j_i(t) \equiv d \left( p^j_i(t) 1_{\{\tau_i > t\}} \right)
= \quad \left( p^j_i(t) \left( 1 - 1_{\{\tau_i < t\}} \right) \right)
= \quad dp^j_i(t) - \left( dM_i^j(t) + \lambda_{ij}(t) p^j_i(t) dt \right).
\]  
(C.112)

Combining equations (C.111)-(C.112), and using the NCJ condition \( \Delta I_i^j(t) \Delta p^j_i(t) = 0 \) (which follows from (A1)), we find

\[
d \left( p^j_i(t) I_i^j(t) \right) = p^j_i(t^{-}) dI_i^j(t) + I_i^j(t^{-}) dp^j_i(t) + \Delta I_i^j(t) \Delta p^j_i(t)
= \quad p^j_i(t^{-}) \left( \lambda_{ij}(t) I_i^j(t) dt + dm_t \right) + I_i^j(t^{-}) \left( dp^j_i(t) - dM_i^j(t) - \lambda_{ij}(t) p^j_i(t) dt \right)
= \quad p^j_i(t^{-}) dm_t + I_i^j(t) dp^j_i(t) - I_i^j(t^{-}) dM_i^j(t),
\]

Hence \( p^j_i(t) I_i^j(t) \) is a P-martingale. Therefore, it follows that:

\[
p^j_i(t) I_i^j(t) = E_t \left[ p^j_i(T) I_i^j(T) \right] = E_t \left[ p^j_i(T) \right],
\]  
(C.113)

(since \( I_i^j(T) = 1 \)), which is the desired result. \( \square \)
C.4: Proof of Lemma 4

We first start by proving a closely related lemma.

**Lemma 6** The process \( H_{k_j}(t) \) defined by

\[
dH_{k_j}(t) \equiv d \left( \xi^j(t) p^j(t) 1_{\{\gamma_k > t\}} \right) + \left( 1 + \tilde{\gamma}_{k_j} \right) \lambda_{k_j}(t) \xi^j(t) p^j(t) 1_{\{\gamma_k > t\}} \right) dt \tag{C.114}
\]

is a \( P \)-martingale.

**Proof:**

Applying Itô’s lemma, noting that \( p^j(t) 1_{\{\gamma_k < t\}} = p^j(t) \left( 1 - 1_{\{\gamma_k > t\}} \right) \), and using Lemma 2 we find:

\[
d \left( \xi^j(t) p^j(t) 1_{\{\gamma_k > t\}} \right)
= \xi^j(t^-) d \left( p^j(t) 1_{\{\gamma_k > t\}} \right) + p^j(t^-) 1_{\{\gamma_k > t\}} d \xi^j(t) + \Delta \left( p^j(t) 1_{\{\gamma_k > t\}} \right) \Delta \xi^j(t)
= \xi^j(t^-) d \left( p^j(t) \left( 1 - 1_{\{\gamma_k < t\}} \right) \right) + p^j(t^-) 1_{\{\gamma_k > t\}} d \xi^j(t) + \Delta \left( p^j(t) 1_{\{\gamma_k > t\}} \right) \Delta \xi^j(t)
= \xi^j(t^-) \left( dp^j(t) - dM^j_k(t) - \lambda_{k_j}(t)p^j(t) 1_{\{\gamma_k > t\}} dt \right) + \sum_{i=1}^{N} \gamma_{ij} d 1_{\{\gamma_i \leq t\}} - \lambda_{ij}(t) \gamma^j_j 1_{\{\gamma_k > t\}} dt + \Delta \left( p^j(t) 1_{\{\gamma_k > t\}} \right) \Delta \xi^j(t). \tag{C.115}
\]

Now note that

\[
\Delta \left( p^j(t) 1_{\{\gamma_k > t\}} \right) = 1_{\{\gamma_k > t\}} \Delta p^j(t) - p^j(t^-) d 1_{\{\gamma_k > t\}}
= p^j(t^-) 1_{\{\gamma_k > t\}} \sum_{i=1}^{N} \gamma_{ij} d 1_{\{\gamma_i \leq t\}} - p^j(t^-) d 1_{\{\gamma_k > t\}}. \tag{C.116}
\]

Thus

\[
\Delta \left( p^j(t) 1_{\{\gamma_k > t\}} \right) \Delta \xi^j(t) = p^j(t^-) 1_{\{\gamma_k > t\}} \xi^j(t^-) \sum_{i=1}^{N} \gamma_{ij} \alpha_i^j(t^-) d 1_{\{\gamma_i \leq t\}} - \xi^j(t^-) p^j(t^-) \gamma_{k_j} d 1_{\{\gamma_k \leq t\}}.
\]

Combining this result with equation (C.115) gives

\[
dH_{k_j}(t) = \xi^j(t^-) \left( dp^j(t) - dM^j_k(t) \right) + \sum_{i=1}^{N} \gamma_{ij} \alpha_i^j(t^-) \left( \gamma_{ij} d 1_{\{\gamma_i \leq t\}} - \gamma_{ji} \lambda_{k_j}(t^-) 1_{\{\gamma_k > t\}} dt \right) \tag{C.117}
\]

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which is clearly a P-martingale.

Lemma 7  The process \( p^j(t) \xi^j(t) \) is a P-martingale.

Proof: Applying Itô’s lemma we find:

\[
d (p^j(t) \xi^j(t)) = \xi^j(t^-) dp^j(t) + p^j(t^-) \xi^j(t^-) \sum_{i=1}^{N} \frac{\lambda_{ij}(t^-)}{\lambda_i(t^-)} \left( \gamma_{ij} \mathbf{1}_{\{\tau_i \leq t\}} - \gamma_{ij} \overline{\lambda}_i(t) dt \right). \tag{C.118}
\]

The result then follows from the fact that the right hand side is the sum of P-martingales.

With these results, we can now provide a Proof of Lemma 4:

From the definitions of \( q^j \), \( \xi \) and \( H_{kj} \), we have

\[
E_t^Q \left[ d \left( q^j(t) \mathbf{1}_{\{\tau_k \leq t\}} \right) \right] = E_t^P \left[ \frac{d \left( \xi(t) q^j(t) \mathbf{1}_{\{\tau_k \leq t\}} \right)}{\xi(t^-)} \right]
\]

\[
= E_t^P \left[ \frac{d \left( \xi^c(t) p^j(t) \xi^j(t) (1 - \mathbf{1}_{\{\tau_k > t\}}) \right)}{\xi(t^-)} \right]
\]

\[
= E_t^P \left[ \frac{d \left( \xi^c(t) \left( p^j(t) \xi^j(t) - H_{kj}(t) \right) \right)}{\xi(t^-)} \right] + (1 + \gamma_{kj}) \lambda_{kj}(t) q^j(t) \mathbf{1}_{\{\tau_k > t\}} dt, \tag{C.119}
\]

The proof is completed by observing that from Lemmas 7 and 6 both \( p^j \xi^j \) and \( H_{kj}(t) \) are quadratic-pure-jump P-martingales, and that, by definition, \( \xi(t) \) is a continuous P-martingale. Since the product of a quadratic pure jump martingale with a continuous (and thus orthogonal) martingale is a martingale the result follows.

C.5:  Proof of Proposition 2

Note that by definition of the \( Q \)-measure:

\[
E_t^Q \left[ \mathbf{1}_{\{\tau_k > T\}} \right] = E_t^P \left[ \frac{\xi(T)}{\xi(t)} \mathbf{1}_{\{\tau_k > T\}} \right]. \tag{C.120}
\]

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Hence, in order to demonstrate equation (39), we need to prove that

$$
\mathbb{E}^P_t \left[ \xi(T) 1_{\{ \tau_k > T \}} \right] = \xi(t) 1_{\{ \tau_k > t \}} \sum_{j=1}^{J} q^j(t) Y_{kj}(t),
$$

(C.121)

where we have defined $Y_{kj}(t) \equiv \mathbb{E}^Q_t \left[ e^{- \int_t^T \lambda^Q_{kj}(s) \, ds} \right]$. Since the risk-neutral probabilities sum to unity, we can rewrite the left hand side of the equation as

$$
\sum_{j=1}^{J} \mathbb{E}^P_t \left[ q^j(T) 1_{\{ \tau_k > T \}} \right] = \sum_{j=1}^{J} \xi(t) 1_{\{ \tau_k > t \}} q^j(t) Y_{kj}(t).
$$

(C.122)

It is convenient to define $Z_{kj}(t) \equiv q^j(t) \xi(t) 1_{\{ \tau_k > t \}} Y_{kj}(t) = \left( p^j(t) \xi(t) 1_{\{ \tau_k > t \}} \right) \left( \xi^c(t) Y_{kj}(t) \right)$. Noting that $Y_{kj}(T) = 1$, we see that it is sufficient to demonstrate that

$$
\sum_{j=1}^{J} \mathbb{E}^P_t \left[ Z_{kj}(T) \right] = \sum_{j=1}^{J} Z_{kj}(t),
$$

(C.123)

which necessarily holds if each individual component satisfies

$$
\mathbb{E}^P_t \left[ Z_{kj}(T) \right] = Z_{kj}(t) \quad \forall j.
$$

(C.124)

Thus to prove the result it is sufficient to show that $Z_{kj}(t)$ is a $P$-martingale for all states $j$. To that effect, note that by definition of the $Q$ measure,

$$
\xi^c(t) Y_{kj}(t) e^{- \int_0^t \lambda^Q_{kj}(s) \, ds} = \mathbb{E}^P_t \left[ \xi^c(T) e^{- \int_0^T \lambda^Q_{kj}(s) \, ds} \right]
$$

is a continuous $P$-martingale. Thus

$$
\frac{d(\xi^c(t) Y_{kj}(t))}{(\xi^c(t^-) Y_{kj}(t^-))} = \lambda^Q_{kj}(t) \, dt + dm^c(t)
$$

(C.125)

for some continuous $P$-martingale $m^c(t)$. Further, using Lemma 6 we have

$$
\frac{d \left( \xi^j(t) p^j(t) 1_{\{ \tau_k > t \}} \right)}{\xi^j(t^-) p^j(t^-) 1_{\{ \tau_k > t^- \}}} = - \left( \lambda^Q_{kj}(t) \right) \, dt + dm^j(t),
$$

(C.126)

where $m^j(t)$ is a quadratic pure jump $P$-martingale (see Protter (1995) for a definition). Combining equations C.125 and C.126 and the fact that quadratic-pure-jump and continuous martingales are orthogonal (their quadratic co-variation process is zero, see Protter (1995)), we get the desired result.

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