A mean-variance benchmark for intertemporal portfolio theory

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Abstract

By simply reinterpreting the symbols, the familiar one period mean-variance portfolio theory can also apply to dynamic and intertemporal problems in incomplete markets, with non-marketed income. This paper shows how. The centerpiece is a dynamic three-fund theorem. Intertemporal investors first hedge non-traded income and preference shocks, and then split their portfolios between a riskless asset and a risky asset. The riskless asset is an indexed perpetuity, which pays a constant real coupon every period. The risky asset is a claim to the aggregate consumption stream. The risky asset, and all investors’ optimal portfolios, lie on a “long run” version of a mean-variance frontier. I calculate long-run mean-variance efficient portfolios in the familiar lognormal iid setting, with a predictable market return, across size, book-to-market and momentum portfolios, across bond portfolios, and I calculate portfolios that hedge labor income.

1 Introduction

The basic idea

This paper studies long-horizon portfolio problems. I allow dynamics in asset returns and dynamic trading; markets are potentially incomplete, and I include non-market income and preference shocks.

I mix two ingredients: First, I focus on the optimal stream of final payoffs or dividends, rather than focusing on the trading strategy that delivers those payoffs, following the Cox and Huang (1989), or, ultimately, Arrow and Debreu (1954) approach. Second, I use the mean-variance approximation that is so useful in one-period problems to easily characterize the dynamic problem with incomplete markets.

The basic idea is simply to treat time and probability symmetrically. This approach maps two-period intuition to multiperiod models directly (treating dates like states) rather than by the conventional method of multiplying together one-period returns. I define an “expectation” that sums over time as well as states of nature,

\[ \mathcal{E}(x) \equiv E \sum_{t=1}^{\infty} \beta^t x_t. \]

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Thus, for example, I write the price of a security in terms of given a discount factor \( m_t \) and a payoff (dividend) stream \( x_t \)

\[
p(x) = E \sum_{t=1}^{\infty} \beta^t m_t x_t = \mathcal{E}(mx).
\]

In this way, we can apply all of asset pricing and portfolio theory that naturally flows from \( p = E(mx) \) in two-period models to a multi-period environment, by simply reinterpreting the symbols.

Applying this idea to portfolio theory, I get an extension of the familiar mean-variance characterization of optimal portfolios. Each investor 1) shorts a hedge portfolio for non-market income or preference-shock risk, and then splits his investment between 2) a “risk free rate,” which is a real perpetuity or the corresponding zero beta security if the perpetuity is not traded, and 3) a payoff that is on the “long-run mean/long-run variance frontier,” more or less so according to the investor’s risk aversion. The “long-run” frontier is defined using \( \mathcal{E} \) and dividends to a one dollar investment in the place of \( E \) and one-period returns. Long-run variance prizes stability over time as well as across states of nature.

If all investors follow this portfolio theory and without non-market income, the common risky portfolio is the market portfolio, which pays aggregate consumption as its dividend. In this case, all investors split their payoffs between the real perpetuity and aggregate consumption. When investors have non-marketed income, the market portfolio loses its special status and is no longer mean-variance efficient. (The basic idea comes from Hansen 1987. Magill and Quinzii 2000 also specify a quadratic utility investor and characterize the optimal portfolio with a nonstochastic bliss point in terms of a “least variable income stream” which minimizes long-run variance.)

I work out the classic mean-variance portfolio problem with \( \mathcal{E} \) notation, interpret the results, and I present five applications 1) the standard iid lognormal environment 2) the choice of value vs. growth 3) optimal market-timing when the market return is predictable 4) optimal bond portfolios for long-run investors interested in real risks but facing nominal bonds 5) optimal hedging of labor / non-market income risks.

Why is this interesting?

Dynamic portfolio theory is important. The fact that returns are not i.i.d. obviously has the potential for a deep impact on portfolio theory. At a minimum, it means that “long-run” and “short-run” investors may opt for different strategies, that asset demands may reflect hedging motives, and it opens the door to dynamic strategies that can get in and out at the right time. Of course, the average investor must hold the market, and it is not obvious which dimensions of investor heterogeneity drive one investor to dynamically invest more in a given security, while another investor optimally invests less. Still, and especially following the tradition of ignoring this general equilibrium consideration, the potential for an important effect of non-iid returns on portfolio theory is there.

Dynamic portfolio theory is hard. To cite one example, consider the question of how should one optimally invest in the market vs. a riskfree rate, given that market returns seem predictable from dividend-price ratios, a classic simple Merton . This problem has been attacked by Kim and Omberg (1996), Brennan, Schwartz and Lagnado (1997), Brennan (1998), Barberis (1999), Brandt (1999), Campbell and Vicera (1999), Brennan and Xia (2000, 2001, 2002, 2002a), Sangvinatsos and Wachter (2003), and Wachter (2003), Liu (2004), and many others. Though they typically only consider two returns, simple AR(1) models for the forecasting variables, and
often utility of terminal wealth with no intermediate consumption, these papers are technically complex.

By contrast, the approach in this paper is easy. I am able to find the optimal payoffs or the optimal dividend stream without having to derive the portfolio strategy that supports those payoffs. The standard approaches, including all of the above-cited papers, must solve both problems at once.

Portfolio theory is easy in complete markets. It’s typically easy to find a discount factor or contingent-claims price \( x_t^* \) that prices assets, i.e. \( p(x_t) = E_0 \sum_{t=1}^{\infty} \beta^t x_t^* x_t \). In a complete market we can then state the portfolio problem as \( \max_{\{c_t\}} E \sum_t \beta^t u(c_t) \) s.t. \( W_0 = p(c_t) = E \sum_t \beta^t x_t^* c_t \) and solve it trivially as \( c_t = u'^{-1}(\lambda x_t^*) \) where \( \lambda \) is the Lagrange multiplier on the wealth constraint. The crucial simplification is that in this construction we do not have to write out the potentially complex dynamic portfolio strategy that supports the optimum consumption \( \{c_t\} \), we only need to verify that the contingent-claim value of the result \( c_t \) is not greater than initial wealth.

Alas, markets are typically not complete in dynamic settings. Innovations to forecasting variable are typically not traded, though the investor would like to hedge them. Though \( \lambda x_t^* \) is a traded payoff, \( u'^{-1}(\lambda x_t^*) \) is not traded, since \( u'^{-1}(\cdot) \) is a nonlinear function.

Quadratic utility provides one answer to this desire to keep the simplicity of the complete-markets approach. With quadratic utility, \( u'(\cdot) \) is a linear function, and so is its inverse. Linear functions of traded payoffs are also traded, so we know that \( u'^{-1}(\lambda x_t^*) \) is the optimum consumption stream. Once again we do not have to actually compute a dynamic portfolio strategy that produces \( c_t \) from the underlying assets.

In a sense, then, the theory in this paper is easy because I stop and declare victory just before the hard part begins. I regard this simplification as a feature, not a bug. Classic one-period mean-variance analysis also stops and declares victory just before the hard part begins. Constructing the mean-variance efficient portfolio is also a difficult problem, resulting from the difficulty of estimating large covariance matrices and saying anything at all about mean returns. It is approached differently for different asset classes, data sets, time horizons, and conditioning information sets and all the other peculiarities of a given application. In fact, though Markowitz (1952) derived the mean-variance frontier more than 50 years ago, we still have no settled and satisfactory way to actually compute that frontier. One can regard much of the money management industry as selling one or another solution to the problem.

This separation of the economic characterization of the final payoffs from the technical problem of constructing a portfolio that achieves those payoffs in a given application is a very useful part of classical portfolio analysis, and accounts for much of its success. It remains useful to say “the investor wants a mean-variance efficient portfolio” and to characterize assets by means and variances, even though we do not in the same breath actually construct mean-variance efficient portfolios. One of my aims is to bring the same useful baseline to dynamic intertemporal portfolio theory, to focus on final payoffs and say “the investor wants a long-run mean-variance efficient portfolio” in a dynamic and incomplete market.

One reason the separation is useful is because the portfolio is not unique, and depends on the details of the application. For example, the optimal portfolio for an investor whose utility drops precipitously at a given level of wealth might consist of an investment in the index plus a protective put. One can achieve that payoff by buying stocks and a protective put, by buying
a bond and a call option, by following a dynamic trading strategy in stock and bond that replicates the option, or by a static investment in an actively-managed fund which offers this payoff, perhaps hedging it in any of these ways. The optimal portfolio is different, and depends whether options or funds are available, as well as the nature of trading costs, institutional restrictions, etc. The standard approach requires one to re-solve the whole problem every time a slightly different asset is added to the mix.

The final-payoff view of portfolio theory is also often clearer and economically more interpretable than the standard dynamic portfolio analysis. For example, when one examines payoffs it is immediately obvious that an indexed perpetuity is the riskless asset for an infinitely lived consumer. In the standard Merton (1971a) intertemporal approach we instead think of an indexed perpetuity as an asset that happens to hedge changes in the intertemporal opportunity set: long bond prices go up when interest rates go down. The fact that a 10 year bond is the riskless asset for an investor with a 10 year horizon, or an indexed perpetuity is the riskless asset for an investor with an infinite horizon, only emerges after a lot of algebra. (See Campbell and Viceira 2001 and Wachter 2003).

One is eventually interested in utility functions more realistic than the quadratic, of course. However, mean-variance analysis survives in one-period contexts in part because it is a good approximation; marginal utility is not that nonlinear over the typical range of outcomes in many interesting problems. It may also survive, and in fact completely dominate all practical or industry analysis because of its conceptual simplicity. Even “portfolio insurance” or the addition of options to one-period portfolios to adapt the portfolio to tastes and prices for disaster insurance, while conceptually and analytically simple and in theory of great value to many investors, is completely absent from standard applied portfolio analysis. The long-horizon mean-variance frontier may be interesting beyond its formal connection to quadratic utility, just as one-year return mean-variance frontiers seem interesting beyond their formal connection to quadratic utility.

Quadratic utility may also be a useful approximation or a good starting point in many dynamic intertemporal problems. Campbell and Viceira (1999, 2001) keep nonlinear utility but linearize the budget constraint, often obtaining excellent approximations. I linearize the utility function keeping the exact budget constraint, which may produce a similar approximation. The vast majority of portfolio theory studies approximate (algebraic or numerical) solutions to exact problems; I study exact solutions to an approximation (quadratic) to the problem, following the example of Hansen and Sargent (2004). As Hansen and Sargent show, the quadratic objective is not nearly as constraining as one might think; all sorts of interesting preferences can technologies can be captured by clever variations. If one is really interested in the nonlinearities of marginal utility, the only answer is to work out the exact solution and then examine how good or bad the quadratic approximation is. And since we are looking at long-horizon questions in which risks may take investors far from a linearization point, one can easily construct examples that will lead quadratic utility to give poor approximations. The quadratic utility results may also represent a good starting point for numerical solution of one-step portfolio problems in incomplete markets with nonquadratic utility.

Nonmarket income

I emphasize the incorporation of risky labor or non-market income in portfolio theory and in an application. For a typical investor, hedging the idiosyncratic (non-priced) component of outside-income risk is the most important task. Furthermore it is free – this portion of the
portfolio is zero-cost, and involves zero expected excess return. It amounts to buying actuarially fair insurance. To the extent that outside income contains priced as well as idiosyncratic components, its presence can profoundly affect the optimal mean-variance allocation. (See for example Heaton and Lucas 2000). Strangely, hedging non-market income is by and large ignored in almost all portfolio theory applications.

Doing a good job of this hedging is not easy. The optimal hedge is investor-specific. It requires an understanding of the nonpriced factors in returns, the factors that do not generate any alpha. Yet all of finance academia and industry is focused on identifying priced factors. In a natural industrial organization of the money management industry, one set of firms would develop expertise in this hedging, selling individual-specific services for reasonable fees. Another set of firms would specialize in the orthogonal skills of finding mean-variance efficient investments that require no personal tailoring, as all investors are the same once they have hedged. It is interesting that this organization has not emerged.

A central problem of including non-market income in the standard return-based portfolio approach is that we observe the stream of income, but not its value. To apply simple portfolio theory, say to compute the covariance matrix of returns, one has to make some unpalatable assumption to turn the stream into a price. For example, Jagannathan and Wang (1996) assume an AR(1) labor income process, and in particular that no variables other than labor income are good for forecasting future labor income, and a constant discount rate. Then, the price of the labor income stream moves one for one with the income stream itself so that labor income growth measures the return to human capital. Campbell (1996) uses multivariate forecasts and assumes that the labor income stream is discounted at the same expected return as stocks.

The alternative approach is to include labor income and a set of variables that forecast labor income as Mertonian state variables to a dynamic portfolio theory. Heaton and Lucas (2000) and Davis, Kubler and Willen (2005) follow this approach. The difficulty of course is that one needs to understand the state variables that govern the dynamics of labor income; one can’t simply throw labor income flows or returns into a portfolio optimizer.

The approach of focusing directly on the stream of payoffs may makes it more practical to bring non-market income into simple portfolio theory. We can compute the required long-run covariance matrix with no information on the value of non-market income streams. We first construct a hedge portfolio for labor income by the long-run analogue to a simple regression. Since it is an asset payoff, we can find the value of this hedge portfolio, and only the hedge portfolio affects the remaining asset allocation decision.

The big picture

This work falls into several larger-scale trends in financial research. At a basic level, the empirical finding that returns are not i.i.d. and that discount rate news rather than cashflow news drives large parts of price variation requires a rewriting of most procedures in finance, including asset pricing, corporate finance such as cost-of-capital calculations, as well as portfolio theory.

Most of finance and a lot of economics amounts to ways of stretching two-period (or two-good) intuition to fit infinite-period (or infinite-good) models. The two-period world with a liquidating dividend is easy to understand. We usually apply this model to infinite periods by finding two-period implications for returns, and then multiplying those returns together. We use the next period’s price plus dividend in place of the “liquidating dividend” of the true two-
period model. This approach works if the world is i.i.d., as then tomorrow’s price moves one for one with tomorrow’s dividend. It continues to work reasonably well if returns are i.i.d., so that tomorrow’s price moves on news of future cashflows rather than tomorrow’s cashflow itself. Alas, if tomorrow’s price moves on discount rate or expected return news, then this standard approach runs into trouble. What sense does it make to treat the covariance of tomorrow’s price (expected return) with tomorrow’s market price (expected return) as exogenous in determining today’s price and expected return? Of course there is no mathematical difficulty, as anything one can say about the long horizon is equivalent to what one can say about the sequence of one-period returns on the way to that long horizon. The difficulty is in coming up with useful simple model for applications.

In this context, I apply two period modeling and intuition to a multiple period problem in a different way. Rather than write a two-period model for returns and then cumulate returns, I treat time and state symmetrically, to use two-period intuition and mechanics directly on prices and the stream of following payoffs. All the beautiful intuition of two period models applies directly.

This focus on the payoff stream in portfolio theory mirrors a renewed interest in payoff streams in asset pricing, for example Menzly, Santos and Veronesi (2004), Hansen, Heaton and Li (2005), Bansal, Dittmar and Lundblad (2004) Bansal and Yaron (2004), Lettau and Wachter (2005) and others. Since beta driven by the movement of tomorrow’s price with factor prices is 99% endogenous, these authors try to link asset pricing with fundamental movements in the underlying cashflows. One can forsee a day that “asset pricing” makes prices the central endogenous variable, rather than an ad-hoc sorting characteristic (book/market ratio), and in which the stream of cashflows is the central exogenous variable, and one period returns are barely mentioned. This approach is nothing new from a pure theory point of view; we have been able to write $p_t = E_t \sum_{j=1}^{\infty} m_{t+j} d_{t+j}$ as long as we have been able to write $1 = E_t(m_{t+1} R_{t+1})$. The challenge lies in specifying workable applications.

## 2 Payoffs and prices

This section sets up the infinite-period interpretation of the standard structure of asset pricing models. I take that standard structure from Cochrane (2004) chapter 4 and originally Hansen and Richard (1986).

### 2.1 Payoffs and prices

$x$ denotes a payoff. In a one-period setting, the payoff is the amount $x_t$ that an investor receives at date 1, in each state of nature, for a time-zero price $p$. If we want to be explicit about uncertainty, we write $x = \{x_1(s)\}$ where $s$ indexes states of nature at time 1. In an infinite period setting, the payoffs are the streams of dividends $\{x_1, x_2, \ldots\}$ or $\{x_1 dt\}$ in continuous time resulting from an initial purchase; $\{x_t(s^t)\}$ if we want to be explicit about states as well as time. In a mixed setting, with intermediate consumption plus a value of terminal wealth, the payoffs are $\{x_1, x_2, \ldots, x_T, W_T\}$.
Returns are price-one payoffs. We can form them by \( x/p \). In a one period setting, the return in each state is the payoff in that state dividend by the initial price

one period: \( R = \{ \hat{R}_1(s) \} = \{ x_1(s)/p_0 \} \).

The many-period counterpart is the payoff in each state and date divided by initial price,

infinite period: \( y = \{ y_t \} = \{ x_t/p_0 \} \)

There is one important difference. In the infinite period model, the “return” to a particular date has the units of a yield, or coupon rate, it is a number like 0.04 not 1.04. I use the notation \( y \) and the word “yield” rather than the word “return” to help keep the typical units in mind.

Excess returns are price-zero payoffs, which we can construct by differencing any two returns.

The natural risk free payoff is one in all states and dates, a perpetuity

\( x^f = \{ 1 \} \).

In a one period model, the risk free assets pays one unit in each state. Now it pays one unit in each state and date. The risk free yield is

\( y^f = 1/p(\{ 1 \}) \)

where \( p(\cdot) \) means “price of.” The riskfree return is also a number like 0.01 not a number like the 1.01. We often divide by \( R^f \) in one-period models, and this operation has small effects on the result. Dividing by \( y^f \) is the same operation, but has larger effects on the numbers.

\( X \) denotes the payoff space of all dividend streams that investors can buy, \( Y \) the set of price-one returns or yields , and \( Y^e \) the set of excess returns,

\( Y \equiv \{ y \in X : p(y) = 1 \} \),

\( Y^e \equiv \{ y^e \in X : p(y^e) = 0 \} \).

I let investors buy any portfolio of payoffs, which means that \( X \) is closed under linear combinations.

\( x, z \in X \rightarrow ax + bz \in X \) (1)

If investors can buy IBM and eat its dividends, and buy Microsoft and eat its dividends, then they can split their money between the two and eat the combined dividend stream.

In an intertemporal context, we also want to allow dynamic trading, or equivalently we want to allow entrepreneurs to sell managed portfolios (the investor doesn’t have to do any dynamic trading if we have a rich enough payoff space). If \( \{ x_t \} \in X \), we also allow

\( \{ x_0, ..., x_{t-1}, x_t + p_t(x) \} \in X \) (2)

\( \{ 0, ..., -p_t(x), x_{t+1}, x_{t+2} ... \} \in X \) (3)

This assumption is related to market completeness. In the one-period model, if investors can change payoffs to sell off claims on any set of states and put the result in another state, we have complete markets. Dynamic trading can similarly lead to complete markets in an intertemporal context. However, the presence of dynamic trading is not enough in general to make markets
complete. If information variables do not correspond to traded assets if the decision to sell an asset in (2) or buy one in (3) is made based on the realization of random variables that are not traded, dynamic trading will not be enough to fully complete the markets.

As usual, we limit dynamic trading so that the investor cannot generate arbitrage opportunities. The time-zero value of wealth must tend to zero \( \lim_{T \to \infty} p(W_T) = 0 \), and continuous time trading strategies must satisfy the standard integrability limitations on trading weights. (See for example Duffie 2001.)

This assumption seems dramatically to expand the set of assets we need to consider. However, we usually condense the thousands of available individual assets to a small number of portfolios that we think adequately capture the cross-section of returns. In a similar fashion, we may condense the infinite variety of trading strategies to a few that we think adequately approximate the range of important opportunities.

2.2 Prices and discount factors

I assume that prices and payoffs follow the “law of one price,” or linearity,

\[
p(ax + bz) = ap(x) + bp(z)
\]  

(4)

I use the notation \( \mathcal{E}(x) \) to denote three operations, depending on context.

One period:  
\[
\mathcal{E}(x) \equiv E_0(x_1) = \sum_s \pi(s)x_1(s)
\]

Infinite period, discrete: \( \mathcal{E}(x) \equiv E_0 \sum_{t=1}^\infty \beta^t x_t = \sum_{s_t} \beta^t \pi(s_t)x_t(s_t) \)

Infinite period, continuous: \( \mathcal{E}(x) \equiv E_0 \int_0^\infty e^{-\rho t} x_t dt \)

(One can write mixed problems with a value of terminal wealth as well.) I treat time and state symmetrically. The \( \mathcal{E} \) operator takes a sum over time, weighted by \( \beta^t \) or \( e^{-\rho t} \) as well as a sum over states, weighted by \( \pi(s) \). The “variance” concept deriving from \( \mathcal{E}(x^2) \) then prizes stability over time as well as stability across states of nature. The constants \( \rho > 0 \) and \( \beta < 1 \) are arbitrary; they ensure that the sums converge even for interesting sets of streams \( x \) that may not converge to zero. It will be useful, but not necessary, to pick them as an agent’s subjective discount factor.

With this notation, I write the fundamental pricing equation as \( p = \mathcal{E}(mx) \), meaning

One period:  
\[
p = \mathcal{E}(mx) = E(mx)
\]

Infinite period, continuous: \( p = \mathcal{E}(mx) = E \int e^{-\rho t} m(t)x(t) dt \)

Infinite period, discrete: \( p = \mathcal{E}(mx) = E \sum_{t=1}^\infty \beta^t m_t x_t \)

where \( m_t \) is the stochastic discount factor.

It’s crucial for our purposes to construct a stochastic discount factor that is also a traded payoff. The standard conditions (Hansen and Richard 1987) on the payoff space \( \mathcal{X} \) that guarantee the existence of such a discount factor apply in this context as well. First, we restrict
attention to square-integrable payoffs

\[ \mathcal{E}(x_t^2) < \infty \forall x_t \in X. \]

This requirement limits us to payoffs that do not grow faster than \( \sqrt{\beta} \), i.e. that do not vary too much over time, as well as limiting variance in the usual sense. Second, we need to assume that investors can form arbitrary portfolios (1) and that the law of one price or linearity of the pricing function (4) holds for those portfolios. Third, the payoff space must be “complete” in the sense that if a sequence of payoffs is in \( X \) and converges, its limit point must also be in \( X \). (“Convergence” here uses \( \mathcal{E}(x_t^2) \) as a norm.) With these assumptions, we can guarantee that there is a unique discount factor \( m = x^* \) in the payoff space,

\[ \exists x^* \in X : p = \mathcal{E}(x^*x). \]

\( x^* \) is a dividend stream that acts as a discount factor process, i.e. \( p = \mathcal{E}(x^*x) \) means

\[ p(\{x_t\}) = E \sum_t \beta^t x^*_t x_t \text{ or } p(\{x_t\}) = E \int e^{-\rho t} x^*_t x_t dt. \]

The standard proof\(^1\) applies.

The conditions for this theorem are actually somewhat restrictive, and we will see cases in which they are violated. However, they are only sufficient, not necessary conditions for a traded discount factor. For example, short sale constraints violate the portfolio assumption (1). A discount factor that is a linear function of asset payoffs may require negative weights and thus not be tradeable. But then again, it might not; we might get lucky and be able to construct a traded discount factor anyway, perhaps for a restricted range of parameters. We will see a similar situation with the completeness assumption.

We can construct a discount factor \( x^* \) easily in the case that the payoff space \( X \) is generated by a finite set of basis dividend streams. Let \( x \) denote the vector of basis dividend streams, with corresponding prices \( p \), i.e.

\[
\begin{bmatrix}
  \{x_1^1(s_t)\} \\
  \{x_2^2(s_t)\} \\
  \vdots \\
  \{x_N^N(s_t)\}
\end{bmatrix},
\]

\[
\begin{bmatrix}
p^1 \\
p^2 \\
\vdots \\
p^N
\end{bmatrix},
\]

then it follows quickly that

\[ x^* = x^' \mathcal{E}(xx^')^{-1} p \]

works as a discount factor.

We can also construct a discount factor \( x^* \) for long-run payoffs from a discount factor for instantaneous returns, since of course payoffs are only correctly priced if all intermediate one-period returns are priced. If there is an underlying set of returns

\[
\begin{align*}
  dr_t &= r^f_t + dr^e_t; \\
  dr^e_t &= \mu_t dt + \sigma_t dz_t,
\end{align*}
\]

\(^1\)Ross (1976) first proved the basic theorem, and Cochrane (2004) presents a simple textbook discussion. See Hansen and Richard (1987) for the completeness assumption, which turns out to matter in these applications.
then we can form
\[
\frac{dx^*}{x^*} = (\rho - \gamma^f) dt - \mu_t^1 \Sigma_t^{-1} \sigma_t dz_t; \quad \Sigma_t \equiv \sigma_t \sigma_t^t
\]
dx^*/x^* is instantaneously mean-variance efficient. dx^*/x^* can be used to represent the instantaneous one-period asset returns, i.e.
\[
E_t(dr_t) = -E_t(dr_t^0 \frac{dx^*_t}{x^*_t})
\]

The resulting x^*_t is a discount factor for all payoffs that result from investing in arbitrary portfolios of the asset returns dr_t and consuming at arbitrary rates (subject to the transversality condition). In this sense, instantaneous and long-run pricing are connected, as instantaneous and long-run mean-variance efficiency are connected.

### 2.3 Mean-Variance Frontier

Since the mathematics are identical to the standard one-period case, we can just write down the characterization of the long-run mean long-run variance frontier, payoffs that solve
\[
\min_{\{y \in \mathbb{Y} \}} \mathcal{E}(y^2) \text{ s.t. } \mathcal{E}(y) = \mu.
\]

I follow the Hansen-Richard (1987) approach (See also Cochrane 2004), but the familiar Lagrangian minimization works just as well. The frontier is generated as
\[
y_{mv} = y^* + wy^e^*.
\]

Here, y^* is the discount-factor mimicking portfolio return,
\[
y^* = \frac{x^*}{p(x^*)} = \frac{x^*}{\mathcal{E}(x^2)}.
\]
y^* has the usual properties familiar from one period models. In particular, y^* is the minimum long-run second moment return,
\[
y^* = \arg \min_{\{y \in \mathbb{Y} \}} \mathcal{E}(y^2);
\]
and since y^* is proportional to x^* it can be used to price other objects (any mean-variance efficient return carries pricing information),
\[
\mathcal{E}(y^* y) = \mathcal{E}(y^{*2}) \forall y \in \mathbb{Y}.
\]
y^e^* is an excess return that carries a portfolio around the frontier as the weight w is varied. It is defined by
\[
y^e^* = \text{proj}(1|\mathbb{Y}^e).
\]
If a riskfree rate is traded (1 \in \mathbb{X}) then it y^e^* is simply
\[
y^e^* = \frac{y^f - y^*}{y^f}.
\]
If no riskfree rate is traded, \( y^* \) is the excess return “closest” to the perpetuity, i.e., its mimicking portfolio. \( y^* \) also has the usual properties as in the one period case: \( y^* \) is orthogonal to \( y^e \) and it generates means as \( y^e \) generates prices,

\[
\mathcal{E}(y^* y^e) = 0, \quad \mathcal{E}(y^{e2}) = \mathcal{E}(y^e) \quad \forall y^e \in \Sigma^e,
\]

In the applications I mostly avoid risk-free rate and inflation issues by focusing on the mean-variance frontier of excess returns,

\[
\min_{\{y^e \in \Sigma^e\}} \mathcal{E}(y^{e2}) \quad \text{s.t.} \quad \mathcal{E}(y^e) = \mu.
\]

This frontier is generated simply by

\[
y^{e,mv} = wy^e \quad w \in \mathbb{R}
\]

When there is a riskfree rate, we can compute \( y^e \) by (9). When there is no riskfree rate, we can calculate \( y^e \) analogously to the calculation (5) of \( y^* \). Start with a vector \( y^e \) of excess yields. Then, either from the definition (8) or the equivalent property (11), construct \( y^e \) by

\[
y^e = \mathcal{E}(y^e) \mathcal{E}(y^e y^e)^{-1} y^e
\]

All of these results can be derived by following exactly the approach in two-period models using \( \mathcal{E} \) in the place of \( E \). First, any yield (return) can be written as \( y^j = y^* + w^jy^e + \eta^j \), with \( \mathcal{E}(\eta^j) = 0, \mathcal{E}(\eta^j y^*) = 0, \mathcal{E}(\eta^j y^e) = 0 \). Second, the mean-variance frontier is clearly the set of returns with \( \eta^j = 0 \).

Of course one can span the frontier with any other two returns as well, and one can extend any standard characterization of the mean-variance frontier to the long-run mean variance frontier by using moments \( \mathcal{E} \) in the place of moments \( E \).

### 3 Portfolio problems

An investor has initial wealth \( W \), labor or non-marketed business income \( \{e_t\} \), and he can buy payoffs \( x = \{x_t\} \in X \) at prices \( p \). I assume no arbitrage in the available prices and payoffs, so there is a discount factor \( m \) that satisfies \( p = \mathcal{E}(mx) \). The investor’s problem is then

\[
\max_{\{x_t \in \Sigma\}} \mathcal{E} \left[ u(c) \right] \quad \text{s.t.} \quad W = \mathcal{E}(mx), \quad c = e + x
\]

As a reminder, we interpret these familiar-looking symbols as long-run portfolio problems,

\[
\max_{\{x_t \in \Sigma\}} E \sum_t \beta^t u(c_t) \quad \text{s.t.} \quad W_0 = p(\{x_t\}) = E \sum_t \beta^t m_t x_t; \quad c_t = e_t + x_t
\]

or

\[
\max_{\{x_t \in \Sigma\}} E \int e^{-\rho t} u(c_t) dt \quad \text{s.t.} \quad W_0 = p(\{x_t\}) = E \int e^{-\rho t} m_t x_t dt; \quad c_t = e_t + x_t.
\]
Again, the statement (13) encompasses these two cases as well as the standard one-period problem. The rest of this section will appear to be a trivial restatement of one-period portfolio theory unless you keep in mind that we are really solving one of these dynamic problems. But of course that triviality is the central point of the paper: a simple reinterpretation of familiar symbols allows us to characterize optimal long-horizon consumption-portfolio decisions in incomplete markets with dynamic trading.

The first order conditions state that at an optimum \( \hat{x} \),

\[
u' (\hat{x} + e) = \lambda m. \quad (15)\]

The Lagrange multiplier \( \lambda \) scales the portfolio up or down so that the initial wealth constraint is satisfied.

We can derive these first order conditions by considering a little more or less of one asset \( x_i \) at a time. Adding a bit more of payoff \( i \) gives us

\[p_i \lambda = \mathcal{E} [u'(\hat{x} + e) x_i].\]

This first order condition means that that marginal utility must be proportional to a discount factor, i.e. for all \( x \in \mathcal{X} \),

\[p = \mathcal{E} \left( \frac{u'(\hat{x} + e)}{\lambda} x \right).\]

Inverting (15), the solution to the portfolio problem is

\[\hat{x} = u^{-1}(\lambda m) - e. \quad (16)\]

**Complete markets**

If markets are complete, the discount factor \( m = x^* \in \mathcal{X} \) is unique, and every payoff is traded, so the construction (16) satisfies the constraint \( \hat{x} \in \mathcal{X} \). Hence, all we have to do is find the Lagrange multiplier \( \lambda \) to satisfy the initial wealth constraint.

The portfolio (16) has a simple intuition: First, each investor shorts a portfolio that matches his idiosyncratic income \( e \). The investor then consumes more in “cheap” (low \( \hat{m} \)) states, and less in “expensive” (high \( m \)) states, with the curvature of \( u' \) dictating how much or little to respond to this relative price. The Lagrange multiplier just scales consumption up and down to match initial wealth plus the gains or losses from hedging outside-income risk.

**Incomplete markets**

If markets are not complete, we have to pay more attention to the constraint \( \hat{x} \in \mathcal{X} \) that the consumer can actually buy the optimal portfolio. There are many discount factors that price assets, and for only (at most) one of them is the inverse marginal utility in the space of traded payoffs \( \mathcal{X} \). Figure 1 illustrates the issue for \( e = 0 \). The space of all discount factors \( m \) consists of \( m = x^* + \varepsilon \), with \( \mathcal{E}(\varepsilon x) = 0 \ \forall x \in \mathcal{X} \). Thus the set of discount factors lies in a hyperplane orthogonal to \( \mathcal{X} \) as shown. Discount factor \( a \) leads to a portfolio that does not lie in the space of marketed payoffs \( \mathcal{X} \). Discount factor \( b \) leads to a portfolio \( \hat{x} \) that does lie in \( \mathcal{X} \) and therefore generates the correct answer. The issue is how to find the correct discount factor \( b \).
Figure 1. Optimal portfolios with incomplete markets. \( X \) gives the space of traded payoffs. \( x^* \) is the unique discount factor in \( X \). \( m = x^* + \varepsilon \) gives the space of all discount factors. It is drawn at right angles to \( X \) since \( \mathcal{E}(\varepsilon x) = 0 \forall x \in X \). The optimal portfolio \( \hat{x} \) satisfies \( u^{-1}(\lambda m) = \hat{x} \) for some \( m \). Case a shows what can go wrong if you pick the wrong \( m \): \( u^{-1}(\lambda m) \) is not in the payoff space \( X \), so it can’t be the optimal portfolio. Case b shows the optimal portfolio: we have chosen the right \( m \) so that \( u^{-1}(\lambda m) \) is in the payoff space \( X \).

**Solution 1: Search**

Obviously, we could search over all possible \( m \) in order to find one whose image \( u'(\lambda m) \) is in the payoff space \( X \). Equivalently, we assume prices for the missing assets (i.e. market prices of risk for missing shocks) and search over those assumed prices until the investor chooses zero loading on the missing securities. (Schroeder and Skiadas 1999).

**Solution 2: Markets that are “complete enough”**

As Figure 1 suggests, markets don’t have to be complete for us to find solutions. \( u^{-1}(\lambda x^*) \) can land in \( X \) if the payoff space \( X \) is closed under some nonlinear transformations, not necessarily all of them. If for every \( x \in X \), we also have \( u^{-1}(x) \in X \), then we have \( \hat{x} = u^{-1}(\lambda x^*) \) and we can again find the optimal portfolio. In one-period models, a full set of options gives closure under nonlinear transformations, and this situation is often referred to as “complete markets,” even though many other shocks are not traded.

**Solution 3: Quadratic utility**

I emphasize here the special case of quadratic utility. In this case, marginal utility is linear. The payoff space \( X \) is closed under linear transformations, so the inverse image of \( x^* \in X \) is guaranteed to be in the space of payoffs \( X \), and this is the optimal portfolio. Analytically, I
suppose utility is quadratic
\[ u(c_t) = -\frac{1}{2} (c_b^t - c_t)^2. \] (17)
c_b^t is a potentially stochastic “bliss point.” Having the bliss point move around over time allows a better approximation to nonlinear utility functions. For example, we can allow growth in consumption over time by having the bliss point increase. A stochastic bliss point is also directly important for applications. Many investors have times and states of the world at which consumption is particularly important, such as when children start going to college.

The basic optimal portfolio is then given by

**Proposition 1.** The optimal payoff for the quadratic utility investor (17) is given by
\[ \hat{x} = (\hat{c}_b^t - \hat{e}_t) - \left[ p(\hat{c}_b^t - \hat{e}_t) - W \right] y^*, \] (18)
where the hedge portfolios \( \hat{c}_b^t - \hat{e}_t \) are the projections of the bliss point and non-market income on the set of available payoffs
\[ \hat{c}_b^t - \hat{e}_t \equiv \text{proj} \left( c_b^t - e | X \right), \]
\( W \) is initial financial wealth and \( y^* \) (7) is the minimum second moment return

**Interpretation of the optimal portfolio**

The investor starts by buying a portfolio \( \hat{c}_b^t - \hat{e}_t \) that gets him as close to the bliss point as traded assets allow. We can also think of the payoff \( \hat{c}_b^t - \hat{e}_t \) as the optimal hedge for preference shocks and labor income risk. I’ll call it the individual hedge portfolio.

Typically, however, complete hedging is not possible. Wealth \( W \) is lower than the cost \( p(\hat{c}_b^t - \hat{e}_t) \) of the hedge portfolio. Thus, the investor shorts an optimal risky portfolio \( y^* \) in order to buy the individual hedge portfolio. \( y^* \) is proportional to contingent claims prices, so by shorting \( y^* \) the investor is shorting the “most expensive” payoff, in order to generate the largest funds possible at minimum risk. \( y^* \) is of course on the mean-variance frontier.

In sum, each investor’s optimal portfolio is a combination of a mimicking portfolio to hedge labor income and preference shock risk, plus an investment in the minimum second moment return, whose size depends on risk aversion or initial wealth.

**Derivation of the optimal portfolio (18)**

With the utility function (17), marginal utility is
\[ u'(c_t) = c_b^t - c_t, \]
and our first order condition (15) reads
\[ c_b^t - \hat{x}_t - e_t = \lambda m_t. \]
Solving,
\[ \hat{x}_t = -\lambda m_t + c_b^t - e_t \] (19)
We have to find the choice of discount factor \( m \) that leaves this portfolio in the space of traded portfolios \( X \). To this end, split up each term on the right as its projection on \( X \) and a residual,
\[ \hat{x} = -\lambda (x^* + \varepsilon_m) + \text{proj} \left( c_b^t - e | X \right) + \varepsilon_{ce} \] (20)
with $E(\varepsilon_m x) = 0$, $E(\varepsilon_c x) = 0 \forall x \in X$. (These conditions define projection.) To get $\hat{x} \in X$ we make the choice of discount factor by choosing $\lambda \varepsilon_m = \varepsilon_c$. Then, we can write the optimal portfolio as

$$\hat{x} = -\lambda x^* + \hat{c}^b - \hat{e}$$

(21)

where $\hat{c}^b - \hat{e}$ denotes a mimicking portfolio for preference shocks and labor income risk,

$$\hat{c}^b - \hat{e} \equiv \text{proj} \left( c^b - e|X \right).$$

(22)

Finally, we can solve for the Lagrange multiplier $\lambda$ in terms of initial wealth. The wealth constraint states

$$W = -\lambda p(x^*) + p(\hat{c}^b - \hat{e}) = -\lambda E(x^{*2}) + p(\hat{c}^b - \hat{e})$$

$$\frac{p(\hat{c}^b - \hat{e}) - W}{E(x^{*2})} = \lambda$$

Substituting into (20), and using $y^* = x^*/p(x^*) = x^*/E(x^{*2})$ we obtain the optimal portfolio (18)

### 3.1 Returns and more traditional statements

Equation (18) is an unusual statement of a familiar result, so it is useful to present the standard statements of the result, and to interpret those statements in a long-run setting. In addition, the expression of the result (18) focuses on the perverse global properties of the quadratic utility problem, and the bliss point in particular. We want an expression of the result that is more sensible as a local approximation to more realistic (nonquadratic) problems. In particular, it is more interesting to characterize investors by their risk aversion rather than their bliss points. Alas, these more traditional and intuitive statements vary depending on whether there is stochastic nontraded income and whether there is a risk free rate, so I have to present several cases.

**No nontraded income or preference shocks; the mean-variance frontier**

We most commonly study a special case: there is no nontraded income, the bliss point is nonstochastic, and a riskfree rate is traded.

**Proposition 2.** When there is no labor income, the bliss point is nonstochastic and a risk free rate is traded, the return of the optimal payoff is

$$\hat{y}^i = y^f + \frac{1}{\gamma_i} \left( y^f - y^* \right),$$

(23)

where $\gamma_i$ is investor i’s coefficient of risk aversion.

In a market composed of such investors, the return of the optimal payoff is a combination of a real perpetuity and the return of the market payoff $\hat{y}^m$, which is a claim to the aggregate consumption stream. The weight of the market return declines as the individual’s risk aversion rises relative to the wealth-weighted average risk aversion coefficient $\gamma^a$ (27)

$$\hat{y}^i = y^f + \frac{\gamma^a}{\gamma_i} \left( \hat{y}^m - y^f \right).$$
Comparing (23) with the characterizations (6) and (9) we see in this case that each investor holds long-run mean-variance efficient investment, with greater weight in risky assets for lower risk aversion.

Derivation of proposition 2.

With no labor income \( e = 0 \) a nonstochastic bliss point \( c^b \) and a traded riskfree rate, the optimal portfolio (18) specializes to

\[
\hat{x} = c^b - \left( \frac{c^b}{y^f} - W \right) y^* \\
\hat{x} = W \left( y^* + \frac{c^b}{y^f W} (y^f - y^*) \right) \\
\hat{y} = \frac{\hat{x}}{W} = y^* + \frac{c^b}{y^f W} (y^f - y^*)
\]

(24)

(25)

The last equation expresses the return (yield) on the investor’s portfolio rather than the level of its payoffs. The mean-variance frontier is composed of all payoffs of the form \( y^* + w(y^f - y^*) \) (see (6)). Therefore, we have proved that investors all hold long-run mean-variance efficient payoffs.

The quantity \( c^b/y^f W \) is related to risk aversion. As \( W \) rises or \( c^b \) declines, the investor becomes more able to finance bliss-point consumption. When \( W \) can finance bliss-point consumption for sure, \( W y^f = c^b \), the investor becomes infinitely risk averse and holds only the riskfree security. As \( W \) declines, the investor becomes less risk averse. Obviously, this character of risk aversion relies on the perverse global implications of quadratic utility, rather than its more sensible local approximation.

It is both interesting and useful to express the portfolio decision in terms of the local risk aversion coefficient rather than in terms of the investor’s ability to finance bliss points. Local risk aversion for quadratic utility is

\[
-\frac{cu''(c)}{u'(c)} = \frac{c}{c^b - c} = \frac{1}{c^b/c - 1}
\]

If we define \( \gamma \) as the risk aversion coefficient when the investor puts wealth \( W \) in the riskfree asset,

\[
\gamma \equiv -\frac{cu''(c)}{u'(c)} \bigg|_{c=y^f W}
\]

we then have

\[
\frac{1}{\gamma} = \frac{c^b - y^f W}{y^f W}.
\]

(26)

Substituting this value in (25), we can write the investor’s portfolio as

\[
\hat{y}^i = y^f + \frac{1}{\gamma^i} (y^f - y^*)
\]

where I have added an \( i \) superscript to emphasize that this is the portfolio for investor \( i \). Each investor holds a mean-variance efficient payoff, with larger investment in the risky asset the lower his risk aversion coefficient.
One can of course express the portfolio using any mean-variance efficient return, not just \( y^* \). Since every mean-variance efficient portfolio is of the form
\[
y^{mv} = y^* + w(y^f - y^*),
\]
we can write
\[
\hat{y}^i = y^f + \frac{w - 1}{\gamma_i} (y^{mv} - y^f)
\]
\((w = 1 \text{ corresponds to } y^f, \text{ so as usual we can generate the frontier using any frontier return except } y^f \text{ itself.})\)

It’s traditional to use the market as reference return on the right hand side. This result requires the additional assumption that all investors have the same mean-variance objective. Imagine therefore an economy filled with quadratic utility investors. Then, the aggregate payoff is
\[
\hat{x}^a = \frac{1}{N} \sum_i \hat{x}^i = \frac{1}{N} \sum_i W^i \left[ y^f + \frac{1}{\gamma_i} (y^f - y^*) \right]
\]
\[
\hat{x}^a = W^a y^f + \left( \frac{1}{N} \sum_i \frac{W^i}{\gamma_i} \right) (y^f - y^*)
\]
\[
\hat{x}^a = W^a y^f + \frac{W^a}{\gamma^a} (y^f - y^*)
\]
where \( \hat{x}^a \) is the aggregate consumption stream, and \( \gamma^a \) is the wealth-weighted average risk aversion coefficient,
\[
\frac{1}{\gamma^a} = \frac{1}{N} \sum_i \frac{W^i}{\gamma^i} \tag{27}
\]

The return on the market portfolio is of course
\[
\hat{y}^m = \frac{\hat{x}^a}{W^a}
\]
\[
\hat{y}^m - y^f = \frac{1}{\gamma^a} (y^f - y^*) .
\]

Now, we can use this market return to get rid of \( y^* \), and write
\[
\hat{y}^i = y^f + \frac{1}{\gamma_i} (y^f - y^*)
\]
\[
\hat{y}^i = y^f + \frac{\gamma^a}{\gamma^i} (\hat{y}^m - y^f)
\]

Non-market income and preference shocks

Non-market income and preference shocks have two effects. First, these shocks may have a systematic component, spanned by the riskfree rate and a mean-variance efficient return. The investor wants to maintain the same overall exposure to the mean-variance efficient / riskfree rate combination, so his exposure in the asset portfolio will change as the systematic component of non-market income and preference shocks changes. Second, these shocks may have
an idiosyncratic components, and the investor wants to hedge as much of this component as possible.

In order to see these effects, then, I break up the $\hat{e} - \hat{c}$ terms of the optimal portfolio (18) into orthogonal “systematic” and “idiosyncratic” components

$$\hat{e} = \beta y^* + \alpha y^e + \varepsilon_e$$

(28)

I define $\hat{c}$ analogously, defining $\beta_c$, $\alpha_c$ and $\varepsilon_c$. $\beta y^* + \alpha y^e$ is the “systematic” component proportional to the mean-variance frontier. $\beta_e$ carries pricing or scale information, $p(\hat{e}) = \beta_e$. $\varepsilon$ is an “idiosyncratic” (yet still traded) part, with zero price and zero mean. As with all these projections, we can calculate the elements straightforwardly as the fitted value of linear (long-run) regressions. Since the right hand variables are orthogonal, the multiple regression coefficients are the same as the single regression coefficients, and we have

$$\beta_e = \frac{E(\hat{e}y^*)}{E(y^e^2)}; \quad \alpha_e = \frac{E[\hat{e}y^e]}{E[y^e^2]}$$

(Since $\eta$ in $e = \hat{e} + \eta$ is orthogonal to all payoffs including $y^*$ and $y^f$, the same expressions hold for $e$ in place of $\hat{e}$.)

When there is a riskfree rate $\alpha$, can also be interpreted as the coefficient of $e$ on a constant payoff in a multiple regression that includes $y^*$. Using $y^{e*} = (y^f - y^*)/y^f$

$$\hat{e} = \alpha_e 1 + \left(\beta_e - \frac{\alpha_e}{y^f}\right) y^* + \varepsilon_e.$$  

Thus, in the special case of nonstochastic $e$, we have $2 \varepsilon_e = 0$ and $\alpha_e = e$.

Now, substituting (28) in the result of proposition 1, (18), we can write the optimal portfolio as

$$\hat{x} = \varepsilon_c - \varepsilon_e + \left(\frac{\alpha_c}{y^f} - \frac{\alpha_e}{y^f}\right) (y^f - y^*) + W y^*.$$  

(29)

(29) shows how the component $\alpha$ generating mean returns of $\hat{c} - \hat{e}$ in (18) affects the mean-variance investment. Next, we need to express (29) in terms of risk aversion and the market portfolio, as we did in the standard case with no labor income. The result is

**Proposition 3.** The optimal payoffs with outside income and preference shocks (18) can be written as a combination of a hedge portfolio, a riskfree investment and a long-run mean-variance efficient investment,

$$\hat{x} = \varepsilon_c - \varepsilon_e + W y^f + \left(W + \frac{\alpha_e}{y^f}\right) \frac{1}{\gamma} (y^f - y^*)$$  

(30)

In this representation, the hedge portfolio $\varepsilon_c - \varepsilon_e$ has mean zero and zero price, it is orthogonal to the riskfree payoff and the mean-variance efficient payoff.

First, the investor hedges the component of labor income and preference shock $\varepsilon_e$, $\varepsilon_c$ that has zero mean, zero cost, and zero expected excess return. Then he allocates financial wealth

$$\alpha_e = \frac{E[eR^{e*}]}{E[R^{e*2}]} = e \frac{E[R^{e*}]}{E[R^{e*2}]} = e.$$  

---

1 More directly,

$$\alpha_e = \frac{E[eR^{e*}]}{E[R^{e*2}]} = e \frac{E[R^{e*}]}{E[R^{e*2}]} = e.$$  

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W to a mean-variance efficient portfolio. The weights in riskfree vs. risky assets are adjusted to reflect the bond-like component of labor income as well as the investor’s risk aversion $\gamma$.

We can read the optimal portfolio after hedging nonsystematic risks $\hat{x} - (\varepsilon_c - \varepsilon_e)$ in several ways. First, the investor starts at the risk free rate, and then moves into the risky asset in accordance with his risk aversion $\gamma$, but at a larger scale, corresponding to someone whose “wealth” reflects the value of his non-market income as measured by $\alpha_e/y^f$ (recall $\alpha_e = e$ if $e$ is nonstochastic) along with financial wealth.

\[
\frac{(\hat{x} - \varepsilon_c - \varepsilon_e)}{W} = y^f + \left(\frac{W + \alpha_e/y^f}{W}\right) \frac{1}{\gamma} (y^f - y^*)
\]

Second, we can think of non-market income as lowering the effective risk aversion by the ratio of financial to total wealth, again giving a larger than usual allocation to the risky asset. Third, we can think of the investor as a traditional mean-variance investor, whose overall payoff is the asset portfolio plus the $\alpha_e$ component of non-market income $\hat{x} + \alpha_e$, and whose overall wealth is financial wealth plus the value of non-market income,

\[
(\hat{x} - \varepsilon_c + \varepsilon_e) + \alpha_e = \left(W + \frac{\alpha_e}{y^f}\right) \left[y^f + \frac{1}{\gamma} (y^f - y^*)\right].
\]

Finally, we can think of the asset portfolio as a short position in the $\alpha_e$ component of non-market income plus the mean-variance efficient portfolio resulting from selling that labor income,

\[
(\hat{x} - \varepsilon_c + \varepsilon_e) = -\alpha_e + \left(\frac{\alpha_e}{y^f} + W\right) \left[y^f + \frac{1}{\gamma} (y^f - y^*)\right].
\]

**Derivation of proposition 3.**

I proceed as before, massaging (29) to have a risk aversion coefficient in place of the constant $\alpha_c$.

\[
\hat{x} &= \varepsilon_c - \varepsilon_e + \left(\frac{\alpha_e}{y^f} - \frac{\alpha_c}{y^f}\right) (y^f - y^*) + W y^*
\]

\[
= \varepsilon_c - \varepsilon_e + W y^f + \left[\frac{\alpha_c}{y^f} - \left(W + \frac{\alpha_e}{y^f}\right)\right] (y^f - y^*)
\]

\[
= \varepsilon_c - \varepsilon_e + W y^f + \left(W + \frac{\alpha_e}{y^f}\right) \frac{\alpha_c - (y^f W + \alpha_e)}{(y^f W + \alpha_e)} (y^f - y^*)
\]

\[
= \varepsilon_c - \varepsilon_e + W y^f + \left(W + \frac{\alpha_e}{y^f}\right) \frac{1}{\gamma} (y^f - y^*)
\]

Here, I define

\[
\frac{1}{\gamma} = \frac{\alpha_c - (W y^f + \alpha_e)}{(W y^f + \alpha_e)}.
\]

Since $\alpha_c = c$ and $\alpha_e = e$ in the case of nonstochastic $e$ and $c$, this definition naturally generalizes (26).
Aggregation and the market portfolio

Aggregating (30), the market portfolio $\hat{x}^a = \frac{1}{N} \sum_i \hat{x}^i$ is

$$
\hat{x}^a = \hat{\varepsilon}^a - \varepsilon^a + W^a y^f + \left( \frac{1}{N} \sum_i \frac{W^i + \alpha^i_a / y^f}{\gamma^i} \right) \left( y^f - y^* \right)
$$

$$
\hat{x}^a = \varepsilon^a - \varepsilon^a + W^a y^f + \left( W^a + \frac{\alpha^a_a}{y^f} \right) \frac{1}{\gamma^a} \left( y^f - y^* \right)
$$

(31)

Here, I define aggregate risk aversion weighted by wealth including the value of the $\alpha_e$ component of non-market income.

$$
\frac{1}{\gamma^a} = \frac{1}{N} \sum_i \frac{W^i + \alpha^i_e / y^f}{\gamma^i}.
$$

If nonmarket income and bliss points are constant, the standard analysis goes through with minor modifications. In this case we have $\varepsilon^a = \varepsilon^c = 0$ and $\alpha_e = \epsilon_e$. The market portfolio is still mean-variance efficient, and we can use it to substitute for $y^f - y^*$ to express portfolios relative to the market index,

$$
y^i = y^f + \left( \frac{W^i + \alpha^i_e / y^f}{W^a + \alpha^a_e / y^f} \right) \frac{\gamma^a}{\gamma^i} \left( y^m - y^f \right)
$$

Now individuals $i$ invest more in the market payoff if they are truly less risk averse ($\gamma^i$) or if they have more bond-like non-market income already $\varepsilon^i$, making them appear less risk averse.

If nonmarket income and bliss points are not constant, however, most of the special nature of the market portfolio disappears. The market payoff is no longer mean-variance efficient, as the elements $\varepsilon^a_e$ and $\varepsilon^a_e$ are present. At best, the market payoff is now “multifactor efficient” (Fama 1996) meaning that it minimizes variance for given mean, and given exposures to $\varepsilon^a_e - \varepsilon^a_e$. As a result, the market payoff is no longer a particularly useful portfolio with which to describe differences across investors. If investor A is more risk averse than investor B, investor A wants more investment in a mean-variance efficient portfolio, not a leveraged investment in the market portfolio. Therefore, we can no longer summarize all investors’ portfolios using just the market portfolio; we must separately identify a mean-variance efficient portfolio.

Since the average investor holds the market, however, it is always useful to describe portfolio decisions by how a particular investor is different from the average rather than in absolute terms. One way to do this, differenting the individual and aggregate payoff (31) is given by

$$
\frac{\hat{x}^i + \alpha^i_e}{W^i + \alpha^i_e / y^f} = \frac{\hat{x}^a + \alpha^a_e}{W^a + \alpha^a_e / y^f} + \left( \frac{\varepsilon^i_c - \varepsilon^i_e}{W^i + \alpha^i_e / y^f} - \frac{\varepsilon^a_c - \varepsilon^a_e}{W^a + \alpha^a_e / y^f} \right) + \left( \frac{1}{\gamma^i} - \frac{1}{\gamma^a} \right) \left( y^f - y^* \right)
$$

On the left, we have the individual asset portfolio plus a constant, the $\alpha_e$ component of labor income, divided by the value of those two components. The units of this term are a return (yield). On the right, we have the same return for the aggregate. If the individual is “average” in all respects, he will hold this aggregate payoff. Next, we have the individual and aggregate hedge portfolios. $\varepsilon^i_c - \varepsilon^i_e$ are zero-cost portfolios; the denominators of these terms simply scale the results by overall wealth. The “average” individual who holds the market $\hat{x}^a$ does no extra hedging. The individual does more or less hedging according to whether his idiosyncratic risks are larger or smaller than average. (One could also express $\frac{\varepsilon^i_c - \varepsilon^i_e}{W^i + \alpha^i_e / y^f} = \beta_i \frac{\varepsilon^a_c - \varepsilon^a_e}{W^a + \alpha^a_e / y^f} + \eta_i$ and thus
express each individual’s hedge payoﬀ as an investment in a single “aggregate hedge payoﬀ” plus a truly idiosyncratic component, or to ﬁnd “factor” portfolios that explain large components of variation in individual hedge payoﬀs.) Finally, individuals who are less risk averse than average will buy more of a mean-variance efﬁcient portfolio and vice versa.

Alternatively, we can use the market portfolio purged of its hedge component \( x^a - \varepsilon^a_c - \varepsilon^a_e \) to describe individual differences, as that portfolio is mean-variance efﬁcient,

\[
\hat{x}^i - (\varepsilon^i_c - \varepsilon^i_e) \quad \text{and} \quad \left( \frac{W^i + \alpha^i / y^f}{W^i} \right) \gamma^a \left( \frac{\hat{x}^a - (\varepsilon^a_c - \varepsilon^a_e)}{W^a} - y^f \right).
\]

Again, the object on the left hand side is a return, since the \( \varepsilon \) have zero price. The individual portfolio after hedging is a more or less leveraged version of the market portfolio after hedging. Of course, understanding \( (\varepsilon^a_c - \varepsilon^a_e) \) is tantamount to understanding the mean-variance efﬁcient payoﬀ itself, so we are still not spared the task of ﬁnding a mean-variance efﬁcient portfolio.

## 4 Applications

In the following sections I pursue several applications of the long-run mean / long-run variance portfolio theory. I do this with some trepidation. After all, one big selling point of the theory is that one can declare victory before the hard part of actually constructing mean-variance efﬁcient portfolios begins. Here, I have to attack that hard part. The approaches I take here are the simplest possible, in order to show that the ideas can in fact be sensibly applied. None of the simpliﬁcations here are essential, and if they are found wanting that should spur work on application rather than a rejection of the basic idea.

The ﬁrst exercise one tries with standard one-period mean-variance optimization is to allocate investment among a small number of ad-hoc portfolios, such as size, value or industry, and ﬁnd the mean-variance frontier, using sample moments for the mean and variance of the portfolios. Of course, one tries to ﬁnd interesting portfolios that one hopes capture most of the investment possibilities of larger collections of assets. Later, one tackles the hard problems of large numbers of assets (there are more securities than data points), time-varying means and correlations (especially of individual securities), the addition of dynamic strategies, the wacky weights that result from small sampling errors in mean returns, and so forth.

Here, I attempt analogous ﬁrst exercises. I start with a small number of rather arbitrary, but, I hope, well-chosen and interesting, payoﬀs. In particular, I choose a few portfolios like value and industry, rather than individual securities; I choose a few (but, again I hope well-chosen and interesting) dynamic strategies, like an investment in the market that rises linearly with a regression forecast of the market expected excess return, and I choose some arbitrary (but, once again, I hope well-chosen and interesting) payout rules for converting accumulated wealth into consumption streams. As in the classic case, if the basic idea survives this scrutiny, it will be well worth the effort to solve the harder problem of maximizing over these large-dimensional parameterizations of the payoﬀ space.

The ﬁrst step in all of this is to construct a discount factor. I follow two approaches. First, I construct a discount factor from one period returns, and iterate that discount factor forwards so that it prices long-term payoﬀs. Second, I construct a discount factor directly from the stream
of payoffs. The second approach is in some ways much more satisfying, as it gets away entirely from modeling one period returns. However, the lessons of the first approach are very useful in setting it up.

4.1 Moments and units

Before calculating long-run means and variances, it’s important to get the units straight. The long-run analogues to returns \( y_t = x_t/p_0 \) have the units of dividend yields or coupon rates, so I call them “yields.” A typical number might be 0.05 annually or 5%. Their long-run mean \( E(y) \) however has the units of a gross return so I call it the long-run mean return. For example, if the return is 5% and the discount factor \( \beta \) is also 5%, then the long-run mean \( E(y) \) is 1.0. A 5% coupon with a 5% discount factor corresponds to “just getting your money back,” in a long-run sense. If the coupons are 10 percent larger (i.e. 5.5% per year), the mean return is 1.10 or 10%. The difference in units stems from the fact that the weights in my definition of \( E \) do not add up to one.

I characterize instead the long-run mean yield, which has the same units as the underlying returns (yields)

\[
E(y) = 1 - \frac{\beta}{\bar{E}(y)} = \frac{1 - \beta}{\beta} E \sum_{j=1}^{\infty} \beta^j y_{t+j}
\]

\[
\text{continuous time : } \bar{E}(y) = \rho E(y) = \rho \int_{t=0}^{\infty} e^{-\rho t} y_t dt.
\]

The adjustment factors \((1 - \beta)/\beta\) or \(\rho\) mean that the weights sum up to one, so the units are the same as those of the object in brackets.

It is especially easy to confuse units with the riskfree rate. The riskfree yield \( y^f \) is a number like 0.05, not a number like 1.05, but \( E(y^f) \) is a number like 1.05. You cannot interchange \( y^f \) with \( E(y^f) \) as you can interchange \( R^f \) with \( E(R^f) \) in one-period calculations. You can interchange \( y^f \) with \( \bar{E}(y^f) \), which makes a case for using mean yields as a set of units.

The natural corresponding definition of long-run yield variance is

\[
\text{yield variance: } \tilde{\sigma}^2(y) \equiv \bar{E} \left( \left[ y - \bar{E}(y) \right]^2 \right) = \bar{E}(y^2) - \bar{E}(y)^2.
\]

For example, if a yield \( y = \bar{y} + 0.01 \) or \( \bar{y} - 0.01 \), each forever, with equal probability, the yield standard deviation is one percentage point.

Writing out the yield variance,

\[
\tilde{\sigma}^2(y) = \frac{1 - \beta}{\beta} \sum_{j=1}^{\infty} \beta^j E \left( \left[ y_{t+j} - \bar{E}(y) \right]^2 \right)
\]

we see that the long-run yield variance is like a discounted sum of the conventional variances of yields. However, we use the same long-run mean \( \bar{E}(y) \) at each date, not the mean yield \( E(y_{t+j}) \).

\[\text{If } \beta = \frac{1}{1+\delta} \text{ and } R = \frac{\delta}{1+\delta} = \frac{(1-\beta)}{\beta}, \text{ then } \bar{E}(R) = \bar{E} \sum_{t=1}^{\infty} \beta^t \frac{(1-\beta)}{\beta} = 1\]
In general \( E(y_{t+j}) \) varies over time; the long run variance of yields counts this variation that the discounted sum of yield variance \( \sum_{j=1}^{\infty} \beta^j E \left( [y_{t+j} - E(y_{t+j})]^2 \right) \) would ignore. To quantify this fact, we can express long-run variance as the sum of variance across states of nature and variance over time\(^4\),

\[
\hat{\sigma}^2(y) = \tilde{E} \left[ \sigma^2(y_t) \right] + \tilde{E} \left\{ [E(y_t) - \tilde{E}(y_t)]^2 \right\}
\]

For example, yields oscillate between \( \bar{y}+0.01 \) and \( \bar{y}-0.01 \) with perfect certainty, we still see a one percent long-run standard deviation of yields, even though the conventionally defined variance of yield is zero. Deterministically growing payoffs also have positive long-run variance.

We can define a long-run \textit{return} variance to go with the definition of long run mean return \( \mathcal{E}(y) \),

\[
\text{return variance: } \left( \frac{\beta}{1-\beta} \right)^2 \tilde{E} \left[ (y - \tilde{E}(y))^2 \right] = \left( \frac{\beta}{1-\beta} \right)^2 \left[ \tilde{E}(y^2) - \mathcal{E}(y)^2 \right].
\]

The \( \beta/(1-\beta) \) factor converts units from squared yields to squared gross returns. The one percent yield standard deviation of the last example corresponds to a \( \beta/(1-\beta) \times 0.01 = 19\% \) \textit{return} standard deviation. The return standard deviation is actually quite large, since the yield itself is typically a small number like five percent.

The choice of units is entirely a question of what numbers conveys more intuition. “Return” and “yield” measures are proportional to each other, and the ratio of mean to standard deviation is the same in each case:

\[
\text{mean yield} = \frac{1-\beta}{\beta} \text{ mean return} \quad \text{yield variance} = \left( \frac{1-\beta}{\beta} \right)^2 \text{return variance}
\]

The long-run mean-variance frontier defined above to minimize \( \mathcal{E}(y^2) \) for given values of \( \mathcal{E}(y) \) also minimizes \( \tilde{E}(y^2) \) for given values of \( \tilde{E}(y) \); it minimizes long-run yield variance for given long-run mean yield, and it minimizes long-run return variance for given long-run mean return.

\(^4\)
Since \( y^e \) defines the mean-variance frontier, the long-run Sharpe ratio or slope of the long-run mean-variance frontier is given by \( \frac{\mathcal{E}(y_t^e)}{\tilde{\sigma}(y_t^e)} \). To evaluate the variance,

\[
\tilde{\sigma}^2(y_t^e) = \frac{\mathcal{E}(y_t^e) - \mathcal{E}(y_t^e)^2}{\mathcal{E}(y_t^e) \left[ 1 - \mathcal{E}(y_t^e) \right]}
\]

The second equality follows from the defining property \( \mathcal{E}(y_t^e)^2 = \mathcal{E}(y_t^e) \). Then we have

\[
\frac{\mathcal{E}(y_t^e)}{\tilde{\sigma}(y_t^e)} = \frac{\mathcal{E}(y_t^e)}{\sqrt{\mathcal{E}(y_t^e) \left[ 1 - \mathcal{E}(y_t^e) \right]}} = \sqrt{\frac{\mathcal{E}(y_t^e)}{1 - \mathcal{E}(y_t^e)}}.
\]

Furthermore, using the definition of \( y^e \) in the case of a riskless payoff, we can write\(^5\)

\[\frac{\mathcal{E}(y_t^e)}{\tilde{\sigma}(y_t^e)} = \frac{\mathcal{E}(x_t^e)}{\mathcal{E}_t} \frac{y^f}{\mathcal{E}(x_t^e)} - 1\]

\(y^f\mathcal{E}(x_t^e)\) is the annuity you can get by selling \( x^e \). \( \mathcal{E}(x_t^e) \) is the average value of the payoffs in \( x^e \). Thus, as usual, the Sharpe ratio represents the (maximal) distortion of prices relative to means discounted at the riskfree rate.

### 5 Iid returns

I start with the natural benchmark for portfolio problems: continuous time, iid returns and a constant interest rate. The investor can rebalance continuously. This familiar benchmark case gives some feeling for how long run means and variances work, and it is a natural springboard for the more complex cases in which the answers are not known.

In particular, there are an enormous variety of payout and portfolio policies that an investor could use to create a yield stream from a set of returns. By studying an environment with a complete answer, we know an important class of such policies to include in other problems.

#### 5.1 Setup and \( x^e \)

There are \( N \) basis assets whose instantaneous excess returns follow

\[
dr_t^e = \mu dt + \sigma dz_t; \quad \sigma \sigma' = \Sigma,
\]

\[
y_t^e = y^f - y^* = 1 - \frac{x^e}{y^f \mathcal{E}(x^*)}
\]

\[
\sqrt{\frac{\mathcal{E}(y_t^e)}{1 - \mathcal{E}(y_t^e)}} = \sqrt{\frac{1 - \frac{\mathcal{E}(x^*)}{y^f \mathcal{E}(x^*)}}{\mathcal{E}_t}} = \sqrt{\frac{y^f \mathcal{E}(x^*) - \mathcal{E}(x^*)}{\mathcal{E}(x^*)}} = \sqrt{\frac{y^f \mathcal{E}(x^*)}{\mathcal{E}(x^*)} - 1}
\]
and there is a constant riskfree rate \( r^f dt \). I use the notation \( d\gamma \) to denote instantaneous returns, i.e. if dividends are paid at rate \( D dt \), then \( d\gamma \equiv dp/p + D/p \, dt \; ; \; d\gamma^e = dr - r^f dt \).

The investor can forms portfolios with instantaneous return
\[
dr^p_t = r^f dt + w_t dr^e_t
\]
He generates payoffs \( x_t \) from initial wealth \( W_0 \) and such portfolios by
\[
dW_t = W_t dr^p_t - x_t dt,
\]
with the usual “transversality condition” that the time-zero value of wealth must eventually tend to zero \( \lim_{T \to \infty} p(W_T) = 0 \). The payoff space \( X \) consists of payoffs generated in this way.

The unique payoff \( x^* \in X \) that generates prices is characterized by
\[
\frac{dx^*_t}{x^*_t} = \left( \rho - r^f \right) dt - \mu' \Sigma^{-1} \sigma dz_t
\]
and thus
\[
x_t^* = e^{\left( \rho - r^f - \frac{1}{2} \mu' \Sigma^{-1} \mu \right) t - \frac{1}{2} \mu' \Sigma^{-1} \sigma \int_0^t dz_t}
\]

By construction, \( x^* \) correctly prices one period returns, i.e.
\[
E_t \left( \frac{dx^*_t}{x^*_t} \right) = \left( \rho - r^f \right) dt
\]
\[
\mu = E_t \left( dr^e_t \right) = -E_t \left( \frac{dx^*_t}{x^*_t} dr^e_t \right).
\]

Then, of course, \( x^* \) also prices any payoff \( x_t \) formed by arbitrary trading strategies in the underlying assets, i.e. it produces
\[
p(x) = E \int_0^\infty e^{-rt} x^*_t x_t dt = \mathcal{E}(x^* x)
\]
for any payoff \( x \).

The price of \( x^* \) itself is
\[
p(x^*) = \mathcal{E}(x^{*2}) = E \int_0^\infty e^{-rt} e^{2\left( \rho - r^f - \frac{1}{2} \mu' \Sigma^{-1} \mu \right) t - 2\mu' \Sigma^{-1} \sigma \int_0^t dz_t} dt
\]
\[
p(x^*) = \frac{1}{2r^f - \rho - \mu' \Sigma^{-1} \mu}
\]
(I consider the possibility that \( 2r^f - \rho - \mu' \Sigma^{-1} \mu \leq 0 \) and thus \( p(x^*) = \infty \) below.) From (33) (or directly from (34)) we have \( E(x^*_t) = e^{(\rho - r^f)t} \) and hence
\[
\mathcal{E}(x^*) = \int_0^\infty e^{-rt} e^{(\rho - r^f)t} = \frac{1}{r^f}.
\]

With \( x^* \) and \( p(x^*) \) in hand, we have all the ingredients to compute the long-run mean-variance frontier and optimal payoffs.
5.2 Consumption-portfolio problems

Since this market is complete, we can write the portfolio problem directly as

$$\max \mathcal{E} [u(c_t)] \quad s.t. \ p(c_t) = \mathcal{E} (x_t^* c_t) = W_0.$$  

The first order conditions give the solution to the portfolio problem,

$$u'(c_t) = \lambda x_t^* \Rightarrow c_t = u'^{-1} (\lambda x_t^*)$$

For power utility

$$u(c_t) = \frac{1}{1 - \gamma} c_t^{1-\gamma},$$

we have

$$c_t^p = (\lambda x_t^*)^{-\frac{1}{\gamma}}.$$  \hspace{1cm} (37)

where the superscript $p$ in $c^p$ reminds us that this is consumption for the power utility investor. For quadratic utility

$$u(c) = -\frac{1}{2} (c_t^b - c_t)^2,$$

we have

$$c_t^q = c_t^b - \lambda x_t^*.$$  \hspace{1cm} (38)

As before, the investor purchases the bliss point, financed by shorting the “most expensive” payoff $x^*$.

Equations (37) and (38) give us the essence of the difference between power and quadratic utility, i.e. long-run mean-variance efficient portfolios. The power utility payoff is a *nonlinear* declining function of $x^*$. The quadratic utility payoff, which corresponds to the long-run mean-variance frontier, is a *linear* declining function of $x^*$, and thus the payoff is itself mean-variance efficient.

To standardize the portfolios to initial wealth, we find Lagrange multipliers from the budget
constraints. For power utility the result is

\[
\frac{c^p}{W_0} = \frac{1}{\gamma} \left[ \rho + (\gamma - 1) \left( r^f + \frac{1}{2} \mu' \Sigma^{-1} \mu \right) \right] x_t^{*-\frac{1}{\gamma}}. \tag{39}
\]

We have already worked out the quadratic utility case. From proposition 2 equation (23), the nicest expression is

\[
\frac{c^q}{W_0} = r^f + \frac{1}{\gamma} \left( r^f - y_t^* \right) \tag{40}
\]

where \(\gamma\) represents a local risk aversion coefficient, controlled by the bliss point \(c^b\).

Figure 1 contrasts the power utility case (39) with the quadratic or mean-variance case (40). Both consumption decision rules slope downward. Lower returns mean higher marginal utility \(x^*\) or \(y^*\) and hence lower consumption. The power utility investor is particularly anxious not to suffer consumption declines; he therefore consumes more in bad times (right hand side of the graph) funding that consumption in good times (middle of the graph) He is also more price-sensitive; if consumption is really cheap on the left side of the graph, he can increase consumption without worrying about exceeding the bliss point. However, figure 1 includes the 1, 5, and 10 year densities of \(x^*\) to make the point that the large divergences between the two decision rules occur in states that are only very infrequently reached.

Figure 2 plots a simulation of power and quadratic consumption from this model. In this particular draw, good and bad luck mostly balance, so \(x^*\) mostly stays in the range that quadratic and power utility give quite similar answers. Very good luck leads to decreases in \(x^*\). Here, the power utility consumption will stay positive, while quadratic utility consumption continues to fall and can even fall below zero. We see this kind of event near year 10 of the simulation. The quadratic utility investor essentially acts like a hedge fund, taking on a set of payoffs that declines drastically in bad times in order to provide “catastrophe insurance,” sales of which fund slightly greater consumption in good times. Of course, since \(x^*\) is a geometric

\[
W_0 = \rho(c_{c^q}) = \mathcal{E} \left[ x_t^* (\lambda x_t^*)^{-\frac{1}{\gamma}} \right]
\]

\[
W_0 = \lambda^{\frac{1}{\gamma}} \mathcal{E} \int_0^\infty e^{-\rho t} e^{\left( \frac{2-\gamma}{\gamma} \right) \left( r^f + \frac{1}{2} \mu' \Sigma^{-1} \mu \right) t} + \left( \frac{2-\gamma}{\gamma} \right) \mu' \Sigma^{-1} \mu} t \right] dt
\]

\[
W_0 = \lambda^{\frac{1}{\gamma}} \int_0^\infty e^{-\rho t} e^{\left( \frac{2-\gamma}{\gamma} \right) \left( r^f + \frac{1}{2} \mu' \Sigma^{-1} \mu \right) t + \left( \frac{2-\gamma}{\gamma} \right) \mu' \Sigma^{-1} \mu} t \right] dt
\]

\[
W_0 = \frac{\lambda^{\frac{1}{\gamma}}}{\frac{1}{\gamma} \left( \rho + (\gamma - 1) \left( r^f + \frac{1}{2} \mu' \Sigma^{-1} \mu \right) \right)}
\]

Substitute this expression for \(\lambda\) in (37).

If \(\gamma < 1\) and enough so that

\[
\rho + (\gamma - 1) \left( r^f + \frac{1}{2} \mu' \Sigma^{-1} \mu \right) < 0
\]

it appears that \(p(c)\) is infinite. In this case, the answer is \(c = 0\). The consumer endlessly puts off consumption since investment is so much more attractive.
Figure 1: Consumption payoffs from power utility and quadratic utility in the iid return case. The x axis is the pricing payoff or marginal utility process $x^\ast$. The y axis gives consumption relative to initial wealth as a function of $x^\ast$. Parameters are risk aversion $\gamma = 2$, discount factor $\rho = 0.01$, riskfree rate $r^f = 0.05$ excess return mean $\mu = 0.04$ and standard deviation $\sigma = 0.20$.

Brownian motion, eventually it always will arrive at a region of the state space on the left or right of Figure 1, in which quadratic and power utility solutions differ by a lot. However, it takes time to do so, and divergence in the far off future is discounted, making little difference in ex-ante utility or the long-run mean-variance properties.

5.3 Long-run mean-variance frontier

Since the whole paper is about long-run mean-variance efficiency, it’s interesting to understand what the frontier looks like in this simple example. The frontier also gives us a natural measure of how far apart the power and quadratic solutions are really, i.e. in their population moments rather than just in one draw.
The slope of the long-run mean-variance frontier (32) is given in this case by

$$\frac{\bar{E}(y_t^*)}{\sigma(y_t^*)} = \sqrt{\frac{(r_f - \rho)^2 + \rho \mu' \Sigma^{-1} \mu}{r_f^2 - (r_f - \rho)^2 - \rho \mu' \Sigma^{-1} \mu}}. \quad (41)$$

The formula is complex because the investor cares about volatility over time as well as across states of nature. To understand it, suppose first that there is no instantaneous Sharpe ratio, \( \mu' \Sigma^{-1} \mu = 0 \). Then there still is a long-run mean-variance frontier,

$$\frac{\bar{E}(y_t^*)}{\sigma(y_t^*)} = \sqrt{\frac{(r_f - \rho)^2}{r_f^2 - (r_f - \rho)^2}}. \quad (42)$$

The only investment is the riskfree rate \( r_f \), but there still is the question of how fast to take money out of wealth place in the riskfree investment, and thus a trade-off between average level and volatility over time. If the consumer takes consumption \( c_t = \alpha W_t \), then wealth grows at \( dW_t/W_t = (r_f - \alpha) \, dt \), and consumption follows \( c_t = W_0 \alpha e^{(r_f - \alpha) t} \). If the consumer chooses

\[ \bar{E}(y_t^*) = \bar{E} \left(1 - \frac{y_t^*}{r_f}\right) = 1 - \frac{\bar{E}(x^*)}{r_f \, p(x^*)} = 1 - \frac{\rho}{r_f^2 \, p(x^*)} \]

\[ = 1 - \frac{\rho}{r_f^2} \left(2r_f - \rho - \mu' \Sigma^{-1} \mu\right) \]

\[ = \frac{(r_f - \rho)^2 + \rho \mu' \Sigma^{-1} \mu}{r_f^2}. \]

(32) then gives the long-run Sharpe ratio based on this quantity.

---

Figure 2: Simulated consumption from power and quadratic utility in iid return case.
\( \alpha = r^f \), he obtains constant consumption and wealth \( W_t = W_0, c_t = r^fW_0 \). This path has zero long-term variance, and long-run mean \( \bar{E}(c) = r^fW_0 \). However, if the consumer chooses a lower consumption/wealth ratio \( \alpha < r^f \), he will start with lower consumption initially, but he will then obtain a rising consumption path. The rising path represents a source of long-term variance, since that measure prizes stability over time as well as over states of nature. If \( \rho = r^f \), the choice of a lower consumption/wealth ratio makes no difference to the long-run mean, since the trade-off in \( r \) is the same as the discounting in \( \rho \), and the formula (42) verifies that with \( \rho = r^f \) the mean-variance trade-off is flat. If \( r^f < \rho \), however, lower initial consumption with higher growth raises the discounted (at the rate \( \rho \)) sum, so the long-run mean rises as well as the long run variance when the consumer chooses a lower consumption-wealth ratio. In this way, the choice of payout rate gives rise to a long-run mean-variance trade-off, captured by formula (42), even when there is no risk at all.

Now, suppose there is risk, but \( \rho = r^f \). The long-run Sharpe ratio simplifies to

\[
\frac{\bar{E}(y^*_t)}{\tilde{\sigma}(y^*_t)} = \sqrt{\frac{\mu^\prime \Sigma^{-1} \mu}{r^f - \mu^\prime \Sigma^{-1} \mu}}
\]

(43)

This function is increasing in the instantaneous sharpe ratio \( \sqrt{\mu^\prime \Sigma^{-1} \mu} \) as we might expect. Now the investor will get more mean, and variance, by taking a larger investment in the risky assets.

Typical numbers for the long-run Sharpe ratio will be a good deal larger than those for the instantaneous Sharpe ratio \( \sqrt{\mu^\prime \Sigma^{-1} \mu} \). That fact simply reflects different units. 10 year Sharpe ratios are about \( \sqrt{10} \) larger than one-year Sharpe ratios, and the long-run frontier in essence characterizes a weighted sum of long-run returns.

The power utility investor’s payoffs will not be long-run mean-variance efficient. To characterize how close the power and quadratic solutions are, I find the long-run mean and long-run variance of the power utility solution. From (39), the long-run mean and standard deviation of consumption \( \bar{E}(c^p_t) \) and \( \tilde{\sigma}(c^p_t) \) are proportional to the long-run mean and standard deviation of
\( x^{* - \frac{1}{2}} \). Evaluating those moments from the \( x^* \) process (34), we have

\[
\tilde{E}(x_t^{* - \frac{1}{2}}) = \frac{\rho}{\rho + \frac{1}{\gamma} (\rho - r^f) - \frac{1}{2\gamma} \left(1 + \frac{1}{2}\right) \mu \Sigma^{-1} \mu}, \quad (44)
\]

\[
\tilde{E}(x_t^{* - \frac{2}{2}}) = \frac{\rho}{\rho + \frac{2}{\gamma} (\rho - r^f) - \frac{1}{2} \left(1 + \frac{2}{2}\right) \mu \Sigma^{-1} \mu}. \quad (45)
\]

The variance and mean/standard deviation ratio of \( x_t^{* - \frac{1}{2}} \) and hence consumption follow, but do not result in simple formulas.

We can locate the long-run mean and variance of the quadratic utility consumption stream— which \textit{point} on the frontier the quadratic-utility investor chooses. From (40)

\[
\frac{c^q_t}{W_0} = r^f + \frac{1}{\gamma} \left(r^f - y_t^*\right)
\]

so the long-run mean and standard deviation are just \( \frac{r^f}{\gamma} \) (a small number) times the mean and standard deviation of \( y_t^* \) calculated above.

Table 1 presents calculations of these long-run means, standard deviations and Sharpe ratios for a variety of parameter values. All cases in Table 1 use \( \rho = r^f \) so at \( \mu = 0 \), both power and quadratic investors choose a constant consumption path with mean equal to the riskfree rate and zero standard deviation. The Sharpe ratio is of course zero in this case.

As we raise the risky asset premium \( \mu \), both mean consumption yield (Panel A) and its standard deviation (Panel B) start to increase over the riskfree rate. Investment opportunities

\[
\tilde{E}(x_t^{* - \frac{1}{2}}) = \rho \int e^{-\rho t} e^{-\frac{1}{2} \left(\rho - r^f + \frac{1}{2} \mu \Sigma^{-1} \mu\right) t + \frac{1}{2} \mu \Sigma^{-1} \mu} dt
\]

\[
= \rho \int e^{-\left(1 + \frac{1}{2}\right) \mu \Sigma^{-1} \mu} \left(1 + \frac{1}{2}\right) \mu \Sigma^{-1} \mu dt
\]

\[
= \frac{\rho}{\rho + \frac{1}{\gamma} (\rho - r^f) - \frac{1}{2\gamma} \left(1 + \frac{1}{2}\right) \mu \Sigma^{-1} \mu}
\]

\[
\tilde{E}(x_t^{* - \frac{2}{2}}) = \rho \int e^{-\rho t} e^{-\frac{1}{2} \left(\rho - r^f + \frac{1}{2} \mu \Sigma^{-1} \mu\right) t + \frac{1}{2} \mu \Sigma^{-1} \mu} dt
\]

\[
= \rho \int e^{-\left(1 + \frac{1}{2}\right) \mu \Sigma^{-1} \mu} \left(1 + \frac{1}{2}\right) \mu \Sigma^{-1} \mu dt
\]

\[
= \frac{\rho}{\rho + \frac{2}{\gamma} (\rho - r^f) - \frac{1}{2} \left(1 + \frac{2}{2}\right) \mu \Sigma^{-1} \mu}
\]
are better, and consumers can afford larger consumption for given initial wealth (higher mean) at the cost of absorbing some standard deviation. For $\mu = 1–2\%$, the mean and standard deviation of consumption yield are almost identical for power and quadratic cases. Mean consumption yields start to diverge substantially at higher means combined with low risk aversion coefficients, and standard deviations diverge slightly faster.

Panel C gives long-run Sharpe ratios. The long-run Sharpe ratio for quadratic utility is just the slope of the long-run mean-variance frontier, and so is the same for any risk aversion coefficient. Power utility investors choose portfolios that are less than frontier, in order to control higher moments. For small risk premia $\mu$ and high risk aversion coefficients $\gamma$, the Sharpe ratios of the power utility solutions are quite similar to the quadratic case. However, for large risk premia and small risk aversion coefficients, the Sharpe ratios for power utility start to decline; here the higher moments start to matter more.

In sum, a characterization that power utility investors desire long-run mean-variance efficient portfolios works best for small risk premia and for large risk aversion.

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<thead>
<tr>
<th>$\gamma$</th>
<th>Power Utility</th>
<th>Quadratic utility</th>
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<td>0.25 1.00 2.25 4.00</td>
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<td>0.22 0.40 0.50 0.40</td>
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<td>2</td>
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<td>0.05 0.10 0.15 0.20</td>
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Table 1. Long run mean, standard deviation, and Sharpe ratio of consumption yields. $\mu$ across the top row gives the assumed instantaneous mean risky asset return, $\gamma$ along the columns gives the assumed risk aversion coefficient. Other parameters are $\rho = r^f = 5\%$, $\sigma = 20\%$. 
5.4 Portfolio strategies

The whole point of the contingent-claim approach is to forget about trading strategies and to focus on final payoffs. However, we can easily find the trading strategies in this case, and it helps intuition and connection to familiar results in this case to follow them.

We generate yields or payoff streams by varying the portfolio weights \( w_t \) in the risky assets and by taking the payoff as a dividend. The value of an investment then follows

\[
dV_t = \left( r^f_t V_t - x_t \right) dt + \omega_t dr^c_t. \tag{46}
\]

\( x_t \) and \( \omega_t \) describe dollar payouts and dollar positions in the excess returns. Often both payoff and portfolio weight are proportional to wealth,

\[
x_t = \alpha_t V_t; \quad w_t = \omega_t V_t
\]

and then we can describe the strategy by payout rates \( \alpha_t \) and portfolio weights \( w_t \).

The power-utility consumption stream has a familiar portfolio characterization. The investor holds a mean-variance efficient portfolio with constant weights and consumes a constant fraction of wealth,

\[
\frac{dW_t}{W_t} = \left( r^f_t - \frac{c^p_t}{W_t} \right) dt + w_t dr^c_t
\]

\[
w_t = \frac{1}{\gamma} \Sigma^{-1} \mu
\]

\[
c^p_t = \frac{1}{\gamma} \left[ \rho + (\gamma - 1) \left( r^f + \frac{1}{2 \gamma} \mu' \Sigma^{-1} \mu \right) \right] W_t.
\]

The quadratic utility investor also holds a mean-variance efficient portfolio. (Equivalently, the long-run mean-variance frontier results from such a portfolio)

\[
w_t = \frac{c^b - r^f W_t}{r^f W_t} \Sigma^{-1} \mu. \tag{47}
\]

The composition of the portfolio is mean-variance efficient. However, the size of the risky asset portfolio changes over time. We can interpret the term before \( \Sigma^{-1} \) as the inverse of a local risk-aversion coefficient. As wealth rises to the value \( r^f W_t = c^b \) that would support bliss-point consumption, the quadratic-utility investor becomes more risk averse. The payout rule is

\[
c_t^q = c_t^b + \left( 2 r^f - \rho - \mu' \Sigma^{-1} \mu \right) \left( W_t - \frac{c^b}{r^f} \right) \tag{48}
\]

Consumption also rises proportionally to wealth, but this time with an intercept. With these two rules, wealth evolves as

\[
dW = \left( r^f W_t - \left[ c^b - \left( 2 r^f - \rho - \mu' \Sigma^{-1} \mu \right) \left( \frac{c^b - r^f W_t}{r^f} \right) \right] \right) dt + \left( \frac{c^b - r^f W_t}{r^f} \right) \Sigma^{-1} \mu.
\]

In sum, the difference between the power and quadratic utility investor is not in the composition of the risky portfolio – both investors hold mean-variance efficient portfolios – but in the payout rate relating consumption to wealth and the portfolio weight or allocation between risky and
riskfree assets. The quadratic utility / long-run mean-variance portfolio adds a constant to the rules of the power utility case.

We can also understand the portfolios and payouts behind the mean-variance efficient payoffs and their characterization in terms of $x^*$ and $y^*$. We can think of $x_t^*$ as a value process that generates the yields $y_t^*$ as the dividend stream, with portfolio weights $-\mu'\Sigma^{-1}$. Start with the diffusion representation for $x^*$ (33) and then rewrite it as a value process in the form (46),

\[
\begin{align*}
\frac{dx^*}{x^*} &= \left( \rho - r^f \right) dt - \mu'\Sigma^{-1}\sigma dz_t \\
\frac{dx^*}{x^*} &= \left[ r^f - \left( 2r^f - \rho - \mu'\Sigma^{-1}\mu \right) \right] dt - \mu'\Sigma^{-1} (\mu dt + \sigma dz_t) \\
x_t^* &= \left( r^f x_t^* - y_t^* \right) dt - x_t^*\mu'\Sigma^{-1}dr_t^e.
\end{align*}
\]

In this way, we can think of the payoff $y_t^*$ as one that comes from a short position in a mean-variance efficient investment, that pays out a constant fraction $(2r^f - \rho - \mu'\Sigma^{-1}\mu)$ of its value $x_t^*$ every period. Scaling things up, of course, we can view $x_t^*$ itself as a payoff as well as a price process. Starting with a value $V_0 = p(x^*) = 1/(2r^f - \rho - \mu'\Sigma^{-1}\mu)$, the same portfolio strategy generates the payoff $x_t^*$. Finally, if one shorts $y^*$ and places the resulting $\$1$ in the riskfree asset, giving a yield $r^f$ each period, that creates a payoff $r^f - y^*$; the excess payoff $y^{**} = (r^f - y^*)/r^f$ is just a larger version of the same operation.

To create a long-run mean-variance efficient yield

\[ y^{mv} = r^f + k \times (r^f - y^*) \]

you do not simply hold a mean-variance efficient instantaneous return $r^f + a\mu'\Sigma^{-1}dr^e$ for some positive $a$ and pay out at some fixed rate. Instead, you hold a short position in a security ($y^*$) that itself cumulates a short position $-\mu'\Sigma^{-1}dr^e$ in a mean-variance efficient portfolio. While the double shorting action cancels itself in instantaneous returns, it does not cancel itself in cumulated returns, since the cumulation process is nonlinear, i.e. giving rise to exponential growth. The mean-variance frontier, and the optimal payoffs for the quadratic utility investor cumulate a short position in the instantaneous mean-variance frontier, or a portfolio on the bottom portion of that frontier. The optimal payoffs for the power utility investor cumulate a long position, from the upper portion of the mean-variance frontier. The results are all driven by the same shock $\int_0^t dz_t$, but the resulting portfolios are different nonlinear functions of that shock, and hence nonlinear functions of each other.

Of course there is nothing essential about this double-short characterization. We can express mean-variance efficient yields as the result of a single value process with time-varying payout and portfolio rules as well – and the quadratic utility payout and portfolio rules (47) and (48) are instances. The riskfree payoff comes from a value process in which you invest $w = 0$ in risky assets and take a payout $\alpha = r^f$ each period. The associated value process is $V^f = 1$ forever. If an investor wants a payoff $y^{mv} = (1 + k)(r^f - ky^*)$ on the upper part of the mean-variance frontier, he shorts $ky^*$ and buys $(1 + k)r^f$. The total value of his portfolio is

\[ W = (1 + k)V^f - kx^* = (1 + k) - kx^* \]
The excess yield payout approaches $W = 0$, the portfolio still pays out a dollar amount so similarly

$$dW_t = \left\{ r^f W_t - \left[ r^f W_t - (1 + k - W_t) \left( r^f - \rho - \mu'\Sigma^{-1}\mu \right) \right]\right\} dt + (1 + k - W_t) \mu'\Sigma^{-1}dr^e.$$

We recognize a payout that is a linear function of $W$ with a constant,

$$y_t^{mv} = r^f W_t - (1 + k - W_t) \left( r^f - \rho - \mu'\Sigma^{-1}\mu \right) \quad (50)$$

and a portfolio that scales up or down a mean-variance efficient investment, again with a rule that is a linear function of $W$ and a constant,

$$w_t^* = (1 + k - W_t) \mu'\Sigma^{-1} \quad (51)$$

Portfolios behind $y^{es}$

When studying frontiers, it is convenient to focus on excess returns and simply to calculate

$$y^{es} = \frac{1}{r^f} \left( r^f - y^* \right).$$

The excess yield $r^f - y^*$ can be generated analogously as a single investment with weights that are linear functions of wealth with a constant, rather than as a short position in proportional payout rules. The value of the underlying portfolio is

$$W = V^f - x^* = 1 - x^*$$

so similarly

$$dW_t = \left\{ r^f W_t - \left[ r^f - (2r^f - \rho - \mu'\Sigma^{-1}\mu) \right] (1 - W_t) \right\} dt + (1 + k - W_t) \mu'\Sigma^{-1}dr^e$$

The payout and portfolio policies are linear functions of $W$ including a constant. In particular at $W = 0$, the portfolio still pays out a dollar amount $\rho - r^f + \mu'\Sigma^{-1}\mu$ and still invests in zero-cost risky assets a dollar amount $\mu'\Sigma^{-1}dr^e$. As $W \to 1$, which can support a payout $r^f$ forever, the payout approaches $r^f$ with zero investment in the risky assets.

---

\(^9\) Using $dW_t = -kdx_t^*$ and (49), that value follows

$$dW_t = \left\{ r^f W_t - \left[ r^f W_t - (1 + k - W_t) \left( r^f - \rho - \mu'\Sigma^{-1}\mu \right) \right]\right\} dt + (1 + k - W_t) \mu'\Sigma^{-1}dr^e.$$

\(^10\) The payout and portfolio policies are linear functions of $W$ including a constant. In particular at $W = 0$, the portfolio still pays out a dollar amount $\rho - r^f + \mu'\Sigma^{-1}\mu$ and still invests in zero-cost risky assets a dollar amount $\mu'\Sigma^{-1}dr^e$. As $W \to 1$, which can support a payout $r^f$ forever, the payout approaches $r^f$ with zero investment in the risky assets.
5.5 Potentially infinite price of $x^*$

Formula (67),

$$p(x^*) = \frac{1}{2r^f - \rho - \mu' \Sigma^{-1} \mu}$$

reveals a technical limitation of this standard setup. If $2r^f - \rho - \mu' \Sigma^{-1} \mu \leq 0$, the price of the payoff $x^*$ is infinite. Then the yield $y^* = x^*/p(x^*)$ is undefined. These parameters are not particularly extreme, at least relative to standard statistics. For example, if one takes a 6% equity premium and an 18% standard deviation of market returns, then $\mu' \Sigma^{-1} \mu = 1/9 = 0.11$. A 6% interest rate $r^f$ and a 1% discount rate $\rho$ are just enough to give an infinite $p(x^*)$, and any lower interest rate or higher discount factor make matters worse. (One might argue that these standard statistics are extreme, which is the entire equity premium puzzle, but that’s another issue.)

Everything seems to explode at $2r^f - \rho - \mu' \Sigma^{-1} \mu = 0$. We can write the Sharpe ratio from (41) as

$$\bar{\mathcal{E}}(y^*_\infty) = \sqrt{\frac{(r^f - \rho)^2 + 2 \rho \mu' \Sigma^{-1} \mu}{r^f - \rho - \mu' \Sigma^{-1} \mu}} = \sqrt{\frac{\rho [2r^f - \rho - \mu' \Sigma^{-1} \mu]}{2r^f - \rho - \mu' \Sigma^{-1} \mu}} - 1,$$

so the slope of the long-run mean-variance frontier rises to infinity. The minimum second-moment yield $y^*_0 = x^*_0/p(x^*)$ is collapses to zero – zero mean and zero standard deviation. A payoff that costs one and delivers zero is a great short opportunity; shorting this payoff and investing in the risk free rate gives higher and higher Sharpe ratios. The consumption decision rule (48) goes simply to consumption at the bliss point, $c_t = c^b$.

These pathologies all seem to indicate an arbitrage opportunity at or beyond the limit $2r^f - \rho - \mu' \Sigma^{-1} \mu = 0$, but they do not. The key restriction on dynamic trading is that the time-zero value of wealth must be zero, $\lim_{T \to \infty} p(W_T) = 0$. Instead, the strategy that seems to provide bliss point consumption and the strategy that seems to provide $-y^*_0 = 0$ despite an initial investment of -1 both involve rolling over debts forever, in such a way that the present value of the debt remains constant for $2r^f - \rho - \mu' \Sigma^{-1} \mu = 0$ and grows with horizon past that point, violating the no-arbitrage limitation on dynamic trading.

Underlying it all, at the limit the discount factor $x^*_t$ ceases to be a traded payoff – it exits the payoff space $X$ which is limited to payoffs generated by trading strategies that do not violate arbitrage. To see this, compute from the $x^*$ process (66) $E(x^*_t) = e^{(\rho - r^f)t}$ and $E(x^*_t^2) = e^{(\rho - 2r^f + \mu' \Sigma^{-1} \mu)t}$. From $E(x^*_t) = e^{(\rho - r^f)t}$ we see that $x^*$ is expected to grow or decay slowly at the difference between discount rate and interest rate, which does not cause much trouble. The trouble occurs from the Ito term in $x^*_t^2$. $\mu' \Sigma^{-1} \mu$ can make $E(x^*_t^2) = e^{(2\rho - 2r^f + \mu' \Sigma^{-1} \mu)t}$ grow faster than $e^{\rho t}$. When this happens, $x^*_t$ no longer obeys $\lim_{t \to \infty} p(x^*_t) = \lim_{t \to \infty} E(x^*_t^2) = 0$. $x^*$ simultaneously starts to violate the no arbitrage condition and it leaves the space of square-integrable $\mathcal{E}(x^*_t) < \infty$ payoffs.

How did $x^*$ leave the space of traded payoffs? Didn’t the law of one price guarantee a discount factor in the payoff space? The answer is that this payoff space does not meet the full set of assumptions required for the existence theorem. It is not “complete.” If you take a sequence of payoffs resulting from positive payout rates, but with a payout rate tending to zero, each of those payoffs is in $X$ – the time-zero value of their limiting wealth is zero – but the limit point is not in the payoff space. For $2r^f - \rho - \mu' \Sigma^{-1} \mu > 0$ we are “lucky”. Even though we have
no guarantee that there will be a discount factor in the payoff space, there happens to be one. Beyond, we are not so lucky.

Therefore, at $2r^f - \rho - \mu'\Sigma^{-1}\mu = 0$ and beyond, this solution technique is no longer valid. It gives answers to the long-run mean-variance frontier and to the quadratic utility investor’s problem that violate arbitrage limits on trading strategies. The discount factor $x^*$ is no longer in the set of traded assets $X$, so $y^*_f$ is undefined ($x^*_f$ has no price to divide by) and we cannot find the long-run mean-variance frontier or the solution to the quadratic utility investor’s problem by taking linear functions of $y^*$.

To solve these problems for $2r^f - \rho - \mu'\Sigma^{-1}\mu \leq 0$, then, one must impose constraints on trading, and one must expect them to bind. I explore below an alternative way of approaching the problem that has this effect. I limit the investor to the choice of a finite number of trading strategies – payout rules and portfolio weights – and all linear combinations of those trading strategies. This space is complete, and bounds the investor away from arbitrage strategies.

There is “almost” an arbitrage opportunity as we approach the limit, and the slope of the long-run mean-variance frontier becomes arbitrarily high. However, the strategies that give these results increasingly rely on wealth that “almost” explodes, rolling over debt that grows astronomically if not quite explosively. While technically correct, such solutions are less and less interesting. The approach below that bounds the investor away from the limit also bounds him away from such less interesting strategies.

Why not....

Why does power utility not run in to similar problems? for $2r^f - \rho - \mu'\Sigma^{-1}\mu \leq 0$, $x^*_f$ is still outside the payoff space and its price is still infinite. However, the power utility solution is $u'(c^*_t) = \lambda x^*_t; c^*_t = (\lambda x^*_t)^{-\frac{1}{\gamma}}$. Even though marginal utility $\lambda x^*_t$ has an infinite price, $(x^*_t)^{-\frac{1}{\gamma}}$ has a finite price so consumption $(\lambda x^*_t)^{-\frac{1}{\gamma}}$ has a finite price. It is not always necessary that marginal utility have a finite price.

This is a “complete market” so the issue has nothing to do with market incompleteness. The payoff space can fail to be “complete” (Cauchy sequences converge) even when the market is “complete” (all contingent claims are traded.)

One might think that this situation can be resolved by giving the quadratic utility investor a growing bliss point, perhaps even driven by shocks $dz$, so that he wants growth as does the power utility investor. This modification will also help to make the quadratic utility setup a better approximation to power or other utility results. Alas, this modification does not solve the problem of potentially infinite $p(x^*)$. From (21) we can write the optimal portfolio $c^a = -\lambda x^* + c^b - \hat{c}$. The investor purchases claims to the bliss point (and labor income) hedge portfolios, by selling claims to $x^*$. The nature of this operation does not change if $c^b$ grows (deterministically or stochastically) over time; if $p(x^*)$ becomes infinite, the shadow value of wealth $\lambda$ seems to go to zero and it seems the investor can finance any $c^b$. If $c^b$ grows so much (in mean or variance) that its value too becomes infinite, we might overcome the problem, but alas two infinities do not cancel.

One might think that the situation can be resolved by raising the discount rate in the $\mathcal{E}$ operator to quash long-run growth. Alas, the situation becomes worse if one chooses a larger (arbitrary) $\rho$ in the definition of $\mathcal{E}$, disconnecting it from the subjective discount factor. Directly, $2r^f - \rho - \mu'\Sigma^{-1}\mu > 0$ requires higher $r^f$ or lower $\mu'\Sigma^{-1}\mu$ for higher $\rho$. More deeply, the pricing
operator $\Lambda_t = e^{-\rho t} x_t^*$ must be the same for any $\rho$. Thus as one chooses to represent prices with a larger $\rho$, the corresponding $x_t^*$ must grow faster so that $e^{-\rho t} x_t^*$ is unaffected.

Intuitively, the lognormal return model gives rise to infrequent “disasters” of extremely low return, low consumption, and thus very high marginal utility. When we price marginal utility, taking $E(x_t^2)$, these high marginal utility states also have very high prices ($x^*$ plays both roles). Since a quadratic utility investor does not care that much about consumption declines, being willing even to tolerate negative consumption with finite marginal utility, he actively sells consumption in these very high-priced states to finance a great deal of consumption in other states. More fundamentally, the result comes from the probability structure imposed by lognormal returns. A probability model with less frequent or less highly valued disaster states will not generate infinite prices.

Infinite values can also pop out of the power utility portfolio. Formulas (44) and (45) assume that $\rho$ dominates expected consumption and squared consumption, so that the long-run first and second moments are finite. This is not necessarily true. For parameter values with $r^f > \rho$ or large Sharpe ratios, mean consumption growth or mean squared consumption growth can grow faster than $\rho$, so one or both numbers are infinite. That just means that the power utility consumer chooses a payoff really far from the long-run mean/variance frontier.

Mean consumption growth for the power utility investor is

$$E\left(\frac{c^p_t}{c^p_0}\right) = \exp\left[\frac{1}{\gamma} \left(r - \rho^f + \frac{1}{2} \left(1 + \frac{1}{\gamma}\right) \mu' \Sigma^{-1} \mu\right) t\right]$$

and mean squared consumption growth is

$$E\left[\left(\frac{c^p_t}{c^p_0}\right)^2\right] = \exp\left[\frac{1}{\gamma} \left(2 \left(\rho - r^f\right) - \left(1 + \frac{2}{\gamma}\right) \mu' \Sigma^{-1} \mu\right) t\right]$$

Even with $\rho = r^f$, mean consumption growth and mean squared consumption growth increase for higher risk premia $\mu' \Sigma^{-1} \mu$, and they do so more for low risk aversion $\gamma$. In the upper right hand corner of Table 1B, we find a parameter region in which mean squared consumption growth rises faster than $e^{-\rho t}$ can discount it so $\int e^{-\rho t} E(c^p_t^2) dt$ is infinite. This is entirely possible. Standard deviations of finite-priced assets can be infinite in one-period models too.

### 6 Predictable returns

Suppose there is a single return, predictable by a mean-reverting state variable,

$$dr_t = r^f dt + dr^e_t$$

$$dr^e_t = \mu_t dt + dz_t$$

$$d\mu_t = \phi(\bar{\mu} - \mu_t) dt + \sqrt{\mu_t} dw_t$$

$$E(dz^2_t) = \sigma^2; E(dw^2_t) = \sigma_w^2; E(dz_t dw_t) = \rho \sigma \sigma_w$$

The shocks $dw_t$ and $dz_t$ are not perfectly correlated, so markets are incomplete in this case. I include a square root in the mean process unlike the standard specification in Kim and Omberg (1997), Brennan, Schwartz and Lagnado, (1997), Campbell and Viceira (1999). It’s just as easy
in this context, and it seems a bit more realistic. It avoids negative conditionally expected excess returns, and it specifies that expected returns are more volatile when they are higher. The results are not strongly affected by the presence of absence of the square root.

The discount factor process is

$$\frac{dx_t^*}{x_t^*} = \left( \rho - r^f \right) dt - \mu_t \sigma^2 d\zeta_t. \tag{55}$$

I ignore any nontraded income, so the optimal portfolio is, from Proposition 1, or directly from

$$u'(c_t) = \lambda x_t^*, \quad c_t^q = \tilde{c}^b - \left[ p(\tilde{c}^b) - W_0 \right] \frac{x_t^*}{p(x_t^*)}.$$

The interesting part of the portfolio follows $x_t^*$.

One may ask, why can’t we solve the power utility problem in this way as well, i.e. from

$$u'(c_t) = c_t^{\gamma} = \lambda x_t^*$$

derive

$$c_t^p = (\lambda x_t^*)^{\frac{1}{\gamma}} \tag{56}$$

where $\lambda$ scales up or down to satisfy the initial wealth constraint? The answer that we cannot, because (56) is a nonlinear function of $x_t^*$. While $x_t^*$ is traded, nonlinear functions of $x_t^*$ are not necessarily traded. We do have continuous trading, and that usually is enough to deliver tradability of nonlinear functions. However, $\mu_t$ appears in the diffusion term of (55) – a time-varying Sharpe ratio requires a conditionally heteroskedastic discount factor. As in stochastic volatility options problems, this feature means that we cannot form nonlinear functions of $x^*$ through dynamic trading. To generate a nonlinear function of $x_t^*$, one must load on the shocks $dw_t$, outside the payoff space. In turn, this fact means that $(\lambda x_t^*)^{\frac{1}{\gamma}}$ is not the solution to the power utility investment problem in the first place. The discount factor $x_t^*$ implies zero market price of the portion of $dw$ orthogonal to $dz$, and at that price the investor would choose to invest in $dw$. The actual solution to the power utility version of the problem (as in Kim and Omberg 1997) implies a larger, time-varying, and, in this case, unknown, market price of $dw$ risk, just sufficient to keep the investor from buying it.

I calibrate a process for $\mu_t$ from standard forecasting regressions as summarized in Cochrane (2004, p. 406). The right hand variables such as D/P that predict returns are typically quite smooth, with autocorrelation coefficients of 0.9, corresponding to $\phi = 0.1$. I use a slightly larger $\phi = 0.2$ so that dynamics are more visible in a reasonably short sample. The size of the variation in expected returns is a more contentious issue. Point estimates of a regression forecasting returns from the dividend price ratio give a standard deviation of expected excess returns of 5%. That surely is overfit, but the general message is that the time-variation in expected returns is of the same order as the level of expected returns. I specify a somewhat smaller $\sigma(\mu_t) = 1\%$ along with a 2% unconditional equity premium $\bar{\mu}$. As in the iid case, these somewhat low equity premia keep $p(x^*)$ finite, in ways I describe in more detail below. From $\sigma(\mu_t)$ I find $\sigma_w$ by inverting the formula for the unconditional variance of a square root process. I specify a correlation between return and expected return shocks equal to -0.7 as in the data. When expected returns rise, prices go down, so one expects a substantial negative correlation between the two shocks.
Figure 3 gives a typical simulation of this central ingredient of the optimal portfolio, $-x^*_t$. The natural question is, how much does predictability alter the optimal portfolio and payoff? To answer this question, Figures 3 compares the optimal payoff to the payoff that would result if the investor ignored the time-variation in mean returns, i.e. it compares (55) to

$$\frac{dx_{iid,t}}{x_{iid,t}} = (\rho - r^f) dt - \mu \sigma^{-2} dz. \quad (57)$$

The $x_{iid,t}^*$ is a traded payoff; the consumer could eat it. It is just suboptimal so he should choose not to. $x_{iid,t}^*$ also is not a discount factor – it does not price assets in this economy.

The top panel of Figure 3 suggests that the payoff or optimal consumption stream is not that much affected by time-varying mean returns. Both payoffs rise and fall together, as stock prices rise and fall through $dz$ shocks. When the expected return $\mu_t$, plotted in the middle panel, is equal to its mean $\bar{\mu}$, the growth in $x^*$ and $x_{iid}^*$ are the same. When the expected return $\mu_t > \bar{\mu}$, $x^*$ moves by more than $x_{iid}^*$, and vice versa. In times of high risk premium, the dynamic portfolio accepts more exposure to risk, in return for the slightly higher overall level of consumption visible in the plot. However, the levels of $x^*$ do not wander that far from each other.

Table 2 collects statistics from a million simulations of this model. As one measure of the overall similarity of $x_t^*$ and $x_{iid,t}^*$, the table contrasts the long-run mean-variance efficiency of the two payoffs. $y^*$ is of course the maximal long-run Sharpe-ratio payoff, attaining a 0.84 long-run Sharpe ratio from a 2.1% mean and a 2.46% long run standard deviation. $y_{iid}^*$ has a slightly lower mean and also a bit less variance, leaving a slightly lower 0.62 long-run Sharpe ratio. The sense that $x^*$ and $x_{iid}^*$ are close then is reflected in the fact that the “static” payoff $x_{iid}^*$ still delivers a respectable long-run Sharpe ratio.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$r^f$</th>
<th>$\bar{\mu}$</th>
<th>$\sigma$</th>
<th>$\sigma(\mu_t)$</th>
<th>$\rho_{dw,dz}$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.05</td>
<td>0.02</td>
<td>0.20</td>
<td>0.01</td>
<td>-0.7</td>
<td>0.2</td>
</tr>
<tr>
<td>$p(x^*)$</td>
<td>$p(x_{iid}^*)$</td>
<td>$p_{iid}(x^*)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>34.15</td>
<td>25.92</td>
<td>25.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tilde{E}(r^f - y^*)$</th>
<th>$\sigma(y^*)$</th>
<th>Sharpe($y^*$)</th>
<th>$\tilde{E}(r^f - y_{iid}^*)$</th>
<th>$\sigma(y_{iid}^*)$</th>
<th>Sharpe($y_{iid}^*$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.10</td>
<td>2.46</td>
<td>0.84</td>
<td>1.18</td>
<td>1.90</td>
<td>0.62</td>
</tr>
</tbody>
</table>

Table 2. Simulation results for the case with time-varying expected excess returns. Expected excess returns $\mu_t = E_t(dr^e)$ follow $d\mu_t = \phi(\bar{\mu} - \mu_t)dt + \sqrt{\mu_t}dw_t$ returns follow $dr_t^e = \mu_t dt + dz_t$, $E_t(dz_t^2) = \sigma^2 dt$, the variance of $dw$ matches $\sigma(\mu_t)$, and corr($dz$, $dw$) = $\rho_{dw,dz}$. I simulate $\mu$ along with $dx^*/x^* = (\rho - r^f) dt - \mu_t \sigma^{-2} dz$ and $dx_{iid}^*/x_{iid}^* = (\rho - r^f) dt - \bar{\mu} \sigma^{-2} dz$, and average over 1,000,000 simulations to find $p(x^*) = E \int e^{-p_t}x_t^2 dt$, $p(x_{iid}^*) = E \int e^{-p_t}x_{iid}^2 dt$, etc. $p_{iid}(x^*)$ is the price of $x^*$ in an iid economy with $\mu_t = \bar{\mu}$. Yields $\tilde{E}(r^f - y^*)$ etc. are in percentage units.

The portfolios supporting $x^*$ and $x_{iid}^*$ differ much more substantially. The portfolio weight on risky assets that generates $x_t^*$ is equal to $\mu_t \sigma^{-2}$. (See (55).) As $\mu_t$ varies a great deal,
Figure 3: Simulated values of $-x^*$, $\mu$ and the optimal portfolio $c^t$ for the model with time-varying mean returns. The top panel contrasts the true $x^*$ with $x^*_{iid}$, which ignores variation in the conditional mean return and holds a constant position in the risky portfolio. The middle panel presents the conditional mean stock excess return $\mu_t$. The bottom panel presents the optimal portfolio when the quadratic utility investor is given a preference shock so that he follows the power-utility solution in the iid case. The optimal payoff in the bottom panel is 

$$
\frac{c^t}{W_0} = p \left( x^*_{iid} \right)^{-\frac{1}{\gamma}} \left\{ x^*_{iid.t} + \frac{(\gamma+1)}{p(x^*_{iid})} - \frac{1}{p(x^*_{iid})} \right\}^{-1} \left( x^*_{iid.t} - \frac{x^*_{iid.t}}{p(x^*_{iid})} - \frac{x^*_{iid}}{p(x^*_{iid})} \right)
$$

investment in the risky asset varies a great deal too. When $\mu$ rises to 4% around year 25, for example, the dynamic portfolio will have twice the investment in stocks as the iid portfolio. This strong variation in portfolios as expected returns vary is standard in the literature that studies this kind of problem.

In sum, the payoffs or optimal consumption streams do not seem much affected by predictable returns, although the portfolios supporting the optimal consumption streams are dramatically affected. This observation may make some sense of why we do not see the aggressive market-timing that portfolio problems prescribe.
6.1 Payoffs; a real portfolio problem

The bottom simulation in Figure 3 computes an actual portfolio payoff. Rather than compute a quadratic utility portfolio, I specify a stochastic bliss point so that in the iid case the quadratic and power utility problems are exactly the same. Then, I calculate the quadratic utility portfolio in the new time-varying mean environment using a risk aversion of two. First I detail the calculation of this payoff, and then discuss the result.

Using a stochastic bliss point so quadratic can mimic power utility

Consider a bliss point of the following form:

\[ c^b_t = c^p_t + k \frac{c^p_t - \gamma t}{p(c^p_t - \gamma)} \]  

(58)

where \( c^p_t \) is the payoff from following the iid-power utility rules,

\[ c^p_t = W_0 \left( \frac{x^*_{t-\frac{1}{2}}}{p(x^*_{t-\frac{1}{2}})} \right) \]

and \( x^*_{iid} \) follows (57). This payoff is traded despite incomplete markets (unlike \( x^*_{t-\frac{1}{2}} \)) since it uses a constant weight on the risky assets.

The quadratic utility solution \( c^q_t \) is

\[ c^q_t = c^b_t - \left[ p(c^b_t) - W_0 \right] \frac{x^*_{t-\frac{1}{2}}}{p(x^*_{t})} \]

By construction, \( p(c^p_t) = W_0 \), so \( p(c^b_t) = W_0 + k \). Substituting for the bliss point \( c^b_t \) from (58), the quadratic utility solution becomes

\[ c^q_t = c^p_t + k \left[ \frac{c^p_t - \gamma t}{p(c^p_t - \gamma)} - \frac{x^*_{t-\frac{1}{2}}}{p(x^*_{t})} \right] = c^p_t + k \left[ \frac{x^*_{iid,t}}{p(x^*_{iid})} - \frac{x^*_{t-\frac{1}{2}}}{p(x^*_{t})} \right]. \]  

(59)

The first term \( c^p_t \) of (59) is the power utility iid solution. If we are in the iid economy, of course, then \( x^*_{iid} = x^* \) so the second term vanishes. Thus, we have shown that for any \( k \), this bliss point generates a quadratic utility optimum equal to that of the power utility solution in the iid economy, \( c^q_t = c^p_t \).

The second term of (59) is a zero-cost portfolio. Here the investor buys less consumption in states that have high contingent-claims prices \( x^* \) relative to the contingent claims prices \( x^*_{iid} \) of the iid solution, and vice versa. This is exactly the sort of modification we want to evaluate.

Suppose \( k \) is very small. Then the bliss point (58) is very close to actual consumption \( c^q_t \), which means the quadratic utility investor displays a very high degree of risk aversion. The parameter \( \gamma \) controls the risk aversion of the power utility investor, whose decisions we are taking as an exogenous baseline or preference shock. It does not control the risk aversion of the quadratic utility investor, i.e. how willing he is to accept payoffs that deviate from that baseline. Since \( k \) controls how far consumption is from the bliss point, the parameter \( k \) instead controls the risk aversion of the quadratic utility investor.
As before, I choose \( k \) so that the quadratic utility investor displays risk aversion \( \gamma \) in a local-approximation sense, by calibrating the difference between initial consumption and bliss point,

\[
\gamma = - \frac{u''(c_0^q)}{c_0^q u'(c_0^q)} = \frac{c_0^q}{c_0^b - c_0^q}
\]

Using (59) and (58) for \( c^q \) and \( c^b \), using \( x_0^q = x_{iid,0}^* = 1 \), and solving for \( k \), we obtain\(^{11}\)

\[
k = \frac{W_0}{p(x_{iid}^* - x_{iid}^*)} \left( \frac{1}{p(x_{iid}^*)} - \frac{1}{p(x_{iid}^*)} \right).
\]

Substituting in for \( k \) in (59) we obtain the final payoff,

\[
c_t^q = \frac{W_0}{p(x_{iid}^* - x_{iid}^*)} \left\{ \frac{1}{x_{iid}^*} \left( (\gamma + 1) \frac{x_{iid,t}^*}{p(x_{iid}^*)} - \frac{1}{p(x_{iid}^*)} \right)^{-1} \left( \frac{x_{iid,t}^*}{p(x_{iid}^*)} - \frac{x_t^*}{p(x_t^*)} \right) \right\}
\]

(60)

The first term simply establishes the scale. As we might expect, consumption scales with wealth. The first term in brackets is the power utility-iid solution. The last term directs the investor to consume more in cheap (low \( x^* \)) states. As \( \gamma \) rises, the \( \gamma^{-1} \) term declines, so the investor deviates less and less from the power utility-iid solution.

To calculate portfolios and the long-run mean-variance frontiers, we need \( \bar{E}(x^*) \), \( p(x^*) = \mathcal{E}(x^*_{iid}) \), \( p(x_{iid}^*) = \mathcal{E}(x^*_{iid}*, \mathcal{E}(x_{iid}^*) \), and, if we want to calibrate to an initial wealth, \( p(x^* - x^*) \). The first quantity is easy. From (55)

\[
E_0(x_t^*) = e^{(\rho - r)t}
\]

\[
\bar{E}(x^*) = \rho \int_0^\infty e^{-\rho t} e^{(\rho - r)t} dt = \frac{\rho}{r}
\]

The prices (second moments) are alas not so easy and the standard approaches do not lead to closed-form expressions. Thus, I evaluate these moments by simply simulating the \( x^* \), \( x_{iid}^* \) and \( \mu \) processes a large number of times.\(^{12}\)

\[
\gamma = \frac{c_0^q + k \left[ \right]}{c_0^q + k \left[ \right]} + \frac{1}{p(x_t^*)} - \frac{1}{p(x_t^*)}
\]

\[
\frac{W_0}{p(x_{iid}^*)} + k \left[ \frac{1}{p(x_{iid}^*)} - \frac{1}{p(x_t^*)} \right] = \frac{W_0}{p(x_{iid}^*)} + \frac{1}{p(x_{iid}^*)} - \frac{1}{p(x_t^*)}
\]

\[
\frac{(\gamma + 1) \frac{1}{p(x_t^*)} - \frac{1}{p(x_t^*)}}{p(x_{iid}^*)} = \frac{W_0}{p(x_{iid}^*)} + \frac{1}{p(x_{iid}^*)} - \frac{1}{p(x_t^*)}
\]

\[
k = \frac{W_0}{p(x_{iid}^*)} \left[ \frac{1}{p(x_{iid}^*)} - \frac{1}{p(x_t^*)} \right] = \frac{W_0}{p(x_{iid}^*)} \left[ \frac{1}{p(x_{iid}^*)} - \frac{1}{p(x_t^*)} \right]
\]

\(^{11}\) I simulate the log of \( x^* \) and \( x_{iid}^* \) i.e.

\[
d \ln x_t = \left( \rho - r - \frac{1}{2} \mu^2 / \sigma^2 \right) dt - \mu_s \sigma^2 dz
\]

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Portfolio results; extreme states and infinite prices again.

The optimal consumption stream in the bottom panel of Figure 3 differs from the iid case by more than the difference between $x_{t \mid t}^*$ and $x^*$ in the top panel and more than a cursory examination of equation (60) suggests. As it turns out, $p(x^*) = 34.15$ is substantially higher than $p(x_{t \mid t}^*) = 25.92$. Thus, even though $x^*$ and $x_{t \mid t}^*$ are close to each other in the top panel, the yields $x_{t \mid t}^*/p(x_{t \mid t}^*)$ and $x_t^*/p(x_t^*)$ are different in equation (60), leading to the larger differences in the bottom panel. The major effect is that the overall level of consumption $c^t$ seems a good deal larger than the power utility - iid counterpart $c^p$ in the bottom panel of Figure 3, which is what we expect from equation (60) – the large $p(x^*)$ means that we are subtracting off a smaller number. Sensibly, the dynamic consumer seems to be doing better on average from his better strategy.

However, the level of consumption $c^p$ cannot always be larger than the power utility benchmark $c^p$. Both are financed by the same initial wealth, so there must be “expensive” states of the world with high $x_t^*$ that the optimal payoff has sold off to afford the larger consumption seen in Figure 3, states that did not to happen in this simulation. The fact that $p(x^*) = E(x^{*2})$ is so much larger than $p(x_{t \mid t}^*) = E(x^*x_{t \mid t}^*)$ suggests the same thing – there are some states with much higher $x^*$ than $x_{t \mid t}^*$. It’s clearly important to understand these states.

Why is $p(x^*) = E(x^{*2})$ so large? Figure 4 plots $e^{-\rho t}x_t^*$ for the draw in the first ten thousand that has the largest value of $\int e^{-\rho t}x_t^{*2}dt$, along with $e^{-\rho t}x_{t \mid t}^{*2}$ and $\mu_t$ for comparison. $x_t^*$ and $x_{t \mid t}^*$ are not mean-reverting processes; the appearance of mean-reversion here is due to the $e^{-\rho t}$ term. The explosion at year 50 is even larger if not discounted, but our objective is to understand how $p(x^*) = E(\int e^{-\rho t}x_t^{*2}dt)$ can get so large, so we are interested in explosions that survive discounting. I plot $e^{-\rho t}x_{t \mid t}^*$ rather than $e^{-\rho t}x_{t \mid t}^{*2}$ just for visibility; the explosion at year 50 is even larger in $e^{-\rho t}x_{t \mid t}^{*2}$ of course. The value of $\int e^{-\rho t}x_t^{*2}dt$ in this simulation is 14,754, dramatically larger than the mean (also $p(x^*)$) of 34.15.

The explosion in $e^{-\rho t}x^*$ around year 50 is impressive. There is a similar smaller event at year 20, which is in some sense coincidental; this simulation produces the largest value of $\int e^{-\rho t}x_t^{*2}dt$ because of the explosion around year 50. The plot makes the source of the explosion clear: Starting at year 40, the mean return $\mu_t$ moves strongly positive, to 8% and 10% despite the unconditional mean of 2%. This large mean means that $x^*$ and the portfolio underlying it take very strong positions $\mu_t\sigma^{-2}$ in the stock portfolio, 4-5 times larger than usual (see (55)). While $x^*$ takes such a strong position, there is a strong decline in stocks – a string of negative realizations of $d\bar{z}_t$. It is the combination of these two events that produces the explosion in $d\mu$ and the square root of $\mu$, i.e.,

$$d\mu = \phi(\bar{\mu} - \mu)dt + \sqrt{\mu}dw$$

$$d\sqrt{\mu} = \frac{1}{2\sqrt{\mu}}(\phi(\bar{\mu} - \mu)dt + \sqrt{\mu}dw) - \frac{1}{8\mu}\mu\sigma^2dw$$

$$d\sqrt{\bar{\mu}} = \left[\frac{\phi}{2\sqrt{\bar{\mu}}}(\bar{\mu} - \bar{\mu}) - \frac{1}{8\sqrt{\bar{\mu}}}\sigma^2\right]dt + \frac{1}{2}dw$$

$$d\sqrt{\bar{\mu}} = \left[\frac{\phi}{2}(\bar{\mu} - \frac{1}{4}\sigma^2) - \frac{1}{\sqrt{\bar{\mu}}} - \phi\sqrt{\bar{\mu}}\right]dt + \frac{1}{2}dw$$

Simulations with constant variance terms gives a better approximation. I use the standard Euler approximation at a monthly interval, after checking that finer intervals make a trivial difference to the results. I use a 100 year time horizon and 1,000,000 simulations.
By contrast, \( x_{iid}^* \) which places a constant weight \( \bar{\mu}\sigma^{-2} \) on the \( dz \) return shocks only rises slightly in this episode. Since the increase in expected returns and the decline in stock prices are, sensibly, correlated via a strong negative correlation of \( dz \) and \( dw \), events such as this are even more likely and more pronounced. Needless to say, consumption nosedives in this episode.

\[
\begin{align*}
\text{Worst offender; } e^{-\rho t}x_t^* \\
\text{dx}_{iid,t}^*/x_{iid,t}^* &= (\rho - r_f)dt - \bar{\mu}\sigma^{-2}dz_t, \\
dx_{iid,t}^*/x_{iid,t}^* &= (\rho - r_f)dt - \bar{\mu}\sigma^{-2}dz_t, \\
d\mu_t &= \phi(\bar{\mu} - \mu)dt + dw_t.
\end{align*}
\]

This is the worst case in 10,000 draws. How frequent are episodes like this? To answer this question, Figure 5 plots the cumulative distribution of \( x^* \) at a 10 year horizon across simulations, and Figure 6 plots the cumulative distribution of \( \int e^{-\rho t}x_t^* dt \) and \( \int e^{-\rho t}x_{iid,t}^* dt \) across simulations. Both plots show the log of the variable, and scale the probabilities on the y axis so that a normal density would plot as a straight line. Thus, deviations from a straight line indicate tails that are fatter or thinner than those of a normal distribution.

Figure 5 shows that the state-price \( x^*_t \) has a right tail of explosions such as shown in Figure 4 that is far fatter than a lognormal. The same figure for \( x_{iid,t}^* \) (not shown) lies almost perfectly on the straight line, confirming the fact that \( x_{iid,t}^* \) is, of course, exactly lognormal. The explosions – the tail above 90% probability shown in Figure 5 is in fact entirely responsible for the large price of \( p(x^*) \).

Summing and squaring, \( \int e^{-\rho t}x_t^* dt \) then has an even bigger right tail, as shown in Figure 6. The corresponding \( \int e^{-\rho t}x_{iid,t}^* dt \) is also a little fatter than a lognormal, even though \( x_{iid,t}^* \) is lognormal. The log distribution and the x axis scale hide quite how dramatic the increase is.
\[ e^9 = 8103 \text{ in the top graph is a lot more than } e^6 = 403 \text{ in the bottom graph.} \]

Figure 5: Cumulative distribution of \( \ln x^* \) at year 10 in simulations. The \( y \) axis is scaled so that a normal distribution plots as a straight line. The series are simulations from \( dx^*_t/x^*_t = (\rho - r^f)dt - \mu \sigma^{-2}dz_t, \ d\mu_t = \phi(\bar{\mu} - \mu)dt + dw_t. \)

This behavior has a practical implication for the simulation. First, it is important to include a very large number of simulations so that the right tail is adequately represented. Second, it is more important than usual to pay attention to the accuracy of simulations, since the size and frequency of infrequent explosions can be altered substantially by small simulation inaccuracies. Third, the right tail can drive \( p(x^*) \) to infinity. Naturally, adding the time-varying mean increases investment opportunities, so \( p(x^*) \) can become infinite even for parameters \( \bar{\mu} \) and \( \sigma \) for which \( p_{\text{iid}}(x^*) \) is finite. To check, I make sure that \( E(e^{-\rho t}x^{*2}_t) \) is a declining function of time.

**Summary and implications**

In sum, the large increase in \( p(x^*) = 34.15 \) relative to \( p(x^*_{\text{iid}}) = 25.92 \) or \( p_{\text{iid}}(x^*_{\text{iid}}) = 25 \) is due to these infrequent and often incredibly large explosions in the state-price \( x^* \). Since \( x^* \) and hence the intertemporally-optimizing investor take larger positions at times of high mean returns, they are especially sensitive to market declines. State-prices, already high in market declines, shoot up astronomically for market declines “in the middle of recessions”, i.e. when expected returns are high and prices are already low.

Investors who follow market-timing strategies, then, sell off consumption at these rare times of extraordinarily high state prices in order to finance larger consumption in normal times. In doing so, they are acting a lot like hedge funds – they are taking on a huge risk during infrequent disasters in order to increase their performance during normal times. This behavior is accentuated by quadratic utility, which allows even negative consumption during times of high
state prices, but the behavior will also characterize power utility solutions, since the distribution of the state-price $x^*$ has a large right tail even relative to the lognormal distribution.

### 7 Avoiding returns

The above analysis maintains a close link to traditional dynamic portfolio theory exercises, but in doing so takes us away from the focus on payoffs rather than one-period returns that motivates the whole approach. After all, it is not all that hard to solve the square root process time-varying mean return problem by numerical dynamic programming techniques, avoiding the unpleasant side effects of quadratic utility all together.

Rather than model one-period returns, it is much more attractive to simply model the final payoffs. After all, in the simplest application of one-period portfolio theory, we take the mean and covariance matrix of a set of returns and find the optimal portfolio. Why not similarly evaluate the long-run mean and long-run covariance matrix of a set of asset yields and directly calculate the optimal payoff? Any dynamics will show up in the difference between long-run and short-run moments, as (say) variance ratios capture autocorrelation. This section implements this simpler idea.
7.1 Payoff streams and payout rules

The first choice in applying the long-run mean / long-run variance idea is what streams of payoffs \( \{x_t\} \) to examine.

It’s tempting simply to define \( \{x_t\} \) as the dividend streams corresponding to the chosen assets. Using dividends directly is not generally such a good choice for portfolio theory, however, as using individual stocks is not such a good choice for one-period portfolio theory. The investor can dynamically reallocate his investment across assets, and he can consume out of capital gains as well as dividends. The optimal portfolio will therefore typically feature a consumption stream that has almost nothing to do with the underlying dividend streams. For example, in the analytic solution to the i.i.d. return case studied above, the investor consumes a proportion of wealth and rebalances continuously, ignoring any distinction between dividends and capital gains.\(^\text{13}\)

Given that the investor can trade dynamically, it’s natural to think of portfolios in terms of their allocations across assets and their payout rates. With underlying assets that follow

\[

dr_t = r^f_t dt + dr^c_t; \\
\]

\[

dr^c_t = \mu_t dt + \sigma_t dz_t,
\]

the value of an investment characterized by a dollar payout \( y_t \) and a dollar portfolio allocation \( \omega_t \) follows

\[

dV_t = \left( r^f_t V_t - y_t \right) dt + \omega^i_t dr^c_t; \quad V_0 = 1.
\]

Since \( V_0 = 1 \), this strategy generates a yield \( y_t \).

Our objective is to find the optimal payoff \( y^*_t \), or equivalently the set of long-run mean / long-run variance efficient payoffs. Rather than construct a parametric model of one-period returns and the discount factor that prices those returns, as in the last two sections, I start here with the long-run properties of a set of such yields.

I start, naturally enough with linear payout rules, i.e. generating yields from asset returns by

\[
\frac{dV^i_t}{V^i_t} = \left( r^f_t - \alpha^i \right) dt + w^i_t dr^c_t; \quad V^i_0 = 1.
\]

and hence

\[
y^i_t = \alpha^i V^i_t
\]

\[
\frac{dy^i_t}{y^i_t} = \left( r^f_t - \alpha^i \right) dt + w^i_t dr^c_t; \quad y_0 = \alpha^i.
\]

\(^{13}\)Examining dividend streams does make sense for long-run asset pricing as in Hansen Heaton and Li (2005), since in that case we are interested in learning how the fundamental cash flows affect value and it is more important to remove future price fluctuation from the “exogenous” variables.

It’s also an attractive idea that firms choose to pay dividend streams that are optimal for consumers. After all, since dividend policy is irrelevant to first order, why not pay dividends that investors want, rather than forcing them to synthesize the dividend stream they want with dynamic trading? If this is true, using dividends would make a lot of sense. I do not use dividends here in part for practical reasons; I tried it with the size and book to market portfolios, but easy rebalancing strategies provided much better long-run mean-variance properties.
with $\alpha$ and $w$ constant over time. The solution to the iid problem is of this form, with $\alpha = 2\gamma^f - \rho - \mu^f\Sigma^{-1}\mu$, so it's important to include them. Given a set of asset returns, we can easily find the sample yields of constant payout and constant-weight strategies and their sample long-run means and variances. One must check of course that the initial payoffs satisfy the no arbitrage condition $\lim_{T \to \infty} p(V_T) = 0$ and $E(y_T^2) < \infty$ (i.e.$\lim_{T \to \infty} e^{-\rho t} E(y_T^2) = 0$). I also add interesting, though ad-hoc dynamic strategies, in which $\alpha$ and $w$ are functions of information at time $t$.

The payoff space is then linear combinations of the payoffs formed by these ad-hoc weighted portfolios. This linear completion adds possibilities. The payoff to a portfolio that is continuously rebalanced to be half stocks and half bonds is not half of a portfolio that is always bonds and half of a portfolio that is always stocks. Similarly, the payoff of a portfolio in which the investor consumes 5% of wealth each year is not the linear combination of the payoff of a portfolio in which he consumes 4% and one in which he consumes 6%. Combining two different constant-payout rules gives rise to a payoff that results from a time-varying portfolio and payout rule. The payout and portfolio weights adjust towards those of the more successful security over time. We have already seen an example in the iid world: the upper portion of the long-run mean variance frontier and the quadratic utility portfolio problem are either described as a linear combination of $y^*$ and $r^f$—or as a time-varying weight and payout rule in (47) and (48).

Though it adds interesting possibilities, this fact also suggests that we need to start with a fairly rich set of payoffs. If the optimum is (say) a constant 55% in risky assets, then that optimum will be poorly approximated by the linear combination of a 0% and 100% constantly-rebalanced portfolio. A richer set of basis assets, perhaps 40%, 50%, 60% weighted in risky assets, may be necessary to get an adequate approximation. On the other hand, small errors may not be that vital, since we discount, tracking errors that build up over long periods of time are less important to the analysis.

With a finite number of basis payoffs, we can then find a discount factor payoff by simple matrix inversion, $x^* = p' E(xx')^{-1} x$. The result will be an approximation, of course, but probably a sensible one.

This approach is exactly the analogue of the standard approach to one period problems. Rather than directly estimate a mean-variance frontier with 6000 individual stocks, plus bonds, options, international stocks, etc., we first summarize the vast cross section by a small number of sensible portfolios, we estimate the means and covariance matrix of the portfolios, and we

14 Suppose two payoffs are generated by

$$\frac{dV_i}{V_i} = (r_i - \alpha_i) \ dt + w_i \ dr^e; i = 1, 2.$$  

Then consider a portfolio $V = V_1 + V_2$. That portfolio obeys

$$dV = (r^f - \alpha_1) V_1 dt + V_1 w_1 \ dr^e + (r^f - \alpha_2) V_2 dt + V_2 w_2 \ dr^e$$

$$dV = [(r^f - \alpha_1) V_1 + (r^f - \alpha_2) V_2] \ dt + [V_1 w_1 + V_2 w_2] \ dr^e$$

$$\frac{dV}{V} = [(r^f - \alpha_1) \frac{V_1}{V} - \alpha_2 \frac{V_2}{V}] \ dt + \left[ \frac{V_1}{V} w_1 + \frac{V_2}{V} w_2 \right] dr^e$$

$$\frac{dV}{V} = (r^f - \alpha_1) \ dt + w_1 \ dr^e$$

Thus, the payout rate $\alpha_t$ and portfolio investment $w_t$ vary over time, tending to whichever of $V_1$ or $V_2$ grows larger than the other.
find the frontier by matrix methods.

Finally, this approach can overcome or at least sidestep the problems caused by \( p(x^*) = \infty \) in the iid analysis. If the set of basis payoffs \( x_t^i \), formed by a set of arbitrary payout and portfolio rules \( \alpha^i, w^i \), each satisfies \( \lim_{T \to \infty} p(V_T^i) = 0 \) and \( \mathcal{E}(x_{T}^{12}) < \infty \) then linear combinations \( x_t = \alpha x_t^i + bx_t^j \) also satisfy these conditions, and the limit points of any convergent sequence of such payoffs also satisfies the conditions and so is a traded payoff. \( x^* = p' \mathcal{E}(x x')^{-1} x \) will be a traded payoff. Not only can we bound payout rules away from payout rates \( \alpha \leq 0 \) that lead to explosive wealth, we can keep the payout rules a sensible distance away to rule out uninteresting as well as illegal wealth explosions.

### 7.2 Fixed payout and portfolio weights in the iid world

Before going to multiple assets and real data, it’s important to check this approach in the (now) well-studied two-asset iid environment. Here we know the mean-variance frontier is generated by \( y^* \) and \( r^f \). If we start with a set of arbitrary fixed portfolio and payout rules that do not include \( y^* \), how close an approximation do we get?

In this environment, we can calculate the long-run mean and long-run variance of such payoffs. (Eventually, we will take these directly from data, as we start standard portfolio calculations with sample means and a covariance matrix.) With a fixed payout rate \( \alpha_i \) and portfolio weight \( w_i \), value follows

\[
\frac{dV_i}{V_i} = (r^f - \alpha_i) dt + w_i dr^e_i
\]

(61)

We generate yields (payoffs to price one securities) with

\[
y_i = \alpha_i V_i; V_0 = 1.
\]

Hence, the \( y \) process follows

\[
\frac{dy_i}{y_i} = (r^f - \alpha_i) dt + w_i dr^e; \quad y_{i0} = \alpha_i
\]

\[
\frac{d\bar{y}_i}{y_i} = (r^f - \alpha_i + w_i \mu) dt + w_i \sigma dz; \quad y_{i0} = \alpha_i
\]

Now we can calculate the long-run mean and long-run covariance matrix of such yields\(^{15} \),

\[
\bar{E}(y_{it}) = \frac{\rho \alpha_i}{\rho - r^f + \alpha_i - w^f \mu}
\]

\[\bar{E}^2(y_{it}) = \frac{\rho \alpha_i}{\rho - r^f + \alpha_i - w^f \mu}
\]

\[dE(y_{it}) = (r^f - \alpha_i + w^f \mu) y_{it} dt
\]

\[E(y_{it}) = \alpha_i e^{(r^f - \alpha_i + w^f \mu)_t}
\]

\[\bar{E}(y_{it}) = \rho \alpha_i \int_0^\infty e^{-\rho t} e^{(r^f - \alpha_i + w^f \mu)_t} = \frac{\rho \alpha_i}{\rho - r^f + \alpha_i - w^f \mu}
\]

\[d(y_{ij}) = y_i dy_j + y_j dy_i + dy_idy_j
\]

\[dE(y_{ij}) = y_i E(dy_{ij}) + y_j E(dy_{ij}) + E(dy_idy_j)
\]

\[= y_i y_j (r^f - \alpha_i + w^f \mu) + (r^f - \alpha_j + w^f \mu) + w^f \Sigma w_j
\]

\[= y_i y_j [2r^f - (\alpha_i + \alpha_j) + (w_i + w_j) \mu + w^f \Sigma w_j]
\]
\[ \tilde{E}(y_{it}y_{jt}) = \frac{\rho \alpha_i \alpha_j}{\rho - 2r^f + (\alpha_i + \alpha_j) - (w_i + w_j)\mu - w_i'\Sigma w_j} \] (63)

Not all payout and portfolio rates are legal – we need to ensure that the payoff is square integrable \( \tilde{E}(y^2) < \infty \) and we need to ensure that the time-zero value of wealth of the underlying portfolio tends to zero \( \lim_{T \to \infty} p(V_T) = 0 \). From (62), the former condition implies

\[ \tilde{E}(y^2) < \infty : \quad \alpha_i > r^f + w_i' \mu + \frac{1}{2} w_i' \Sigma w_j - \frac{1}{2} \rho \]

and the latter only requires\(^\text{16}\)

\[ \lim_{T \to \infty} p(V_T) = 0 : \quad \alpha_i > 0. \]

Both conditions ensure that we pull money out of the investment fast enough as wealth rises, or put money back in fast enough if wealth becomes large and negative.

If we allow all \( \alpha > 0 \) and \( w \in R^N \), the mean-variance frontier is spanned by the risk free rate and the minimum second moment portfolio,

\[ \min_{\{\alpha, w\}} \mathcal{E}(y_t^2) = \min_{\{\alpha, w\}} \frac{\alpha^2}{\rho - 2r^f + 2\alpha - 2w' \mu - w' \Sigma w} \]

which gives

\[ w = -\Sigma^{-1} \mu \]

\[ E(y_{it}y_{jt}) = \alpha_i \alpha_j e^{[2r^f-(\alpha_i+\alpha_j)+(w_i+w_j)'\mu+w_i'\Sigma w]t} \]

\[ \tilde{E}(y_{it}y_{jt}) = \rho \alpha_i \alpha_j \int e^{-\rho t} e^{[2r^f-(\alpha_i+\alpha_j)+(w_i+w_j)'\mu+w_i'\Sigma w]t} dt = \frac{\rho \alpha_i \alpha_j}{\rho - 2r^f + (\alpha_i + \alpha_j) - (w_i + w_j)' \mu - w_i' \Sigma w_j} \]

\(^{16}\)The value of the strategy that delivers \( y_{it} \) is simply \( x_{it} \) with

\[ \frac{dx_i}{x_i} = (r^f - \alpha_i) dt + w_i' dr^e; \quad x_{i0} = 1 \]

We have

\[ p(x_{it}) = e^{-\rho t} E(x_{it}^*x_{it}) \]

\[ \frac{d(x_{it}^*x_{it})}{x_{it}^*x_{it}} = \frac{dx_i^*}{x_i^*} + \frac{dx_{it}}{x_{it}} + \frac{dx_i^* dx_{it}}{x_i^* x_{it}} \]

\[ \frac{d(x_{it}^*x_{it})}{x_{it}^*x_{it}} = (\rho - r^f) dt - \mu' \Sigma^{-1} \sigma dz + (r^f - \alpha_i + w_i' \mu) dt + w_i' \sigma dz - \mu' w_i dt \]

\[ \frac{dE(x_{it}^*x_{it})}{x_{it}^*x_{it}} = (\rho - \alpha_i) dt \]

\[ e^{-\rho t} E(x_{it}^*x_{it}) = e^{-(\rho - \alpha_i)t} = e^{-\alpha_i t}. \]

Thus,

\[ \lim_{T \to \infty} p(x_{it}) = 0 \text{ iff } \alpha_i > 0. \]

Both conditions also hold if we add constants to the payout rules, i.e.

\[ dV = \left[ r^f V - (\alpha + \alpha V) \right] dt + [\omega + wV'] dr^e, \]

i.e. only the slopes \( \alpha, w \) on \( V \) matter for square-integrability and no arbitrage.
This is just our friend $y^*$ with

$$
\mathcal{E}(y^*_t) = \frac{\rho}{r_f} \left( 2r_f - \rho - \mu'\Sigma^{-1}\mu \right)
$$

$$
\mathcal{E}(y^*_t^2) = \rho \left( 2r_f - \rho - \mu'\Sigma^{-1}\mu \right)
$$

Figure 7: Long-run mean-variance frontiers. The dots are the means and variances of portfolio strategies with constant payout rates $\alpha$ and constantly rebalanced weights $w$ in the risky asset. The left panel connects portfolios with the same portfolio weight $w$, the right panel connects portfolios with the same payout rate $\alpha$. The red star gives the minimum second-moment portfolio $y^*$. Parameters $\mu = 0.02$, $\rho = r_f = 0.05$, $\sigma = 0.20$.

Figure 7 presents long-run means and standard deviations in the iid economy, for a variety of fixed portfolio allocation $w$ and payout rules $\alpha$. The frontier in Figure Figure 7 is generated by combinations of the risk free rate together with $y^*$ which here occurs at $w = -0.50$, $\alpha = 0.04$.

The figure shows that many portfolios are close to mean-variance efficient. In particular, when the payout rate $\alpha \approx 0.04$, a wide variety of portfolio rules generate very efficient portfolios. This suggests that a relatively sparse sprinkling of ad-hoc portfolio weights will do a good job of spanning the full set.

To examine this question in more detail, suppose we start with a set of payout rules and weights that does not include the optimum $y^*$. How close to that optimum will we get with linear combinations of a few arbitrary payoffs, and then construct an approximate $y^*$ as a linear combination of those arbitrary payoffs by

$$
y^* = \frac{1}{1'\mathcal{E}(yy')^{-1}y}{1'\mathcal{E}(yy')^{-1}y'}
$$

(64)
Figure 8 presents a calculation. The basis assets here have 3% and 5% payout rates and portfolio weights -1, -0.75, -0.25, 0, in both cases avoiding the exact optimum of a 4% payout and -0.50 portfolio weight. The blue dots give the long run means and standard deviations of these basis payoffs. The blue triangle gives the mean-variance efficient portfolio formed from these basis assets by (64). The red line gives the exact mean-variance efficient frontier generated by a 4% payout rate and -0.5 portfolio weight. As the graph shows, the approximate calculation gives almost exactly the same mean-variance trade-off as the exact calculation. The approximate Sharpe ratio is 0.494 rather than 0.500. The numbers next to each blue dot give the weight of each basis asset in the mean-variance efficient portfolio. The two closest portfolios ($\alpha = 0.05, w = -0.25$ and $w = -0.75$) each give about 1/2 weight with smaller weights on the other portfolios.

Figure 8: Long-run mean-variance frontier computed from a few fixed payout portfolios. The blue dots give the long run mean and standard deviation of the basis portfolios. The basis portfolios have payout rates of 3% and 5% and portfolio weights -1, -0.75, -0.25, 0. The blue triangle gives the mean-variance efficient portfolio formed from these basis assets by $y^* = \gamma'(yy')^{-1}y / [\gamma'(yy')^{-1}]$. The numbers give the weight of each basis asset in the mean-variance efficient portfolio. The red line gives the exact mean-variance efficient frontier generated by a 4% payout rate and -0.5 portfolio weight. Parameters $\rho = r^f = 0.05, \mu = 0.02, \sigma = 0.20$.

It’s a little uncomfortable to focus so much on short positions in “maximally inefficient” portfolios on the lower half of the mean-variance frontier. That issue needs to be understood.

A first lesson of Figure 7 is how poorly portfolios do with constant positive weights and payout rates – all the portfolios in the upper half of the mean-variance frontier. Portfolios with
$w = 1$, the standard “market portfolio”, barely fit on the graph. The $w = 1$ portfolio that does fit has a large payout rate $\alpha = 0.09$. The most natural portfolio with $w = 1$, $\alpha = r^f = 0.05$ isn’t even in the payoff space; its conditional variance grows faster than $e^{-\rho t}$ its long-term variance is infinite. The reason is that geometric Brownian motion can explode in the positive direction, but the long-run variance criterion prizes stability in the level, not the log.

To illustrate, Figure 9 plots a simulation of the yield from three portfolios with $\alpha = 0.04$, one with risky-asset weight $w = +0.5$, one with $w = 0$ and one (the mean-variance efficient payoff $y^*$) with $w = -0.5$. You can see the volatility of the $w = +0.5$ payoff; it is just what a geometric Brownian should look like. $w = 0$ gives steady 1% growth, since in this case $r^f = 0.05$. The payoff that is short the market $w = -0.5$ trundles off downward, as one would expect of a short position in a positive expected return security. But trundling off downward has much less arithmetic variance than exploding upward. Thus, the last portfolio graphed in the Figure, formed by a short position in this short portfolio, $r^f + 1.67(r^f - y^*)$, which is on the upper portion of the mean-variance frontier, looks quite good. By construction (the 1.67) it matches the long-run mean of the $w = +0.5$ portfolio, but you can see the dramatically lower long-run variance since it varies much less across time as well as across states of nature.

Figure 9: Draw of three portfolio-payout rules. I simulate 50 years of a random normal, then cumulate $dV = (r^f - \alpha)V dt + w'(\mu dt + \sigma dz)$. The payout is $y = \alpha V$. The first three portfolios have a constant $\alpha = 0.04$ payout and weights $w = -0.5, 0,$ and $0.5$ in the risky assets. The final portfolio is $r^f + 5/3(r^f - y^*)$ where $y^*$ is the payoff with $w = -0.5$. Parameters $\rho = 0.05$, $r^f = 0.05$ $\mu = 0.02$, $\sigma = 0.20$.

Rather than understand the efficient portfolio $r^f + 1.67(r^f - y^*)$ as a weird short position in a portfolio that it short the market with constant weight $w = -0.5$ and payout $\alpha = 0.04$, it
may make more intuitive sense to understand it as a portfolio with a time-varying payout and portfolio weight. From (50) and (51), we can instead think of it as the payout of a portfolio that starts with $1 investment, then pays a dollar amount
\[ y_t = 0.0267 + 0.04W_t = 0.05W_t + 0.01 \times (2.67 - W_t) \]
and maintains a dollar investment in the risky excess return
\[ W_t w_t = 1.33 - 0.5 \times W_t = 0.5 \times (2.67 - W_t) \].

Compared to the \( \alpha = 0.04 \), \( w = 0.5 \) payoff, the portfolio is initially more aggressive; at \( W_0 = 1 \) paying \( y = 0.0667 \) rather than \( y = 0.4 \) and investing $0.835 in risky assets rather than $0.5. As wealth increases, however, the efficient portfolio becomes more cautious; at \( W = 2.67 \) paying out and investing only in the risk free rate.

Intuitively, if long-run variance is a concern, it makes sense for the portfolio to shift more towards riskfree assets as wealth increases.

As it is interesting to characterize the payoff on the frontier in this way, it can also be more intuitive to construct basis payoffs that in the upper portion of the frontier, rather than the slightly weird basis payoffs in the lower portion shown in Figure 8. At the same time, I shift to excess (price-zero) yields which are the cleanest way to express the mean-variance frontier.

The upper portion of the frontier here is generated by portfolios with payout rules and weights that include constants, which can be generated by short positions in constant-weight portfolios (61) plus long positions in the riskless asset (see (50) and (51)). If one wants to see portfolios on the upper portion of the mean-variance frontier expressed as positive weights of underlying assets, then the underlying assets must also be generalized to have constants in the portfolio and payout rules, i.e.
\[ dV_i = \left( r^f V_i - (a + \alpha_i V_i) \right) dt + (\omega_i + V_i w_i)' dr_t^e \]
in place of (61).

Mechanically, we can just take the yields calculated so far and form \( y^e = k \left( r^f - y \right) \). The point however is to interpret these payoffs as the basis assets. Starting with value processes of the form (61)
\[ dV^i = (r^f - \alpha)V^i dt + V^i w^i dr^e; V^i_0 = 1 \]

\[ y^{mv}_t = r^f W_t + (1 + k - W_t) \left( \mu' \Sigma^{-1} \mu \right) \]
\[ = 0.05W_t + (2.67 - W_t) \left( \frac{0.02^2}{0.20^2} \right) \]
\[ = 0.05W_t + 0.01 \times (2.67 - W_t) \]
\[ = 0.0267 + 0.04W_t \]
\[ w'_t = (1 + k - W_t) \mu' \Sigma^{-1} \]
\[ w'_t = (2.67 - W_t) \frac{0.02}{0.20^2} \]
\[ = (2.67 - W_t) 0.5 \]
\[ w_t = 1.33 - 0.5 \times W_t. \]

\[ \text{55} \]
the excess yields come from a value process
\[ V = k(1 - V^i); V_0 = 0 \]

The value process follows
\[
\begin{align*}
dV & = -kdV^i = -kV^i(r^f - \omega)dt - kV^i w^e dr^e \\
dV & = (V - k)(r^f - \omega)dt + (k - V)w^e dr^e \\
dV & = \left\{ r^f V - \left[ k r^f - \omega (k - V) \right] \right\} dt - (k - V)w^e dr^e
\end{align*}
\]
i.e. a payout rule
\[ y^e = k r^f - \omega (k - V) \]
and a portfolio allocation rule
\[ -(k - V) w. \]

Since \( k > V \), previously short weights are now long positions.

Figure 10 presents the results of this calculation, and should look more comfortably familiar. Again, I delete the choices \( \alpha = 0.04 \) and \( w = 0.5 \) so we can see how well approximating the payoff space with a smaller number of assets works.

The basis payoffs represented by blue dots have positive payouts and portfolio weights, though a somewhat different interpretation. They are zero-cost portfolios. They have payouts \( y^{ei} = r^j - y^j = \left[ r^j - \omega i (1 - V^i) \right] \) and portfolio positions \( \omega^i = (1 - V^i)w^i \), where \( V_i \) is the value of each underlying portfolio. Value starts at \( V_0^i = 0 \) and accumulates by \( dV^i = \left\{ r^i V^i - y^{ei} \right\} dt - \omega^i dr^e \). Thus a “4% payout rate” now means that the payout changes by 4% of the value of the portfolio; if the portfolio rises by $100, you get $4 more. Similarly, a 50% portfolio weight means that if value increases by $100, $50 more will be put in the risky assets. But both payout and portfolio rules have constants.

The blue triangle gives the mean-variance efficient payoff formed from these basis assets by \( r^f y^{e*} = r^f \mathcal{E}(y^e)\mathcal{E}(y^e y^e)^{-1}y^e \). The nearly indistinguishable red square gives the exact mean-variance payoff computed from \( r^f y^{e*} = r^f - y^* \) using all assets and \( \alpha = 4\% \), \( w = 0.5 \) in particular, and the lines give the corresponding frontiers. The numbers \( wt \) give the weight of each basis asset in the mean-variance efficient portfolio, i.e. the values \( r^f \mathcal{E}(y^e y^e)^{-1}\mathcal{E}(y^e) \). These do not sum to 100 because as zero cost portfolios they do not have to do so.

\[ dV = (r^f - \alpha) (V - k)dt + (V - k)w^e dr^e \]
\[ y = kr^f + \alpha (V - k) \]

Thus they obey
\[ dy = \alpha V \]
\[ dy = \alpha \left\{ r^f V - \left[ kr^f + \alpha (V - k) \right] \right\} dt + \alpha(V - k)w^e dr^e \]
\[ = \alpha (r^f - \alpha) (V - k) dt + \alpha(V - k)w^e dr^e \]
\[ dy = (r^f - \alpha) (y - kr^f) dt + (y - kr^f) w^e dr^e; y_0 = k(r^f - \alpha) \]
i.e. \( y - kr^f \) also follows a Geometric Brownian motion.
Figure 10:
8 Size and Value

Now, for a real problem. How much should a mean-variance investor tilt towards small stocks, value stocks, and momentum strategies? Perhaps dynamics in the portfolios give a different answer to this question for long horizon investors than for one-period investors. To this end, I examine the value-weighted market, and the Fama-French smb, hml, and umd factor portfolios. Larger collections such as the 25 size and book/market portfolios are well described by the factor portfolios, so there is little point in expanding the set of underlying portfolios.

Table 1 presents statistics on annual returns, and Figure 11 presents the usual one-year horizon mean-variance frontier for excess returns. As usual, the size, value and momentum strategies appear to improve markedly on simply holding the market portfolio. Our objective is to do the analogous simple mean-variance exercise for a long-run investor. In particular, dynamics in the returns have the potential to change the patterns. For example, a negatively autocorrelated return can have much lower long-term variance than its short term variance.

![Figure 11: Mean-variance frontier for annual excess returns on market (rmrf) small (smb) value (hml) and momentum (umd) portfolio. “Optimum” is the mean-variance efficient portfolio at the mean market return. Percentages give each portfolio’s weight in the optimum portfolio.](image)

Figure 11: Mean-variance frontier for annual excess returns on market (rmrf) small (smb) value (hml) and momentum (umd) portfolios. “Optimum” is the mean-variance efficient portfolio at the mean market return. Percentages give each portfolio’s weight in the optimum portfolio.
Figure 12: Long-run mean-variance frontier of excess yields. The percentages give the weight of each portfolio in the optimum.

<table>
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<tr>
<th></th>
<th>rmrf</th>
<th>smb</th>
<th>hml</th>
<th>umd</th>
<th>1/4 ea.</th>
<th>opt.</th>
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<td>3.6</td>
<td>5.0</td>
<td>10.7</td>
<td>8.3</td>
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<td></td>
<td></td>
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<tr>
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<td>8.5</td>
<td>28.7</td>
<td>54.2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Mean-variance statistics for excess returns, 1927-2004. “Optimal” is the mean-variance efficient portfolio with the same mean as the value weighted market return rmrf.

The first choice in applying the long-run mean / long-run variance idea is what streams of
payoffs \( \{x_t\} \) to examine. It’s tempting simply to define \( \{x_t\} \) as the dividend streams corresponding to the chosen assets such as the size and book/market portfolios. Using dividends directly is not generally such a good choice for portfolio theory, however. The investor can dynamically reallocate his investment across assets, and he can consume out of capital gains as well as dividends. The optimal portfolio will therefore typically feature a consumption stream that has almost nothing to do with the underlying dividend streams. For example, in the i.i.d. return case, we know the solution analytically and in this case the investor consumes a proportion of wealth and rebalances continuously, ignoring any distinction between dividends and capital gains.

Given my approach of maximizing over a few representative portfolios rather than attempting the full (infinite-dimensional) exact dynamic optimization over assets and dynamic strategies, therefore, I start by constructing payoffs \( x_t \) from returns alone, ignoring the dividend/capital gain split, via sensible but ad-hoc payout rules.

I start with constant payout-portfolio rules, cumulating an initial one dollar investment at the three-month treasury bill rate. For given payout \( \alpha \) and weight \( w \) in [rmrf, smb, hml, umd], I construct

\[
\begin{align*}
\frac{dV}{V} &= \left( \mu - \alpha \right) dt - w'dr^e; \quad V_0 = 1 \\
y_t &= \alpha V_t
\end{align*}
\]
I construct one such yield with a zero weight on all assets,
\[ \frac{dV^0}{V^0} = \left(r_t^f - \alpha\right) dt; \quad y_t^0 = \alpha V_t^0 \]

Then I construct excess yields by
\[ y_t^e = y_t^0 - y_t \]

Again, as above, we can view these as short positions in portfolios that are short the risky assets, or as portfolios that are long the risky assets with a constant in the dollar payout rule; the iid model suggests that they will perform better in a long-run mean-variance sense than standard constant weight portfolios.

The yields so constructed respond sensibly to the fortunes of the various investments. Figure 14 presents the yield and excess yield so constructed starting with a one dollar investment in 1926. The lines represent the yields from using a 5% payout rate and a weight in rmrf of 0, 0.25, 0.5, 0.75 and 1.00 respectively. The smooth line is a 5% payout with a constant investment in treasury bills. The value slowly declines because interest rates were typically below 5% through this period. The other yields cumulate short positions and pay out proportionally to value, so you see them suffer in the runup of the late 1920s, recover a bit in the crash of 1929, and then decline swiftly as the market recovers. The excess yield is a short position in these yields, and a long position in the smooth yield from the top graph. Thus, they do well and poorly as we would expect through the market rise in the late 20s, the crash, and then recovery, and so forth.

It’s tempting to worry a lot about tail behavior of these portfolios, but remember that the tails all get multiplied by \( \beta^t \), so matter little to long-run means and variances.

To estimate long run means and variances, I recreate these value and yield processes at each date. Thus denote \( y_{t,t+j} \) the yield at time \( t + j \) due to a one dollar investment at time \( t \). To estimate long-run means and variances, I start by estimating \( E_0(y_j) \) by averaging all the \( j \)-step ahead yields, \( E_0(y_j) = \frac{1}{T-j} \sum_{t=1}^{T-j} (y_{t,t+j}) \) and similarly for second moments \( E(y_j y^2_j) \). I test that \( \beta^j E_0(y_j) \) and \( \beta^j E_0(y^2_j) \) are converging to zero and report infinity for those cases that are not converging. Then I calculate \( \bar{E}(y) = \sum \beta^j E_0(y_j) \) and similarly for the long-run second moments. I calculate \( \bar{E}(y) = \frac{1-\beta}{\beta(1-\beta^j)} \bar{E}(y) \). Standard deviations are \( \bar{\sigma}(y) = \left[ \bar{E}(y^2) - \bar{E}(y)^2 \right]^{0.5} \). Denoting by \( y^e \) the vector of all excess yields in consideration, the mean-variance efficient excess yield is \( y^{e*} \) given by\(^\text{19}\) \( y^{e*} = \bar{E}(y^e)/\bar{E}(y^e y^e)^{-1} y^e \).

Figure 13 presents the long-run means and variances for a variety of such portfolios. I standardize on a payout rate \( \alpha = 0.05 \) corresponding to the discount factor \( \beta = 0.95 \) throughout. Figure 13 then presents long-run means and variances of portfolios that put weight 0, 0.25, 0.5, 0.75, and 1.0 in each of rmrf, smb, hml, and umd respectively with zero weight in the other assets. I also present a portfolio with weight \( \mu \Sigma^{-1} \), where \( \mu \) and \( \Sigma \) are the mean and covariance matrix of one period returns. This portfolio is long-run mean-variance efficient in the iid world, so it’s a good one to try here as well.

Figure 13 says that the choice of scale does not matter much. As we increase the weight in a given risky asset portfolio, long-run means and standard deviations move out along a ray of

\(^{19}\) With large payoff spaces of potentially very highly correlated assets, second moment matrices can be close to singular. This is particularly a problem when one generates assets as I do here by varying parameters on a fairly fine grid – the yield generated from a 5% payout is likely to be very correlated with the yield generated from a 4.9% payout! I use a pseudo-inverse to address this problem, raising the tolerance level until the results are no longer sensitive to that choice.
constant Sharpe ratio. This fact means we can standardize on one weight, pruning considerably the number of basis portfolios that need to be included. Interestingly the weight $\mu'\Sigma^{-1}$ does not do much better than the umd portfolio alone.

Since scale does not seem to matter, Figure 12 and Table 1 consider portfolios with weight one in each of rmrf, smb, hml, and umd, along with a portfolio that places even weight $w' = [0.25 \ 0.25 \ 0.25 \ 0.25]$ across all four assets. Again, a portfolio of a yield that is 100% invested in rmrf and one that is 100% invested in hml is not the same thing as a portfolio that is constantly rebalanced to 50%-50%, so it’s interesting to include such portfolios.

Comparing the long-run mean-variance frontier in Figure 12 with its short-run counterpart in Figure 11, and comparing the corresponding numbers in Table 1, we see a lot of commonality but also some interesting differences between short and long-run mean-variance frontiers. The relative positions are about the same; hml gives about the same Sharpe ratio as the market at a lower mean and variance, smb decidedly less, and umd substantially more.

The main difference is that the improvement in Sharpe ratio is considerably less in the long-run case. The one-year mean-variance optimum gives the same mean return with only one third the standard deviation (7%, not 20%) resulting in a three times higher 1.16 Sharpe ratio rather than “only” the 0.4 Sharpe ratio that launched the equity premium puzzle. By contrast, the long-run optimal portfolio has a yield standard deviation of 0.64% rather than the 1% of the market portfolio, so the Sharpe ratio only increases by 50% from 1.59 to 2.47. That’s still a big

Figure 14: Yield (top) and excess yield (bottom) resulting from one dollar investment in 1926. All portfolios use a 5% payout rate, and weights 0, 0.25, 0.5, 0.75 and 1.0 in the market excess return rmrf.
increase in Sharpe ratio, but nothing like a factor of three!

Underlying the relatively smaller benefits of moving away from the market portfolio are larger betas. The long-run betas of hml and umd (calculated from long-run covariance divided by long-run variance) are 0.3 and 0.4 rather than essentially zero. This greater correlation results in less diversification benefit.

9 Bonds

Bond portfolios are a natural place to use a payoff view, since we know bond returns have interesting dynamics. In a pure mean-variance sense, we know that long-term bonds look dreadfully worse than short term bonds in a one-month mean-variance frontier, but the pattern is reversed in a (say) ten year mean-variance frontier. We should see the same patterns here.

10 Hedging labor income

To investigate the possibilities for hedging labor or outside business income, I try two easily available data sources. Figure 15 presents real employee compensation per employee, and Figure 16 presents real proprietor’s income. Both series are far from perfect of course. Compensation of all employees ignores life-cycle variation in the compensation of a given employee, and more importantly it ignores the risks to an individual employee of becoming unemployed. Also, one can see the suspicious change in the behavior of the time series in the late 1960s. Proprietor’s income is an aggregate, and hence even less representative of the income to an individual proprietor. Nonetheless, these series show important cyclical variation and they are an easy place to start in order to see if there is any hope for hedging labor income.

How well can we replicate these time series using asset yields? To what extent do the hedge portfolios load on priced assets, leading to a distortion of the asset allocation decision?

I first construct the labor income hedge portfolio. I start with the familiar treasury bill (rf), market (rmrf), size (smb), book/market (hml) and momentum (umd) portfolios. I form 5 payoffs with a 4% payout rate, one simply invested in treasury bills and the others invested -50% in each risky asset in turn. These portfolios do a good job of capturing the spectrum of priced assets, though not necessarily a good job of capturing the spectrum of unpriced assets, and of course the latter are just as important as the former if not more so for hedging. Later, I include industry portfolios into the analysis to capture the latter dimension.

Using all employee compensation (the “all” line of Figure 15), I find the long-run regression coefficient $b = \mathbb{E}(yy^T)^{-1}\mathbb{E}(ey)$ where $e$ denotes employee compensation and $y$ denotes the vector of yields on the basis assets (rf,rmrf,smb,hml,umd). I estimate $\mathbb{E}(ey)$ in the same way I estimate $\mathbb{E}(yy^T)$ and other long-run moments above. I estimate $E_0(e_jy_j)$ by simulating forward value processes starting with a one dollar investment at each date, producing a yield series $y_{t,t+j}$. Then I average $y_{t,t+j}e_{t+j}/e_t$ across initial dates $t$, $E(e_jy_j) = \frac{1}{T-t} \sum_{t=0}^{T-j} y_{t,t+j} e_{t+j}/e_t$ and finally I sum to form $\mathbb{E}(ey) = \frac{1-\beta}{1-\beta T} \sum_{j=1}^{T} \beta^j E(e_jy_j)$

Figure 17 presents the first observation of the best hedge payoff $b \times y_{1,t}$ along with the original employee compensation series. The long run $R^2$ of this regression – $\mathbb{E}[(b'y)^2]/\mathbb{E}(e^2)$ – is 89%.
Figure 15: Employee compensation. This is “compensation of employees” divided by number of full-time equivalent employees and deflated by the consumer price index.

Figure 16: Real proprietor’s income.
Figure 17: Real employee compensation and payoff of hedge portfolio. The hedge portfolio is a long-run regression of the employee compensation series on yields corresponding to investments in the treasury bill, market return rmrf, small smb, value hml and momentum umd portfolios.

While this seems remarkably good, of course a large part of the good fit is the match to the change in trend from before 1970 to afterwards, which is perhaps not a particularly interesting feature of the data.

The fitted value of this regression by is a payoff closest to the income process e. Dividing \( \hat{b} = b / \sum b_i \) we obtain the yield of the hedge payoff. We can characterize this yield by its long-run mean-variance decomposition,

\[
\hat{e} = \hat{b}'y = y^* + we^* + \varepsilon
\]

1 \times y^* establishes scale; this is a yield or price-1 payoff. \( we^* \) is the systematic or priced component. The residual \( \varepsilon \) is a zero price, zero expected yield payoff, the truly idiosyncratic part of the labor income payoff.

Figures 18 and 19 present the mean-variance frontier for yields and for excess yields respectively, including the income hedge portfolio. Excess yields are the yield for the riskfree investment less the risky asset yields.

It’s initially surprising that the minimum variance payoff in Figure 18 is so large, and that the variance of the payoff marked “rf” is so large. \( rf \) is the real payoff of a portfolio that invests in treasury bills, and pays out a constant 4% of its value. Both the nominal risk free rate and inflation change a lot over time, and the payout rate is not engineered to match the long-run growth rate of the portfolio, so it has both a trend and substantial variation over time. Synthesizing an indexed perpetuity is not so easy! Clearly, though, one can do a better job of
this synthesis and that will extend the payoff space to the left. At a minimum, 0 payoff is always available!

The hedge portfolio in both figures has a huge systematic component. It costs a lot to form the payoff graphed in Figure 17; That payoff loads strongly on assets such as hml and umd that pay very high average returns. If this is reality, investors have good reasons to avoid asset market investments with similarly high premia.

The hedge portfolio has a much smaller idiosyncratic component, so there is little an investor with this outside income series can do to hedge without paying a premium or affecting his or her optimal asset portfolio. However, the set of basis payoffs here focuses on priced payoffs; adding industry portfolios and other dimensions of unpriced risk could substantially increase the idiosyncratic component of the hedge portfolio.

Figure 18: Long run mean-variance frontier. The underlying yields are formed from investments in the riskfree rate (rf) the market excess return (rmrf) and small (smb) value (hml) and momentum (umd) portfolios. Each payoff is formed from a 4% payout rate and a 50% short position in the irisky assets. Payoffs are real.
Figure 19: Long run mean-variance frontier for excess yields. The underlying yields are formed from investments in the riskfree rate (rf) the market excess return (rmrf) and small (smb) value (hml) and momentum (umd) portfolios. Each payoff is formed from a 4% payout rate and a 50% short position in the irisky assets. Payoffs are real.
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