Illiquid Assets and Self-Control.

Sebastian Ludmer*

Princeton University
Department of Economics

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Abstract

This paper analyzes the general equilibrium impact of satisfying time-inconsistent ("hyperbolic-discounting") individuals' demand for commitment devices. The first finding is that the availability of such devices decreases welfare from the perspective of initial tastes (which are the ones that commitment purportedly favors), even though commitment is individually desirable, *ceteris paribus*. The second main result of this paper is a characterization of equilibrium (prices and portfolio holdings) when some assets are illiquid and individuals face two opposing concerns: changing preferences and uncertainty about future tastes. I show how the interaction between these two forces affect the prices of liquid and illiquid assets in an economy with a fixed supply of those assets. If taste shocks are sufficiently important, equilibrium prices display a liquidity premium and ex-ante identical agents hold the same portfolio. Illiquid assets decrease welfare as they hinder the agents' ability to accommodate the shocks. Conversely, if time-inconsistency is sufficiently important relative to the taste shocks, prices display an illiquidity premium. The equilibrium is asymmetric, with ex-ante identical agents sorting themselves between fully committing to a future consumption profile or not committing at all. Even though illiquid assets trade at a premium, consumers are made worse off by their availability. I establish conditions under which the equilibrium must take either of those two forms.

Keywords: Asset pricing, commitment, hyperbolic discounting, illiquid assets, liquidity constraints, portfolio choice.

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1 Introduction

The present study proposes to analyze the equilibrium welfare properties of commitment devices when individuals face changing preferences. Decision makers whose tastes change over time will demand commitment devices restricting future choices. Making such devices available will affect the economy's fundamentals. However, the following question remains unanswered: What are the resulting general equilibrium welfare effects?[1]

I address this issue by studying an economy in which commitment devices are available to "hyperbolic-discounting" agents. In addition, I characterize equilibrium when some assets are illiquid and individuals face two opposing concerns: changing preferences and uncertainty about future tastes.

A hyperbolic discounting individual will implement an excessively impatient future consumption profile. If part of her wealth were illiquid, future overconsumption would be limited by the amount of liquid resources available. This implies that illiquid assets allow pre-committing to higher future savings, avoiding the anticipated bias. *Ceteris paribus*, time-inconsistent decision makers would benefit from the availability of these assets.

Observing agents' expressed demand for commitment devices, a benevolent policy maker might be tempted to satisfy it. A number of measures could achieve this goal. For example, consider forbidding the collateralization of "family homestead". Many states have family homestead exemptions of varying a reach, which protect family property (including housing) against bankruptcy.[2] Until 1998, it was illegal in Texas to use claimed family homestead as collateral for home equity loans. This would prevent families from borrowing against assets that are costly to sell down; such self-handicapping a measure would have been rationalized by a desire for commitment. Introducing a restriction on households' access to mortgage credit like the mentioned one can make housing an illiquid investment, and hence a commitment device. Other examples of available interventions include instituting 401K or IRA accounts that penalize early consumption of designated savings; Laibson (1996) discusses other revenue-neutral policy measures.[3]

What will be the welfare effect of such measures when general equilibrium feedback is accounted for? Hyperbolic-discounting individuals will demand those commitment devices, strongly suggesting a welfare-improving effect by supplying them. But will the equilibrium analysis ratify the choice-theoretic intuition that they are helpful? In the model analyzed in this paper, the answer is no: When commitment devices are available, individuals are worse off than if none exist. Agents cannot help making use of those devices, which are always individually beneficial *ceteris paribus*, but this is welfare-deteriorating in equilibrium.

[1] When preferences rationalizing choice change over time, the appropriate welfare measure is ambiguous under some interpretations (see Gul and Pesendorfer, 2002). However, it is natural to focus on initial tastes, given that these are the ones generating demand for commitment devices. In this paper, each and every reference to welfare should be understood to refer to initial tastes.

[2] Property covered by homestead provisions cannot be seized by creditors in the event of bankruptcy. While in most states there is limited or no protection to housing, Florida and Texas provide generous coverage: Housing up to a specified size is fully protected, regardless of monetary value. Items generally covered by the homestead exemptions, depending on the state, include clothing, books and burial plots.

[3] For a non-intra-personal interpretation, notice that parents make substantial wealth transfers to their children in the form of education, a non-collateralizable, irreversible investment. Alternatively, they could put the money in a bank account and let children make their own inter-temporal decision.

I first address this issue by studying a general equilibrium asset pricing economy in which claims on the aggregate deterministic endowment are sequentially traded. Of the total claims on future goods, a fraction of them is exogenously specified to be liquid and the rest illiquid. The former can be re-traded in intermediate periods, while the latter cannot. Moreover, individuals cannot borrow against their illiquid asset holdings nor short-sell the liquid ones.

In addition to the welfare impact of illiquid assets, the link between commitment and risk is studied in this model. An idiosyncratic discount factor shock which affects both initial and later preferences but whose value is realized only after tastes have changed is introduced as well. Each agent's shock realization is unobservable private information and therefore uninsurable.

The timing unfolds as follows. First, agents make a portfolio decision to allocate their wealth between liquid and illiquid assets with the objective of maximizing initial utility and in anticipation of future behavior. After this, taste shocks are realized and trading for current consumption and liquid claims on future goods takes place.

A decision maker investing in illiquid claims locks in a minimum future consumption level which cannot be subsequently cut back. This is desirable due to the preference change. But on the other hand, it hinders the individual's ability to accommodate the taste shock. The initial portfolio decision will weigh the gains from commitment against the losses from foregoing flexibility.

A simple characterization is derived for the polar cases when either commitment or flexibility is most valuable to the agent. These cases allow a full characterization of equilibrium prices and portfolio holdings and they are exhaustive when the taste shock follows a power distribution. I also examine how the model is solved for arbitrary alternative specifications.

If the time-inconsistency problem is relatively small, liquid assets trade at a premium and ex-ante identical agents with same wealth and preferences acquire identical initial portfolios. Later, people realizing impatient tastes sell liquid claims on future goods to finance current consumption; this is limited by the non-selling constraint on illiquid assets. Conversely, agents who realize patient tastes sell claims on current consumption in exchange for claims on future goods. If some assets are illiquid, equilibrium welfare is lower than if all claims are re-tradeable. Time-inconsistency is weak, and consumers behave like neoclassical agents; illiquid assets are harmful in this environment since accommodating the shock is always optimal.

When the time-inconsistency problem is most severe, illiquid assets trade at a premium compared to liquid counterparts of identical payoff structure. Moreover, ex-ante identical individuals implement asymmetric portfolio holdings. Some of them follow a "full flexibility" strategy, holding only liquid claims. The remaining agents, meanwhile, pursue a "full commitment" strategy in which future consumption is fully chosen in advance. Full flexibility offers no protection against time-inconsistency but it allows the agent to purchase relatively inexpensive liquid claims. This course of action implies a higher level of lifetime consumption, which compensates foregoing valuable commitment. Individuals pursuing the full commitment strategy, instead, overcome the time-inconsistency problem but at the cost of consuming less on average.

Individuals' strong demand for commitment is reflected in an equilibrium price premium on illiquid assets. Such overprice be interpreted as evidence of the desirability of providing illiquidity. Contrary to this intuition, I show that consumers' welfare is maximal when all assets are liquid. If illiquid assets are in positive supply, equilibrium utility is strictly lower. A benevolent interventionist authority taking into account general equilibrium feedback effects would rather strive to provide for full re-trading of all assets.

Equilibrium features specialization, with some individuals fully pre-committing and others acquiring no illiquid assets at all. The optimal commitment decision is an all-or-nothing choice. Notice that future savings will only take place when the individual is not constrained by the inherited stock of illiquid wealth. However, illiquid assets trade at a premium. Paying this premium and not constraining future decisions would be a waste of resources. Therefore, an optimal purchase of illiquid assets must imply no future additional savings. Finally, some agents must be saving in the future, or the market for liquid claims on future goods will be in excess supply. This explains why specialization needs to take place in equilibrium. One interpretation would suggest that there is limited availability of commitment resources (illiquid assets) and so not all agents can pre-commit in equilibrium.

To highlight the driving factors behind the negative welfare result, I next analyze a deterministic economy with production and where the use of commitment devices is endogenous. In a production economy, providing commitment to a fraction of the population will change the time profile of aggregate output, as pre-committed agents save more. A central authority concerned with insufficient aggregate savings might thus consider providing commitment possibilities to the population.

Given that trading illiquid assets induced a welfare loss, a natural question is whether endogeneizing the availability of commitment devices could eliminate this inefficient trading. In the unique equilibrium, agents make use of the commitment technology. However, utility is strictly lower when the technology is available than when it is not. Thus, the negative welfare result from the availability of commitment is robust to the introduction of aggregate production. More surprisingly, a welfare-decreasing use of the commitment technology arises in equilibrium. When commitment devices exist, the economy is led to an inefficient equilibrium in which they are used.

The equilibrium welfare result is again contrary to the choice-theoretic intuition. Pre-committing to future plans is always desirable to an individual given the profile of inter-temporal prices that she faces. However, introducing commitment devices into the economy modifies equilibrium prices and allocations in an adverse way, precisely because of the role played by commitment. People who acquire illiquid assets overcome their time-inconsistency and save more, driving interest rates down. Inter-temporal prices thus become more favorable to overconsumption, which makes the non-committed plans worse. More abstractly, when illiquid assets are available agents trade themselves to an inefficient equilibrium allocation.

There is a clash between individuals' desire for commitment and the aggregate effects from facilitating it. Individuals value commitment, but would rather live in an economy in which no commitment devices were available at all. When the market for commitment opens, individuals cannot avoid engaging in welfare-decreasing trading.

The previous literature on this topic discusses positive and normative implications of time-varying tastes. This paper studies the unexplored welfare consequences of satisfying individuals' demand for commitment. A related reference on markets and efficiency is Luttmer and Mariotti (2002) who extended the classical first welfare theorem to economies with changing tastes, but without commitment devices. Imrohoroglu et al. (2003) study the welfare properties of unfunded social security in a calibrated general equilibrium model.[4] In their paper, the use of the commitment technology (the social security system) is not an individual choice governed by price incentives. Krusell et al. $(2000, 2002a)$ focus on welfare consequences of tax policy in a general equilibrium, preference-changes model.

The treatment of illiquid assets in the present paper relates to Laibson (1997) and Kocherlakota (2001). Building on intuition developed by Strotz in 1956, the former was the first to model how such assets could fulfill a self-control role. He concluded that restrictions on the re-trading of some assets could improve the welfare of the representative agent. An important assumption behind this result is that liquid and illiquid assets of identical payoff structures trade at the same price, and that there are no limitations to the liquid/illiquid breakdown of the representative agent's portfolio. I discuss why this assumption is crucial in Section 3.

Kocherlakota (2001) focused on the positive properties of equilibrium when illiquid assets are available, albeit in limited supply. He showed that in a deterministic setup, illiquid assets will command an equilibrium price premium due to the valuable commitment services they provide. Moreover, the economy does not admit a representative agent construct, as ex-ante identical agents pursue asymmetric equilibrium strategies. My analysis of illiquid assets, besides posing the welfare question (which was not addressed before), adds idiosyncratic risk to this environment.

A trade-off between commitment and flexibility similar to the one analyzed in the illiquid assets model, was studied before by Bénabou and Tirole (2002) and Amador et al. (2004). The two papers concern individual decision-making but not general equilibrium. The latter analyzes the general problem of optimal mechanism design to induce future actions that are in line with initial preferences, given that more information is learned over time but tastes change. On the contrary, in the former, as well as in this paper, the structure of the commitment mechanism is a restriction from the environment and not an optimal choice.

Bénabou and Tirole (building on intuition developed by Carrillo and Mariotti, 2000) rationalize "self-confidence" as strategic information acquisition and retention to induce oneself to undertake tasks with delayed reward. A person may choose not to learn the random return from a given task, in order to avoid a situation in which undertaking the task is optimal from the initial point of view but not carried out due to preference changes. An interpretation of the choice not to learn before acting is that the person pre-commits to undertaking the task without accommodating to the received information. The structure of this commitment mechanism is analogous to illiquid assets analyzed in this paper.

More generally, the change-in-tastes framework motivating this research was prompted by evidence from psychology documenting departures of human behavior from neoclassical postulates (see Rabin, 1998, for a survey). Evidence initially came from experiments (see

[4] The welfare properties of a social security system were also studied by Feldstein (1985).

for instance Thaler, 1981), and later from field data: standardized contracts (DellaVigna and Malmendier 2003, 2004), job search (DellaVigna and Paserman, 2004), and surveys (Ameriks et al., 2004, Thaler, 1981), among other places. Some studies have tried to evaluate the hyperbolic discounting model through partial-equilibrium calibration exercises (Angeletos et al., 2001, Harris and Laibson, 2001*b*, and Laibson et al. 2003, 2004).

Macroeconomic evidence of time-inconsistent preferences is still lacking, to some extent owing to an identification hurdle. Barro (1999) noticed that the standard neoclassical growth model and its hyperbolic variation are observationally equivalent; this was later generalized by Luttmer and Mariotti (2003) to asset pricing in an economy with aggregate uncertainty. Laibson and Yaariv (2004) showed that any observed sequence of equilibrium liquid asset prices and inter-temporal choices of individuals with "non-standard preferences" can be rationalized by an appropriately chosen set of standard neoclassical preferences. Kocherlakota (2001) suggested that this observational equivalence could be broken by focusing on commitment assets.

The paper is structured as follows. Section 2 analyzes the illiquid assets model, Section 3 the endogenous commitment model with production and Section 4 concludes. All proofs are in the Appendix.

2 Illiquid Assets

In this section, I study an economy in which individuals have time-inconsistent preferences and illiquid assets are available to them. A three-period world is populated by a continuum of agents indexed by $i \in I \equiv [0, 1]$. The aggregate endowment is deterministic and individuals are identical in preferences and initial wealth. Consumption takes place in periods 2 and 3, and in period 1 each agent makes an asset allocation decision. The aggregate endowment of period-2 and period-3 goods are denoted $\overline{c}_2$ and $\overline{c}_3$, respectively.

Preferences change over time. I represent this by assuming that individuals are "hyperbolic discounting": The weight given to period-3 consumption utility is higher in period 1 than in period 2. In addition, a discount factor shock that affects both period-1 and period-2 tastes is realized and learned in period 2.

Taking prices as given, agents act to maximize utility subject to the sequential budget and liquidity constraints. Equilibrium requires that all agents follow optimal plans at all times, and that all markets clear.

2.1 Markets and budgets

There are three productive assets in the economy. The first one pays off $\overline{c}_2 > 0$ units of period-2 goods. The second one, $\alpha\overline{c}_3$ units of period-3 goods, where $\overline{c}_3 > 0$ and $\alpha \in [0, 1]$. And the third asset pays off $(1 - \alpha)\,\overline{c}_3$ units of period-3 goods.

Individuals are initially endowed with representative claims on these assets. To simplify the exposition (so that all assets have unitary nominal return), let there be $\overline{c}_2$ claims on the asset paying off in period 2, and a total of $\overline{c}_3$ claims on the two assets paying off in period 3. The latter total, in turn, breaks down into $\alpha\overline{c}_3$ and $(1 - \alpha)\,\overline{c}_3$ units, respectively, of each of the two assets paying off in period 3. I will refer to claims on the asset paying off in period 2 as "short term assets". Claims on the two types of assets paying off in period 3 are called "long term assets". Specifically, the first long term asset (whose net supply is $\alpha\overline{c}_3$) is called the "illiquid" long term asset, while the other one is the "liquid" long term asset.

In period 1, the three assets are traded. In period 2, the short-term asset pays off and matures. There is trading of period-2 consumption goods and liquid long term assets only. Although they are still outstanding, illiquid long term assets are not traded in period 2. In period 3, both long term assets pay off and mature. Each individual consumes her received dividends.

The following notation and normalizations will be used throughout for prices and asset holdings.

In period 1, the price of the short term asset is normalized to 1. The price of liquid long term assets is denoted by q_{13}, and that of illiquid long term assets, by p. A given individual's end-of-period-1 short term asset holdings are denoted by a_{12}; liquid long term asset holdings by a_{13}, and illiquid assets by k. For variables referring to liquid assets, the first entry in the sub-index is the period of reference, and the second one, the period of maturity.

In period 2, the price of period-2 consumption goods is normalized to 1 and the price of the liquid long term asset is denoted by q_{23}. A given individual's period-2 consumption is denoted by c_2, and her end-of-period-2 liquid long term asset position, a_{23}. Illiquidity of the other long term asset implies that the end-of-period-2 holding must be equal to k.

Each individual's period-3 consumption is denoted by c_3, and equals $a_{23} + k$.

All asset holdings are restricted to be non-negative: $a_{12} \geq 0, a_{13} \geq 0, a_{23} \geq 0, k \geq 0$.

The economy's liquidity is parameterized by α, the fraction of period-3 goods that are procured through illiquid assets. The non-negativity constraint $k \geq 0$ plays the role of ensuring that α is determined exogenously. Individuals cannot short-sell human capital or a house. Allowing k to be negative would be equivalent to letting agents offer commitment contracts to others for free. An alternative way of making α endogenous would be to allow such contracting, but at a cost.

The other non-negativity assumptions are needed for illiquid assets to indeed imply an illiquidity constraint. It is essential that the individuals face borrowing constraints in period 2 (the restriction $a_{23} \geq 0$ could be generalized to $a_{23} \geq -b$, for $b > 0$). Otherwise, individuals would be able to bring forward illiquid resources in period 2 and hence, the investment in illiquid assets would not be irreversible.

To some degree, a central authority can implement legislation fostering illiquidity. Consider for instance forbidding the use of claimed "family homestead" as collateral for mortgage loans (as was the case in Texas until 1998): Individuals face costs to sell down houses and cannot borrow against them. The aggregate stock of such "family homestead" illiquid assets is limited by the actual number of houses, and this is captured in this model by the exogenously given α.

The market structure determines the sequential budget constraints that each agent faces. Let $w \equiv \overline{c}_2 + q_{13}(1-\alpha)\overline{c}_3 + p\alpha\overline{c}_3$ denote the market value of the representative stock of initial wealth. In period 1, the individual makes a portfolio decision subject to the following budget constraint:

$$a_{12} + q_{13}a_{13} + pk \leq w. \tag{1}$$

Suppose that the period-1 portfolio choice is given by a triple (a_{12}, a_{13}, k). Then in period-2, the individual makes a consumption and savings decision subject to the following budget constraint:

$$c_2 + q_{23}a_{23} \leq a_{12} + q_{23}a_{13}. \tag{2}$$

Because a_{23} is restricted to be non-negative, period-2 consumption is capped by the period-2 value of liquid assets. The individual is forbidden from borrowing against her stock of illiquid assets and is forced to consume at least k units in period 3.

2.2 Preferences

Let $\delta \in \left[\underline{\delta}, \overline{\delta}\right]$ be a random variable with absolutely continuous distribution function $F(\delta)$. Let (c_2, c_3) denote a profile of period-2 and period-3 consumption. Conditional on the realization of δ, period-1 and period-2 preferences over (c_2, c_3) are given by

$$\begin{aligned} v_1(c_2, c_3; \delta) &= u(c_2) + \delta u(c_3), \\ v_2(c_2, c_3; \delta) &= u(c_2) + \beta\delta u(c_3). \end{aligned}$$

$u(c)$ is a strictly increasing, strictly concave, everywhere twice differentiable function. In period 1, $\beta < 1$ is anticipated, but δ is not known.[5] The realization of each agent's idiosyncratic taste shock is unobservable, private information of the individual. This precludes insurance, due to incentive compatibility constraints. No contract conditioning on an individual's shock realization can be enforced in this economy, and this is why markets are incomplete.

Let $c(\delta) \equiv (c_2(\delta), c_3(\delta))$ be a measurable function of period-2 and period-3 consumption amounts. Period 1 preferences over this stochastic consumption profile are given by the expected utility formulation

$$U = E\left[v_1(c_2(\delta), c_3(\delta); \delta)\right].$$

The period-1 portfolio decision is made under uncertainty about the person's own future preferences, but in anticipation of the preference change.

Because tastes change, the point of reference is important when discussing welfare. In what follows, I will use the terms "period-1 optimal" and "period-2 optimal" to refer to utility maximizing consumption profiles from the point of view of period-1 and period-2 preferences, respectively. When there is no ambiguity, I will use the term "optimal" as a shortcut for "period-1 optimal".

2.3 Choice

People act sequentially to maximize utility subject to the budget constraints. Period-1 and period-2 preferences over inter-temporal consumption profiles differ, and this is correctly

[5] Krusell et al. (2002*b*) study the opposite case in which agents experience an urge to over-save: $\beta > 1$.

anticipated in period 1. For this reason, it is convenient to start with the period-2 decision problem.

Period-2 choice is made under knowledge of δ. Suppose that the period-1 portfolio choice is given by a triple (a_{12}, a_{13}, k). Then, the period-2 decision problem is

$$\begin{array}{cl} \max\limits_{c_2, a_{23}} & \{u(c_2) + \beta\delta u(c_3)\} \\ s.t. & c_2 + q_{23}a_{23} \le a_{12} + q_{23}a_{13} \\ & a_{23} \ge 0, \quad c_3 = a_{23} + k \end{array} \tag{3}$$

Let $c_2(\delta), a_{23}(\delta)$ denote the solutions to this problem. In period 2, the individual chooses how to allocate available wealth into period-2 and period-3 consumption. Because she cannot borrow against her stock of illiquid assets, she faces a potential liquidity constraint that may hinder implementation of an optimal plan. This decision is made with knowledge of δ, but with the additional discount factor $\beta < 1$.

In anticipation of the δ-contingent period-2 policy rule, the period 1 problem is

$$\begin{array}{rcl} V = & \max\limits_{a_{12}, a_{13}, k} & \left\{\int_{\underline{\delta}}^{\overline{\delta}} \left(u(c_2(\delta)) + \delta u(c_3(\delta))\right)\ f(\delta)\ d\delta\right\} \\ & s.t. & a_{12} + q_{13}a_{13} + pk \le w,\ a_{12} \ge 0, a_{13} \ge 0, k \ge 0, \\ & & c_3(\delta) = a_{23}(\delta) + k, \\ & & \forall\delta : (c_2(\delta), a_{23}(\delta)) \text{ solve (3).} \end{array} \tag{4}$$

The period-1 choice consists of allocating initial wealth into liquid and illiquid assets. This is made under uncertainty over future preferences, and in anticipation that the period-2 decision will be made under knowledge of δ but with the additional discounting $\beta < 1$.

2.4 Equilibrium

Definition 1 *A competitive equilibrium is a price vector* $(p, q_{12}, q_{13}, q_{23})$ *together with individual choices* $\left(a_{12}^i, a_{13}^i, k^i, a_{23}^i(\delta^i), c_2^i(\delta^i), c_3^i(\delta^i)\right)_{i \in I}$ *such that:*

1. *For each* i, $\left(a_{12}^i, a_{13}^i, k^i, a_{23}^i(\delta^i), c_2^i(\delta^i), c_3^i(\delta^i)\right)$ *are measurable and solve the consumer's period-*1 *and period-*2 *utility maximization problems given prices.*

2. *Markets clear:*

$$\begin{aligned} \int_{i \in I} a_{12}^i\ di &= \overline{c}_2, \quad \int_{i \in I} a_{13}^i\ di = (1-\alpha)\overline{c}_3, \quad \int_{i \in I} k^i\ di = \alpha\overline{c}_3, \\ \int_{i \in I} c_2^i(\delta^i)\ di &= \overline{c}_2, \quad \int_{i \in I} a_{23}^i(\delta^i)\ di = (1-\alpha)\overline{c}_3, \\ \int_{i \in I} c_3^i(\delta^i)\ di &= \overline{c}_3. \end{aligned}$$

It is useful to state, at this early stage, a result that is used to simplify the exposition below. The assumed asset structure allows two ways of carrying liquid wealth into period 2, either by purchasing short term assets, or by purchasing liquid long term assets. In equilibrium, the two must imply the same cost per unit of period-2 liquid wealth. Otherwise, either of the two assets will be in excess supply.[6]

Proposition 1 *Equilibrium prices of liquid assets admit no dynamic arbitrage:* $q_{13} = q_{23}$.

For expositional purposes, let $q \equiv q_{13} = q_{23}$ denote the price of liquid long term assets for the rest of the paper.

2.5 Individual Optimization

The period-1 portfolio choice depends on expected period-2 behavior; therefore I begin by studying the latter. The general solution of the period-1 problem is in the Appendix; here I focus on the two polar cases in "C" and "F", in which commitment and flexibility, respectively, are most valuable to the agent. Finally, I give conditions on $F(\delta)$ and β for these two cases to hold.

2.5.1 The period-2 problem

Imagine that a given triple (a_{12}, a_{13}, k) was chosen in period 2. This portfolio choice implies that the amount of liquid resources available in period 2 is $w_2 \equiv a_{12} + qa_{13}$. Given this, the solution to problem (3) is determined by the necessary and sufficient first order condition

$$\frac{u'(c_2)}{u'(c_3)} \geq \frac{\beta\delta}{q}, \tag{5}$$

together with the period-2 budget restriction- and the no-borrowing constraint $a_{23} \geq 0$. Specifically, the Euler equation will hold with inequality only if $a_{23} = 0$.

The period-2 decision is liquidity-constrained if and only if the realized δ is smaller than the threshold $\widetilde{\delta}$ defined by

$$\widetilde{\delta} \equiv \frac{qu'(w_2)}{\beta u'(k)}.$$

Lemma 1 *Suppose that* $k > 0$. *For* $\delta \leq \widetilde{\delta}$, *the implemented choice is* $c_2 = w_2, c_3 = k$; *while for* $\delta > \widetilde{\delta}$, *(5) holds with equality and the implemented consumption is* $c_2 < w_2, c_3 > k$.

Consider an individual who enters period 2 with a positive stock of illiquid assets $k > 0$. This stock caps period-2 consumption, as illiquid assets cannot be sold down to finance consumption. So, suppose that the realized period-2 discount factor $\beta\delta$ is low, and therefore that large period-2 consumption is desirable. The individual will want to allocate as much wealth as possible into period-2 consumption. She will sell down all her liquid long term assets, and will be willing but unable to sell down her illiquid long term assets. She will find herself restricted by the constraint $a_{23} \geq 0$.

[6] I make the standard assumption that the realization of the individual random variables, $(\delta^i)_{i \in I}$, is measurable, and that $F(\delta)$ is the realized measure of the set $\{i \in I : \delta^i \leq \delta\}$. This implies that all prices are deterministic.

Therefore, for low realizations of δ the individual behaves "hand-to-mouth" in period 2, allocating all inherited liquid resources into period-2 consumption.

Instead, if the realized $\beta\delta$ is high enough, the individual will actually want to consume period-3 goods beyond her inherited stock of illiquid claims. In that case, she will find it optimal to purchase liquid long term assets, and the liquidity constraint $a_{23} \geq 0$ will not bind.

2.5.2 The structure of period-1 strategies

To understand period-1 optimal choice, it is useful to divide the possible solutions to the period-1 problem (4) into three classes.[7]

First, the consumer could find optimal not to carry any illiquid assets at all, and hence not restrict period-2 choice. I call this the "full flexibility strategy". Second, the optimal policy could be the other extreme: a "full commitment strategy", in which the period-2 choice is liquidity-constrained for any realization of δ. This is achieved with a sufficiently illiquid portfolio and it implies a deterministic consumption profile, since the consumer always behaves "hand-to-mouth" in period 2. Finally, the intermediate case is a "partial commitment strategy", in which the period-2 choice is liquidity-constrained only for low realizations of the discount factor.

For $k > 0$ given, Lemma 2 identifies a threshold δ^* such that period-2 behavior is hand-to-mouth for lower realized discount factors. Equivalently, to each value δ^*, there is a unique illiquid assets choice k that yields δ^* as threshold. The larger the threshold δ^*, the larger the stock of illiquid assets necessary to achieve it.

This property can be used to think of the period-1 liquid/illiquid assets decision as one of choosing a cutoff level δ^*: The implemented ratio of marginal utilities $\frac{u'(c_2)}{u'(c_3)}$ is $\frac{\beta\delta^*}{q}$ for all realizations $\delta \leq \delta^*$; and it equals $\frac{\beta\delta}{q}$ for realizations $\delta \geq \delta^*$. In other words, when $\delta \leq \delta^*$, the consumer acts as if her discount factor were $\beta\delta^*$.

So, consider an arbitrary cutoff δ^*. Conditional on $\delta \leq \delta^*$, the expected period-1 discount factor is $E[\delta|\,\delta \leq \delta^*]$, but the consumer acts as if it were $\beta\delta^*$. Suppose that $E\left[\delta|\,\delta \leq \delta^*\right]$ is smaller than $\beta\delta^*$. Then choosing δ^* induces too patient a period-2 behavior, from the period-1 perspective, even in spite of the dynamic inconsistency. At such a δ^*, lack of flexibility is a stronger distortion to optimal choice than excessive future impatience.

Conversely, when $\beta\delta^* < E\left[\delta|\,\delta \leq \delta^*\right]$, choosing δ^* as cutoff value will induce a more impatient behavior than the conditionally expected period-1 optimal one. Conditional on such a δ^*, excessive period-2 impatience is a stronger distortion to optimal choice than lack of flexibility. Notice that $\psi\left(\delta^*\right)$ will be positive-valued whenever $E\left[\delta|\,\delta \leq \delta^*\right] < \beta\delta^*$.

The period-1 portfolio decision will hinge on sign and monotonicity properties of the following function:

$$\psi\left(\delta^*\right) \equiv \left(E\left[\delta|\,\delta \leq \delta^*\right] - \beta\delta^*\right)\frac{F\left(\delta^*\right)}{\delta^*}. \tag{6}$$

[7] In this three-period model, the structure of the commitment device is formally equivalent to a time-to-sell condition on illiquid assets like the one in Laibson (1997). In that paper, a decision to sell down illiquid assets at time t translates into resources available at $t+1$, and, as a consequence, does not appease the period-t temptation to overconsume in period t.

A sharp characterization of the optimal portfolio choice is possible under the following two mutually exclusive assumptions on (6).

Assumption C. $\psi(\delta^*)$ is positive and strictly increasing in δ^*.

Assumption F. $\psi(\delta^*)$ is negative and strictly decreasing in δ^*.

"C" stands for commitment, and "F" for flexibility. When Assumption C holds, the commitment concern will drive the period-1 decision. Instead, when Assumption F is satisfied, the flexibility concern will. Under an important special case, these two assumptions exhaust all possibilities. For exponential distributions (with the normalization $\underline{\delta} = 0$), either C or F must hold: C if β is low, and F if it is high.

2.5.3 Assumption C

Assume that $\psi(\delta^*)$ is positive and monotonically increasing in δ^*. Under this assumption, the optimal period-1 choice is either a full commitment or a full flexibility strategy. Under this assumption, partial commitment is never optimal. The individual will choose to fully commit unless illiquid assets are too expensive, in which case none will be acquired.

The optimal commitment decision is an all-or-nothing choice. If illiquid assets are less expensive than liquid ones, then fully committing is optimal given that commitment is more valuable than flexibility. Instead, when illiquid assets trade at a premium, paying the premium and failing to constrain future choice implies a waste of resources. In either case, an optimal purchase of illiquid assets must imply no future additional savings.

To characterize the optimal portfolio choice, let k^* be the unique value satisfying

$$\frac{u'(w - pk^*)}{u'(k^*)} = \frac{E[\delta]}{p}. \tag{7}$$

Define expected period-1 utility from the full commitment ("FC") and full flexibility ("FF") strategies, respectively, by:

$$\begin{aligned} V_{FC} &= u(w - pk^*) + E[\delta]\, u(k^*), \\ V_{FF} &= \int_{\underline{\delta}}^{\overline{\delta}} \left(u(c_2(\delta)) + \delta u(c_3(\delta))\right) f(\delta)\ d\delta, \\ &\text{where } c_2(\delta), c_3(\delta) \text{ solve (3) for } k = 0. \end{aligned}$$

For a fixed q, let $\widetilde{p}$ be the (unique) price that makes $V_{FC} = V_{FF}$.

Proposition 2 *Under Assumption C, for each q a unique threshold $\widetilde{p}$ exists. When the price p of illiquid assets verifies $p \leq \widetilde{p}$, full commitment is optimal, while if $p \geq \widetilde{p}$, full flexibility is. $\widetilde{p}$ is strictly larger than q, increasing in q and decreasing in β.*

When Assumption C holds, the implemented ratio of marginal utilities is always too low from the period-1 perspective; moreover, the gap increases in δ^*. The more liquidity-constrained the individual is, the more such constraints are valued from the period-1 perspective. This implies that the period-1 problem must have a corner solution, either fully committing or not committing at all. Either full flexibility or full commitment (or both) always strictly dominate partial commitment.

Full commitment is equivalent to a deterministic consumption profile. All period-3 consumption is procured through illiquid assets purchased in period 1, while all liquid wealth goes into period-2 consumption. (7) is the necessary and sufficient first order condition for the optimal deterministic consumption profile when the price of period-3 consumption is p. That is, k^* is the best affordable full commitment plan. Notice, however, that a smaller illiquid assets stock could suffice to liquidity-constrain period-2 choice for all values of δ. Analogously, $k > k^*$ would also achieve full commitment, but would be suboptimal.

Given q, a unique illiquid assets price $\widetilde{p}$ exists such that the consumer is indifferent between full commitment and full flexibility; otherwise the two strategies are strictly ranked. The indifference price $\widetilde{p}$ is always larger than q, since, under Assumption C, the consumer values commitment more than flexibility.

2.5.4 Assumption F

Suppose that $\psi(\delta^*)$ is negative and monotonically decreasing in δ^*. A partial commitment strategy will be optimal in this case when the relative price of illiquid assets falls in an intermediate region. In order to characterize the optimal partial commitment choice, fix δ and let $k(\delta)$ be the unique value satisfying

$$\frac{u'(w - pk(\delta))}{u'(k(\delta))} = \frac{\beta\delta}{q}.$$

Notice that $k(\delta)$ is strictly increasing in δ. Using this, define the value from choosing cutoff $\widetilde{\delta}$ as

$$V(\widetilde{\delta}) = \left(u(w - pk(\widetilde{\delta})) + E[\delta|\, \delta \leq \widetilde{\delta}]\ u(k(\widetilde{\delta}))\ \right) F(\widetilde{\delta}) + \int_{\widetilde{\delta}}^{\overline{\delta}} \left(u\left(c_2\left(\delta\right)\right) + \delta u\left(c_3\left(\delta\right)\right)\right)\ f\left(\delta\right)\ d\delta,$$

$$\text{where } c_2(\delta), c_3(\delta) \text{ solve (3) for } k = k(\widetilde{\delta}), \delta \geq \widetilde{\delta}.$$

The partial commitment value ("PC") is defined as

$$V_{PC} = \max_{\widetilde{\delta}\in[\underline{\delta},\overline{\delta}]} V(\widetilde{\delta}). \tag{8}$$

Let δ^* denote the solution to problem V_{PC}. Under Assumption F, this solution is unique, and is identified by the first order conditions associated with program V_{PC}. Finally, let $\overline{\rho} \equiv E[\delta]/\overline{\delta}\beta$; under Assumption F, this is strictly smaller than 1.

Proposition 3 *Under Assumption F, the optimal illiquid assets choice, as a function of the price ratio p/q, is*

$$\begin{array}{ll} k = k^* & \text{if } \frac{p}{q} \leq \overline{\rho} \\ k = k(\delta^*) & \text{if } \overline{\rho} < \frac{p}{q} < 1 \\ k = 0 & \text{if } 1 < \frac{p}{q}, \end{array}$$

At $p = q$, the consumer is indifferent over all portfolios containing $k \in [0, k(\underline{\delta})]$.
For $\overline{\rho} < p/q < 1$, the optimal k is decreasing in β.

Consider a given cutoff value δ^* such that $\psi(\delta^*) < 0$. As explained before, this negative value holds if $\beta\delta^* > E[\delta|\,\delta \leq \delta^*]$. By construction, $\beta\delta^*$ is the "implemented" discount factor whenever the realization of δ is smaller than δ^*: In these states of the world, the ratio $qu'(c_2)/u'(c_3)$ equals $\beta\delta^*$.

So $\psi(\delta^*) < 0$ means that, conditional on $\delta \leq \delta^*$, the implemented discount factor is larger than the expected optimal one. The distortion stemming from being unable to accommodate the taste shock is thereby stronger than the benefit from commitment. Consequently, holding illiquid assets can only be optimal if they offer a capital gain. When $p > q$, the consumer has no reason to hold them. But if $p < q$, loosing valuable flexibility (in exchange for not-so-valuable self-control) is compensated by the capital gain from buying assets at a discounted price. The optimal choice trades off these two effects. If the price of illiquid assets is small enough, the consumer will choose a deterministic consumption profile.

2.5.5 Sufficient conditions for C and F

The previous analysis has focused on assumptions on the function $\psi(\delta^*)$. In this section, I give conditions on the distribution function $F(\delta)$ and the hyperbolic factor β for Assumptions C and F to hold. In order to do so, let $\varepsilon(\delta) \equiv d\ln f(\delta)/d\ln\delta$ denote the point elasticity of the density function. What the assumptions do is link β to this elasticity. Sufficient conditions for them to hold are given in the following proposition.

Proposition 4 *Assumption C holds if $\beta \leq (1+\varepsilon(\delta))/(2+\varepsilon(\delta))$ and $\varepsilon(\delta) \geq -1$ are true for all δ.*

Assumption F holds if $\underline{\delta} f(\underline{\delta}) = 0$ and, for all δ, either $\beta \geq (1+\varepsilon(\delta))/(2+\varepsilon(\delta))$ or $\varepsilon(\delta) \leq -1$. The boundary condition $\underline{\delta} f(\underline{\delta}) = 0$ is necessary for F.

Consider a threshold level δ^*; the conditional expected value of the discount factor is $E[\delta|\,\delta \leq \delta^*]$. If the point elasticity of the density function is high at δ^*, then the period-1 expected discount factor locally increases faster than $\beta\delta^*$. The lower β, the lower the density elasticity necessary to make this true.

Assumption F will hold if β is sufficiently high, or if the elasticity is everywhere low enough. In addition, assumption F requires a boundary condition on the support of $F(\delta)$, which ensures that $\psi(\delta^*)$ is non-increasing at $\underline{\delta}$.

Notice that whenever the elasticity is bounded above, there exists a $\widetilde{\beta} \in (0,1)$ such that $\beta \geq (1+\varepsilon(\delta))/(2+\varepsilon(\delta))$ is satisfied for $\beta \geq \widetilde{\beta}$. Analogously, whenever the elasticity is bounded below by -1, there will exist a level $\widetilde{\beta} \in (0,1)$ such that C holds if $\beta \leq \widetilde{\beta}$.

The proposition is stated in terms of conditions on β given ε. Alternatively, given β, C imposes a lower bound of $(2\beta-1)/(1-\beta)$ on the elasticity $\varepsilon(\delta)$, while F imposes an upper bound of $(2\beta-1)/(1-\beta)$, plus a boundary condition.

General distribution functions need not satisfy either C or F for all parameter values. For instance, if $\delta \in [0,1]$ has distribution $F(\delta) = \left(e^{\delta}-1\right)/(e-1)$, then C is satisfied for $\beta \leq 1/2$, F for $\beta \geq 1-e^{-1}$, and none of the two hold for the intermediate values.

However, when the distribution function is a power function, and the lower bound $\underline{\delta}$ equals 0, then Assumptions C and F are exhaustive. There is a threshold $\widetilde{\beta}$ such that for C is satisfied if β is below the threshold, and F is satisfied otherwise.

Proposition 5 *Suppose* $\underline{\delta} = 0$ *and* $F(\delta) = \delta^x / \overline{\delta}^x$*, for some* $x > 0$*. Then C is satisfied if and only if* $\beta < \widetilde{\beta} \equiv \overline{\delta}^x x / (1 + x)$ *while F is satisfied if and only if* $\beta > \widetilde{\beta}$*.*

For example, suppose that δ is uniformly distributed on $[0, 1]$. Then, C holds for $\beta < 1/2$ and F for $\beta > 1/2$.

At the boundary value $\widetilde{\beta}$, $\psi(\delta^*)$ is everywhere equal to 0. The self-control and flexibility forces exactly offset each other in this case.

Amador et al. (2004) show that when β is high enough relative to the elasticity of the shock density function, the optimal commitment mechanism takes the form of a minimum savings rule, such as the one implemented by illiquid assets. Bénabou and Tirole (2002) show that when time-inconsistency is strong enough (β low) and an elasticity condition on the shock distribution function is met, it is optimal in period 1 to prevent period-2 state-contingent action. In other words, commitment to a specified course of action is preferred to acting on realized information. When β is high, instead (and a condition on the elasticity is met), it is period-1 optimal to allow state-contingent period-2 action, in spite of time-inconsistency.

2.6 Equilibrium

Given individual choice rules, the next step is to analyze the competitive equilibrium in this economy. Because of time-inconsistency, and the presence of irreversible investment, textbook arguments for existence of equilibrium cannot be directly applied. I show that equilibrium always exists, nevertheless.

Proposition 6 *A general equilibrium always exists in this economy.*

A sharp characterization of equilibrium is feasible under assumptions C and F. Without these, alternative assumptions must be imposed on $\psi(\delta^*)$ to be able to describe the structure of equilibrium. When $\underline{\delta} = 0$ and the distribution $F(\delta)$ is a power function, cases C and F are the only possible ones.

Assume that C holds. By Proposition 3, for every price pair (p, q), the individual optimal period-1 choice is either a full commitment or a full flexibility strategy, and possibly both. Full commitment involves an illiquid assets level $k^* > 0$ and hand-to-mouth period-2 behavior with probability 1. The latter consists of acquiring zero illiquid assets.

Everyone following a full flexibility strategy is not an admissible equilibrium, since the net supply of illiquid assets is strictly positive.

Suppose instead that all agents follow a full commitment strategy, so that everyone behaves hand-to-mouth in period 2. This implies that no one will hold liquid long term assets at the end of period 2. But the net supply of these assets is strictly positive, and therefore this is not an admissible equilibrium configuration either.

The only way markets could clear is if some individuals adopted full flexibility, and others full commitment. Fully committed agents absorb the supply of illiquid long term assets; fully flexible people the period-2 supply of liquid long term assets.

This is only possible if agents are indifferent between those two strategies, which will happen if and only if $p = \widetilde{p}$. As shown above, this indifference price is always larger than q.

Proposition 7 *Under Assumption C, any equilibrium involves a fraction $\lambda^* \in (0, 1)$ of the population pursuing a full commitment strategy, and the remaining $1 - \lambda^*$ a full flexibility one. Equilibrium prices verify $p = \widetilde{p} > q$: Illiquid assets command an equilibrium price premium.*

When C is satisfied, the distortion coming from dynamic inconsistency is always stronger than the distortion from not accommodating the taste shock at low δ. This is a "preference for commitment" economy and illiquid assets are desirable at the individual level. The equilibrium is similar to the deterministic case studied by Kocherlakota (2001), and displays specialization of savings strategies. At the end of period 2, a cross section of the population will show that individuals either hold only illiquid assets, or only liquid assets. Those holding illiquid assets will be liquidity-constrained.

If F holds, instead, the distortion coming from not accommodating the taste shock at low δ is always stronger than any distortion from tastes change. Consumers' behavior is similar to that of neoclassical agents, and illiquid assets are not desirable. This is compensated by illiquid assets trading at a discount, thanks to which carrying a positive stock is optimal. The equilibrium is symmetric, with all agents acquiring the same amount of liquid and illiquid assets in period 1. The economy is observationally equivalent to the neoclassical case.

Proposition 8 *Under Assumption F, $p \leq q$ in equilibrium.*

If $p < q$, then all agents acquire identical portfolios (equal, therefore, to the representative one) in period 1. There exists a cutoff δ^* such that agents are liquidity-constrained in period 2 if and only if the realized δ is smaller than δ^*.

If $p = q$, then individual initial asset holdings are undetermined, but no agent is ever liquidity-constrained in period 2.

An equilibrium with $p = q$ can exist for α low enough; it is ruled out if $\underline{\delta} = 0$. Such an equilibrium will occur whenever the amount of illiquid assets in the representative portfolio is smaller than individual period-3 consumption, for any δ.

For clarity, consider the case $\underline{\delta} = 0$. When $\underline{\delta} = 0$, there is a positive probability of δ being very low. And if the realized δ is very low, desired period-3 consumption is close to 0. Therefore, a stock of illiquid assets will liquidity-constrain the period-2 choice with positive probability, no matter how small the stock is. But such constraint is never desirable under F. Therefore, the price of illiquid assets must have a discount, reflecting the loss occurred when the realized δ is very low.

If $\underline{\delta}$ is strictly positive and the representative portfolio has few illiquid assets, it is possible that the latter never liquidity-constrains period-2 choice. Specifically, this will happen if at $\delta = \underline{\delta}$, desired period-3 consumption is larger than $\alpha\overline{c}_3$ (the stock of illiquid assets in the representative portfolio). In such a case, liquid and illiquid long term assets are indistinguishable, and hence have the same price.

2.7 Equilibrium Welfare

To a large extent, research on departures from neoclassical assumptions is motivated by the welfare consequences of behavioral biases. Benevolent tampering with the market structure could be desirable for a central authority concerned about individuals' well-being.

I focus on period-1 utility. This is consistent with the "temptation and self-control" interpretation of modeled behavior: in period 2, individuals "agree" with period-1 preferences, but a craving to overconsume interferes with the implementation of optimal plans.

The "split-self" approach, instead, interprets behavior as a true change in tastes. Agents at each point in time are treated as distinct persons, and the Pareto criterion is given an intra-personal extension. A change is welfare-improving only if it makes all individuals (weakly) better off from the perspective of all periods. Notice that both views coincide on the undesirability of a period-1 utility loss.

Proposition 9 *Assume that either C or F holds, and let $u(c) = \log c$. Then for every α, there is a unique equilibrium. Period-1 expected utility is maximized at $\alpha = 0$, and is strictly lower for all $\alpha \in (0, 1]$.*

Any level of illiquidity is period-1 harmful in this economy. Even when the dynamic inconsistency distortion is maximal, individuals are better off if no illiquid assets are available at all. This aggregate results shows a sharp contrast with preferences at the individual level. Individually, each person finds illiquid assets valuable in this economy, and pays a premium for them.

Under Assumption C, illiquid assets trade at an equilibrium price premium, which could suggest that a positive net supply of these assets is desirable. An outside observer performing a thought experiment based on the observed choice behavior could be misled to believe that it is optimal to forbid re-trading of some assets. An example is prohibiting individuals to borrow against their 401K or IRA accounts, or allowing them to declare that their houses are inalienable homestead which cannot be mortgaged (as in Texas prior to 1998). A broader interpretation of this model is awareness. Drawing people's attention towards available commitment devices could be construed as an increase in α.

2.8 Discussion

The general solution to the period-1 portfolio problem is in the Appendix. I show that if $p > q$, then a partial commitment strategy can only be optimal if it implies a cutoff δ^* for which $\psi(\delta^*)$ is positive. Conversely, when $p < q$, $\psi(\delta^*)$ negative must be true at the cutoff δ^*.

Suppose that at $p > q$, illiquid assets are purchased to implement a partial commitment strategy. If the realized δ is high, the period-2 liquidity constraint will not bind. Acquiring illiquid assets will have been wasteful ex-post as they were more expensive than liquid assets but did not serve a self-control purpose.[8] Therefore, acquiring illiquid assets can only be optimal if, for δ low, they generate an expected period-1 utility gain. Conditional on $\delta \leq \delta^*$, they will generate such a gain if the self-control that illiquid assets help achieve is more valuable than the inability to accommodate the taste shock. And this, in turn, is true if and only if $\psi(\delta^*) > 0$.

When $p < q$, for δ high, illiquid asset yield a capital gain and hence a period-1 utility increase. If, conditional on δ being low, a utility gain is also achieved because $\psi(\delta^*) > 0$, then the consumer will gain by increasing the stock of illiquid assets. So such a stock of illiquid assets cannot be an optimal purchase. A partial commitment strategy can only be optimal in this case if $\psi(\delta^*) < 0$, and the capital gain when δ is high is offset by the flexibility loss when δ is low.

If $p > q$ but $\psi(\delta^*)$ is everywhere negative, full flexibility is the optimal strategy. Conversely, if $p < q$ and $\psi(\delta^*)$ is everywhere positive, full commitment is optimal. For arbitrary $\psi(\delta^*)$ functions, more than one strategy could be optimal, including several partial commitment plans involving different cutoff values for δ^*. Further characterizations require more structure on $\psi(\delta^*)$, such as assuming that it is a strictly quasiconcave or quasiconvex function.

An interpretation is that the illiquid assets problem consists of assigning the inter-temporal wealth allocation decision, given the trade-off between uncertainty and self-control. Full flexibility implies leaving the inter-temporal wealth allocation decision for period 2, without impinging on it. Under full commitment, the inter-temporal decision is wholly made in period 1 (thereby losing the ability to condition on δ) and no active choice is left for period 2. Partial commitment, finally, is the intermediate case in which the wealth allocation decision for low δ states is, in effect, made in period 1, whereas when δ is high, the allocation is decided in period 2.

Alternatively, the period-2 problem can be thought of as involving non-convex adjustment costs. The consumer inherits a stock $k > 0$ of claims on period-3 consumption goods, whose sell price is effectively 0; whereas the buy price for period-3 consumption goods is q. This wedge between prices implies that the optimal policy rule as a function of δ involves inaction if δ is low.

This suggests the generalization of this model to finite transaction costs, and sheds light on the nature of illiquid assets as commitment devices. Suppose that, instead of being a completely irreversible investment, these assets could be sold down at a fraction $\gamma < 1$ of q. The period-2 problem would have non-convex adjustment costs (see Dixit and Pindyck, 1994). Optimal period-2 behavior as a function of δ would involve a region of inactivity $[\delta_L, \delta_H]$. For $\delta < \delta_L$, the consumer would partially sell down the inherited claims on period-3 goods at γq_{23}; while for $\delta > \delta_H$ she would acquire additional claims at q. For $\delta \in [\delta_L, \delta_H]$, the consumer would only be willing to sell at the buy price or buy at the sell price; none possible. This is studied in companion work.

So think of a pre-specified future consumption profile. The commitment mechanism underlying illiquid assets is a non-convex cost schedule for adjustments of that profile. In

[8] Due to the time-separable and strictly concave utility assumption, utility is strictly increasing in period-2 wealth The reason is that both goods are normal under those assumptions. If $u(\cdot)$ has linear portions, examples can be constructed for which it is optimal in period 1 to "burn money": reduce the total amount of resources available in period 2.

the limit, as the costs of bringing consumption forward becomes infinitely large (complete irreversibility), they end up implementing a minimum savings rule.

3 Commitment and Welfare

The striking welfare result goes against the intuition that if individuals have a demand for commitment, satisfying it should make them better off. The finding is not specific to illiquid assets, but rather a result of the availability of commitment devices. In this section, I introduce production and an abstract commitment technology which allows individuals to implement period-1 optimal plans; the use of this technology is endogenously determined. Individual preferences are as before, but $\delta = 1$ with probability 1: This is a deterministic preference-for-commitment economy. Consumption takes place in periods 2 and 3, and agents are more impatient in period 2 than in period 1.

The illiquid assets pricing model specified an exogenous aggregate endowment of goods. A production economy is studied next, in which the time-profile of aggregate output is fully endogenous (an endowment economy is obtained as a limit case). There is an initial endowment of period-2 goods. These can be either consumed in period 2 or invested in a technology transforming them into period-3 goods. In a production economy, providing commitment to a fraction of the population will change the time profile of aggregate output, as committed agents save more. A central authority concerned with insufficient aggregate savings due to changing tastes might consider providing commitment possibilities to the population, in order to streamline the goods endowment time path.

Given that trading of illiquid assets induced an equilibrium welfare loss, a natural question is whether endogeneizing the availability of commitment devices could eliminate this inefficient trading. To address this issue, I introduce a commitment technology which need not be used in equilibrium.

In the model's unique equilibrium, agents make use of the commitment technology. However, for a wide range of parameter values, utility is strictly lower when the technology is available than when it is not. Thus, the negative welfare result from the availability of commitment is robust to the introduction of aggregate production. More intriguingly, a welfare-decreasing use of the commitment technology arises in equilibrium. Once commitment devices exist, the economy is led to an equilibrium in which they are used, even when this is welfare-decreasing.

There is a clash between individuals' desire for commitment and the aggregate effects from facilitating it. An individual's choice to make use of a commitment technology is a ceteris paribus decision. No matter what the inter-temporal prices are, she is always willing to pay a premium to avoid time-inconsistency. In spite of the negative effects from satisfying it, demand for commitment persists, and this is why it is misleading as an indicator of equilibrium welfare. When a market for commitment is open, individuals cannot avoid trading themselves to a welfare-dominated equilibrium.

3.1 The model

A unit-measure continuum of ex-ante identical agents lives for three periods. Consumption takes place in periods 2 and 3, while period 1 is an ex-ante, "temptation-free" stage in which welfare evaluations of future consumption profiles are made. Each agent's period-1 and period-2 preferences over a consumption pair (c_2, c_3) are, respectively, represented by

$$
\begin{aligned}
u_1(c_2, c_3) &= \log c_2 + \log c_3, \\
u_2(c_2, c_3) &= \log c_2 + \beta \log c_3,
\end{aligned}
$$

where $\beta < 1$. An interpretation is that period-2 consumption is a tempting good.

Trading for period-2 and period-3 consumption goods takes place in period 2. Each agent holds the representative wealth w, the price of period-2 goods is normalized to 1, and the price of period-3 goods is denoted q.

A self-control technology allowing individuals to overcome their preference changes exists. In period 1, each agent decides whether to use it or not; if she does, she incurs a period-1 utility cost γ. Specifically, if an agent consumes (c_2, c_3) and uses the self-control technology, her period-1 utility becomes

$$v_1(c_2, c_3) = u_1(c_2, c_3) - \gamma.$$

Let x be an indicator variable which takes on value 1 if and only if the individual uses the self-control technology, 0 otherwise. Then each person's period-1 decision problem is the following:

$$
\begin{aligned}
V = \max_{x \in \{0,1\}} \quad & \log c_2 + \log c_3 - x\gamma \\
s.t. \quad & (c_2, c_3) \arg\max_{c_2, c_3} \{\log c_2 + (x + (1 - x)\beta) \log c_3\} \\
& c_2 + qc_3 \leq w
\end{aligned}
\tag{9}
$$

If the person does not use the self-control technology ($x = 0$), she avoids the utility cost. However, because of the preference change, the implemented consumption profile must be $u_2(c_2, c_3)$-optimal. Instead, if the technology is used ($x = 1$), the implemented consumption profile will maximize $u_1(c_2, c_3)$. The interpretation is that the individual anticipates yielding to the temptation to overconsume in period 2. By incurring the utility cost γ, the person avoids this temptation.

Production

Each agent is initially endowed with one unit of period-2 goods, which can be consumed or sold to firms. A unit measure of firms indexed by $j \in J \equiv [0, 1]$ operates a production technology transforming k units of period-2 goods into k^θ units of period-3 goods, $\theta \in [0, 1]$. Notice that when $\theta = 0$, this is an endowment economy with a unitary endowment of goods in each period.

The commitment technology

I assume that the commitment technology has decreasing returns to scale, so that the cost to each individual is increasing in the measure of agents who make use of it. Self-control devices are a scarce resource and individuals impose a crowding externality on each other.

Let $\kappa > 0$ and $\lambda \equiv \int_i x^i di$ be the measure of individuals using the technology. Then the utility cost to each one of the agents using the technology is $\gamma = \kappa\lambda$. As more agents acquire the commitment services, the cost of providing them to each individual person increases.

Define

$$\overline{\lambda} \equiv \frac{1}{\kappa} \ln \frac{(1+\beta)^2}{4\beta}.$$

$\overline{\lambda}$ measures the efficiency of the commitment sector. When $\overline{\lambda} = 0$, acquiring commitment is infinitely costly – hence, not available. As $\overline{\lambda}$ increases, it becomes cheaper to provide it to each person. When $\overline{\lambda} \to \infty$, the commitment technology is free. An example is a personal finance workshop aimed at training the agents to save more, where the larger the crowd, the smaller the attention the speaker can pay to each attendee. For a broader interpretation, imagine a consulting group offering contracts whereby individuals delegate their inter-temporal wealth management for a fee.

Finally, all consumers and all firms are price-takers in this economy. The competitive equilibrium requires that everyone pursues optimal plans given price incentives, and that markets clear.

Definition 2 *An equilibrium in this economy is a price q together with a set $\left(x^i, c_2^i, c_3^i\right)_{i \in I}$ of individual decisions and a set of firm input decisions $\left(k_j\right)_{j \in J}$ such that*

1. *For each i, $\left(x^i, c_2^i, c_3^i\right)$ solves problem (9) given q.*
2. *For each firm j, k_j is optimal given q.*
3. *Markets clear:*

$$\int_i c_2^i di + \int_j k^j dj = 1, \quad \int_i c_3^i di = \int_j k_j^\theta dj.$$

3.2 Equilibrium welfare

The fraction of individuals using the commitment technology in equilibrium is endogenous, and depends on the efficiency parameter $\overline{\lambda}$. Agents will make use of the available commitment devices even if doing so decreases everyone's equilibrium welfare. Commitment is individually valuable, ceteris paribus, but its availability is harmful unless it is sufficiently inexpensive.

Proposition 10 *There is a unique equilibrium in this economy, in which the fraction of agents using the commitment technology is $\lambda^* = \min\left\{\overline{\lambda}, 1\right\}$. For $\theta < 1$, equilibrium welfare is strictly decreasing in $\overline{\lambda}$ for $\overline{\lambda} < 1$, and strictly increasing for $\overline{\lambda} > 1$. There exists a threshold value $\overline{\overline{\lambda}} > 1$ such that when $\overline{\lambda} = \overline{\overline{\lambda}}$, period-1 utility is the same as when $\overline{\lambda} = 0$; it is strictly lower for $\overline{\lambda} \in (0, \overline{\overline{\lambda}})$, and strictly higher for $\overline{\lambda} > \overline{\overline{\lambda}}$. The threshold $\overline{\overline{\lambda}}$ is strictly decreasing in θ; it takes value 1 when $\theta = 1$, and converges to ∞ when $\theta \to 0$.*

The assumption that the individual cost of using the commitment technology is increasing in the measure of agents doing so delivers a unique equilibrium but is not essential for the welfare result. Without this assumption, there would exist multiple welfare-ranked equilibria, with utility-decreasing use of the commitment technology.

Suppose that $\overline{\lambda} < 1$. Then a fraction $\lambda^* = \overline{\lambda}$ of the population will make use of the technology and achieve self-control in equilibrium. Whenever the fraction λ of individuals using the techonology equals $\overline{\lambda}$, each agent is indifferent between $x = 0$ and $x = 1$.[9] When $\lambda < \overline{\lambda}$, the individual utility cost is low, and therefore each agent strictly prefers to make use of the self-control technology. When $\lambda > \overline{\lambda}$, instead, the individual utility cost is too large, and each agent strictly prefers to choose $x = 0$. Therefore, equilibrium requires that $\lambda^* = \overline{\lambda}$.

When the commitment technology is available, individuals voluntarily use it, revealing that it is valuable. Agents who use the commitment technology implement a period-1 optimal consumption profile. As the fraction of committed, high-saving agents grows, the aggregate production profile gets closer to the period-1 optimal one. The revealed demand for commitment and the improved aggregate output profile could suggest a positive welfare role for the commitment devices. However, this is contradicted by the actual equilibrium welfare computation: Any $0 < \overline{\lambda} < 1$ implies a welfare loss. Perversely, as the commitment technology becomes more efficient and $\overline{\lambda}$ increases towards 1, the welfare loss to everyone becomes larger.

When $\overline{\lambda} \geq 1$, the commitment technology is so inexpensive that all agents will make use of it in equilibrium. When $\overline{\lambda} > 1$, the measure of agents acquiring commitment does not change (the whole population does), but each agent faces a smaller cost from using the technology. Still, this does not guarantee a welfare-improving role for the device: For $1 \leq \overline{\lambda} < \overline{\overline{\lambda}}$, equilibrium period-1 utility is strictly lower than if no commitment devices were available. Only as $\overline{\lambda}$ surpasses $\overline{\overline{\lambda}}$ does the commitment technology result in a welfare increase.

The threshold $\overline{\overline{\lambda}}$ depends on θ, the curvature of the production function. In the limit case when $\theta = 0$ (so that this is an endowment economy), the commitment technology always generates a welfare loss, no matter how efficient it is. As the production function becomes closer to linear, the threshold $\overline{\overline{\lambda}}$ decreases towards 1. In the limit, as $\overline{\lambda}$ becomes unboundedly large, committing is free, and individuals are better off in equilibrium (as there is in fact no dynamic inconsistency at all). Cost-free commitment was a key assumption in Laibson's (1997) analysis, which concluded a positive welfare role for commitment devices in a production economy.

An intuition for why commitment lowers welfare is the following. When a fraction of the population acquires the ability to pre-commit to period-1 optimal plans, the aggregate demand function of period-3 goods shifts up. This shift is absorbed partially by an output change and partially by an increase in the relative price of period-3 goods. This price effect favors consumption of tempting period-2 goods, which worsens the prospects for consumers who do not make use of the technology. This group of individuals is thus hurt by the change in market conditions. In turn, agents who acquire commitment enjoy a better consumption profile, but this is offset by the cost of acquiring self-control. In equilibrium, they are indifferent between committing or not, which means that on net they are also made worse off by the availability of commitment devices.

[9] The fact that prices do not enter this indifference level is due to the logarithmic utility assumption. With a more general specification (for instance, CRRA utility), the indifference value γ will depend on the intertemporal price q. Multiple equilibria will be possible, with different levels of use of the commitment technology. This is studied in companion work.

The less curvature the production function has, the more the demand shift is absorbed by changes in aggregate output rather than prices; hence, the smaller the welfare loss for non-committed consumers. In the endowment economy case, all the demand change impact goes into prices. However, only in the linear limit does the welfare loss disappear.

Alternatively, consider consumption profiles (c_2, c_3) and think of an agent as being in either of two "states": committed or non-committed. Committed individuals compete for period-1-optimal (c_2, c_3) pairs. Switching people from non-committed to committed intensifies this competition. Only those who move from one state to the other experience a utility gain, while everyone else is made worse off. But these gains are eliminated, in equilibrium, by the cost of switching states.

Imagine the extreme situation in which a given individual is the only self-controlled person in an economy where everyone else is time-inconsistent. This agent will face a favorable price profile in which the future goods are cheap due to other people's excessive impatience.[10] If more agents acquired self-control, this individual would face increased competition for future goods and suffer a welfare decrease. However, even as aggregate conditions deteriorated due to more self-controlled competitors entering the market, the decision-maker would still value commitment and want to incur a cost to achieve it.

The anticipated preference change always makes individuals willing to bear a cost to pre-commit, given prices. But the welfare effects from providing commitment depend on how doing so affects equilibrium prices. This explains why self-control devices will be used in equilibrium whenever they are available, even when they should not be.

Finally, a corollary from this analysis is that if free commitment devices are offered to a fraction of the population, agents fortunate enough to benefit from the giveaway will experience a welfare increase while the rest will be made strictly worse off. Only if the production function is linear will the intervention not harm those agents who are not reached by it.

4 Conclusions

This paper proposes studying the general equilibrium welfare implications from the availability of commitment devices, and characterizes equilibrium prices and portfolio holdings when liquid and illiquid assets exist.

When tastes change over time, the link between behavior and welfare is severed as action is rationalized by a set of preferences but evaluated with another. The obvious choice-theoretic intuition is that this is true whenever the action timing differs from the reference point. This research reveals that a version of this divergence can translate to the link between choice and the preferences rationalizing it. The fact that individuals display a demand for commitment devices does not imply a welfare-improving role for them. On the contrary, a benevolent policy maker, even if tempted to do so, should refrain from satisfying this demand.

[10] Thinking of this situation in terms of split selves, an interpretation of pre-committing is that the person's early self is active in trading. Instead, if an individual is not pre-committed, her future self is the one arranging for inter-temporal trades. From trade theory, current selves benefit from trading with other people's future selves and vice-versa.

A natural next step would be to extend the analysis to allow for agent heterogeneity. In related work I show that if agents are homogeneous in β but possibly heterogeneous in all other respect, an inefficiency from introducing commitment devices persists. Whether such an interventionist policy leads to welfare worsening or improvement will depend on parameter values. The assumption that might lead to a reversal of the welfare proposition is heterogeneity in β, given that the equilibrium without commitment devices is not efficient in that case. If agents differed in their degrees of time-inconsistency, trading in commitment devices might achieve gains for all.

The analysis of illiquid assets when individuals face both changing preferences and time-inconsistency sheds light on the relation between commitment and risk. The evaluation of commitment devices hinges on the probability distribution of the states of nature in which they impose binding constraints on their future actions. Building on this intuition, an open research avenue is to study portfolio decisions when assets have different dividend processes and varying liquidity.

5 Appendix.

5.1 Proposition 1.

Proposition 1. *Equilibrium prices admit no dynamic arbitrage:* $q_{13} = q_{23}$.

Proof. Suppose $q_{13} > q_{23}$. For the assets market to clear in period 1, some agents must carry a positive amount of re-tradeable long-term liquid assets into period 2. Let $w_2^i = a_{12}^i + q_{23}a_{13}^i$ denote the period-2 value of the end-of-period-1 liquid portfolio carried by any such agent. The period-1 cost of this portfolio is $\omega_2^i \equiv a_{12}^i + q_{13}a_{13}^i$, since the period-1 price of short-term liquid assets is normalized to 1. Then acquiring $a_{12}^i = \omega_2^i$ costs the same, and yields $w_2^{i\prime} = \omega_2^i$ resources in period-2 Since $a_{13}^i > 0$, $q_{13} > q_{23}$ implies that $\omega_2^i > w_2^i$.

To see that this increases period-1 welfare, suppose that this agent was liquidity-constrained for $\delta \leq \delta^* \in \left(\underline{\delta}, \overline{\delta}\right)$. It is clear that the new cutoff $\delta^{*\prime}$ associated to $w_2^{i\prime} = \omega_2^i$ will be smaller than δ^*. Therefore for $\delta \leq \delta^{*\prime}$, the increase in ω_2^i implies an increase in period-2 consumption; while for $\delta \geq \delta^{*\prime}$, the increase implies an increase in both period-2 and period-3 consumption (since both goods are normal from the period-2 perspective). Therefore period-1 utility must increase by carrying $w_2^{i\prime} = \omega_2^i$. Even though this need not be the optimal re-assignment of period-1 resources, this guarantees that maximal period-1 utility will also increase by switching portfolios.

The implication is that when $q_{13} > q_{23}$, every agent must hold $a_{13}^i = 0$ in an optimal plan, hence aggregate demand for the re-tradeable long-term asset is null and falls short of the aggregate supply $(1-\alpha)\,\overline{c}_3 > 0$.

For $q_{13} < q_{23}$, repeat the same argument for agents who hold $a_{12}^i > 0$. Any such agent will carry an amount $w_2^i = a_{12}^i + qa_{13}^i$ of liquid wealth into period-2, for a cost of $\omega_2^i = a_{12}^i + q_{13}a_{13}^i$. Acquiring $a_{13}^i = \frac{\omega_2^i}{q_{13}}$ then costs the same and yields $w_2^{i\prime} = \frac{q_{23}}{q_{13}}\omega_2^i = \frac{q_{23}}{q_{13}}a_{12}^i + q_{23}a_{13}^i > w_2^i$. As before, these additional period-2 resources, together with the normal goods assumption implicit in strict concavity of $u\,(c)$, imply that for each δ, either $c_2\,(\delta)$, $c_3\,(\delta)$, or both, will increase, while none of the two will decrease. This implies that period-1 utility will be higher with the alternative portfolio choice, even without considering an optimal re-assignment of wealth. The conclusion is that at $q_{13} > q_{23}$, period-1 aggregate demand for short term assets will be null, thereby falling short of the aggregate supply, equal to $\overline{c}_2 > 0$. ■

5.2 Solution to the consumer problem.

In this sub-section, I lay out the full solution to the individual portfolio decision problem. In doing so, I will establish Lemma 1, and Propositions 2, 3, 4 and 5. Other characterization results which are not included in the main body of the paper are here for completeness.

5.2.1 Lemma 1.

Lemma 1. *Suppose that* $k > 0$*. For* $\delta \leq \widetilde{\delta}$*, the implemented choice is* $c_2 = w_2, c_3 = k$*; while for* $\delta > \widetilde{\delta}$*, (5) holds with equality and the implemented consumption is* $c_2 < w_2, c_3 > k$*.*

Proof. The period-2 Lagrangian function is

$$\mathcal{L}(c_2, c_3, \lambda, \mu) = u(c_2) + \beta\delta u(c_3) + \lambda\left(w_2^i + qk^i - c_2 - qc_3\right) + \mu(c_3 - k).$$

The first-order and Kuhn-Tucker conditions for this problem are

$$\begin{aligned} u'(c_2) - \lambda &= 0, \beta\delta u'(c_3) - \lambda q_{23} + \mu = 0, \\ \mu &\geq 0, \mu(c_3 - k). \end{aligned}$$

Under the assumption that $\mu = 0$, the optimal c_3 purchase is strictly increasing in δ. This monotonicity implies that for $\delta < \widetilde{\delta}$ the unconstrained-optimal c_3 purchase is smaller than k. Since this is not feasible, the constrained-optimal period-2 choice is to behave hand-to-mouth at any such δ, and

$$\frac{u'\left(w_2^i\right)}{u'(k)} > \frac{\beta\delta}{q}$$

will be true. When $\delta > \widetilde{\delta}$, instead, $\mu = 0$ yields $\frac{u'(c_2)}{u'(c_3)} = \frac{\beta\delta}{q}$ at the optimal choice. ■

Remark 1 *Under strict concavity, period-*2 *and period-*3 *consumption goods are both normal goods from the period-*2 *perspective. Let* $\omega_2^i = w_2^i + qk^i$ *denote the total amount of resources available in period* 2 *(both liquid and illiquid). Then for all* δ*, an increase in* ω_2^i *results in an increase in* $c_2(\delta)$*,* $c_3(\delta)$*, or both, and a decrease in none. Since period-*1 *utility is strictly increasing in* $c_2(\delta)$ *and* $c_3(\delta)$ *for all* δ*, the conclusion is that an increase in* ω_2^i *always results in an increase in period-*1 *utility. Strict concavity ensures that money-burning is never optimal.*

5.2.2 Existence of solution to the period-1 problem.

Proposition 11 *The correspondence of maximizers* (w_2, k) *of problem (4) is non-empty, compact-valued, and upper hemi-continuous.*

Proof. *Existence follows from the Maximum Theorem (see Stokey and Lucas,* 1989*). Clearly the period-*2 *problem always has a solution; the Theorem implies that the policy rules* $c_2(\delta)$ *and* $c_3(\delta)$ *are upper hemi-continuous in* (w_2, k, δ, q)*; strict concavity of* $u(\cdot)$ *makes them single-valued (and hence continuous). As a composition of continuous functions, so is* $u(c_2(\delta)) + \delta u(c_3(\delta))$ *for each value of* δ *(see, for instance, Apostol,* 1974*, Theorem* 4.17*). The objective function inherits continuity in* (w_2, k, q) *from being a weighted average of continuous functions (see, for instance, Apostol,* 1974*, Theorem* 7.39*). The choice set is a budget set – a non-empty, compact-valued and continuous correspondence. Therefore the conditions for the Theorem are satisfied and the proposition follows.* ■

In a sequential model with more than 3 periods, existence of the solution will hinge on choosing the right tie-breaking rule whenever more than 1 alternative are optimal in a given period (see Gul and Pesendorfer 2004, Peleg and Yaari 1973). Existence will be guaranteed, generically, by adopting a strategic approach to the problem (as in Phelps and Pollack, 1968) and imposing curvature assumptions on the utility function (see Harris and Laibson 2001*a*).

5.2.3 Characterization of optimal choice.

It is convenient to change the notation to emphasize the dependence of some values on prices and initial wealth. For a given wealth, liquid and illiquid assets prices (w, q, p), let

$$\begin{aligned} V_{FC}(w;p) &\equiv V_{FC},\ V_{FF}(w;q) \equiv V_{FF}, \\ V_{PC}(\delta, q, p) &\equiv V(\delta),\ V_{PC}(q,p) \equiv V_{PC}. \end{aligned}$$

Also, let $k(\delta; q, p)$ denote the unique value of illiquid assets k which, given δ, sets

$$\frac{u'(w-pk)}{u'(k)} = \frac{\beta\delta}{q}.$$

And finally, let $k^*(p)$ denote the unique level that satisfies $\frac{u'(w-pk)}{u'(k)} = \frac{E[\delta]}{p}$.

The solution procedure is the following. First I show that the consumer's period-1 choice is solved by three possible illiquid asset strategies: $k = 0$, $k = k^*(p)$, or $k = k(\delta; q, p)$ for $\delta \in [\underline{\delta}, \overline{\delta}]$. These asset levels do not exhaust feasible alternatives, but the optimal choice must belong to that set. Second, I derive conditions for $k = k(\delta; q, p)$ for $\delta \in [\underline{\delta}, \overline{\delta}]$ being a candidate for an optimum.

A. Period-1 choice set.

The set of feasible illiquid assets in period 1 is the interval $\left[0, \frac{w}{p}\right]$. If $\underline{\delta} > 0$, then $k(\underline{\delta}; p, q) > 0$. Under the assumption that $\lim_{c\to 0} u'(c) = \infty$, $k(\overline{\delta}; p, q) < \frac{w}{p}$ for any finite q. Let $k^*(p)$ denote the solution to the first-best deterministic consumption problem $V_{FC}(w;p)$ given price p.

Lemma 2 *If $\rho \neq 1$, the period-1 optimal $k \in \left\{ \left[k(\underline{\delta}; p, q), k(\overline{\delta}; p, q)\right] \cup \{0, k^*(p)\} \right\}$.*

Proof. *Suppose $k \in (0, k(\underline{\delta}; p, q))$. Then the period-2 choice is never liquidity-constrained. If $p > q$, more period-2 resources are achieved by setting $k = 0$, without affecting the profile of liquidity constrains. By Remark 1, this implies that period-1 utility increases. Therefore, $k \in (0, k(\underline{\delta}; p, q))$ is not optimal. If $p < q$, then more period-2 resources are achieved by setting $k = k(\underline{\delta}; p, q)$. Again, this yields an increase in available period-2 resources while still never liquidity-constraining period-2 choice. This must result in a period-1 utility increase, and optimality of $k \in (0, k(\underline{\delta}; p, q))$ is ruled out again.*

$k^*(p)$ implements the first-best deterministic consumption profile. By construction, $k(\overline{\delta}; p, q)$ also implements a deterministic consumption profile, since, at this k level, the period-2 choice is liquidity-constrained for every realized δ. However, by definition of $k^*(p)$, whenever $k(\overline{\delta}; p, q) \neq k^*(p)$, the period-1 value from implementing $k = k(\overline{\delta}; p, q)$ must be strictly smaller than $V_{FC}(w;p)$. Therefore, $\left[k(\underline{\delta}; p, q), k(\overline{\delta}; p, q)\right]$ may not exhaust the set of candidates for an optimal period-1 choice.

Specifically, when $k(\overline{\delta}; p, q) < k^*(p)$, true if and only if $\rho < \overline{\rho} \equiv \frac{E[\delta]}{\beta\overline{\delta}}$, the period-1 problem could be solved by $k = k^*(p)$. No $k \in \left(k(\overline{\delta}; p, q), k^*(p)\right)$ or $k > k^*(p)$ could ever be optimal, since any such k would implement a deterministic consumption profile, but with lower payoff than $V_{FC}(w;p)$.

Finally, let $k(\overline{\delta}; p, q) > k^*(p)$. Any $k > k(\overline{\delta}; p, q)$ would implement a deterministic consumption profile, but with excessive period-3 consumption, and therefore would be dominated by $k = k(\overline{\delta}; p, q)$. ■

Choosing $k \in (0, k(\underline{\delta}; p, q))$ would be like "burning money", which is never optimal due to the strict concavity plus time separability assumptions.

In the borderline case $\rho = 1$, all asset levels $k \in [0, k(\underline{\delta}; p, q)]$ are indistinguishable, since they imply exactly the same implemented period-2 policy rules, and hence the same period-1 value $V_{FF}(w; q)$. The period-1 problem, can be thought of as only including $k = 0$ as an alternative, without loss of generality given the continuum of consumers.

Notice that if $k^*(p) \in \left(k(\underline{\delta}; p, q), k(\overline{\delta}; p, q)\right)$, then $k = k^*(p)$ would not yield period-1 expected utility of $V_{FC}(w; p)$. The reason is that if $k^*(p)$ units of illiquid assets were acquired in period 1, there would exist, by construction, a level $\widehat{\delta} \in \left[\underline{\delta}, \overline{\delta}\right]$ such that the period-2 choice is liquidity constrained only for $\delta < \widehat{\delta}$. But then for $\delta > \widehat{\delta}$, the period-2 optimal decision would involve consuming less than total liquid wealth in period 2, and more than $k^*(p)$ in period 3. The implemented consumption profile would thus depend on δ, and therefore would be stochastic. Hence the period-1 utility from $k = k^*(p)$ would differ from $V_{FC}(w; p)$, which presupposes deterministic consumption.

B. Candidate solutions as a function of prices.

The next step is to link the set of solutions to prices. Let $\overline{\rho} \equiv \frac{E[\delta]}{\overline{\delta}\beta}$.

Proposition 12 *The set of candidates for solutions to the period-1 portfolio problem depends on prices p and q as follows:*

1. *If $\overline{\rho} < 1$, then any optimal k satisfies*

$$\begin{array}{ll} k \in \{k(\delta^*) : \delta^* \text{ solves } V_{PC}\} \text{ or } k = k^* & \text{if } \frac{p}{q} < \overline{\rho} \\ k \in \{k(\delta^*) : \delta^* \text{ solves } V_{PC}\} & \text{if } \overline{\rho} \leq \frac{p}{q} < 1 \\ k = 0 \text{ or } k \in \{k(\delta^*) : \delta^* \text{ solves } V_{PC}\} & \text{if } 1 < \frac{p}{q} \end{array}$$

2. *If $\overline{\rho} > 1$, then any optimal k satisfies*

$$\begin{array}{ll} k \in \{k(\delta^*) : \delta^* \text{ solves } V_{PC}\} \text{ or } k = k^* & \text{if } \frac{p}{q} < 1 \\ k = 0,\ k \in \{k(\delta^*) : \delta^* \text{ solves } V_{PC}\} \text{ or } k = k^* & \text{if } 1 < \frac{p}{q} \leq \overline{\rho} \\ k = 0 \text{ or } k \in \{k(\delta^*) : \delta^* \text{ solves } V_{PC}\} & \text{if } \overline{\rho} < \frac{p}{q} \end{array}$$

Proof. First notice that when $\frac{p}{q} \leq 1$, the value $V_{FF}(w; q)$ from implementing $k = 0$ is not relevant for the description of the period-1 problem. The reason is that at this price configuration, $k = k(\underline{\delta}; p, q)$ yields either a strict increase in period-2 resources (if $p < q$) or no change at all ($\rho = 1$), without affecting the profile of period-2 liquidity constraints. Hence $V_{FF}(w; q) \leq V^i_{PC}(w; p, q)$.

If $\overline{\rho} < 1$, there are three relevant cases.

1. When $\frac{p}{q} < \overline{\rho}$, $k^*(p) > k\left(\overline{\delta}; p, q\right)$; therefore implementing $k = k^*(p)$ attains period-1 value $V_{FC}(w; p)$, which may be higher or lower than $V^i_{PC}(w; p, q)$. On the other hand, since $p < q$, $k = 0$ could never be an optimal choice: Implementing $k = k(\underline{\delta}; p, q)$ yields an increase in period-2 resources without altering the set of period-2 liquidity constraints; and such an increase is period-1 valuable.

2. If $\overline{\rho} \leq \frac{p}{q} \leq 1$, $V^i_{PC}(w; p, q)$ must be the period-1 value function. First, because $k^*(p) < k\left(\overline{\delta}; p, q\right)$, implementing $k = k^*(p)$ will not attain $V_{FC}(w; p)$, and second, because $V_{FF}(w; q)$ is not relevant when $\frac{p}{q} \leq 1$, as discussed above.

3. When $\frac{p}{q} > 1 > \overline{\rho}$, $V_{FC}(w; p)$ is not relevant for the same reasons as (2). But now implementing $k = k(\underline{\delta}; p, q)$ yields strictly lower value than $k = 0$, because the latter achieves a higher level of period-2 resources than the former, while both imply the same profile of liquidity constraints. Therefore $V_{FF}(w; q) \leq V^i_{PC}(w; p, q)$ is no longer guaranteed.

If $\overline{\rho} > 1$, there are three relevant cases.

1. When $\frac{p}{q} \leq 1$, $k^*(p) > k\left(\overline{\delta}; p, q\right)$; hence $V_{FC}(w; p)$ is a candidate for period-1 utility, achieved by $k = k^*(p)$; but $V_{FF}(w; q)$ is not relevant.

2. When $1 < \frac{p}{q} < \overline{\rho}$, the first inequality implies that $V_{FF}(w; q) \leq V^i_{PC}(w; p, q)$ need not be true, and therefore $V_{FF}(w; q)$ is a candidate value function. The second inequality implies that $V_{FC}(w; p)$ also is.

3. When $\frac{p}{q} \geq \overline{\rho} > 1$, $k^*(p) < k\left(\overline{\delta}; p, q\right)$ implies that $V_{FC}(w; p)$ is no longer relevant, since it will not be the period-1 expected value from implementing $k = k^*(p)$. On the other hand, $\frac{p}{q} > 1$ implies that $V_{FF}(w; q)$ is still a candidate value.

■

When $p = q$, the consumer is indifferent between any value $k \in [0, k(\underline{\delta})]$. Any such value leaves the period-2 choice unconstrained for any realization of δ, and has no effect on the amount of resources passed on.

Notice that, by construction, program V_{PC} only implies illiquid asset values on the interval $\left[k(\underline{\delta}), k\left(\overline{\delta}\right)\right]$. These are the set of choices that will liquidity-constraint the period-2 decision for $\delta \leq \widetilde{\delta}$, given a target $\widetilde{\delta} \in \left[\underline{\delta}, \overline{\delta}\right]$. But this need not coincide with the set of feasible choices. From the period-1 budget constraint, the feasible set of illiquid asset purchases is $[0, w/p]$.

C. Solution to the partial commitment problem.

To characterize the solution to V_{PC} we compute the first order conditions associated to that program.

Proposition 13 *The first order condition of program V_{PC} is satisfied at $\delta^* \in \left[\underline{\delta}, \overline{\delta}\right]$ if and only if*

$$\psi(\delta^*) + (1 - \rho)\left(\beta F(\delta^*) + \int_{\delta^*}^{\overline{\delta}} \mu(\beta, \delta; q) \frac{u'(c_2(\delta))}{u'(c_2(\delta^*))} f(\delta)\, d\delta\right) = 0, \qquad (10)$$

where $\mu(\beta, \delta; q) \equiv \frac{q^2 u''(c_2(\delta)) + \beta^2 \delta u''(c_3(\delta))}{\beta q^2 u''(c_2(\delta)) + \beta^2 \delta u''(c_3(\delta))} > 1$ and $c_2(\delta)$ solves problem (3).

Proof. The objective function is:

$$V(\delta^*; p, q) \equiv u(w - pk(\delta^*; p, q)) \cdot F(\delta^*) + \int_{\underline{\delta}}^{\delta^*} \delta f(\delta)\, d\delta \cdot u(k(\delta^*; p, q))$$
$$+ \int_{\delta^*}^{\overline{\delta}} (u(c_2(\delta)) + \delta u(c_3(\delta))) f(\delta)\, d\delta.$$

where, $c_2(\delta)$ and $c_3(\delta)$ solve problem (3) for each δ.

Notice that, by construction, $k(\delta^*; p, q) = c_3(\delta^*)$, and $w - pk(\delta^*; p, q) = c_2(\delta^*)$. Therefore, the first term in the objective function can be re-written in terms of $c_2(\delta^*)$ and $c_3(\delta^*)$. Using this, the first order condition is the derivative with respect to δ^* :

$$\frac{\partial}{\partial \delta^*} \{u(c_2(\delta^*)) F(\delta^*) + E[\delta | \delta \leq \delta^*] u(c_3(\delta^*)) F(\delta^*)\} +$$
$$+ \frac{\partial}{\partial \delta^*} \left\{ \int_{\delta^*}^{\overline{\delta}} (u(c_2(\delta)) + \delta u(c_3(\delta))) f(\delta)\, d\delta \right\} = 0.$$

Let $\omega_2 \equiv w + (q - p) k(\delta^*)$ denote the market value of period-2 resources, imputing a price q to illiquid assets. When $\delta > \delta^*$, the consumer acts as if total available resources were ω_2, ignoring the fact that some of this is in the form of illiquid assets. Using $c_3(\delta^*) = k(\delta^*)$, $\omega_2 = w + (q - p) c_3(\delta^*)$.

$$\frac{\partial \omega_2}{\partial \delta^*} = (q - p) \frac{\partial c_3(\delta^*)}{\partial \delta^*}.$$

The derivative of the integrand term is

$$\frac{\partial (u(c_2(\delta)) + \delta u(c_3(\delta)))}{\partial \delta^*} = u'(c_2(\delta)) \frac{\partial c_2(\delta)}{\partial \delta^*} + \delta u'(c_3(\delta)) \frac{\partial c_3(\delta)}{\partial \delta^*}. \tag{11}$$

Because $\delta > \delta^*$, $c_2(\delta)$ and $c_3(\delta)$ only depend on δ^* through its effect on total resources ω_2. In other words, conditional on $\delta > \delta^*$, acquiring illiquid assets only generates a capital effect – a capital gain if $q > p$, and a loss otherwise. This fact implies that the chain rule can be applied on derivative (11) :

$$\begin{aligned} \frac{\partial (u(c_2(\delta)) + \delta u(c_3(\delta)))}{\partial \delta^*} &= \left(u'(c_2(\delta)) \frac{\partial c_2(\delta)}{\partial \omega_2} + \delta u'(c_3(\delta)) \frac{\partial c_3(\delta)}{\partial \omega_2} \right) \frac{\partial \omega_2}{\partial \delta^*} \\ &= \left(u'(c_2(\delta)) \frac{\partial c_2(\delta)}{\partial \omega_2} + \delta u'(c_3(\delta)) \frac{\partial c_3(\delta)}{\partial \omega_2} \right) (q - p) \frac{\partial c_3(\delta^*)}{\partial \delta^*}. \end{aligned}$$

Since the Euler equation holds with equality at $\delta > \delta^*$, $\frac{\partial c_2(\delta)}{\partial \omega}$ and $\frac{\partial c_3(\delta)}{\partial \omega}$ can be backed up by the implicit function formula, to get

$$\frac{\partial c_2(\delta)}{\partial \omega_2} = \frac{\beta \delta u''(c_3)}{q u''(c_2) + \beta \delta u''(c_3)}, \frac{\partial c_3(\delta)}{\partial \omega_2} = \frac{q u''(c_2)}{q u''(c_2) + \beta \delta u''(c_3)}.$$

Plugging into (11),

$$\frac{\partial (u(c_2(\delta)) + \delta u(c_3(\delta)))}{\partial \delta^*} = \frac{\beta \delta^* u''(c_3(\delta)) u'(c_2(\delta)) + \delta q_{23} u''(c_2(\delta)) u'(c_3(\delta))}{q u''(c_2(\delta)) + \beta \delta^* u''(c_3(\delta))} (q - p) \frac{\partial c_3(\delta^*)}{\partial \delta^*}.$$

Using the Euler equation to replace for $\frac{u'(c_2(\delta))}{u'(c_3(\delta))}$, and simplifying terms, this becomes

$$\frac{\partial(u(c_2(\delta))+\delta u(c_3(\delta)))}{\partial\delta^*} = \frac{qu''(c_2(\delta))+\beta^2\delta u''(c_3(\delta))}{qu''(c_2(\delta))+\beta\delta u''(c_3(\delta))}\frac{1-\rho}{\beta}u'\left(c_2\left(\delta\right)\right)q\frac{\partial c_3(\delta^*)}{\partial\delta^*}.$$

This result, applied to the first order conditions, yields

$$u'\left(c_2\left(\delta^*\right)\right)F\left(\delta^*\right)\frac{\partial c_2\left(\delta^*\right)}{\partial\delta^*}+u'\left(c_3\left(\delta^*\right)\right)E\left[\delta\middle|\,\delta\leq\delta^*\right]F\left(\delta^*\right)\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}$$
$$+\frac{1-\rho}{\beta}q\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}\int_{\delta^*}^{\overline{\delta}}\frac{qu''\left(c_2\left(\delta\right)\right)+\beta^2\delta u''\left(c_3\left(\delta\right)\right)}{qu''\left(c_2\left(\delta\right)\right)+\beta\delta u''\left(c_3\left(\delta\right)\right)}u'\left(c_2\left(\delta\right)\right)f\left(\delta\right)d\delta=0.$$

Using the implicit function formula on the Euler equation, $\frac{\partial c_2(\delta^*)}{\partial\delta^*}=\frac{\partial c_2(\delta^*)}{\partial c_3(\delta^*)}\frac{\partial c_3(\delta^*)}{\partial\delta^*}$. Using this,

$$u'\left(c_2\left(\delta^*\right)\right)F\left(\delta^*\right)\frac{\partial c_2\left(\delta^*\right)}{\partial c_3\left(\delta^*\right)}\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}+u'\left(c_3\left(\delta^*\right)\right)E\left[\delta\middle|\,\delta\leq\delta^*\right]F\left(\delta^*\right)\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}$$
$$+\frac{1-\rho}{\beta}q\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}\int_{\delta^*}^{\overline{\delta}}\frac{qu''\left(c_2\left(\delta\right)\right)+\beta^2\delta u''\left(c_3\left(\delta\right)\right)}{qu''\left(c_2\left(\delta\right)\right)+\beta\delta u''\left(c_3\left(\delta\right)\right)}u'\left(c_2\left(\delta\right)\right)f\left(\delta\right)d\delta=0.$$

From the period-1 budget constraint, and by definition of $c_2\left(\delta^*\right), c_3\left(\delta^*\right)$,

$$c_2\left(\delta^*\right)+pc_3\left(\delta^*\right)=w\Rightarrow\frac{\partial c_2\left(\delta^*\right)}{\partial c_3\left(\delta^*\right)}=-p.$$

With this, and re-arranging,

$$\left(E\left[\delta\middle|\,\delta\leq\delta^*\right]-p\frac{u'\left(c_2\left(\delta^*\right)\right)}{u'\left(c_3\left(\delta^*\right)\right)}\right)u'\left(c_3\left(\delta^*\right)\right)F\left(\delta^*\right)\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}$$
$$+\frac{1-\rho}{\beta}q\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}\int_{\delta^*}^{1}\frac{qu''\left(c_2\left(\delta\right)\right)+\beta^2\delta u''\left(c_3\left(\delta\right)\right)}{qu''\left(c_2\left(\delta\right)\right)+\beta\delta u''\left(c_3\left(\delta\right)\right)}u'\left(c_2\left(\delta\right)\right)f\left(\delta\right)d\delta=0.$$

Which becomes, with some algebra,

$$\left(\frac{E\left[\delta\middle|\,\delta\leq\delta^*\right]}{\delta^*}-\rho\beta\right)u'\left(c_3\left(\delta^*\right)\right)\delta^*F\left(\delta^*\right)\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}$$
$$+\frac{1-\rho}{\beta}q\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}\int_{\delta^*}^{\overline{\delta}}\frac{qu''\left(c_2\left(\delta\right)\right)+\beta^2\delta u''\left(c_3\left(\delta\right)\right)}{qu''\left(c_2\left(\delta\right)\right)+\beta\delta u''\left(c_3\left(\delta\right)\right)}u'\left(c_2\left(\delta\right)\right)f\left(\delta\right)d\delta=0.$$

The positive common factor $\frac{\partial c_3(\delta^*)}{\partial\delta^*}\delta^*u'\left(c_3\left(\delta^*\right)\right)$ can be taken out, so that the first order conditions are

$$\frac{\partial c_3\left(\delta^*\right)}{\partial\delta^*}\delta^*u'\left(c_3\left(\delta^*\right)\right)\left(\left(\frac{E\left[\delta\middle|\,\delta\leq\delta^*\right]}{\delta^*}-\rho\beta\right)F\left(\delta^*\right)\right.$$
$$\left.+\left(1-\rho\right)\frac{q}{\beta\delta^*}\int_{\delta^*}^{\overline{\delta}}\frac{qu''\left(c_2\left(\delta\right)\right)+\beta^2\delta u''\left(c_3\left(\delta\right)\right)}{qu''\left(c_2\left(\delta\right)\right)+\beta\delta u''\left(c_3\left(\delta\right)\right)}\frac{u'\left(c_2\left(\delta\right)\right)}{u'\left(c_3\left(\delta^*\right)\right)}f\left(\delta\right)d\delta\right)=0.$$

Finally, since $\frac{u'(c_2(\delta^*))}{u'(c_3(\delta^*))} = \frac{\beta\delta^*}{q}$, we get

$$\frac{\partial c_3(\delta^*)}{\partial \delta^*}\delta^* u'(c_3(\delta^*))\Bigg(\psi(\delta^*) + (1-\rho)\left(\beta F(\delta^*) + \int_{\delta^*}^{\overline{\delta}} \frac{qu''(c_2(\delta)) + \beta^2\delta u''(c_3(\delta))}{qu''(c_2(\delta)) + \beta\delta u''(c_3(\delta))}\frac{u'(c_2(\delta))}{u'(c_2(\delta^*))} f(\delta)\,d\delta\right)\Bigg) = 0.$$

Or, letting $\mu(\beta,\delta;q) \equiv \frac{qu''(c_2(\delta))+\beta^2\delta u''(c_3(\delta))}{\beta q_{23}^2 u''(c_2(\delta))+\beta^2\delta u''(c_3(\delta))} > 1$, define

$$\begin{aligned}
\zeta(\delta^*) &\equiv \frac{\partial c_3(\delta^*)}{\partial \delta^*}\delta^* u'(c_3(\delta^*)), \\
P(\delta^*) &\equiv \int_{\delta^*}^{1} \mu(\beta,\delta;q)\frac{u'(c_2(\delta))}{u'(c_2(\delta^*))} f(\delta)\,d\delta, \\
\xi(\delta^*) &\equiv \psi(\delta^*) + (1-\rho)(\beta F(\delta^*) + P(\delta^*)).
\end{aligned} \tag{12}$$

And the first order conditions require that the product of $\zeta(\delta^*)$ with $\xi(\delta^*)$ be null:

$$\zeta(\delta^*) \cdot \xi(\delta^*) = 0. \tag{13}$$

But $\zeta(\delta^*) > 0$. Clearly, δ^* and $u'(c_3(\delta^*))$ are strictly positive. $\frac{\partial c_3(\delta^*)}{\partial \delta^*}$ also is, since $c_3(\delta^*) = k(\delta^*)$ is strictly increasing in δ^*. Therefore the first order conditions are satisfied if and only if $\xi(\delta^*) = 0$. This establishes the Proposition. ■

Corollary 1 *An interior optimum of* (8) *must involve*

$$E[\delta|\,\delta \le \delta^*] - \beta\delta^* > 0 \Leftrightarrow p > q.$$

To see this, suppose $p < q$, then the second term in $\xi(\delta^*)$ is positive; hence $\xi(\delta^*) = 0$ can only be satisfied if the first one is negative:

$$\frac{E[\delta|\,\delta \le \delta^*]}{\delta^*} < \beta. \tag{14}$$

Conversely, if $p > q$, then the second term will be negative; $\xi(\delta^*) = 0$ requires the first term to be positive, and the sign of (14) is reversed.

The next step is to characterize second order conditions. Program V_{PC} need not be a quasiconcave program, and first order conditions need not identify even local maxima. The next Proposition shows that if at a critical point δ^* associated to $p > q$, $\psi(\delta^*)$ is increasing in δ^*, the first order conditions will identify a local minimum, rather than maximum. To see why, notice that a sufficient condition for $\psi(\delta^*)$ to be increasing is that $\frac{E[\delta|\delta\le\delta^*]}{\delta^*}$ is; suppose this was true. If the threshold δ^* was increased by a small $\varepsilon > 0$, the wedge between period-1 desired and period-2 implemented marginal utility ratios would become larger, hence the self-control concern, too. This implies that if illiquid assets were desirable at δ^*, they would be even more so at $\delta^* + \varepsilon$. But then δ^* could not be a utility-maximizing cutoff value given price ρ; it must be a local minimizer.

Lemma 3 *When $p < q$, $(1-\rho)(\beta F(\delta^*) + P(\delta^*))$ is positive and strictly decreasing in δ^*; while $p > q$ implies that it is negative and strictly increasing.*

Proof. *This follows from the fact that $\beta F(\delta^*) + P(\delta^*)$ is positive and strictly decreasing in δ^*:*

$$\frac{\partial}{\partial \delta^*}\{\beta F(\delta^*) + P(\delta^*)\} = -\frac{u''(c_2(\delta^*))}{u'(c_2(\delta^*))}\frac{\partial c_2(\delta^*)}{\partial \delta^*}P(\delta^*) - (\mu(\beta, \delta^*; q) - \beta) f(\delta^*) < 0.$$

Because $\frac{\partial c_2(\delta^*)}{\partial \delta^*} < 0$ and $u''(c_2(\delta^*)) < 0$, while all other terms are positive, the first term is negative. The second one also is, since $f(\delta^*) > 0$ and $\mu(\beta, \delta^*; q) > 1 > \beta$. The Lemma then follows from combining this with the sign of $(1-\rho)$. ∎

Proposition 14 *Let δ^* be a critical point of program V_{PC}.*

Suppose $p < q$. If $\psi(\delta^*)$ is decreasing in δ^*, δ^* is a local maximum.

Suppose $p > q$. If $\psi(\delta^*)$ is increasing in δ^*, δ^* is a local minimum.

Proof. The second derivative of the left-hand side of (13) with respect to δ^* is

$$\frac{\partial}{\partial \delta^*}\{\zeta(\delta^*) \cdot \xi(\delta^*)\} = \xi(\delta^*)\frac{\partial \zeta(\delta^*)}{\partial \delta^*} + \zeta(\delta^*)\frac{\partial \xi(\delta^*)}{\partial \delta^*} = \zeta(\delta^*)\frac{\partial \xi(\delta^*)}{\partial \delta^*}.$$

The last equality uses the fact that the first order condition is exactly $\xi(\delta^*) = 0$. Since $\zeta(\delta^*) > 0$, the sign of this derivative is determined by the sign of $\frac{\partial \xi(\delta^*)}{\partial \delta^*}$.

The Proposition then follows immediately from Lemma 3. $\psi(\delta^*)$ is the first term in (12), while $(1-\rho)(\beta F(\delta^*) + P(\delta^*))$ is the second term. When $p < q$, the latter is decreasing in δ^*. If, at a critical point δ^*, the former also is, then $\xi(\delta^*)$ will be locally decreasing around that critical point, and hence this will identify a local maximum of the objective function.

Conversely, when $p > q$ the second term in (12) is strictly increasing, by the previous lemma. So if the first one also is at the critical point δ^*, this will be a local minimum of the objective function. ∎

5.3 Proposition 2.

Proposition 2. *Under Assumption C, for each q a unique threshold $\widetilde{p}$ exists. When the price p of illiquid assets verifies $p \leq \widetilde{p}$, full commitment is optimal, while if $p \geq \widetilde{p}$, full flexibility is. $\widetilde{p}$ is strictly larger than q, increasing in q and decreasing in β.*

Proof. The proof of this proposition proceeds by iteratively eliminating candidate solutions to the consumer's period-1 problem.

Lemma 4 *$k \in \left(k(\underline{\delta}; p, q), k(\overline{\delta}; p, q)\right)$ cannot be a solution to the period-1 problem.*

Proof. *Suppose it was. This is equivalent to stating that program $V^i_{PC}(w; p, q)$ has interior solution $\delta^* \in \left(\underline{\delta}, \overline{\delta}\right)$; such an interior optimum must perforce be a critical point. By Assumption C, $\psi(\delta^*)$ is positive. From Corollary 1, this implies that δ^* can be a critical point only if accompanied by $p > q$. By Assumption c, $\psi(\delta^*)$ is increasing. By Proposition 15, the last two facts imply that δ^* is a local minimum, and therefore not optimal.* ∎

This reduces the set of solution candidates to $k \in \{0, k^*(p), k(\underline{\delta}; p, q), k(\overline{\delta}; p, q)\}$. The next step is to rule out $k(\overline{\delta}; p, q)$; we can do so if and only if $k^*(p) > k(\overline{\delta}; p, q)$. To see this, suppose that the inequality was true. Then $k^*(p)$ would liquidity-constrain the period-2 choice for all realizations of δ; thereby indeed implementing a deterministic consumption profile. $k(\overline{\delta}; p, q)$ also implements a deterministic consumption profile. Of the two, $k^*(p)$ is optimal by definition.

Conversely, suppose that $k^*(p) < k(\overline{\delta}; p, q)$. Then $k^*(p)$ would fail to implement a deterministic consumption profile, and it would hence cease to be relevant for the solution of the consumer's problem.

For the proof it is convenient to define the price $\widehat{p}(q)$ such that $V_{PC}(\underline{\delta}; \widehat{p}(q), q) = V_{PC}(\overline{\delta}; \widehat{p}(q), q)$. It is also useful to recall that $\widetilde{p}(q)$ is the unique price that sets $V_{FC}(w; p) = V_{FF}(w; q)$, given q, and that $\overline{\rho} \equiv \frac{E[\delta]}{\overline{\delta}\beta}$. Finally, Assumption C states that $\psi(\delta^*)$ is positive and increasing, which yields $\frac{E[\delta]}{\overline{\delta}} > \beta$. Notice this implies $\overline{\rho} > 1$.

Lemma 5 $p < q \Rightarrow V_{PC}^i(w; p, q) = V_{PC}(\overline{\delta}; p, q) > V_{PC}(\underline{\delta}; p, q)$.

Proof. *Under Assumption C, the derivative $\xi(\delta^*)$ is easily verified to be strictly positive at all δ^* when $p < q$. Therefore the optimal choice for program $V_{PC}^i(w; p, q)$ is $k(\overline{\delta}; p, q)$.* ■

Lemma 6 $\rho > \overline{\rho} \Rightarrow V_{PC}^i(w; p, q) = V_{PC}(\overline{\delta}; p, q) > V_{PC}(\overline{\delta}; p, q)$.

Proof. *Consider the objective function of program $V_{PC}^i(w; p, q)$ and evaluate its first derivative at $\delta^* = \overline{\delta}$. The sign of this derivative is given by the sign of*

$$\xi(\overline{\delta}) = \frac{E[\delta]}{\overline{\delta}} - \beta + (1 - \rho)\beta = \frac{E[\delta]}{\overline{\delta}} - \rho\beta.$$

When $\rho > \overline{\rho}$, this derivative is negative and $\overline{\delta} = 1$ can not be optimal; undercutting to $\delta^* = \overline{\delta} - \varepsilon$ improves period-1 expected utility. Since under F this program always has a corner solution, we conclude that $\underline{\delta} = 0$ is optimal, and $V_{PC}^i(w; p, q) = V_{PC}(\underline{\delta}; p, q) > V_{PC}(\overline{\delta}; p, q)$. ■

Corollary. $1 < \frac{\widehat{p}(q)}{q} < \overline{\rho}$.

Lemma 7 $\frac{\widetilde{p}(q)}{q} \leq \overline{\rho}$.

Proof. *To obtain a contradiction, I show that $\frac{\widetilde{p}(q)}{q} > \overline{\rho}$ implies $\frac{\widehat{p}(q)}{q} > \overline{\rho}$. Since the latter was already proven false, the contradiction follows.*

So let $\frac{\widetilde{p}(q)}{q} > \overline{\rho}$ and notice that $\frac{\widetilde{p}(q)}{q} > \overline{\rho}$ means that at the price ratio $\frac{p}{q} = \overline{\rho} > 1$,

$$V_{FC}(w; q\overline{\rho}) > V_{FC}(w; \widetilde{p}(q)) = V_{FF}(w; q) > V_{PC}(\underline{\delta}; q\overline{\rho}, q). \quad (15)$$

The first inequality follows from the fact that $V_{FC}(w; p)$ is decreasing in p, and the assumption $\frac{\widetilde{p}(q)}{q} > \overline{\rho}$. The last inequality follows from the fact that $V_{FF}(w; q)$ and $V_{PC}(\underline{\delta}; q\overline{\rho}, q)$ both presuppose no commitment role for illiquid assets, since the period-2 choice is never liquidity-constrained. At $p > q$, carrying $k = k(\underline{\delta}; p, q)$ thus yields a gratuitous decrease in period-2 available resources, and hence a period-1 welfare loss.

Notice next that at price $\rho = \overline{\rho}$, $k^*(p) = k\left(\overline{\delta}; p, q\right)$; hence

$$V_{PC}\left(\overline{\delta}; q\overline{\rho}, q\right) = V_{FC}\left(w; q\overline{\rho}\right).$$

Combining this with the string of inequalities (15), we conclude that at $\rho = \overline{\rho}$,

$$V_{PC}\left(\overline{\delta}; q\overline{\rho}, q\right) > V_{FF}\left(w; q\right) > V_{PC}\left(\underline{\delta}; q\overline{\rho}, q\right).$$

Fix q. Decreasing p to any value $p \in [q, q\overline{\rho}]$ would only increase $V_{PC}\left(\overline{\delta}; p, q\right)$ while keeping $V_{FF}(w; q)$ unchanged; and while keeping the inequality $V_{FF}(w; q) \geq V_{PC}\left(\underline{\delta}; p, q\right)$ still true. Hence $\widehat{p}(q)$ can not lie in the interval $[q, q\overline{\rho}]$. But $\widehat{p}(q) > q$ was shown above. And yields the absurd conclusion that $\frac{\widehat{p}(q)}{q} > \overline{\rho}$, as desired. ■

Lemma 8 $1 \leq \frac{\widetilde{p}(q)}{q}$.

Proof. *Suppose not. Then the following chain of inequalities is true:*

$$V_{FC}\left(w; \widetilde{p}(q)\right) = V_{FF}\left(w; q\right) < V_{PC}\left(\underline{\delta}; \widetilde{p}(q), q\right) < V_{PC}\left(\overline{\delta}; \widetilde{p}(q), q\right). \tag{16}$$

The first one follows from the contradictory assumption that $\frac{\widetilde{p}(q)}{q} < 1$. This implies that carrying $k = k\left(\underline{\delta}; p, q\right)$ yields a capital gain, compared to $k = 0$, without affecting the profile of liquidity constraints – and this results in a period-1 utility gain. The second inequality was shown to be true when proving that $1 \leq \frac{\widehat{p}(q)}{q}$.

But (16) is absurd, since $V_{FC}(w; p) > V_{PC}\left(\overline{\delta}; p, q\right)$ is always true when $\frac{p}{q} \leq \overline{\rho}$, and $\frac{\widetilde{p}(q)}{q} \leq \overline{\rho}$ has already been established. ■

Putting these results together:

$\rho < 1:$ $V_{FC}(w; p) > V_{PC}\left(\overline{\delta}; p, q\right) > V_{PC}\left(\underline{\delta}; p, q\right) > V_{FF}(w; q)$.

The first inequality follows from $\rho < p < q < \overline{\rho}$; the following two, from $p < q$.

$1 \leq \rho < \frac{\widetilde{p}(q_{23})}{q_{23}}:$ $V_{FC}(w; p) > V_{FF}(w; q) > V_{PC}\left(\underline{\delta}; p, q\right)$.

The first one follows from $\rho < \frac{\widetilde{p}(q)}{q}$, and the second one, from $p > q$.

In addition, $V_{FC}(w; p) > V_{PC}\left(\overline{\delta}; p, q\right)$ follows too, from $\rho < \frac{\widetilde{p}(q)}{q} \leq \overline{\rho}$.

$\frac{\widetilde{p}(q_{23})}{q_{23}} < \rho < \overline{\rho}:$ $V_{FF}(w; q) > V_{FC}(w; p) > V_{PC}\left(\overline{\delta}; p, q\right)$.

The first one follows from $\rho > \frac{\widetilde{p}(q)}{q}$, and the second one from $\rho < \overline{\rho}$.

In addition, $V_{FF}(w; q) > V_{PC}\left(\underline{\delta}; p, q\right)$ follows from $p > q$.

$\overline{\rho} < \rho:$ $V_{FF}(w; q) > V_{PC}\left(\underline{\delta}; p, q\right) > V_{PC}\left(\overline{\delta}; p, q\right)$.

The first inequality follows from $p > q$; the second one from $\rho > \overline{\rho}$.

In addition, $V_{FF}(w; q) > V_{FC}(w; p)$ follows from $\rho > \frac{\widetilde{p}(q)}{q}$.

This concludes that for $p < \widetilde{p}(q)$, period-1 maximized utility is $V_{FC}(w;p)$; while for $p > \widetilde{p}(q)$, it is $V_{FF}(w;q)$, thereby establishing that either $k = k^*(p)$ or $k = 0$ can be optimal.

Next, notice that neither β nor q enter $V_{FC}(w;p)$; while p does not enter $V_{FF}(w;q)$. Because both period's consumption goods are normal, $V_{FC}(w;p)$ is strictly decreasing in p while $V_{FF}(w;q)$ is strictly decreasing in q. This implies that the indifference price $\widetilde{p}$ must exist and be unique; and that it is increasing in q.

To see that $V_{FF}(w;q)$ is strictly decreasing in β, notice from the Euler equation that decreases in β are necessarily accompanied by period-2 tilting of the implemented consumption profile (which cannot remain unchanged in β), for each δ. Diagrammatically, this means that as β falls below 1, the implemented consumption profile moves along the budget line towards the c_2-axis. This necessarily decreases period-1 utility, conditional on each value of δ; hence it also does in expected terms. This finalizes the proof of Proposition 2. ■

5.4 Proposition 3.

Proposition 3. *Under Assumption F, the optimal illiquid assets choice as a function of the price ratio* $\frac{p}{q}$ *is*

$$\begin{array}{ll} k = k^* & \text{if } \frac{p}{q} \leq \overline{\rho} \\ k = k(\delta^*) & \text{if } \overline{\rho} < \frac{p}{q} < 1 \\ k = 0 & \text{if } 1 < \frac{p}{q}, \end{array}$$

At $p = q$*, the consumer is indifferent over all portfolios containing* $k \in [0, k(\underline{\delta})]$*.*

For $\overline{\rho} < \frac{p}{q} < 1$*, the optimal* k *is decreasing in* β*.*

Proof. Notice that under Assumption F, $\psi(\delta^*) < 0$ and decreasing implies that $\frac{E[\delta]}{\overline{\delta}} - \beta < 0$; hence $\overline{\rho} < 1$. From Corollary 1, program V_{PC} will have critical points only if $p < q$. Moreover, from Lemma 3 any critical point of V_{PC} will be a maximum; hence, there can be a unique critical point.

Let $\rho \leq \overline{\rho} < 1$ and notice that $\xi\left(\overline{\delta}\right) = \frac{E[\delta]}{\overline{\delta}} - \rho\beta \geq 0$. Because $\rho < 1$, Lemma 18 implies that $\xi(\delta^*)$ is strictly decreasing in δ^*. So $\xi\left(\overline{\delta}\right) \geq 0$ means that $\xi(\delta^*) > 0$ holds for all $\delta^* < \overline{\delta}$. Therefore, the solution to problem V_{PC} must involve choosing cutoff value $\overline{\delta}$; so that $V_{PC} = V\left(\overline{\delta}\right)$. In addition, when $\rho < \overline{\rho}$, $V_{FC}(w;p) > V\left(\overline{\delta}\right)$; they coincide at $\rho = \overline{\rho}$. And this concludes that when $\rho < \overline{\rho}$,

$$V_{FC}(w;p) > V_{PC} = V\left(\overline{\delta}\right) > V\left(\underline{\delta}\right).$$

Finally, since $\rho \leq \overline{\rho} < 1$ implies $V(\underline{\delta}) > V_{FF}(w;q)$, it follows that the maximal period-1 utility value is $V_{FC}(w;p)$. The optimal choice is $k = k^*(p)$.

Next, suppose $\overline{\rho} < p < q$. At any such price ratio, $k^*(p) < k\left(\overline{\delta}\right)$; hence $V_{FC}(w;p)$ is no longer relevant for the consumer's problem. In addition, $p < q$ implies $V(\underline{\delta}) > V_{FF}(w;q)$. Therefore, the solution to problem V_{PC} is also the overall solution to the period-1 problem. The optimal δ^* is wholly identified by the necessary and sufficient first order conditions $\xi(\delta^*) = 0$.

Since $\frac{\partial \xi(\delta^*)}{\partial \rho} = -\left(\beta F\left(\delta^*\right) + P\left(\delta^*\right)\right) < 0$, as ρ increases towards 1, the optimal δ^* decreases towards $\underline{\delta}$, reaching it exactly at $\rho = 1$. To see this, notice that when $\rho = 1$, $\xi\left(\underline{\delta}\right) = 0$, while for $p < q$, $\xi\left(\underline{\delta}\right) > 0$.

At $\rho = 1$, the solution to V_{PC} involves $k = k\left(\underline{\delta}\right)$. Any illiquid assets level $k \in \left[0, k\left(\underline{\delta}\right)\right]$ implies the same amount of period-2 resources, and the same profile of period-2 liquidity constraints. Therefore, any such value is a solution to the period-1 problem.

Finally, for $p > q$ problem V_{PC} has no interior solution, and therefore it must be solved by $\delta^* = \underline{\delta}$ (this last statement follows from the decreasing property $\frac{\partial \delta^*}{\partial \rho} < 0$). However, at $p > q$, $V_{FF}\left(w; q\right) > V\left(\underline{\delta}\right)$. Hence the global solution to the consumer's problem is $k = 0$.

To see that the optimal k is decreasing in β, fix a δ^* for which the first order condition (10) is satisfied, and take derivatives with respect to β :

$$\begin{aligned} \frac{\partial \xi\left(\delta^*\right)}{\partial \beta} &= -\rho F\left(\delta^*\right) + (1-\rho) \int_{\delta^*}^{\overline{\delta}} \frac{\partial \mu\left(\beta, \delta; q\right)}{\partial \beta} \frac{u'\left(c_2\left(\delta\right)\right)}{u'\left(c_2\left(\delta^*\right)\right)} f\left(\delta\right) d\delta \\ &\quad + (1-\rho) \int_{\delta^*}^{\overline{\delta}} \mu\left(\beta, \delta; q\right) \frac{\partial}{\partial \beta} \left\{ \frac{u'\left(c_2\left(\delta\right)\right)}{u'\left(c_2\left(\delta^*\right)\right)} \right\} f\left(\delta\right) d\delta. \end{aligned}$$

The derivatives are:

$$\begin{aligned} \frac{\partial \mu\left(\beta, \delta; q\right)}{\partial \beta} &= -u''\left(c_2\left(\delta\right)\right) \frac{q^2}{\beta^2} \frac{q^2 u''\left(c_2\left(\delta\right)\right) + (2-\beta)\beta\delta u''\left(c_3\left(\delta\right)\right)}{\left(q^2 u''\left(c_2\left(\delta\right)\right) + \beta\delta u''\left(c_3\left(\delta\right)\right)\right)^2}, \\ \frac{\partial}{\partial \beta} \left\{ \frac{u'\left(c_2\left(\delta\right)\right)}{u'\left(c_2\left(\delta^*\right)\right)} \right\} &= \frac{u'\left(c_2\left(\delta\right)\right)}{\beta u'\left(c_2\left(\delta^*\right)\right)} \left(\frac{q^2 u''\left(c_2\left(\delta\right)\right)}{q^2 u''\left(c_2\left(\delta\right)\right) + \beta\delta u''\left(c_3\left(\delta\right)\right)} - \frac{q^2 u''\left(c_2\left(\delta^*\right)\right)}{q^2 u''\left(c_2\left(\delta^*\right)\right) + \beta\delta^* u''\left(c_3\left(\delta^*\right)\right)} \right). \end{aligned}$$

The second one uses the implicit function theorem on the Euler equation to solve for $\frac{\partial c_2(\delta)}{\partial \beta}$.

Using these two expressions, and with some algebra,

$$\begin{aligned} \frac{\partial \xi\left(\delta^*\right)}{\partial \beta} &= -\rho F\left(\delta^*\right) \\ &\quad - (1-\rho) \int_{\delta^*}^{\overline{\delta}} \mu\left(\beta, \delta; q\right) \frac{u'\left(c_2\left(\delta\right)\right)}{\beta u'\left(c_2\left(\delta^*\right)\right)} \frac{q^2 u''\left(c_2\left(\delta^*\right)\right)}{q^2 u''\left(c_2\left(\delta^*\right)\right) + \beta\delta^* u''\left(c_3\left(\delta^*\right)\right)} f\left(\delta\right) d\delta \\ &\quad - (1-\rho)\, 2q^2\delta \left(\frac{1}{\beta} - 1\right) \int_{\delta^*}^{\overline{\delta}} \frac{u'\left(c_2\left(\delta\right)\right)}{u'\left(c_2\left(\delta^*\right)\right)} \frac{u''\left(c_2\left(\delta\right)\right) u''\left(c_2\left(\delta\right)\right)}{\left(q^2 u''\left(c_2\left(\delta\right)\right) + \beta\delta u''\left(c_3\left(\delta\right)\right)\right)^2} f\left(\delta\right) d\delta. \end{aligned}$$

This is negative. So if δ^* is a critical point, an increase in β makes $\xi\left(\delta^*\right)$ negative. Under Assumption F, $\xi\left(\delta^*\right)$ is decreasing in δ^*. Therefore equality can only be restored if δ^* decreases.

This shows that at an interior optimum of V_{PC}, δ^* is decreasing in β. Hence, so is $k\left(\delta^*\right)$. And this establishes Proposition 3. ■

5.5 Proposition 4.

Proposition 4. *Assumption C holds if, for all δ, $\beta \leq \frac{1+\varepsilon(\delta)}{2+\varepsilon(\delta)}$ and $\varepsilon\left(\delta\right) \geq -1$.*

Assumption F holds if $\underline{\delta} f\left(\underline{\delta}\right) = 0$ and, for all δ, either $\beta \geq \frac{1+\varepsilon(\delta)}{2+\varepsilon(\delta)}$ or $\varepsilon\left(\delta\right) \leq -1$. The boundary condition $\underline{\delta} f\left(\underline{\delta}\right) = 0$ is necessary for F.

Proof. Let $\Phi(\delta^*) \equiv (E[\delta|\,\delta \leq \delta^*] - \delta^*\beta) F(\delta^*)$, so that $\psi(\delta^*) = \frac{\Phi(\delta^*)}{\delta^*}$. The derivative is $\frac{d\Phi(\delta^*)}{d\delta^*} = \phi(\delta^*)$, and since $\Phi(\underline{\delta}) = 0$, $\left.\frac{\Phi(\delta^*)}{\delta^*}\right|_{\delta^*=\underline{\delta}} = 0$ (when $\underline{\delta} = 0$, this is also true by L'Hôpital's rule). When $\phi(\delta)$ is increasing, $\Phi(\delta^*)$ will be positive, increasing in convex. Convexity implies that

$$\frac{\Phi(\delta^*)}{\delta^*} \geq \frac{\Phi(\underline{\delta}) + (\delta^* - \underline{\delta})\,\phi(\underline{\delta})}{\delta^*} = \frac{\delta^* - \underline{\delta}}{\delta^*}\phi(\underline{\delta}),$$

and since $\phi(\underline{\delta}) = (1-\beta)\,\underline{\delta} f(\underline{\delta}) \geq 0$, it follows that $\frac{\Phi(\delta^*)}{\delta^*}$ is positive. To see that $\frac{\Phi(\delta^*)}{\delta^*}$ is increasing, compute the derivative $\frac{d}{d\delta^*}\left\{\frac{\Phi(\delta^*)}{\delta^*}\right\} = \frac{\delta^*\phi(\delta^*) - \Phi(\delta^*)}{\delta^{*2}}$. The numerator is non-negative at $\delta^* = \underline{\delta}$, and increasing in δ^*. To see this, suppose that $\phi(\delta^*)$ is differentiable, then the derivative of the numerator reduces to $\delta^*\phi'(\delta^*)$, which is positive if $\phi(\delta^*)$ is increasing.

The proof of the statement about F is identical, only with the signs reversed and using $\phi(\underline{\delta}) = 0$, which is ensured by the boundary condition $\underline{\delta} f(\underline{\delta}) = 0$.

In the case of twice-differentiability of $F(\delta)$, $\phi'(\delta) = (1-\beta)\,\delta f'(\delta) - (2\beta - 1) f(\delta)$. This is positive if

$$\frac{\delta f'(\delta)}{f(\delta)} > \frac{2\beta - 1}{1 - \beta}, \tag{17}$$

zero if the two terms are equal, and negative otherwise. The conditions in Proposition 4 follow from re-arranging (17). ■

5.6 Proposition 5.

Proposition 5. *Suppose* $\underline{\delta} = 0$ *and* $F(\delta) = \frac{\delta^x}{\overline{\delta}^x}$, *for some* $x > 0$. *Then C is satisfied if and only if* $\beta < \widetilde{\beta} \equiv \frac{x}{x+1}\overline{\delta}^x$; *while F is satisfied if and only if* $\beta > \widetilde{\beta}$.

Proof. With this functional form,

$$\psi(\delta^*) = \left(\frac{x}{x+1}\overline{\delta}^x - \beta\right)\frac{\delta^{*x}}{\overline{\delta}^x}.$$

So $\psi(\delta^*)$ is positive and increasing if and only if $\frac{x}{x+1}\overline{\delta}^x \geq \beta$, negative and decreasing otherwise. ■

5.7 Proposition 6.

Proposition 6. *A general equilibrium always exists in this economy.*

Proof. To deal with existence of equilibrium, it is convenient to drop the normalization used before. It is also useful to study a slightly simpler market structure. Instead of three periods, let this be a two-period economy, but, in order not to modify the notation too much, I call the first period "period 2", and the second period "period 3". There are three commodities: Period-2 and period-3 consumption goods, and a third kind of good, called "illiquid assets". This commodity is only traded in an ex-ante phase in which no other good is traded. Stocks in this third good simply add to period-3 consumption. I will prove existence for this economy, and then show that any competitive equilibrium of this economy must be a competitive equilibrium of the original economy.

Let q_2 and q_3 denote the prices of period-2 and period-3 consumption goods, respectively. Let π denote the price of illiquid assets. The aggregate endowment in this economy is $\overline{c}_2$ unit of period-2 goods, $(1-\alpha)\,\overline{c}_3$ units of period-3 goods, and $\alpha\overline{c}_3$ units of the "illiquid assets" commodity. Let $w = \alpha\pi + q_2 + (1-\alpha)\,q_3$ denote the aggregate value of wealth in this economy. Let $q \equiv (\pi, q_2, q_3)$ be in the three-dimensional simplex $q \in \Delta_{3+}$.

For generality, I will establish existence of equilibrium in this economy allowing for agent heterogeneity. Let J be a finite positive integer, and $\{\mu_j\}_{j=1}^{N}$ a collection of positive numbers such that $\sum_{j=1}^{J}\mu_j = 1$.

I assume that there are J types of agents, indexed by $1 \le j \le J$, and a measure μ_j of type j. For each j, let $\delta_j \in [\underline{\delta}_j, \overline{\delta}_j]$ be a random variable with absolutely continuous distribution function $F_j(\delta)$. Let (k, c_2, c_3) denote a profile of "illiquid assets", period-2 and period-3 consumption goods. Conditional on the realization of δ_j, a type j agent's period-1 and period-2 preferences over such profiles are given respectively by

$$v_1^j(k, c_2, c_3; \delta_j) = u^j(c_2) + \delta_j u^j(k + c_3), \tag{18}$$
$$v_2^j(k, c_2, c_3; \delta_j) = u^j(c_2) + \beta_j \delta_j u^j(k + c_3). \tag{19}$$

$\beta^j < 1$ is the dynamic inconsistency factor, known and anticipated in period 1. Let $c(\delta) \equiv (k, c_2(\delta), c_3(\delta))$ denote a non-negative vector of δ-contingent (and Borel-measurable) consumption vectors (notice k is not contingent). Period 1 preferences over such contingent consumption vectors are given by the expected utility formulation

$$V^j = E_j[v_1^j(k, c_2(\delta), c_3(\delta); \delta^j)]. \tag{20}$$

For every j, $u^j(c)$ is a strictly increasing, strictly concave, everywhere twice differentiable function.

Finally, agents are also allowed to be heterogeneous in wealth; type j's initial endowment $(\overline{k}^j, \overline{c}_2^j, \overline{c}_3^j)$ is assumed non-negative and non-zero; type j's initial wealth market value is denoted by w_j.

The timing is the following: First individuals choose whether to purchase "illiquid assets" goods at price π, with the objective of maximizing (20). If k illiquid assets are purchased, remaining available resources equal $w^j - \pi k^i$. After this, values for δ are learned and individuals trade for period-2 and period-3 goods with the objective to maximize (19). Any acquired stocks of "illiquid assets" goods cannot be sold down nor borrowed against.

Let k^i denote the stock of illiquid assets purchased by individual i, and $w_2^i \equiv w^j - \pi k^i$. Define $x^i \equiv (w_2^i, k^i)$ and $y^i \equiv (x^i, q_2, q_3)$. Let $c_2^j(y^i;\delta), c_3^j(y^i;\delta)$ denote the period-2 policy rules given y^i and the realized δ. Notice that these are restricted to be non-negative. Denote the net period-2 goods demand conditional on δ by

$$c^j(y^i;\delta) \equiv \left(c_2^j(y^i;\delta), c_3^j(y^i;\delta)\right).$$

Notice that although the value of y^i is a matter of individual choice, the demand function $c^j(\cdot;\delta)$ is common to all individuals of type j, since it depends on type-j parameters.

At the "ex-ante" trading phase, each type-j individual i chooses illiquid asset purchases to solve

$$\begin{array}{rll} V_j(q) = & \max\limits_{k} & \int_{\underline{\delta}_j}^{\overline{\delta}_j} \left(u^j(c_2(\delta)) + \delta_j u^j(c_3(\delta))\right) f_j(\delta)\, d\delta \\ & s.t. & 0 \leq \pi k^i \leq w^j, \\ & & c_2(\delta) = c_2^j\left(y^i;\delta\right), c_3(\delta) = c_3^j\left(y^i;\delta\right) + k. \end{array} \tag{21}$$

For each δ, $c^j\left(y^i;\delta\right)$ is single-valued, due to the strictly concave utility assumption. Moreover, it is continuous in y^i, from the Theorem of the Maximum.

Let $k^j(q)$ denote the correspondence of optimal k choices given price vector q. By Proposition 12, this is non-empty, compact-valued and upper-hemicontinuous. The correspondence of optimal c choices associated with prices q and optimal illiquid asset purchases $k^j(q)$ is given by

$$\psi^j(q,\delta) \equiv \left\{c \in \mathbb{R}_+^2 : c = c\left(w - \pi k^i, k, q_2, q_3; \delta\right) \text{ for some } k \in k^j(q)\right\}.$$

Let $\overline{k}^i(q) \in k^j(q)$ denote an element of $k^j(q)$, and let $\overline{c}^j\left(w - \pi\overline{k}^i(q), \overline{k}^i(q), q_2, q_3; \delta\right)$ be the associated element in $\psi^j(q,\delta)$, for each δ (notice that this is single-valued, from the strict concavity assumption). Define

$$C^j\left(\overline{k}^i(q), q\right) = \int_{\underline{\delta}_j}^{\overline{\delta}_j} \overline{c}^j\left(w - \pi\overline{k}^i(q), \overline{k}^i(q), q_2, q_3; \delta\right) f_j(\delta)\, d\delta.$$

This is expected period-2 demand associated with $\overline{k}^i(q)$. Whenever all type-j agents implement choice $\overline{k}^i(q)$, their period-2 demands will be, by assumption, exactly $\mu_j \cdot C^j\left(\overline{k}^i(q), q\right)$. Let

$$z^j(q) = \left\{(k, c_2, c_3) \in \mathbb{R}_+^3 : k \in k^j(q), (c_2, c_3) = \mu_j \cdot C^j(k, q)\right\}$$

Lemma 9 *$z^j(q)$ is non-empty, compact-valued and upper hemi-continuous.*

Proof. *First, the correspondence of maximizers $k^j(q)$ is non-empty, compact-valued and upper hemi-continuous for each j, by Proposition* 11*. Second, because $c^j(w - pk, k, q_2, q_3; \delta)$ is continuous in k and (q_2, q_3), $C^j\left(\overline{k}(q), q\right)$ is also a compact-valued, continuous function (hence upper-hemicontinuous correspondence). $z^j(q)$ is the Cartesian product of two non-empty, compact-valued and upper-hemicontinuous correspondences, and therefore also is (see Stokey and Lucas,* 1989*).* ■

Finally, type j agents' demand correspondence is the convex hull of $z^j(q)$; and aggregate demand is

$$Z(q) = \sum_{j=1}^{J} co\left[z^j(q)\right].$$

The infinite number of agents allows the "convexification" of aggregate demand $Z(q)$. By construction, if there are more than two elements in $k^j(q)$ then all type-j agents will be indifferent (in period 1) between all the portfolio options in $k^j(q)$, which will all be optimal. This indifference implies that they will be willing to randomize between those portfolios, and this achieves convex-valuedness of the aggregate demand correspondence. As a sum of non-empty, compact- and convex-valued, and upper hemi-continuous correspondences, so is

The net aggregate demand correspondence is

$$\vartheta(q) \equiv Z(q) - (\alpha\bar{c}_3, \bar{c}_2, (1-\alpha)\bar{c}_3).$$

A general equilibrium in this economy is a price vector q^* such that $0 \in \vartheta(q^*)$. By construction, $Z(q)$ is always the set of points at which individuals maximize utility (in both periods), given prices. And when $0 \in Z(q^*)$, the additional requirement of market clearing is also satisfied by the appropriately chosen profile of individual demands in $z^j(q^*)$.

$\vartheta(q)$ inherits non-emptyness, compact-valuedness and upper hemi-continuity from $Z(q)$.

Boundary properties of $z^j(q)$ are all that need to be checked, in order to apply a standard existence of equilibrium theorem (notice that Walras' law is satisfied since every consumer always exhausts her wealth). I show next that at least some type j's demand for some good must explode if the price vector converges to the boundary of the price simplex. Consequently, aggregate demand will explode as well.

Lemma 10 *Let $\{q^n\}_{n=1}^{\infty}$ be a sequence in Δ_{3+} converging to a point $q \in Bound[\Delta_{3+}]$ in the boundary of the price simplex. Then $\exists j : \lim_{n\to\infty}\{\max z^j(q^n)\} = \infty$.*

Proof. *There are four cases. Since the market value of aggregate wealth $w^n \equiv q_2^n + (1-\alpha)q_3^n + \alpha\pi^n$ is bounded away from 0 for any $q^n \in \Delta_{3+}$; there must exist a j such that w_j^n also is. The proof strategy consists of showing that whenever a price goes to 0, period-1 utility arbitrarily close to the relevant (as discussed below) upper bound of type j's utility can be attained. This, together with the fact that a solution to problem (21) always exists, implies that the period-1 value is arbitrarily close to that upper bound. Consequently, the agent must be implementing an unbounded consumption profile (remember that utility is strictly increasing), as desired.*

1) $q^n \to (0, q_2, q_3)$; q_2 may be 0. Let type j be such that $w_j^n \to \underline{w} > 0$. Let $\overline{u}^j \equiv \lim_{c\to\infty} u^j(c)$.

Consider the sequence $k^n = \frac{w_j^n}{\sqrt{\pi^n}}$; $k^n \to \infty$ and $w_j^n - \pi^n k^n = \left(1-\sqrt{\pi_n}\right) w_j^n$ converges to $\underline{w}$.

Suppose $\overline{u}^j = \infty$. Notice that either $\lim_{c\to 0} u'^j(c) = \infty$ and hence the minimum over δ of $c_2^n(\delta)$ is bounded away from 0 in a period-2 optimal choice (since $\overline{\delta}$ is finite), or $u^j(0) > -\infty$. In either case, unbounded utility is achieved as $n \to \infty$ by the proposed plan.

If $\overline{u}^j < \infty$, then $u(c)$ strictly concave and increasing implies that $\lim_{c\to\infty} u'^j(c) = 0$. For each δ_j, period-2 choice is liquidity-constrained if and only if δ_j is not larger than the number $\delta_{j,n}$ satisfying

$$\frac{u^{j\prime}\left(\frac{(1-\sqrt{\pi_n})w_j^n}{q_2^n}\right)}{u^{j\prime}\left(\frac{w_j^n}{\sqrt{\pi^n}}\right)} = \frac{\beta_j \delta_{j,n}}{q_3^n/q_2^n}. \tag{22}$$

The left-hand side of (22) converges to ∞ *as* $n \to \infty$*; the right-hand side is finite (and converges to* 0 *if* $q_2 = 0$*). Hence the consumer is, eventually, liquidity constrained for all values of* δ *by the proposed plan, and behaves hand-to-mouth. That is, the implemented consumption profile is eventually deterministic and converges to* $c_2 = \underline{w}$ *and unbounded period-*3 *consumption. This yields infinite period-*1 *expected utility.*

2) $q^n \to (\pi, 0, q_3)$*. Let type* j *be such that* $w_j^n \to \underline{w} > 0$*. Let* $\overline{u}^j \equiv \lim_{c\to\infty} u^j(c)$*.*

Consider the sequence $k^n = \frac{w_j^n - \sqrt{q_2^n}}{\pi^n}$*;* $k^n \to \frac{\underline{w}}{\pi}$ *and* $w_j^n - \pi^n k^n = \sqrt{q_2^n}$ *converges to* 0*; while* $\frac{w_j^n - \pi^n k^n}{q_2^n} = \frac{1}{\sqrt{q_2^n}}$ *converges to* ∞*. Notice that utility arbitrarily close to* $\overline{u}^j + \beta_j \delta_j u^j(\underline{w})$ *is feasible in any period-*2 *optimal plan, for any* δ_j*; hence period-*2 *utility must converge to that number. But this implies that* c_2^n *converges to* ∞ *for every* δ*. Therefore, period-*1 *value converges to* $\overline{u}^j + E_j[\delta]\, u^j\left(\frac{\underline{w}}{\pi}\right)$*.*

3) $q^n \to (\pi, q_2, 0)$*;* π *may be* 0*. Let type* j *be such that* $w_j^n \to \underline{w} > 0$*. Let* $\overline{u}^j \equiv \lim_{c\to\infty} u^j(c)$*. Consider the sequence* $k^n = 0$*. Period-*2 *utility arbitrarily close to* $u^j\left(\frac{\underline{w}}{q_2}\right) + \beta_j \delta_j \overline{u}^j$ *is feasible for* n *large enough; hence it must be obtained in the limit;* c_2^n *converges to* $\frac{\underline{w}}{q_2}$ *for all* δ_j*, and* c_3^n *explodes to infinity. This implies that limit period-*1 *expected utility of* $u^j\left(\frac{\underline{w}}{q_2}\right) + E_j[\delta]\, \overline{u}^j$ *and therefore must be achieved.*

4) $q^n \to (\pi, 0, 0)$*. Let type* j *be such that* $w_j^n \to \underline{w} > 0$*. Let* $\overline{u}^j \equiv \lim_{c\to\infty} u^j(c)$*.*

Suppose $\frac{q_2^n}{q_3^n} \to 0$*, and consider the sequence* $k^n = \frac{w_j^n - \sqrt{q_2^n}}{\pi^n}$*;* $k^n \to \frac{\underline{w}}{\pi}$ *and* $w_j^n - \pi^n k^n = \sqrt{q_2^n}$ *converges to* 0*; while* $\frac{w_j^n - \pi^n k^n}{q_2^n} = \frac{1}{\sqrt{q_2^n}}$ *converges to* ∞*. The fact that* $\frac{q_2^n}{q_3^n} \to 0$ *implies that* c_2^n *must converge to infinity for every* δ_j*. Therefore, period-*1 *utility is at least arbitrarily close to* $\overline{u}^j + E_j[\delta]\, u^j\left(\frac{\underline{w}}{\pi}\right)$ *– and may be higher.*

Suppose $\frac{q_2^n}{q_3^n} \to \infty$*, and consider the sequence* $k^n = 0$*. Period-*2 *utility arbitrarily close to* $u^j\left(\frac{\underline{w}}{q_2}\right) + \beta_j \delta_j \overline{u}^j$ *is feasible for* n *large enough; hence it must be obtained in the limit;* c_2^n *converges to* $\frac{\underline{w}}{q_2}$ *for all* δ_j*, and* c_3^n *explodes to infinity. This implies that limit period-*1 *expected utility of* $u^j\left(\frac{\underline{w}}{q_2}\right) + E_j[\delta]\, \overline{u}^j$ *and therefore must be achieved.*

*For all six possible boundaries, I have shown that an upper-bound period-*1 *expected utility is feasible when* q^n *converges to a boundary point in the price simplex. Specifically, utility equal to allocating all wealth to the non-zero-priced good while enjoying unbounded consumption of the zero-priced good can be achieved.*

Therefore, no sequence of optimal plans associated to q^n *can involve utility converging to a number below that bound. Notice, on the other hand, that for every* q^n *the period-*1 *problem has a solution. The two facts together imply that solutions must always exist with utility approaching the upper bound. Since this is only with exploding consumption profiles, it follows that the maximum element in the demand correspondence must explode as well. And this completes the proof.* ■

Finally, to back up equilibrium in the original economy, simply change the normalization to $q_2 = 1$, by letting $q \equiv \frac{q_3}{q_2}$ and $p \equiv \frac{\pi}{q_2}$. If (π^*, q_2^*, q_3^*) is a competitive equilibrium price vector of the economy studied in this section, so is $(p^*, 1, q)$ in the original economy; with the same allocation of assets and goods. ■

5.8 Proposition 7.

Proposition 7. *Under Assumption C, any equilibrium involves a fraction $\lambda^* \in (0,1)$ of the population pursuing a full commitment strategy, and the remaining $1-\lambda^*$ a full flexibility one. Equilibrium prices verify $p = \widetilde{p} > q$.*

Proof.

Lemma 11 *In equilibrium, $\alpha \in (0,1) \Rightarrow p = \widetilde{p}(q)$.*

Proof. *Consider $p > \widetilde{p}(q)$. By Proposition 3, all agents implement $k = 0$. Aggregate demand for illiquid assets is thus null, and falls short of aggregate supply of $\alpha\overline{c}_3 > 0$. Suppose instead that $p < \widetilde{p}(q)$. Then all agents implement deterministic consumption profiles. This implies that the period-2 choice is liquidity-constrained for all realizations of δ, and therefore that period-2 behavior is always hand-to-mouth. Consequently, no-one demands liquid long-term assets at the end of period 2; hence the market for this asset remains in excess supply.* ■

Corollary 2 *For market clearing, a fraction $\lambda^* \in (0,1)$ of agents must choose a deterministic, full commitment plan, and the remaining $1-\lambda^*$, a full flexibility strategy.*

Corollary 3 *In equilibrium, $\rho^* > 1$.*

The Corollary follows from the fact that $\widetilde{p}(q) > q$ is true under C, as shown in the proof of Proposition 2.

It is clear that when $\alpha = 1$, there is a unique equilibrium in which every agent consumes the representative endowment in each period. ■

5.9 Proposition 8.

Proposition 8. *Under Assumption F, $p \leq q$ in equilibrium.*

If $p < q$, then all agents acquire identical portfolios (equal, therefore, to the representative one) in period 1. There exists a cutoff δ^* such that agents are liquidity-constrained in period 2 if and only if the realized δ is smaller than δ^*.

If $p = q$, then individual initial asset holdings are undetermined, but no agent is ever liquidity-constrained in period 2.

Proof.

Lemma 12 *In equilibrium, $\alpha \in (0,1) \Rightarrow \overline{\rho} < \rho \leq 1$.*

Proof. *Let $p > q$. By Proposition 3, every agent's problem is solved by setting $k = 0$; aggregate demand for illiquid assets is null and therefore falls short of aggregate supply. Suppose, conversely, that $\rho \leq \overline{\rho}$. Then every agent's optimal plan involves deterministic* *consumption, by Proposition 4. Therefore, every agent is liquidity-constrained in period 2 and behaves hand-to-mouth. This implies that aggregate period-2 demand for liquid long term assets is null, thereby falling short of aggregate supply.* ■

Corollary 4 *If, in equilibrium, $\overline{\rho} < p < q$, then all individuals carry identical end-of-period-1 portfolios. There is a common cutoff value $\delta^* \in \left(\underline{\delta}, \overline{\delta}\right)$ such that all agents are liquidity-constrained in period 2 if and only if $\delta \leq \delta^*$.*

The Corollary follows from the fact that at $\overline{\rho} < p < q$, every individual's period-1 problem has identical solution, and that the solution involves a strictly interior δ^*. Both facts are established in the proof of Proposition 3. ■

5.10 Proposition 9.

Proposition 9. *Assume that either C or F holds, and let $u(c) = \log c$. Then for every α, there is a unique equilibrium. Period-1 expected utility is maximized at $\alpha = 0$, and is strictly lower for all $\alpha \in (0, 1]$.*

Proof. I divide the proof into the two cases, beginning with C.

5.10.1 Assumption C.

By Proposition 7, the equilibrium price of illiquid assets is $\widetilde{p}$, given the price q of liquid assets. With logarithmic preferences, $\widetilde{p}$ is a linear function of q. Specifically,

$$\frac{\widetilde{p}}{q} = \rho^* = E[\delta] \exp\left(- \frac{\int_{\underline{\delta}}^{\overline{\delta}} \left((1+\delta) \log \frac{1+E[\delta]}{1+\beta\delta} + \delta \log \delta\beta \right) f(\delta)\, d\delta}{E[\delta]} \right).$$

Lemma 13 *An equilibrium $\lambda^* \in (0, 1)$ exists and is unique.*

Proof. *If a fraction λ^* of agents carry illiquid assets, each one of those must acquire $k^* = \frac{\alpha\overline{c}_3}{\lambda^*}$ for the illiquid assets market to clear. Moreover, the solution to the V_{FC} problem with logarithmic utility is*

$$k^* = \frac{E[\delta]}{1+E[\delta]} \frac{w}{\rho^* q}. \tag{23}$$

The implemented period-3 consumption of people who acquire $k = 0$, conditional on the realized discount factor, is $c_3(\delta) = \frac{w}{1+\beta\delta}\frac{\beta\delta}{q}$.

Let

$$\Lambda(\lambda^*) \equiv \alpha\overline{c}_3 + (1-\lambda^*) \int_0^1 \frac{w}{1+\beta\delta} \frac{\beta\delta}{q}\ f(\delta)\ d\delta.$$

Then market clearing of the period-3 consumption goods market is equivalent to $\Lambda(\lambda^*) = \overline{c}_3$. Let $D \equiv \int_0^1 \frac{\beta\delta}{1+\beta\delta}\ f(\delta)\ d\delta$. Using (23) in combination with $k^* = \frac{\alpha\overline{c}_3}{\lambda^*}$, we get that $\frac{w}{q} = \frac{\alpha\overline{c}_3\rho^*}{\lambda^*}\frac{1+E[\delta]}{E[\delta]}$:

$$\Lambda(\lambda^*) = \alpha\overline{c}_3 + \frac{1-\lambda^*}{\lambda^*} \frac{1+E[\delta]}{E[\delta]} \alpha\overline{c}_3\rho^* D.$$

The unique $\lambda^* \in (0,1)$ setting $\Lambda(\lambda^*) = \overline{c}_3$ is solved for

$$\lambda^* = \frac{\alpha\rho^*(1+E[\delta])D}{(1-\alpha)E[\delta]+\alpha\rho^*(1+E[\delta])D}. \tag{24}$$

■

Notice that λ^* is independent of the endowment process. The unique equilibrium value of q can be solved for.

Using $k^* = \frac{\alpha\overline{c}_3}{\lambda^*}$, (24), and the period-1 budget constraint, the (deterministic) consumption profile implemented by agents who acquire illiquid assets is, as a function of α :

$$\begin{aligned} c_2^*(\alpha) &= \frac{(1-\alpha)E[\delta]+\alpha\rho^* D(1+E[\delta])}{(1+E[\delta])((1-\alpha)(1-D)E[\delta]+\alpha\rho^* D)}\overline{c}_2, \\ c_3^*(\alpha) &= \left(\alpha + \frac{(1-\alpha)E[\delta]}{\rho^*(1+E[\delta])D}\right)\overline{c}_3. \end{aligned} \tag{25}$$

When $\alpha = 1$, these reduce to $c_2^*(\alpha) = \overline{c}_2$ and $c_3^*(\alpha) = \overline{c}_3$, as expected. Because in equilibrium agents are indifferent between the consumption profile (25) and the full flexibility strategy, equilibrium utility can be derived by evaluating this profile.

When $\alpha = 1$, equilibrium expected utility is $V^*(1) = \log\overline{c}_2 + E[\delta]\log\overline{c}_3$. For $\alpha < 1$, it is

$$\begin{aligned} V^*(\alpha) &= V^*(1) + \log\frac{(1-\alpha)E[\delta]+\alpha\rho^* D(1+E[\delta])}{(1+E[\delta])((1-\alpha)(1-D)E[\delta]+\alpha\rho^* D)} \\ &\quad + E[\delta]\log\left(\alpha + \frac{(1-\alpha)E[\delta]}{\rho^*(1+E[\delta])D}\right). \end{aligned} \tag{26}$$

Lemma 14 $V^*(0) - V^*(1) > 0$.

Proof. *From (26),*

$$V^*(0) = V^*(1) + \log\frac{1}{(1+E[\delta])(1-D)} + E[\delta]\log\frac{E[\delta]}{\rho^*(1+E[\delta])D}.$$

Therefore, $V^*(0) - V^*(1)$ is strictly decreasing in ρ^*, and equals 0 if and only if

$$\rho^* = \widehat{\rho} \equiv \frac{E[\delta]}{(1+E[\delta])^{1+1/E[\delta]}(1-D)^{1/E[\delta]}D}.$$

So $V^*(0) > V^*(1)$ will be established if $\rho^* < \widehat{\rho}$ is proven. Using the expression displayed above,

$$\rho^* = (1+E[\delta])^{-\frac{1+E[\delta]}{E[\delta]}}E[\delta]\exp\left(-\frac{\int_{\underline{\delta}}^{\overline{\delta}}\left((1+\delta)\log\frac{1}{1+\beta\delta}+\delta\log\delta\beta\right)f(\delta)\,d\delta}{E[\delta]}\right).$$

Therefore, $V^*(0) - V^*(1)$ will be strictly positive if

$$\exp\left(-\frac{\int_{\underline{\delta}}^{\overline{\delta}}\left((1+\delta)\log\frac{1}{1+\beta\delta}+\delta\log\delta\beta\right)f(\delta)\,d\delta}{E[\delta]}\right) < \frac{1}{(1-D)^{1/E[\delta]}D}.$$

This inequality is equivalent to

$$E[\delta]\log(1-D)^{1/E[\delta]}D-\int_{\underline{\delta}}^{\overline{\delta}}\left((1+\delta)\log\frac{1}{1+\beta\delta}+\delta\log\delta\beta\right)f(\delta)\,d\delta<0. \tag{27}$$

Notice that the integrand expression is strictly convex:

$$\frac{\partial^2}{\partial\delta^2}\left\{(1+\delta)\log\frac{1}{1+\beta\delta}+\delta\log\delta\beta\right\}=\frac{\beta^2\delta+1}{(1+\beta\delta)^2\delta}>0.$$

By Jensen's inequality, this implies

$$-\int_{\underline{\delta}}^{\overline{\delta}}\left((1+\delta)\log\frac{1}{1+\beta\delta}+\delta\log\delta\beta\right)f(\delta)\,d\delta<-(1+E[\delta])\log\frac{1}{1+\beta E[\delta]}-E[\delta]\log E[\delta]\beta.$$

So a sufficient condition for (27) to hold is that

$$\log(1-D)+E[\delta]\log D-(1+E[\delta])\log\frac{1}{1+\beta E[\delta]}-E[\delta]\log E[\delta]\beta<0. \tag{28}$$

The left-hand side of (28) is strictly concave in D, evaluates to $-\infty$ as D converges to both 0 and 1, and reaches its maximum value at $D=\frac{E[\delta]}{1+E[\delta]}$.

On the other hand, remember that $D\equiv\int_{\underline{\delta}}^{\overline{\delta}}\frac{\beta\delta}{1+\beta\delta}\ f(\delta)\ d\delta$. Since the integrand is a concave function, by Jensen's inequality we have $D<\overline{D}\equiv\frac{\beta E[\delta]}{1+\beta E[\delta]}$. Notice that $\overline{D}<\frac{E[\delta]}{1+E[\delta]}$; this implies that for the range of possible values of D, the left-hand side of (28) will be a strictly increasing function.

Finally, when evaluated at $D=\overline{D}$, the left-hand side of (28) equals exactly 0. This shows that $\rho^*<\widehat{\rho}$, and therefore that $V^*(0)>V^*(1)$. ■

Notice that the proof does not use properties of $F(\delta)$; therefore this result is valid for any distribution function. The underlying procedure of the proof has been to find the deterministic consumption profile which yields the same utility as the equilibrium allocation when $\alpha=0$, and show that this deterministic profile delivers higher utility than the one consumed when $\alpha=1$. This utility comparison is valid regardless of the distribution.

Proposition 15 *$V^*(\alpha)$ is a strictly convex function, which attains its maximum at $\alpha=0$, and its minimum at an interior value $\alpha\in(0,1)$.*

Proof. *I show that $V^*(\alpha)$, which is a continuously differentiable function on $\alpha\in(0,1)$, has a unique point of null derivative, and has positive slope at $\alpha=1$. So $V^*(0)>V^*(1)$ necessarily implies that $V^*(\alpha)$ is convex and that the critical point, which is a global minimum, lies on the unit interval.*

First, checking that $V^*(\alpha)$ has a unique critical point in α is simply a matter of algebra. It can be verified that

$$\frac{\partial V^*(\alpha)}{\partial\alpha}=0$$
$$\Leftrightarrow\alpha=\frac{(D-(1-D)E[\delta])\rho^*D}{((1-\rho^*D)E[\delta]-\rho^*D)((1-D)E[\delta]-\rho^*D)}+\frac{(1-D)E[\delta]}{(1-D)E[\delta]-\rho^*D}.$$

Second, the derivative at $\alpha = 1$ is

$$\left.\frac{\partial V^*(\alpha)}{\partial \alpha}\right|_{\alpha=1} = \frac{\rho^* - 1}{\rho^*} E[\delta] > 0.$$

■

Corollary 5 *$V^*(\alpha) < V^*(0)$ for all $0 < \alpha \leq 1$. There exists a threshold $\overline{\alpha} \in (0,1)$ such that $V^*(\alpha)$ is strictly decreasing in α for $\alpha < \overline{\alpha}$ and strictly increasing for $\alpha > \overline{\alpha}$. Moreover, $V^*(\overline{\alpha}) < 0$.*

This establishes that when C holds and utility is logarithmic, for each α there is a unique competitive equilibrium. Equilibrium period-1 expected utility is maximized at $\alpha = 0$, and is strictly smaller for any $\alpha > 0$. ■

5.10.2 Assumption F.

Equilibrium under this assumption takes the following form. Agents do not trade in period 1, keeping the originally received liquid/illiquid wealth split. This implies a cutoff value δ^* such that agents who realize a discount factor $\delta \leq \delta^*$ behave hand-to-mouth in period 2, consuming all liquid wealth in that period. Instead, agents realizing $\delta \geq \delta^*$ are not liquidity-constrained.

In period 3, therefore, a fraction $F(\delta^*)$ of the population consumes $\alpha\overline{c}_3$ units. These individuals would like to have consumed less in period 3, but this would have required selling down their stock of illiquid assets, which is not feasible. The remaining agents implement their period-2 optimal consumption plan, which involves an amount $c_3(\delta) = \frac{w+(q-p)\alpha\overline{c}_3}{1+\beta\delta}\frac{\beta\delta}{q}$. To understand this, notice that these individuals behave like standard neoclassical consumers with logarithmic utility and discount factor $\beta\delta$. They assign a fraction $\frac{1}{1+\beta\delta}$ of wealth to period-2 consumption, and implement a consumption growth rate equal to $\frac{\beta\delta}{q}$. Finally, available wealth for these individuals is the market value of inherited liquid wealth $(w - p\alpha\overline{c}_3)$ plus their illiquid assets stock, valued at the imputed price q.

Using the fact that at $\delta = \delta^*$, the period-2 Euler equation holds with equality when behaving hand-to-mouth, $\frac{\alpha\overline{c}_3}{w-p\alpha\overline{c}_3} = \frac{\beta\delta^*}{q}$, we can re-write

$$c_3(\delta) = \frac{(1+\beta\delta^*)\,\delta}{(1+\beta\delta)\,\delta^*}\alpha\overline{c}_3.$$

For $x \in \left[\underline{\delta}, \overline{\delta}\right]$, let

$$\Lambda(x) \equiv \alpha F(x) + \int_x^{\overline{\delta}} \frac{(1+\beta x)\,\delta}{(1+\beta\delta)\,x}\alpha f(\delta)\,d\delta,$$

and the period-3 market clearing condition is equivalent to $\Lambda(\delta^*) = 1$. It can be verified that $\Lambda(x)$ is a strictly decreasing, continuous function. This implies that the equilibrium δ^* is unique.

Once δ^* is obtained, the supporting prices can be backed out. In particular,

$$\begin{aligned} q^* &= \frac{\beta\delta^*}{\alpha - (1-\alpha)\beta\delta^*}\frac{\overline{c}_2}{\overline{c}_3}, \\ \frac{p^*}{q^*} &= \frac{1 + \mathbb{E}[\delta] - F(\delta^*) + \frac{1}{\beta\delta^*}\int_{\underline{\delta}}^{\delta^*}\delta f(\delta)\,d\delta}{1 + \mathbb{E}[\delta] + \beta\delta^* F(\delta^*) - \int_{\underline{\delta}}^{\delta^*}\delta f(\delta)\,d\delta}. \end{aligned}$$

The liquid assets price q^* (which can be verified to be positive by using $\Lambda(x)$) comes from $\frac{\alpha\overline{c}_3}{w - p\alpha\overline{c}_3} = \frac{\beta\delta^*}{q}$ and the period-1 budget constraint. The equilibrium price ratio ρ^* is the value that supports δ^* as the individuals' optimal choice, by satisfying their first order conditions.

Using the implicit function formula and the chain rule where applicable, the following comparative statics can be derived

$$\begin{aligned} \frac{\partial\delta^*}{\partial\alpha} &= \frac{\delta^*(1+\beta\delta^*)}{\alpha}\frac{1}{1-\alpha F(\delta^*)}, \\ \frac{\partial q^*}{\partial\alpha} &= \frac{q^*}{\alpha - (1-\alpha)\beta\delta^*}\frac{(1+\beta\delta^*)\alpha F(\delta^*)}{1-\alpha F(\delta^*)}. \end{aligned}$$

In addition, from the market clearing conditions it follows that if $\alpha = 0$, $q^* < \frac{\overline{c}_2}{\overline{c}_3}\beta E[\delta]$; while for $\alpha \to 1$, q^* converges to $q = \frac{\overline{c}_2}{\overline{c}_3}\beta\overline{\delta}$.

The equilibrium period-1 expected utility is

$$V^*(\alpha) = \int_{\underline{\delta}}^{\overline{\delta}} (\log c_2(\delta) + \delta \log c_3(\delta)) f(\delta)\,d\delta.$$

It is useful to notice that, by construction, when $\delta \leq \delta^*$ consumption is

$$(c_2(\delta), c_3(\delta)) = \left(\frac{w + (q^* - p^*)\alpha\overline{c}_3}{1+\beta\delta^*}, \frac{w + (q^* - p^*)\alpha\overline{c}_3}{1+\beta\delta^*}\frac{\beta\delta^*}{q^*}\right).$$

Finally, use $w + (q^* - p^*)\alpha\overline{c}_3 = \overline{c}_2 + q^*\overline{c}_3$ and compute expected utility:

$$\begin{aligned} V^*(\alpha) &= \int_{\underline{\delta}}^{\delta^*}\left(\log\frac{\overline{c}_2 + q^*\overline{c}_3}{1+\beta\delta^*} + \delta\log\frac{\overline{c}_2 + q^*\overline{c}_3}{1+\beta\delta^*}\frac{\beta\delta^*}{q^*}\right) f(\delta)\,d\delta \\ &+ \int_{\delta^*}^{\overline{\delta}}\left(\log\frac{\overline{c}_2 + q^*\overline{c}_3}{1+\beta\delta} + \delta\log\frac{\overline{c}_2 + q^*\overline{c}_3}{1+\beta\delta}\frac{\beta\delta}{q^*}\right) f(\delta)\,d\delta. \end{aligned}$$

α affects expected utility through the equilibrium price q^* and cutoff discount factor δ^*:

$$\frac{\partial V^*(\alpha)}{\partial\alpha} = -\frac{\beta\delta^* - E[\delta|\,\delta \leq \delta^*]}{(1+\beta\delta^*)\delta^*}F(\delta^*)\frac{\partial\delta^*}{\partial\alpha} + \frac{\overline{c}_3}{\overline{c}_2 + q^*\overline{c}_3}\left(1 - \frac{E[\delta]\,\overline{c}_2}{q^*\overline{c}_3}\right)\frac{\partial q^*}{\partial\alpha}.$$

The first term is

$$-\frac{\beta\delta^* - E\left[\delta \middle| \delta \leq \delta^*\right]}{(1+\beta\delta^*)\,\delta^*} F(\delta^*) \frac{\partial \delta^*}{\partial \alpha} = -\frac{\beta\delta^* - E\left[\delta \middle| \delta \leq \delta^*\right]}{(1+\beta\delta^*)\,\delta^*} F(\delta^*) \frac{\delta^*(1+\beta\delta^*)}{\alpha} \frac{1}{1-\alpha F(\delta^*)}$$
$$= -\frac{\beta\delta^* - E\left[\delta \middle| \delta \leq \delta^*\right]}{\delta^*} F(\delta^*) \frac{\delta^*}{\alpha} \frac{1}{1-\alpha F(\delta^*)}$$
$$= \psi(\delta^*) \frac{\delta^*}{\alpha} \frac{1}{1-\alpha F(\delta^*)}.$$

The second one is

$$\frac{\overline{c}_3}{\overline{c}_2 + q^*\overline{c}_3} \left(1 - \frac{E[\delta]\,\overline{c}_2}{q^*\overline{c}_3}\right) \frac{\partial q^*}{\partial \alpha} = \frac{\overline{c}_3}{\overline{c}_2 + q^*\overline{c}_3} \left(1 - \frac{E[\delta]\,\overline{c}_2}{q^*\overline{c}_3}\right) \frac{q^*}{\alpha - (1-\alpha)\beta\delta^*} \frac{(1+\beta\delta^*)\,\alpha F(\delta^*)}{1-\alpha F(\delta^*)}.$$

Using the expression for q^* found above,

$$\frac{q^*\overline{c}_3}{\overline{c}_2 + q^*\overline{c}_3} = \frac{\beta\delta^*}{\alpha(1+\beta\delta^*)},$$

so

$$\frac{\overline{c}_3}{\overline{c}_2 + q^*\overline{c}_3} \left(1 - \frac{E[\delta]\,\overline{c}_2}{q^*\overline{c}_3}\right) \frac{\partial q^*}{\partial \alpha} = \left(1 - \frac{E[\delta]\,\overline{c}_2}{q^*\overline{c}_3}\right) \frac{\beta\delta^*}{\alpha - (1-\alpha)\beta\delta^*} \frac{F(\delta^*)}{1-\alpha F(\delta^*)}$$
$$= \left(q^*\frac{\overline{c}_3}{\overline{c}_2} - E[\delta]\right) \frac{F(\delta^*)}{1-\alpha F(\delta^*)}.$$

Putting these results together, the derivative is

$$\frac{\partial V^*(\alpha)}{\partial \alpha} = \psi(\delta^*) \frac{\delta^*}{\alpha} \frac{1}{1-\alpha F(\delta^*)} + \left(q^*\frac{\overline{c}_3}{\overline{c}_2} - E[\delta]\right) \frac{F(\delta^*)}{1-\alpha F(\delta^*)}.$$

The sign is given by the sign of

$$\psi(\delta^*)\,\delta^* + \left(q^*\frac{\overline{c}_3}{\overline{c}_2} - E[\delta]\right) \alpha F(\delta^*) \equiv A(\alpha) + B(\alpha).$$

$A(\alpha)$ and $B(\alpha)$ are functions of α since δ^* is an increasing function of α.

By the chain rule, $A(\alpha)$ is the negative, decreasing function $\psi(\delta^*)$ multiplied by the positive, increasing function δ^* – therefore, it is a negative, decreasing function.

$B(\alpha)$ is negative for α low and positive for α high, as $q^*\overline{c}_3/\overline{c}_2$ increases from being strictly smaller than $\beta E[\delta]$ to $\beta\overline{\delta}$ (and the latter is larger than $E[\delta]$ from assumption F). The term between parenthesis is therefore first negative and then positive, always increasing. It is multiplied by $\alpha F(\delta^*)$ which is positive and increasing. Therefore, a α^* exists such that $B(\alpha)$ is negative for $\alpha < \alpha^*$, and it is positive and increasing for $\alpha > \alpha^*$.

The two facts together imply that $A(\alpha)+B(\alpha)$ is negative for $\alpha < \alpha^*$, and for $\alpha > \alpha^*$ it is the sum of a negative, decreasing function and a positive, increasing one. These can have at most one crossing. In other words, either $A(\alpha) + B(\alpha)$ is negative everywhere, or exactly one level $\alpha^{**} > \alpha^*$ exists such that $A(\alpha) + B(\alpha)$ is negative for $\alpha < \alpha^{**}$ and positive for $\alpha > \alpha^{**}$.

This proves that $\partial V^*(\alpha)/\delta\alpha$ is either always negative, or a unique level α^{**} exists such that $\partial V^*(\alpha)/\delta\alpha$ is negative for $\alpha < \alpha^{**}$ and positive for $\alpha > \alpha^{**}$. In turn, this means that $V^*(\alpha)$ is either always decreasing (and therefore maximized at $\alpha = 0$), or decreasing for $\alpha < \alpha^{**}$ and increasing for $\alpha > \alpha^{**}$. Finally, the fact that $V^*(0) > V^*(1)$, found above to hold for any β and distribution $F(\delta)$ implies that conclude that in the latter case, $V^*(\alpha) < V^*(0)$ is true for all α.

5.11 Proposition 10.

Proposition 10. *There is a unique equilibrium in this economy, in which the fraction of agents using the commitment technology is* $\lambda^* = \min\left\{\overline{\lambda}, 1\right\}$. *For* $\theta < 1$, *equilibrium welfare is strictly decreasing in* $\overline{\lambda}$ *for* $\overline{\lambda} < 1$, *and strictly increasing for* $\overline{\lambda} > 1$. *There exists a threshold value* $\overline{\overline{\lambda}} > 1$ *such that when* $\overline{\lambda} = \overline{\overline{\lambda}}$, *period-*1 *utility is the same as when* $\overline{\lambda} = 0$; *it is strictly lower for* $\overline{\lambda} \in (0, \overline{\overline{\lambda}})$, *and strictly higher for* $\overline{\lambda} > \overline{\overline{\lambda}}$. *The threshold* $\overline{\overline{\lambda}}$ *is strictly decreasing in* θ; *it takes value* 1 *when* $\theta = 1$, *and converges to* ∞ *when* $\theta \to 0$.

Proof. Let w denote the mark value of the representative wealth stock. This is composed by the initial endowment of 1 unit, plus firms' net output. Let i denote aggregate investment; let q denote the price of period-3 goods. Then $w = 1 - i + qi^{\alpha}$.

When λ individuals act on period-1 preferences and the rest on period-2, aggregate investment is

$$i = 1 - \left(\frac{\lambda}{2} + \frac{1-\lambda}{1+\beta}\right) w \equiv 1 - \gamma w.$$

On the other hand, the firms' optimal demand of capital k satisfies $\alpha q k^{\theta} = k$. Equilibrium in the goods markets requires that i satisfies this last equality. This solves for

$$i = \theta \frac{1-\gamma}{\theta + (1-\theta)\gamma}.$$

Notice that i is strictly increasing in λ. The values of prices and wealth are:

$$q = \left(\frac{1-\gamma}{\theta + (1-\theta)\gamma}\right)^{1-\theta} \theta^{-\theta}, \ w = \frac{1}{\theta + (1-\theta)\gamma}.$$

Both q and w are strictly increasing in λ. Equilibrium utility for non-committed agents is

$$V_{NC} = \log \frac{w}{1+\beta} + \log \frac{w}{q} \frac{\beta}{1+\beta}.$$

Replacing for the equilibrium quantities found above,

$$\frac{\partial V_{NC}}{\partial \lambda} = -\frac{(1-\theta)(1-\lambda)(1-\beta)^2}{2(1-\gamma)(\theta + (1-\theta)\gamma)(1+\beta)^2} < 0.$$

When $\theta = 1$, the derivative is null.

Equilibrium utility for committed agents is

$$V_C = 2\log \frac{w}{2} + \log \frac{1}{q} - \gamma.$$

Replacing for equilibrium quantities, the difference between V_C and V_{NC} only depends on λ through γ :

$$V_C - V_{NC} = \ln \frac{(1+\beta)^2}{4\beta} - \gamma.$$

$\lambda^* = \overline{\lambda}$ is the unique value that makes this difference 0, and therefore agents indifferent between committing or not. If λ^* was smaller than $\overline{\lambda} \leq 1$, then every agent would strictly prefer to commit, so $\lambda^* < \overline{\lambda}$ could not be an equilibrium value. Analogously, $\lambda^* > \overline{\lambda}$ could not be an equilibrium value since every agent would strictly prefer not to commit. This is why the equilibrium fraction of the population acquiring commitment must equal $\overline{\lambda}$ if this number is not larger than 1. If $\overline{\lambda} > 1$, then in equilibrium all agents use the commitment technology and strictly prefer to do so.

Finally, let

$$\overline{\overline{\kappa}}(\theta) = \left(2\log \frac{w}{2} + \log \frac{1}{q} \bigg|_{\lambda=1} \right) - \left(\log \frac{w}{1+\beta} + \log \frac{w}{q} \frac{\beta}{1+\beta} \bigg|_{\lambda=0} \right). \tag{29}$$

The first term is the utility value from the period-1 optimal consumption plan under the assumption that all individuals pursue this plan. The second term is the period-1 utility value from the period-2 optimal plan, under the assumption that all individuals pursue this plan. Thus, (29) is the utility difference achieved by making all individuals follow a pre-committed plan.

If and only if the commitment technology's cost is lower than $\overline{\overline{\kappa}}(\theta)$ will individuals gain from switching from a no-commitment economy to one in which everyone pre-commits. Evaluating the equilibrium prices and wealth, this is

$$\overline{\overline{\kappa}}(\theta) = (1+\theta) \ln \frac{1+\theta\beta}{\beta+\theta\beta} + \ln \beta.$$

Equivalently, this solves for

$$\overline{\overline{\lambda}}(\theta) \equiv \frac{1}{(1+\theta)\ln \frac{1+\theta\beta}{\beta+\theta\beta} + \ln \beta} \ln \frac{(1+\beta)^2}{4\beta}$$

Notice that $\overline{\overline{\lambda}}(\theta)$ is strictly decreasing in θ; it equals 1 for $\theta = 1$ and converges to ∞ as θ approaches 0.

This establishes Proposition 10. ■

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