# A Continuous-Time Agency Model of Optimal Contracting and Capital Structure†

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**ABSTRACT.** We explore optimal financing in a setting when the agent can conceal and divert cash flows from a project, and investors' only means of forcing repayment is the threat of termination. DeMarzo and Fishman (2003) show that that an optimal contract in this setting is a combination of credit line, debt and equity. The credit line gives the agent flexibility to run the project when it temporarily generates losses, and an equity stake gives the agent incentives not to divert cash. We look at optimal securities in detail using the transparency of continuous-time characterizations. We explore how the optimal credit limit depends on a specific project, and how market values of securities change for the duration of the project. We consider an extension in which the mean of cash flows depends on the agent's effort choice. The model provides a simple dynamic theory of security design and optimal capital structure.

# 1. Introduction

In this paper, we consider a dynamic contracting environment in which a risk-neutral agent or entrepreneur with limited resources manages an investment activity. While the investment is profitable, it is also risky, and in the short-run can generate large losses. The agent will need outside financial support to cover these losses and continue the project. The difficulty is that while the distribution of the cash flows is publicly known, the agent may distort these cash flows by taking a hidden action that leads to a private

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benefit. Specifically, the agent may (i) conceal and divert cash flows for his own consumption, and/or (ii) stop providing costly effort, which reduces the mean of the cash flows. Therefore, from the perspective of the principal or investors funding the project, there is the concern that a low cash flow realization may be a result of the agent's actions, rather than the project fundamentals. To provide the agent with appropriate incentives, investors control the agent's wage, and may withdraw their financial support for the project and force its early termination. We seek to characterize an optimal contract in this framework and relate it to the firm's choice of capital structure.

DeMarzo and Fishman (2003), hereafter denoted DF, consider a discrete-time model of this sort. Using a dynamic-programming approach DF show that an optimal contract is a combination of a credit line, debt and equity. Dividends are paid when cash flows exceed debt payments and the credit line is paid off. If debt service payments are not made or the credit line is overdrawn, the project is terminated with a probability that depends on the size of the cash shortfall. The defining feature of this contract is a credit limit, which can be found by computing a fixed point of a Bellman equation in a discrete-time model. The continuous-time model of this paper has an alternative convenient way to compute an optimal contract using an ordinary differential equation. Using a continuous-time characterization, we explore the optimal contract in more detail. We are able to show how the optimal credit limit depends on the distribution of the project's cash flows and the consequences of liquidation. We describe the dynamics of security prices. In addition, we also derive results about how optimal project selection depends on the credit line balance. In all cases our differential equation characterization proves very useful for the analysis.

In our continuous-time setting the cumulative cash flows generated by the project follow a Brownian motion with a positive drift. We derive the optimal contract using continuous-time techniques introduced by Sannikov (2003). Two differences emerge between the optimal contracts in discrete and continuous time. First, termination is no longer stochastic in continuous time, but occurs the moment the credit line is overdrawn or there is a default on the long-term debt. Second, because the project can generate large short-term losses, projects that are very risky will not use long-term debt but instead require a compensating balance with the credit line. (A compensating balance is a cash deposit that the firm must hold with the lender to maintain the credit line.) The compensating balance serves two roles: it allows for a larger credit line, which is valuable given the risk of the project; and it provides an inflow of interest payments to the project that can be used to somewhat offset operating losses. The model therefore provides an explanation for why firms might hold substantial cash balances at low interest rates while simultaneously borrowing at higher rates.

For the bulk of our analysis, we focus on the case in which the agent can conceal and divert cash flows. We show in Section 4 that the characterization of the optimal contract is unchanged if the agent makes a hidden binary effort choice. We also consider the possibility of contract renegotiation in Section 5, and characterize the optimal renegotiation-proof contract.

## 1.1. Related Literature.

Our paper is part of a growing literature on dynamic optimal contracting models using recursive techniques that began with Green (1987), Spear and Srivastava (1987), Phelan and Townsend (1991), and Atkeson (1991) among others. (See, for example, the text by Ljungqvist and Sargent (2000) for a description of many of these models.) As previously mentioned, this paper builds directly on the model of DeMarzo and Fishman (2003). Other recent work developing optimal dynamic agency models of the firm includes Albuquerque and Hopenhayn (2001), Clementi and Hopenhayn (2000), DeMarzo and Fishman (2003b), and Quadrini (2001). With the exception of DeMarzo and Fishman (2003), these papers do not share our focus on an optimal capital structure. In addition, none of these models are formulated in continuous time.

While discrete time models are adequate conceptually, in many cases a continuous-time setting may prove to be much simpler and more convenient analytically. An important example of this is the principal-agent model of Holmstrom and Milgrom (1987), hereafter HM, in which the optimal continuous-time contract is shown to be linear. Schattler and Sung (1993) develop a more general mathematical framework for analyzing agency problems of this sort in continuous time, and Sung (1995) allows the agent to control volatility as well. Hellwig and Schmidt (2002) look at the conditions for a discrete-time principal-agent model to converge to the HM solution. See also Bolton and Harris (2001), Ou-yang (2003), Detemple, Govindaraj and Loewenstein (2001), Cadenillas, Cvitannic and Zapatero (2003) for further generalization and analysis of the HM setting.

Several features distinguish our model from the HM problem: the investor's ability to terminate the project, the agent's consumption while the project is running, and the nature of the agency problem. In HM, the agent runs the project until date *T*, and then receives compensation. In our model, the agent receives compensation many times while the project is running, until the contract calls for the agent’s termination. Also, HM analyze a setting in which the agent takes hidden actions. In our main setting the agent observes private payoff-relevant information; we also consider the possibility of a binary hidden action choice. Unlike HM, the termination decision is a key feature of the optimal contract in our setting. Here, as in DF, we demonstrate how this decision can be implemented through bankruptcy.[1]

Sannikov (2003) and Williams (2004) analyze principal-agent models, in which the principal and the agent interact dynamically. Their interaction is characterized by evolving state variables. In their models, the agent continuously chooses actions (e.g. hidden effort) that are not directly observable to the principal, and the principal takes actions (e.g. payments to the agent) that affect the agent's payoff. Besides having a dynamic nature in the spirit of Sannikov (2003) and Williams (2004), our paper develops a new method to deal with the problem of private observations in continuous time. Also, unlike in Sannikov (2003) and Williams (2004), hidden savings do not pose any additional difficulties in our model. We derive an optimal contract in a setting without hidden savings, and verify that it remains incentive compatible even when the agent can save secretly.

In contemporaneous work, Biais et al. (2004) consider a dynamic principal-agent problem in which the agent’s effort choice is binary (work or shirk). While they do not formulate the problem in continuous time, they do exam the continuous limit of the

[1] Spear and Wang (2002) also analyze the decision of when to fire an agent in a discrete-time model. They do not consider the implementation of the decision through standard securities.

discrete-time model and focus on the implications for the firm's balance sheet. As we show in Section 4, their setting is a special case of our model and our characterization of the optimal contract applies.

This paper is organized as follows. Section 2 presents a continuous-time model. After that, it derives an optimal contract and its implementation with standard securities: credit line, debt and equity. In Section 3 we analyze the properties of an optimal contract, providing characterizations that cannot be obtained in a discrete-time setting. Sections 4 shows the optimality of our contract with hidden binary effort. Section 5 analyzes renegotiation-proof contracts and the issue of robustness. Section 6 concludes the paper.

# 2. The Setting and the Optimal Contract

In this section we describe a continuous-time formulation of the contracting problem that arises when the agent privately observes the cash flows of the project that he manages on investors' behalf. We then solve the model and derive an optimal contract. We use a dynamic programming approach. The ultimate form of an optimal contract is analogous to that in discrete time, but the techniques to derive it are somewhat different. The derivation employs a HJB equation, which is analogous to the Bellman equation in discrete time, subject to incentive compatibility and promise keeping conditions.

The main contribution of this section is methodological: to formulate the model in continuous time and derive an optimal contract. We then show how to implement the optimal contract through a choice of capital structure, where we allow the agent to control the firm's payout policy. While the form of an optimal contract and its implementation turns out to be similar to that in the discrete-time model of DeMarzo and Fishman (2003), we will see that an optimal contract has a sharper characterization in continuous time, which can be exploited to derive comparative statics results and analyze extensions in the following sections.

## 2.1. The Dynamic Agency Model

The agent manages a project that generates stochastic cash flows with mean $\mu$ and variance $\sigma^2$

$$dY_t = \mu\, dt + \sigma\, dZ_t,$$

where $Z$ is a standard Brownian motion. The agent observes the actual cash flows $Y$, but the principal does not. The agent makes a report $\{\hat{Y}_t; t \geq 0\}$ of the realized cash flows to the principal. The principal does not know whether the agent is lying or telling the truth. The principal receives the reported cash flows $d\hat{Y}_t$ from the agent and gives him back transfers of $dI_t$ that are based on the agent's reports. Formally, the agent's income process $I_t$ is non-decreasing and $\hat{Y}$-measurable. If the agent underreports realized cash flows, he steals the difference. Stealing may be costly: the agent is able to enjoy only a fraction $\lambda \in$

(0,1] of what he steals. Also, the agent can over-report and put his own money back into the project. As a result, the agent receives a flow of income of[2]

$$[dY_t - d\hat{Y}_t]^{\lambda} + dI_t\,, \quad \text{where } [dY_t - d\hat{Y}_t]^{\lambda} \equiv \underbrace{\lambda(dY_t - d\hat{Y}_t)^+}_{\text{stealing}} - \underbrace{(dY_t - d\hat{Y}_t)^-}_{\text{over-reporting}} \tag{1}$$

To make sure that the agent does not receive income of minus infinity, we assume that process $\hat{Y}_t - Y_t$ has to have bounded variation.

The agent is risk-neutral and discounts his consumption at rate $\gamma$. This continues until a termination time $\tau$ that is contractually specified by the principal. The agent maintains a private savings account, from which he consumes and into which he deposits his income. The principal cannot observe the balance of the agent's savings account. The agent's balance $S_t$ grows at interest rate $\rho < \gamma$:

$$dS_t = \rho S_t dt + [dY_t - d\hat{Y}_t]^{\lambda} + dI_t - dC_t\,, \tag{2}$$

where $dC_t$ is the agent's consumption at time $t$, which must be nonnegative. The agent must maintain a nonnegative balance on his account, i.e. $S_t \geq 0$.

Once the contract is terminated, the agent receives payoff $R \geq 0$ from an outside option. Therefore, the agent's total expected payoff from the contract at date 0 is given by[3]

$$W_0 = E\left[\int_0^{\tau} e^{-\gamma s} dC_s + e^{-\gamma\tau} R\right]. \tag{3}$$

The principal discounts cash flows at rate $r$, such that $\gamma > r \geq \rho$.[4] Once the contract is terminated, she receives expected liquidation payoff $L \geq 0$. The principal's total expected profit at date 0 is

$$b_0 = E\left[\int_0^{\tau} e^{-rs}(d\hat{Y}_s - dI_s) + e^{-r\tau} L\right].$$

The project requires external capital of $K \geq 0$ to be started. The principal offers to contribute this capital in exchange for a contract $(\tau, I)$ which specifies a termination time $\tau$ and payments $\{I_t;\ 0 \leq t \leq \tau\}$ that are based on reports $\hat{Y}$. Formally, $I$ is a $\hat{Y}$-measurable continuous process, and $\tau$ is a $\hat{Y}$-measurable stopping time.

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[2] Note that (1) implies that the agent pays a proportional cost $(1-\lambda)$ to divert funds, but does not recover the cost if the funds are put back into the firm. We could also allow the agent to conceal and save funds within the firm, avoiding the cost $(1-\lambda)$ if the funds are ultimately used to boost future reported cash flows (i.e., the cost is only paid if the funds are diverted to the agent's personal consumption). This change would not alter the results in any way.

[3] We can ignore consumption beyond date $\tau$ because $\gamma \geq r$ implies it is optimal for the agent to consume all savings at termination (i.e., $S_\tau = 0$).

[4] Typically for a borrowing-constrained agent the intertemporal marginal rate of substitution is greater than the market interest rate $r$. To represent the idea that the agent is borrowing-constrained in a risk-neutral setting, we assume that $\gamma > r$. (The case $\gamma = r$ requires either a finite horizon setting or introducing a bound on the magnitude of the project's per period operating losses; otherwise it is optimal to postpone the agent's consumption indefinitely.)

In response to a contract ($\tau$, $I$), the agent chooses a strategy. A feasible strategy is a pair of processes ($C$, $\hat{Y}$) adapted to $Y$, such that

(i) process $Y_t - \hat{Y}_t$ has bounded variation,

(ii) process $C_t$ is nondecreasing, and

(iii) the savings process, defined by (2), stays nonnegative.

The agent chooses a feasible strategy to maximize his expected payoff. Therefore, the agent's strategy ($C$, $\hat{Y}$) is *incentive compatible* if it maximizes his total expected payoff $W_0$ given a contract ($\tau$, $I$). An *incentive compatible contract* refers to a quadruple ($\tau$, $I$, $C$, $\hat{Y}$) that includes the agent's recommended strategies.

We have not explicitly modeled the agent's option to quit and receive the outside option $R$ at any time. We could incorporate this by including an individual rationality constraint requiring that the agent's future payoff from continuing at date $t$, $W_t$, is no worse than his outside option $R$ for all $t$. However, in our setting this is not necessary as the individual rationality constraint will never bind. The agent can always under-report and steal at rate $\gamma R$ until termination to obtain a payoff of $R$ or greater. Thus any incentive compatible strategy yields the agent at least $R$.

The optimal contracting problem is to find an incentive-compatible contract ($\tau$, $I$, $C$, $\hat{Y}$) that maximizes the principal's profit subject to delivering to the agent an initial required payoff $W_0$. By varying $W_0$ we can use this solution to consider different divisions of bargaining power between the agent and the investors.

**Remark.** For simplicity, we have specified the contract assuming the agent's income $I$ and the termination time $\tau$ are determined uniquely by the agent's report. While this assumption rules out public randomization, because the principal's value function turns out to be concave (Proposition 1), public randomization would not improve the contract and this assumption is without loss of generality. In section 5, however, we introduce public randomization when considering renegotiation-proof contracts.

## 2.2. Derivation of the Optimal Contract

We solve the problem of finding an optimal contract in several steps. First, we show that it is sufficient to look for an optimal contract within a smaller class of contracts, namely contracts in which the agent chooses to report cash flows truthfully and maintain zero savings. Thus we consider a relaxed problem by ignoring the possibility that the agent can save secretly. Our derivation of an optimal contract for the relaxed problem follows the framework of the discrete-time optimal contracting literature. Along the way we explain the techniques from stochastic calculus that we need in continuous time. Finally, we show that the contract is fully incentive compatible even when the agent can save secretly.

We begin with a revelation principle type of result:

**Lemma A.** *There exists an optimal contract in which the agent chooses to tell the truth, and maintains zero savings.*

**PROOF:** See Appendix. ◆

The intuition for this result is straightforward – it is inefficient for the agent to conceal and divert cash flows ($\lambda \le 1$) or to save them ($\rho \le r$). We can improve the contract by having the investors save and make direct payments to the agent. Thus, we can look for an optimal contract in which truth telling and zero savings is incentive compatible.

## The Optimal Contract without Saving

Note that if the agent could not save, then he would not be able to over-report cash flows and would consume all income as it is received. Thus,

$$dC_t = dI_t + \lambda(dY_t - d\hat{Y}_t). \tag{4}$$

We can relax the problem by restricting the agent's savings so that (4) holds. After we find an optimal contract for the relaxed problem, we show that it remains incentive-compatible even if the agent can save secretly.

One difficulty with working in a dynamic setting is the complexity of the contract space. The contract can depend on the entire path of reported cash flows $\hat{Y}$, making it difficult to evaluate the agent's incentives in a tractable way. Our first task is to find a convenient representation for the agent's incentives. To do so, define the agent's promised value $W_t(\hat{Y})$ after a history of reports ($\hat{Y}_s$, $0 \le s \le t$) to be the total expected payoff the agent receives, from transfers and termination utility, if he tells the truth after time $t$:

$$W_t(\hat{Y}) = E_t\left[\int_t^\tau e^{-\gamma(s-t)} dI_s + e^{-\gamma(\tau-t)} R\right]$$

The following result provides a useful representation for $W_t(\hat{Y})$.

**LEMMA B.** *At any moment of time $t \le \tau$ there is a sensitivity $\beta_t(\hat{Y})$ of the agent's continuation value towards his report such that*

$$dW_t = \gamma W_t dt - dI_t + \beta_t(\hat{Y})(d\hat{Y}_t - \mu dt) \tag{5}$$

*This sensitivity $\beta_t(\hat{Y})$ is determined by the agent's past reports $\hat{Y}_s$, $0 \le s \le t$.*

**PROOF:** Note that $W_t(\hat{Y})$ is also the agent's promised value if $\hat{Y}_s$, $0 \le s \le t$ were the true cash flows and the agent reported truthfully. Therefore, without loss of generality we can prove (5) for the case when the agent truthfully reports $\hat{Y} = Y$. In that case,

$$V_t = \int_0^t e^{-\gamma s} dI_s(Y) + e^{-\gamma t} W_t(Y) \tag{6}$$

is a martingale and by the martingale representation theorem there is a process $\beta$ such that $dV_t = e^{-\gamma t}\, \beta_t(Y)\, (dY_t - \mu\, dt)$, where $dY_t - \mu\, dt$ is a multiple of the standard Brownian motion. Differentiating (6) with respect to $t$ we find

$$dV_t = e^{-\gamma t}\beta_t(Y)(Y_t - \mu dt) = e^{-\gamma t} dI_t(Y) - \gamma e^{-\gamma t} W_t(Y) dt + e^{-\gamma t} dW_t(Y)$$

and thus (5) holds. ◆

Informally, the agent has incentives not to steal cash flows if he gets at least $\lambda$ of promised value for each reported dollar, i.e. if $\beta_t \geq \lambda$. If this condition holds for all $t$ then the agent's payoff will always integrate to less than his promised value if he deviates. If this condition fails on a set of positive measure, the agent can obtain at least a little bit more than his promised value if he underreports cash when $\beta_t < \lambda$. We summarize our conclusions in the following proposition.

**LEMMA C.** *If the agent cannot save, truth-telling is incentive compatible if and only if* $\beta_t \geq \lambda$ *for all* $t \leq \tau$.

**PROOF:** If the agent steals $dY_t - d\hat{Y}_t$ at time $t$, he gains immediate income of $\lambda(dY_t - d\hat{Y}_t)$ but loses $\beta_t(dY_t - d\hat{Y}_t)$ in continuation payoff. Therefore, the payoff from reporting strategy $\hat{Y}$ gives the agent the payoff of

$$W_0 + E\left[\int_0^\tau e^{-\gamma t}\lambda(dY_t - d\hat{Y}_t) + \int_0^\tau e^{-\gamma t}\beta_t(dY_t - d\hat{Y}_t)\right], \tag{7}$$

where $W_0$ denotes the agent's payoff under truth-telling. We see that if $\beta_t \geq \lambda$ for all $t$ then (7) is maximized when the agent chooses $d\hat{Y}_t = dY_t$, since the agent cannot over-report cash flows. If $\beta_t < \lambda$ on a set of positive measure, then the agent is better off underreporting on this set than always telling the truth.[5] ◆

Now we use the dynamic programming approach to determine the most profitable way for the principal to deliver to the agent any value $W$. We present an informal argument, which is formalized in the proof of Proposition 1. Denote by $b(W)$ the principal's value function (the highest profit to the principal that can be obtained from a contract that provides the agent with payoff $W$). To facilitate our derivation of $b$, we assume $b$ is concave. In fact, we could always ensure that $b$ is concave by allowing public randomization, but at the end of our intuitive argument we will see that public randomization is not needed in an optimal contract.[6]

Because the principal has the option to provide the agent with $W$ by paying a lump-sum transfer of $dI > 0$ and moving to the optimal contract with payoff $W - dI$,

$$b(W) \geq b(W - dI) - dI\ . \tag{8}$$

Equation (8) implies that $b'(W) \geq -1$ for all $W$; that is, the marginal cost of compensating the agent can never exceed the cost of an immediate transfer. Define $W^1$ as the lowest value such that $b'(W^1) = -1$. Then it is optimal to pay the agent according to

$$dI = \max(W - W^1, 0) \tag{9}$$

[5] The agent's report affects $\beta$. How do we know that lying does not alter $\beta$ to be always greater than or equal to $\lambda$, whereas we had $\beta < \lambda$ on a set of positive measure under truthtelling? One way the agent can lie is by reporting $d\hat{Y}_t = dY_t - dt$ when $\beta < \lambda$ and telling the truth when $\beta \geq \lambda$. Then the probability measure over the agent's reports has the same positive probability events as the measure over the true cash flows, so $\beta(\hat{Y}) \geq \lambda$ on set of positive measure even after a deviation.

[6] Given the linearity of the incentive compatibility condition, public randomization would only be useful for allowing stochastic termination of the contract.

These transfers, and the option to terminate, keep the agent's promised value between $R$ and $W^1$. Within this range, equation (5) implies that the agent's promised value evolves according to $dW_t = \gamma W_t dt + \beta_t \sigma dZ_t$ when the agent is telling the truth. We need to determine the sensitivity β of the agent's value to reported cash flows. Using Ito's lemma, the principal's expected cash flows and changes in contract value are given by

$$E[dY + db(W)] = \left(\mu + \gamma W b'(W) + \tfrac{1}{2}\beta^2\sigma^2 b''(W)\right)dt$$

Because at the optimum the principal should earn an instantaneous total return equal to the discount rate, $r$, we have the following Bellman equation for the value function:

$$rb(W) = \max_{\beta \geq \lambda} \quad \mu + \gamma W b'(W) + \tfrac{1}{2}\beta^2\sigma^2 b''(W) \tag{10}$$

Given the concavity of $b$, $b''(W) \leq 0$ and so $\beta = \lambda$ is optimal.[7] Intuitively, because the inefficiency in this model results from early termination, reducing the risk to the agent lowers the probability that the agent's promised value falls to $R$.

The principal's value function therefore satisfies the following second-order ordinary differential equation:

$$rb(W) = \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W), \quad R \leq W \leq W^1, \tag{11}$$

with $b(W) = b(W^1) - (W - W^1)$ for $W > W^1$.

We need three boundary conditions to pin down a solution to this equation and the boundary $W^1$. The first boundary condition arises because the principal must terminate the contract to hold the agent's value to $R$, so $b(R) = L$. The second boundary condition is the usual "smooth pasting" condition – the first derivatives must agree at the boundary, and so $b'(W^1) = -1$.[8]

The final boundary condition is the "super contact" condition for the *optimality* of $W^1$, which requires that the second derivatives match at the boundary. This condition implies that $b''(W^1) = 0$, or equivalently, using equation (11),

$$rb(W^1) + \gamma W^1 = \mu. \tag{12}$$

This boundary condition has a natural interpretation. It is beneficial to postpone payment to the agent by making $W^1$ larger because it reduces the risk of early termination. Postponing payment is sensible until the boundary (12), when the principal and agent's required expected returns exhaust the available expected cash flows.[9] An example of the value function is shown in Figure 1.

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[7] In fact, we show in the proof that $b(W)$ is strictly concave for $W \leq W^1$ (see also footnote 9), so that $\beta = \lambda$ is the unique optimum.

[8] Roughly speaking, if there were a kink at $W^1$, $b''(W^1) = -\infty$ and (11) could not be satisfied.

[9] A similar argument can be used to show that public randomization is not useful. If it were, then $b$ would be linear ($b'' = 0$) in the region in which it is used. At a boundary $w$ of this region, the super contact condition would require $b''(w) = 0$ and so $rb(w) = \mu + \gamma w b'(w)$. But $b'(w) \leq -1$ implies $rb(w) + \gamma w \geq \mu$, and thus $w > W^1$. Thus, $b$ is strictly concave on $[R, W^1]$ and there is no role for public randomization.

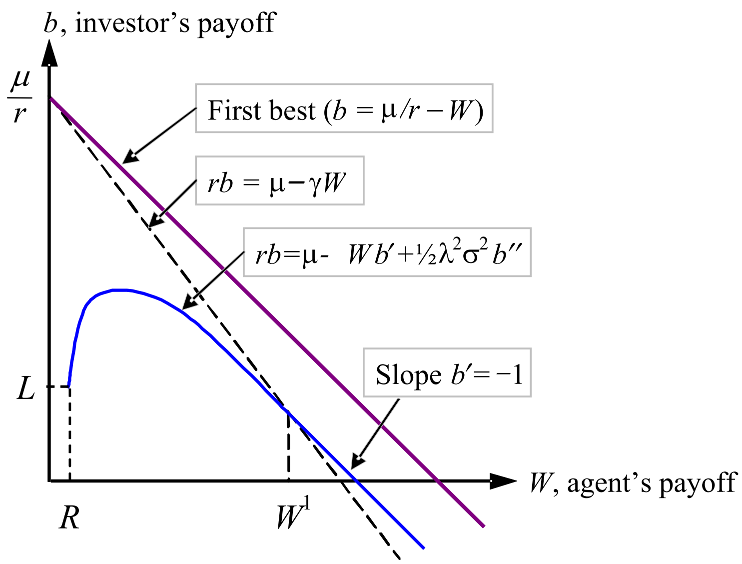


**Figure 1:** The Principal's Value Function $b(W)$

The following proposition formalizes our findings:

**PROPOSITION 1.** *The contract that maximizes the principal's profit and delivers to the agent value $W_0 \in [R, W^1]$ takes the following form: $W_t$ evolves as*

$$dW_t = \gamma W_t dt - dI_t + \lambda(d\hat{Y}_t - \mu\, dt). \tag{13}$$

*When $W_t \in [R, W^1)$, $dI_t = 0$. When $W_t = W^1$, payments $dI_t$ cause $W_t$ to reflect at $W^1$. If $W_0 > W^1$, an immediate payment $W_0 - W^1$ is made. The contract is terminated at time $\tau$ when $W_t$ hits $R$. The principal's expected payoff at any point is given by a concave function $b(W_t)$, which satisfies*

$$rb(W) = \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W) \tag{14}$$

*on the interval $[R, W^1]$ and $b'(W) = -1$ for $W \geq W^1$, with boundary conditions $b(R) = L$ and $rb(W^1) = \mu - \gamma W^1$.*

**PROOF:** See Appendix.

## Hidden Savings

Thus far, we have restricted the agent from saving. We now show that the contract of Proposition 1 remains incentive compatible even when we relax this restriction. The intuition for the result is that because the marginal benefit to the agent of reporting or consuming cash is constant over time, and since private savings grow at rate $\rho < \gamma$, there is no incentive to delay reporting or consumption. In fact, in the proof we show that this result holds even if the agent can save within the firm without paying the diversion cost.

**PROPOSITION 2.** *Suppose the process $W_t$ is bounded above and solves*

$$dW_t = \gamma W_t dt - dI_t dt + \lambda(d\hat{Y}_t - \mu dt). \tag{15}$$

*until stopping time* $\tau = \min\{t \mid W_t = R\}$. *Then the agent earns payoff of at most* $W_0$ *from any feasible strategy in response to a contract* $(\tau, I)$. *Furthermore, payoff* $W_0$ *is attained if the agent reports truthfully and maintains zero savings.*

**PROOF:** See Appendix. ◆

This result confirms that contracts from a broad class, including the optimal contract of Proposition 1, remain incentive-compatible even if the agent has access to hidden savings. Proposition 2 will help us characterize incentive-compatible capital structures in the next subsection.

## 2.3. Capital Structure Implementation

The optimal contract in our setting depends upon the history of reported cash flows. This history dependence is captured through the promised payoff $W$ to the agent. In this section, we show that the optimal contract can be implemented using standard securities: equity, long-term debt, and a credit line. We begin by describing these securities.

*Equity.* Equity holders receive dividend payments made by the firm. Dividends are paid from the firm's available cash or credit, and are at the discretion of the agent.

*Long-term Debt.* Long-term debt is a consol bond that pays continuous coupons at rate $x$. Without loss of generality, we let the coupon rate be $r$, so that the face value of the debt is $D = x/r$. If the firm defaults on a coupon payment, debt holders force termination of the project.

*Credit Line*. A revolving credit line provides the firm with available credit up to a limit $C^L$. Balances on the credit line are charged a fixed interest rate $r^c$. The firm borrows and repays funds on the credit line at the discretion of the agent. If the balance on the credit line exceeds $C^L$, the firm defaults and the project is terminated.

We now show that the optimal contract can be implemented using a capital structure based on these three securities. While the implementation is not unique (e.g., one could always use the single contract derived in Section 2.2, or strip the long-term debt into zero-coupon bonds), it provides a natural interpretation. It also demonstrates how the contract can be decentralized into limited liability securities (equity and debt) that can be widely held by investors. Finally, it shows that the optimal contract is consistent with a capital structure in which, in addition to the ability to steal the cash flows, the agent has wide discretion regarding the firm's leverage and payout policy – the agent can choose when to draw on or repay the credit line, the amount of dividends, and whether to accumulate cash balances (earning interest $r$) within the firm.

Before stating our main result, we note that while it will be important for the pricing of the securities, for purposes of implementation it is not necessary to specify the prioritization of the securities over the liquidation payoff $L$ in the event of termination. We will, however, compensate the agent with equity, and it is important that the agent does not receive part of the liquidation payoff. Thus, we define *inside equity* as identical to equity, but with the provision that it is worthless in the event of termination.[10] (With

[10] Inside equity could correspond to a stock grant to the agent combined with a zero interest loan due upon termination that equals or exceeds the liquidation value of the equity.

absolute priority, this distinction will often be unnecessary, as debt holders claims will typically exhaust $L$.)

**PROPOSITION 3.** *Consider a capital structure in which the agent holds inside equity for fraction* $\lambda$ *of the firm, the credit line has interest rate* $r^c = \gamma$, *and debt satisfies*

$$rD = \mu - \gamma R / \lambda - \gamma C^L. \tag{16}$$

*Then it is incentive compatible for the agent to refrain from stealing, and to use the project cash flows to pay the debt coupons and credit line before issuing dividends. Once the credit line is fully repaid, all excess cash flows are issued as dividends. With this capital structure, the agent's expected future payoff* $W_t$ *is determined by the current draw* $M_t$ *on the credit line:*

$$W_t = R + \lambda\left(C^L - M_t\right). \tag{17}$$

*This capital structure implements the optimal contract if, in addition, the credit limit satisfies*

$$C^L = \lambda^{-1}(W^1 - R). \tag{18}$$

**PROOF:** See Appendix. ◆

The intuition for the incentive compatibility of this capital structure is as follows. First, providing the agent with the fraction $\lambda$ of the equity eliminates his incentive to steal cash flows because he can do as well by paying dividends. But how can we ensure that the agent does not pay dividends prematurely by, for example, drawing down the credit line immediately and paying a large dividend? Given balance $M_t$ on the credit line, the agent can pay a dividend of $C^L - M_t$ and then default. But if (17) holds, the payoff from deviating in this way is equal to the payoff $W_t$ that the agent receives from paying off the credit line before paying dividends, and so there is no incentive to deviate. Finally, because the agent earns interest at his discount rate $\gamma$ paying off the credit line, but earns interest at rate $r < \gamma$ on accumulated cash, the agent has the incentive to pay dividends once the credit line is repaid.

The role of the long-term debt, defined by (16), is to adjust the profit rate of the firm so that the agent's payoff does indeed satisfy equation (17).[11] If the debt were too high, the agent's payoff would be below the amount in (17), and the agent would draw down the credit line immediately. If the debt is too low and the firm's profit rate too high, the agent would build up cash reserves after the credit line was paid off in order to reduce the risk of termination. Thus, as long as (16) holds, we say the capital structure is *incentive compatible* – the agent will not steal and will pay dividends if and only if the credit line is fully repaid.

Under what conditions does this capital structure implement the optimal contract of section 2.2? Note that the history dependence of the optimal contract is implemented

[11] One can rewrite (16) as $\lambda\,(\mu - rD - \gamma C^L) = \gamma R$, which states that the agent's share of the firm's profit rate (after interest payments) matches the agent's outside option when the credit line is exhausted.

through the credit line, with the balance on the credit line acting as the "memory" device to track the agent's payoff $W_t$. In the optimal contract, the agent is paid in order to keep the promised payoff from exceeding $W^1$. Here, dividends are paid when the balance on the credit line $M_t = 0$. For the capital structure to implement the optimal contract, these conditions must coincide. Solving equation (17) for $C^L$ leads to the optimality condition $C^L = \lambda^{-1}(W^1 - R)$.

There is no guarantee that in this capital structure the debt required by equation (16) is positive. If $D < 0$, we interpret the debt as a *compensating balance.* A compensating balance is a cash deposit required by the bank issuing the credit line. The firm earns interest on this balance at rate $r$, and the interest supplements the firm's cash flows. The firm cannot withdraw this cash, and it is seized by creditors in the event of default. We will examine the settings in which a compensating balance arises in the next section.

The implementation here is very similar to the implementation shown in the discrete-time model of DeMarzo and Fishman (2003).[12] There are three important distinctions. First, because cash flows arrive in discrete portions, the termination decision is stochastic in the discrete-time setting (i.e. the principal randomizes when the agent defaults). Second, because cash flows may be arbitrarily negative in a continuous-time setting, the contract may involve a compensating balance requirement as opposed to debt. Lastly, the discrete-time framework does not allow for a characterization of the incentive compatibility condition for the capital structure in terms of the primitives of the model, as we do here.

# 3. Optimal Capital Structure and Security Prices

The capital structure implementation of the optimal contract inspires many interesting questions. What factors determine the amount that the agent borrows? When will the agent borrow for initial consumption? When is there a compensating balance? What is the optimal length of the credit line? How do market values of securities involved in the contract depend on the firm's remaining credit? In this section, we exploit the continuous-time machinery to answer these questions and provide new insights.

## 3.1. The Debt Choice

A key feature of the optimal capital structure is its use of both fixed long-term debt and a revolving credit line. In this section we develop further intuition for how the amount of long-term debt, the size of the credit line, and the initial draw on the credit line are determined.

To simplify the analysis, we focus on the case $\lambda = 1$ in which there is no cost to diverting cash flows. In this case, the agent holds the equity of the firm, and finances the firm solely through debt. While this case might appear restrictive, the following result shows that the optimal debt structure with lower levels of $\lambda$ can be determined by considering an appropriate change to the termination payoffs.

[12] An alternative implementation is given in Shim (2004) and Biais et al. (2004) for a specialized setting. Rather than a credit line, they suppose the firm retains a cash reserve and that the coupon payment on the debt varies contractually with the level of the cash reserves.

**PROPOSITION 4.** *The optimal debt and credit line with agency parameter and termination payoffs* $(\lambda, R, L)$ *are the same as with parameters* $(1, R^\lambda, L^\lambda)$ *where*

$$R^\lambda = \tfrac{1}{\lambda} R \quad and \quad L^\lambda = \tfrac{1}{\lambda} L + (1 - \tfrac{1}{\lambda}) \tfrac{\mu}{r}.$$

**PROOF:** See Appendix. ◆

When $\lambda = 1$, the optimal credit limit is $C^L = W^1 - R$. The optimal level of debt is then determined by (16), which in this case can be written

$$rD = \mu - \gamma R - \gamma C^L = \mu - \gamma W^1$$

Recall also that in the optimal contract, $W^1$ is determined by the boundary condition (12):

$$rb(W^1) + \gamma W^1 = \mu$$

Combining these two results implies that the optimal face value of debt is $D = b(W^1)$. Figure 2 shows an example, illustrating the size of the credit line and the debt face value when the cash flow volatility is low. From the figure, $D > L$, so the debt is risky.

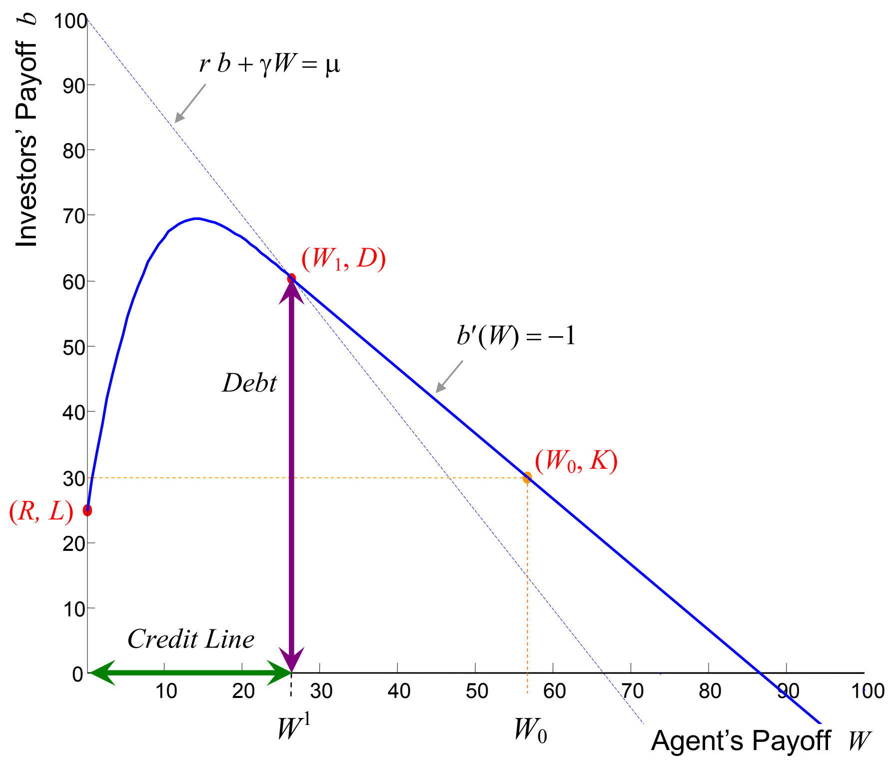


**Figure 2:** The Optimal Contract with Low Volatility
($L = 25$, $R = 0$, $\mu = 10$, $\sigma = 5$, $r = 10\%$, $\gamma = 15\%$, $\lambda = 1$, $K = 30$)

Note that the optimal capital structure for the firm does not depend on the external capital $K$ that is required. However, the initial payoffs of the agent and the investors depend upon $K$ as well as the parties' relative bargaining power. For example, if investors are competitive, the agent's initial payoff is the maximal payoff $W_0$ such that $b(W_0) = K$ as illustrated in Figure 2. In this example, $W_0 > W^1$. This payoff is achieved by giving the agent an initial cash payment of $W_0 - W^1$, and starting the firm with zero balance on the credit line (providing the agent with continuation payoff $W^1$). In other words, the firm initiates the credit line and issues the long-term debt. The capital raised is used to fund the project and pay an initial dividend of $W_0 - W^1$. The credit line is then used as needed to cover operating losses.

Thus, the initial capital that is raised from investors is $b(W^1)$, which is equal to the face value of the debt $D$. However, the debt is risky ($D > L$) and so, given coupon rate $r$, trades at a discount to its face value. How does the firm raise the additional capital to make up for this discount? Given the high interest rate $\gamma$ on the credit line, the lender earns an expected profit from the credit line, and so will pay this to the firm upfront. This payment exactly offsets the initial discount on the long-term debt due to credit risk.

Recall that the optimal credit line results from the following trade-off: a large credit line delays the agent's consumption, but also gives more flexibility to delay termination. Payments on debt are chosen to give the agent incentives to report truthfully: if payments on debt were too burdensome, the agent would draw down the credit line immediately and quit the firm; if they were too small, the agent would delay termination by saving excess cash flows when the credit line is paid off.

In Figure 3, we illustrate how these intuitive considerations affect the optimal contract for different levels of volatility. With an increase in volatility, the principal's profit function drops. Riskier cash flows require more financial flexibility, so the credit line becomes longer. Given the higher interest burden of the longer credit line, the optimal level of debt shrinks.

With medium volatility (as shown in the left panel of Figure 3), the face value of the debt is below the liquidation value of the firm ($D < L$). Thus, if the long-term debt has priority in default, it is now riskless. The firm will therefore raise $D$ through the long-term debt issue. However, in this case $D < K$. The additional capital needed to initiate the project is raised through an initial draw on the credit line of $W^1 - W_0$. Because $b' > -1$ on $(W_0, W_1)$, the draw on the credit line exceeds $K - D$. The difference can be interpreted as an initial fee charged by the lender to open the credit line with this initial balance; this fee compensates the lender for the negative NPV of the credit line due to the firm's greater credit risk.

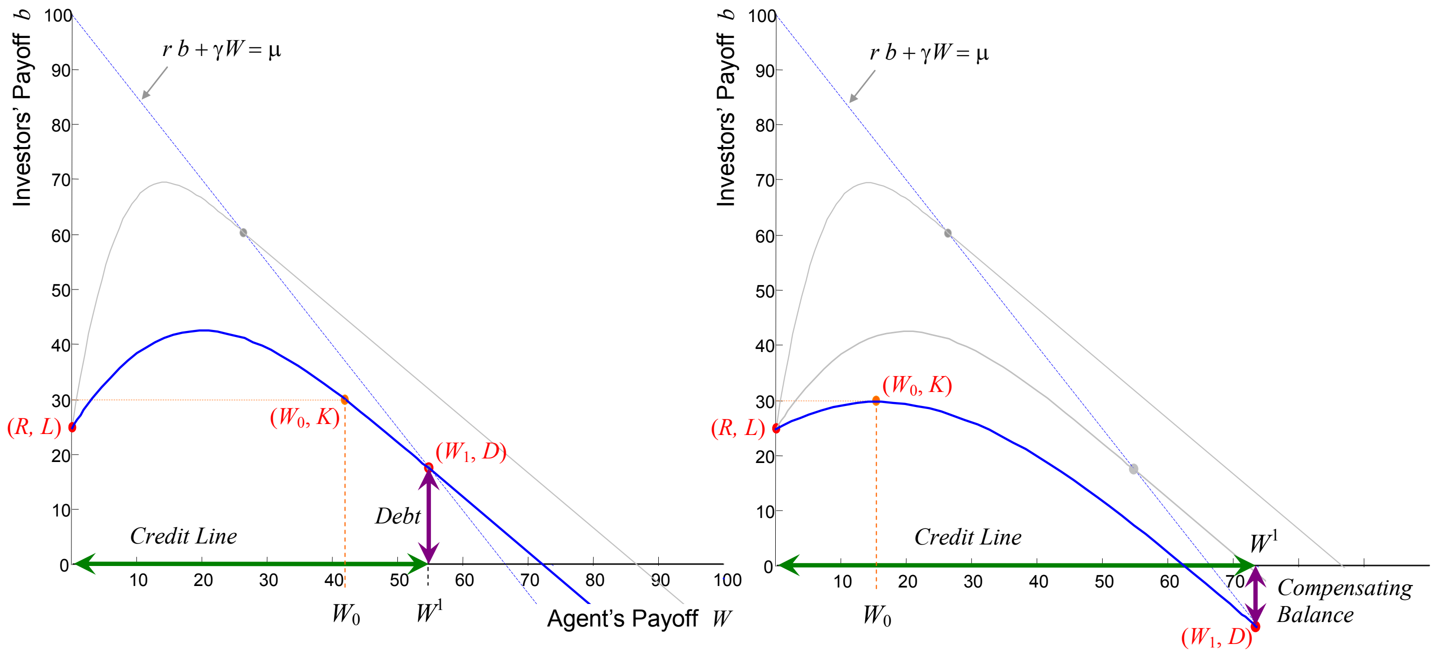


**Figure 3:** The Optimal Contract with Medium and High Volatility ($\sigma = 12.5$, $\sigma = 19.07$)

With high volatility (as shown in the right panel of Figure 3), the principal's profit falls further. This very risky project requires a very long credit line. Note that in this case $D = b(W^1) < 0$. Thus, the credit line has a compensating balance requirement – the firm must hold cash in the bank equal to $-D$ as a condition of the credit line. Both the required capital $K$ and the compensating balance $-D$ are funded through a large initial draw of $W^1 - W_0$ on the credit line. Given this large initial draw, substantial profits must be earned before dividends will be paid.

The compensating balance provides additional operating income of $rD$ to the firm. This income increases the attractiveness of the project to the agent, preventing the agent from leaving the firm when the balance on the credit line is high. By funding the compensating balance upfront, investors are committed to providing the firm with income $rD$ even when the credit line is paid off. This commitment is necessary since investors' continuation payoff at $W^1$ is negative, which would violate their limited liability. The compensating balance therefore serves to tie the agent and the investors to the firm in an optimal way.

Finally, note that if we increase volatility further in this example, the maximal profit for the principal falls below $K$. Thus, while such a project is positive NPV, it cannot be financed due to the incentive constraints.

**Remark.** While we have derived the agent's initial payoff assuming investors are competitive, other possibilities are straightforward to consider. For example, if the principal were a monopolist hiring the agent to run the firm, the contract would be initiated at the value $W^*$ that maximizes the principal's payoff $b(W^*)$. This would not change the optimal debt and credit limit, but in this case the firm would always start with a draw on the credit line. Interestingly, as can be seen in by comparing Figure 2 and

Figure 3, while higher volatility decreases $b(W^*)$, the effect on the agent's payoff $W^*$ is not monotonic. Thus the agent might prefer to manage a higher risk project.

## 3.2. Comparative Statics

How do the credit line, debt, and the agent's and investors' initial payoff depend on the parameters of the model? In the discrete-time setting, many of these comparative statics are analytically intractable, and must be computed for a specific example. A key advantage of the continuous time framework is that we can use the differential equation that characterizes the optimal contract to compute these comparative statics analytically.

Here we outline a new methodology for explicitly calculating comparatives statics. Details are in the Appendix. First, we derive the effect of parameters on the principal's profit. We start with the HJB equation for the principal's profit for a fixed credit line, which is represented by the interval $[R, W^1]$:

$$rb(W) = \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W)$$

The effect of any parameter θ on the principal's profit can be found by differentiating the HJB equation and its boundary conditions with respect to θ. During differentiation we keep $W^1$ fixed, which is justified by the envelope theorem. As a result, we get an ordinary differential equation for $\partial b(W)/\partial\theta$ with appropriate boundary conditions. We apply a generalization of the Feynman-Kac formula to write the solution as an expectation

$$\frac{\partial b(W)}{\partial\theta} = E\left[\int_0^\tau e^{-rt}\left(\frac{\partial\mu}{\partial\theta} + \frac{\partial\gamma}{\partial\theta}W_t b'(W_t) + \frac{1}{2}\frac{\partial(\lambda^2\sigma^2)}{\partial\theta}b''(W_t)\right)dt + e^{-r\tau}\frac{\partial L}{\partial\theta}\;\middle|\; W_0 = W\right] \quad (19)$$

where $dW_t = \gamma W_t dt - dI_t + \lambda dZ_t$ as before. Intuitively, equation (19) counts how much profit is gained or lost on the path of $W_t$ due to the modification of parameters. For example,

$$\frac{\partial b(W)}{\partial L} = E\left[e^{-r\tau}\;\middle|\; W_0 = W\right],$$

which is expected discounted value of a dollar at liquidation time.

Once we know the effect of parameters on the principal's profit, we deduce their effect on the debt and credit line by differentiating the boundary condition $rb(W^1) + \gamma W^1 = \mu$, and on the agent's starting value by differentiating $b(W_0) = K$ (or $b'(W^*) = 0$ when the principal is a monopolist). For example, the effect of $L$ is found as follows:

$$r\left(\frac{\partial b(W^1)}{\partial L} + \underbrace{b'(W^1)}_{-1}\frac{\partial W^1}{\partial L}\right) + \gamma\frac{\partial W^1}{\partial L} = 0 \quad\Rightarrow\quad \frac{\partial W^1}{\partial L} = -\frac{r}{\gamma - r}E\left[e^{-r\tau} \mid W_0 = W^1\right] < 0.$$

As $L$ increases, inefficiency of liquidation declines, so a shorter credit line optimally provides less financial flexibility for the project. By similar methods, we can quantify the impact of the model parameters on the main features of an optimal contract. The derivations are carried out in the appendix.

|  | $dC^L/$ | $dD/$ | $dW_0/$ | $dW^*/$ | $db(W^*)/$ |
|---|---|---|---|---|---|
| $dL$ | $-$ | $+$ | $+$ | $-$ | $+$ |
| $dR$[13] | $-$ | $-$ | $-$ | $+$ | $-$ |
| $d\gamma$ | $-$ | $\pm$ | $-$ | $-$ | $-$ |
| $d\mu$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| $d\sigma^2$ | $+$ | $-$ | $-$ | $\pm$ | $-$ |
| $d\lambda$ | $\pm$[14] | $+$ | $-$ | $\pm$ | $-$ |

**Table 1:** Comparative Statics for the Optimal Contract

While the analytic derivation of these results is technically involved, the intuition behind them is clear. Consider the effect of parameters on the credit line and debt. We already know that the credit line decreases as $L$ increases, because it makes liquidation less inefficient. This reduces the agent's temptation to draw the entire credit line and default, so the principal can extract greater coupon payments on debt. If the agent's outside option $R$ increases, the agent becomes more tempted to draw down the credit line. The length of the credit line decreases to reduce this temptation, and payments on debt decrease to make it more attractive for the agent to run the project, as opposed to taking the outside option. If the mean of cash flows $\mu$ increases, the credit line increases to delay termination and debt increases because the principal can extract more cash flows from the agent. If the agent's discount rate $\gamma$ increases, then the credit line decreases because it becomes costlier to delay the agent's consumption. On the other hand, the amount of debt could move either way due to two effects. For small $\gamma$ debt increases in $\gamma$ because the agent is able to borrow more through debt when the credit line is smaller. When $\gamma$ becomes large, the project becomes less profitable due to the agent's impatience, so the agent is able to borrow less through debt. We already saw in Section 3.1 why the credit limit increases and the debt decreases with volatility $\sigma$ – riskier projects require longer credit line and therefore the agent is able to borrow less through debt.

In Table 1, $b(W^*)$ is the maximum of the profitability of the project, i.e. the maximal amount of capital that the project can raise. When $L$ or $\mu$ increase the project becomes more profitable, so it can potentially raise more capital. When risk of the project $\sigma^2$ or the agent's impatience $\gamma$ increases, the project becomes less profitable. Finally, higher outside option $R$ makes it more difficult for investors to punish the agent, so overall profitability of the project decreases.

We conclude by computing the quantitative effect of the parameters on the debt choice of the firm for a specific example in Figure 4. Note for example that a compensating balance is required if $\sigma$ is high (to mitigate risk), if $R$ is high or $\mu$ is low (to increase the profit rate of the firm to maintain the agent's incentive to stay), or if $\lambda$ is very low (when

[13] These are found for the case when the project is profitable even if the agent does not have any initial cash, which implies that $b'(R) > 0$.

[14] $dC^L/d\lambda$ is negative if $R = 0$.

the agency problem is small, a smaller threat of termination is needed, and thus the credit line expands and debt shrinks). (Though not visible in the figure, it is also true as $\gamma \to r$.)

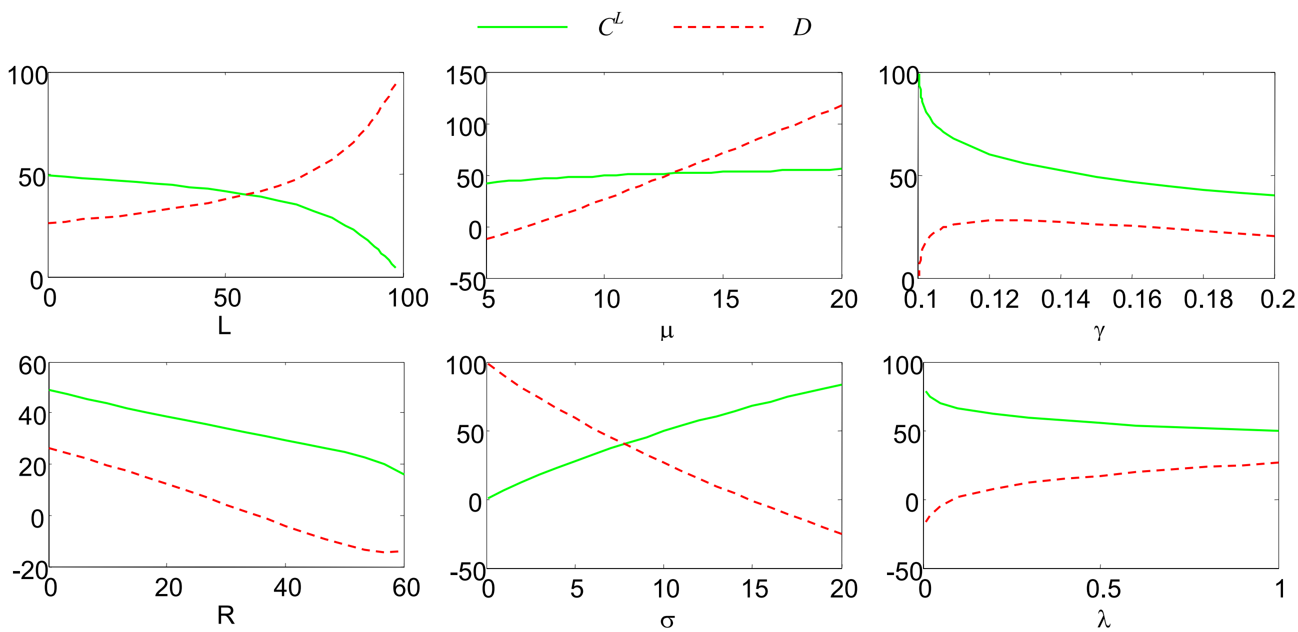


**Figure 4:** Comparative Statics (base case: $L = 0$, $R = 0$, $\mu = 10$, $\sigma = 10$, $r = 10\%$, $\gamma = 15\%$, $\lambda = 1$)

## 3.3. Security Market Values

We now consider the market values of the credit line, long-term debt and equity that implement the optimal contract. For this we need to make an assumption regarding the prioritization of the debt in default. We assume that the long-term debt is senior to the credit line; similar calculations could be performed for different assumptions regarding seniority.[15] With this assumption, the long-term debtholders get $L_D = \min(L, D)$ upon termination. The market value of long-term debt is therefore

$$V_D(M) = E\left[\int_0^\tau e^{-rt} x\, dt + e^{-r\tau} L_D \,\middle|\, M\right]$$

Note that we compute the expected discounted payoff for the debt conditional on the current draw $M$ on the credit line, which measures the firm's "distance to default" in our implementation.

Until termination, the equity holders get total dividends of $dDiv_t = dI_t / \lambda$, with the agent receiving fraction $\lambda$. At termination, the *outside* equity holders receive the remaining part of liquidation value, $L_E = \max(0, L - D - C^L)/(1-\lambda)$ per share, after the debt and credit line have been paid off.[16] The value of equity (per share) to outside equity holders is then

[15] Recall that only the aggregate payments to investors matter for incentives; the division of the payments between the securities is only relevant for pricing.

[16] Lemma E in the Appendix shows that $L < D + C^L$ when $\lambda = 1$ and there are no outside equity holders, so in that case we can set $L_E = 0$ to compute the "shadow price" of outside equity.

$$V_E(M) = E\left[\int_0^\tau e^{-rt}\, dDiv_t + e^{-r\tau} L_E \,\middle|\, M\right]$$

Finally, the market value of the credit line is

$$V_C(M) = E\left[\int_0^\tau e^{-rt}\,(dY_t - x\,dt - dDiv_t) + e^{-r\tau} L_C \,\middle|\, M\right]$$

where $L_C = \min(C^L, L - L_D)$. For the optimal capital structure, the aggregate value of the outside securities equals the principal's continuation payoff. That is, from (17),

$$b(R + \lambda(C^L - M)) = V_D(M) + V_C(M) + (1-\lambda)\, V_E(M).$$

We show in the appendix how to represent these market values in terms of an ordinary differential equation, so that they may be computed easily. See Figure 5 for an example. In this example, $L < D$ so that the long-term debt is risky. Note that the market value of debt is decreasing towards $L$ as the balance on the credit line increases towards the credit limit. Similarly the value of equity declines to 0 at the point of default. The figure also shows that the initial value of the credit line is positive – the lender earns a profit by charging interest rate $\gamma > r$. However, as the distance to default diminishes, additional draws on the credit line result in losses for the lender (for each dollar drawn, the value of the credit line goes up by less than one dollar, and eventually declines).

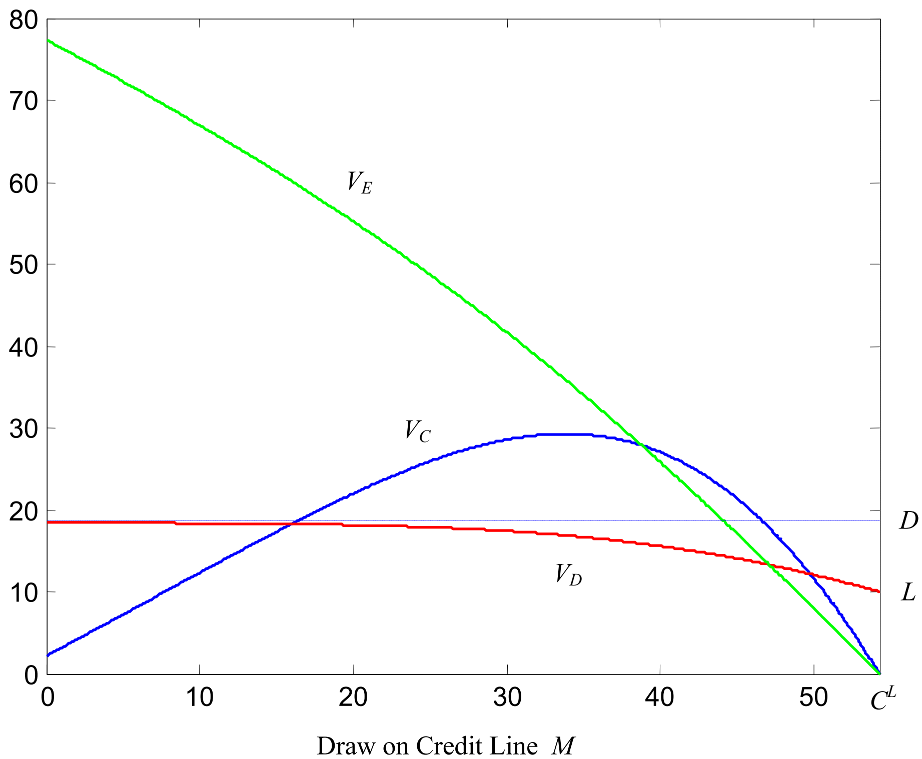


**Figure 5:** Market Values of Securities for $\mu = 10$, $\sigma = 10$, $\lambda = 50\%$, $r = 10\%$, $\gamma = 15\%$, $L = 10$, $R = 0$

Figure 5 also illustrates several other interesting properties of the security values. Note, for example, that the leverage ratio of the firm is not constant over time. When cash flows are high, the firm will pay off the credit line and its leverage ratio will decline. On the other hand, during times of low profitability, the firm increases its leverage. This pattern is broadly consistent with the empirical behavior of leverage.

One surprising observation from Figure 5: the value of equity is concave in the credit line balance, which implies that the value of equity would decline if the cash flow volatility were to increase. In fact, we can show:

**PROPOSITION 5.** *When debt is risky* ($L < D$)*, for the optimal capital structure the value of equity decreases if cash flow volatility increases. Thus, equity holders would prefer to reduce volatility.*

**PROOF:** See appendix. ◆

This is counter to the usual presumption that risky debt implies that equity holders benefit from an increase in volatility due to their option to default. That is, in our setting, there is no "asset substitution problem" related to leverage. Note also that the agent's payoff is linear in the credit line balance, so that the agent is indifferent regarding changes to volatility.

# 4. Hidden Effort

Throughout our analysis we have concentrated on the setting in which the cash flows are privately observed, and the agent may divert them for his own consumption. In this section we discuss the relationship between this model and a standard principal-agent model in which the agent makes a hidden binary effort choice. This model is also studied by Biais et al. (2004) in contemporaneous work. Our main result is that, subject to natural parameter restrictions, the solutions are identical for both models. Thus, all of our results apply to both settings.

In the standard principal-agent model with hidden effort, the principal observes the cash flows. Based on the cash flows, the principal decides how to compensate the agent, and whether to continue the project. Thus, there are only two key changes to our model. First, since cash flows are observed, there is no issue of the agent saving and using the savings to over-report future cash flows. Second, we assume that at each point in time, the agent can choose to shirk or work. Depending on this decision, the resulting cash flow process is

$$d\hat{Y}_t = dY_t + a\,dt\,, \text{ where } a = \begin{cases} 0 & \text{if the agent works} \\ -A & \text{if the agent shirks} \end{cases}$$

We assume that working is costly for the agent, or equivalently that shirking results in a private benefit.[17] Specifically, we suppose the agent receives an additional flow of utility equal to $\lambda A\, dt$ if he shirks. The agent cannot misreport the cash flows, since $r < \gamma$ the agent will consume all payments immediately. Thus, if the agent shirks,

$$dC_t = dI_t + \lambda\, A\, dt\,.$$

[17] The difference between the two interpretations amounts to shifting the agent's utility by a constant.

Again, $\lambda$ parameterizes the cost of effort and therefore the degree of the moral hazard problem. We assume $\lambda \leq 1$ so that working is efficient.

Our first result establishes the equivalence between this setting and our prior model:

**PROPOSITION 6.** *The optimal Principal-Agent contract implementing high effort is the optimal contract of Section 2.*

**PROOF:** The incentive compatibility condition in Lemma C is unchanged: to implement high effort at all times, we must have $\beta_t \geq \lambda\,\sigma$. But then Proposition 1 shows that our contract is the optimal contract subject to this constraint. ◆

It is not surprising that our original contract is incentive compatible in this setting, since shirking is equivalent stealing cash flows at a fixed rate. What is perhaps more surprising is that the additional flexibility the agent has in the cash flow diversion model does not require a "stricter" contract.

Of course, Proposition 6 does assume that implementing high effort at all times is optimal. Under what circumstances is this assumption correct? If a contract were to call for the agent to shirk after some history, the project cash flows would be diminished, but it would not be necessary to provide the agent with incentives.[18] Therefore, in these states the agent's continuation payoff would no longer need to be sensitive to the realized cash flows, and the agent's promised payoff would evolve as

$$dW_t = \begin{cases} \gamma W_t dt - dI_t + \lambda(d\hat{Y}_t - \mu\, dt) & \text{if } a = 0 \\ \gamma W_t dt - (dI_t + \lambda A\, dt) & \text{if } a = -A \end{cases}$$

Because the principal's continuation function is concave, this reduction in the volatility of $W_t$ could be beneficial. For that not to be the case, and for high effort to remain optimal, it must be that for all $W$,

$$rb(W) \geq (\mu - A) + (\gamma W - \lambda A)b'(W) \tag{20}$$

Intuitively, this equation states that the principal's payoff rate from having the agent shirk would be less than under our existing contract.[19] Define

$w^s = \lambda A/\gamma$ and $b^s = (\mu - A)/r = (\mu - \gamma w^s/\lambda)/r$,

the agent and principal's payoff if the agent shirks forever and receives no other payment. Then we have the following necessary and sufficient condition, as well as a simple sufficient condition, for high effort to remain optimal at all times:

**PROPOSITION 7.** *Implementing high effort at all times is optimal in the Principal-Agent setting if and only if* $b^s \leq f\left(w^s\right)$ *where* $f(z) \equiv \min_w b(w) + \frac{\gamma}{r}(z - w)b'(w)$. *A simpler sufficient condition is*

[18] Specifically, in Lemma C we can set $\beta_t = 0$ in states where the contract called for the agent to shirk.
[19] Formally, condition (20) is required in the proof of Proposition 1 for $G_t$ to remain a supermartingale for either effort choice.

$$b^s \leq \frac{\gamma}{r} b\left(w^s\right) + \left(1 - \frac{\gamma}{r}\right) b(W^*) \tag{21}$$

*Given λ, both of these conditions imply a lower bound on A, or equivalently, $w^s$.*

**PROOF:** See Appendix. ◆

We can interpret Proposition 7 as follows. The point $(w^s, b^s)$ represents the agent's and principal's payoff if the agent shirks forever. Thus, shirking is never optimal if and only if this point lies below the function *f*. The function *f* is concave and below *b*, with equality only at the maximum, as shown in Figure 6. The factor $\gamma/r$ increases the steepness of *f* relative to *b*; when $\gamma = r$, *f* and *b* coincide. As can be seen from the figure, Proposition 7 puts a lower bound on $w^s$, or equivalently on *A*, the magnitude of the cash flow impact of shirking. For example, in Figure 6, if $w^s \geq \underline{w}^s$, then high effort is always optimal. This is the case for $(w_1^s, b_1^s)$.

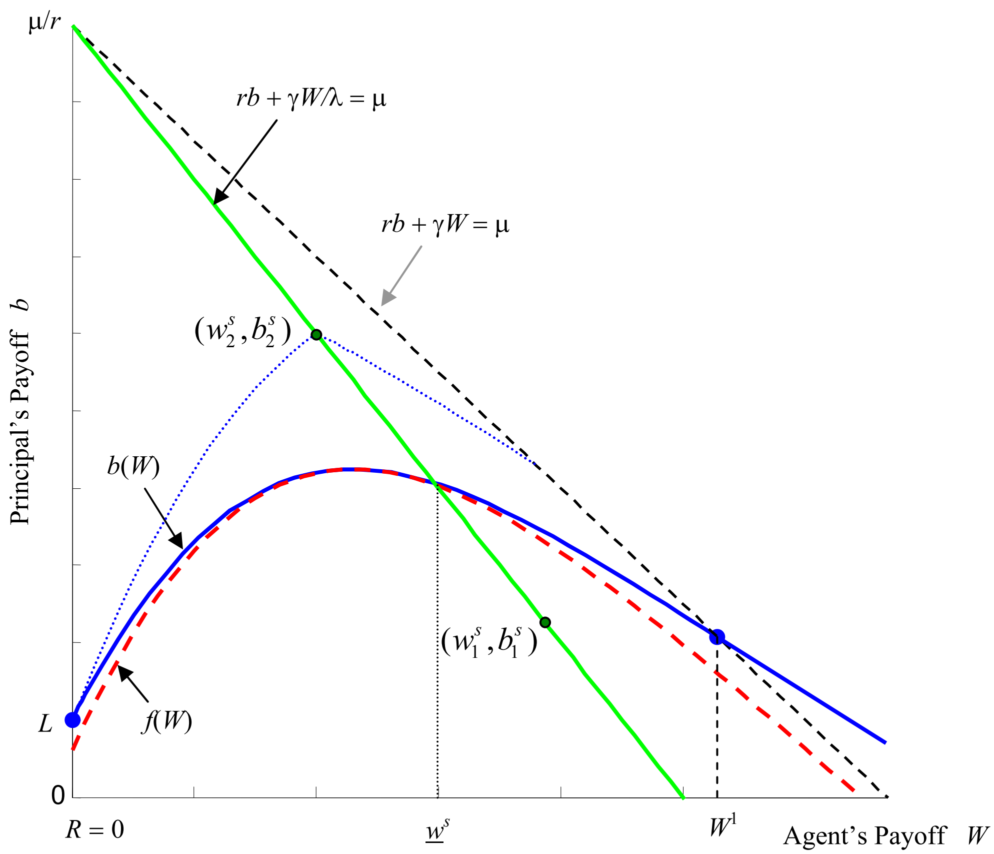


**Figure 6:** Example showing Optimality of High Effort

On the other hand, if *A* is too small so that $w^s < \underline{w}^s$, then the optimal principal-agent contract will involve shirking after some histories. However, in some cases the optimal contracting techniques of this paper may still apply. For example, see $(w_2^s, b_2^s)$ in Figure 6. In this case, the optimal contract calls for high effort until the point $(w_2^s, b_2^s)$ is reached; once it is reached the agent is paid a fixed wage and shirks forever. Thus, the optimal contract is again as in our model, but with a fixed wage and shirking in place of termination so that $(R, L) = (w_2^s, b_2^s)$.[20]

**Remark.** We can also consider a hybrid model, in which the agent can both divert cash flows and choose whether to work or shirk. In this case, let $\lambda_d$ parameterize the benefit the agent receives from diverting cash flows, and let $\lambda_a$ represent the benefit from shirking. Then we can show that the optimal contract implementing high effort is the optimal contract of Section 2 with $\lambda = \max(\lambda_d, \lambda_a)$.

# 5. Further Extensions of the Model

In this section we consider various extensions of the basic model. First, we allow the termination payoffs $(R, L)$ to be determined endogenously by either the principal's option to hire a new agent or the agent's option to start a new project. Second, we consider the construction of an optimal renegotiation-proof contract. Third, we consider the case in which the agent and principal disagree about key parameters of the model, such as the project's profitability, or the agent's impatience.

## 5.1. Endogenously Determined Termination Payoffs.

Thus far, we have treated the termination payoffs $(R, L)$ as exogenous. Suppose, however, that they are endogenously determined as in the following to examples:

**Unique Assets, Replaceable Agent:** The assets of the firm are unique, but the agent can be fired and replaced at cost $c_a$ to the principal/investors. The agent's termination payoff if fired is $R$, but the investor's payoff on firing the agent is

$$L = b(W^*) - c_a \tag{22}$$

**Unique Agent, Replaceable Assets:** Suppose the agent can quit the firm a start a new firm by raising external capital $K$ from new investors. If the agent quits, the old investors liquidate and receive $L$, while the agent receives

$$R = e^{-\gamma \Delta t} W_0 \tag{23}$$

where $\Delta t$ is the time required to start a new firm and $W_0$ satisfies $b(W_0) = K$.[21]

The optimal contract in either case takes exactly the same form as described in Section 2. The only change is that now the boundary condition (22) or (23) replaces $b(R) = L$. The solution is illustrated in Figure 7. Because $db(W^*)/dL < 1$, when assets are unique the liquidation value $L$ is decreasing in $c_a$. From the results of Section 3.2, the credit line increases and the debt decreases in $c_a$. This is intuitive, because the project requires more financial flexibility when it is more difficult to replace the agent. Similarly, when the

[20] This will be the case whenever shirking yields the highest possible payoff for the investors; i.e., when $A$ is sufficiently small. For intermediate values of $A$, an optimal contract calls for shirking only temporarily. In that case, a more complicated contract than the one described in this paper will be necessary to achieve optimality.

[21] This setting is similar to Hart and Moore's (1994) notion of "inalienable human capital" and its relationship to optimal debt structure.

agent is unique, as $\Delta t$ falls and it becomes easier for the agent to start a new firm, $R$ rises. This leads to a decrease in both the credit line and in debt. Note that as $\Delta t \to \infty$ and starting a new firm becomes impossible, $R \to 0$, and as $\Delta t \to 0$ and restarting is costless, $R \to R^*$, the point at which $b'(R^*) = 0$.

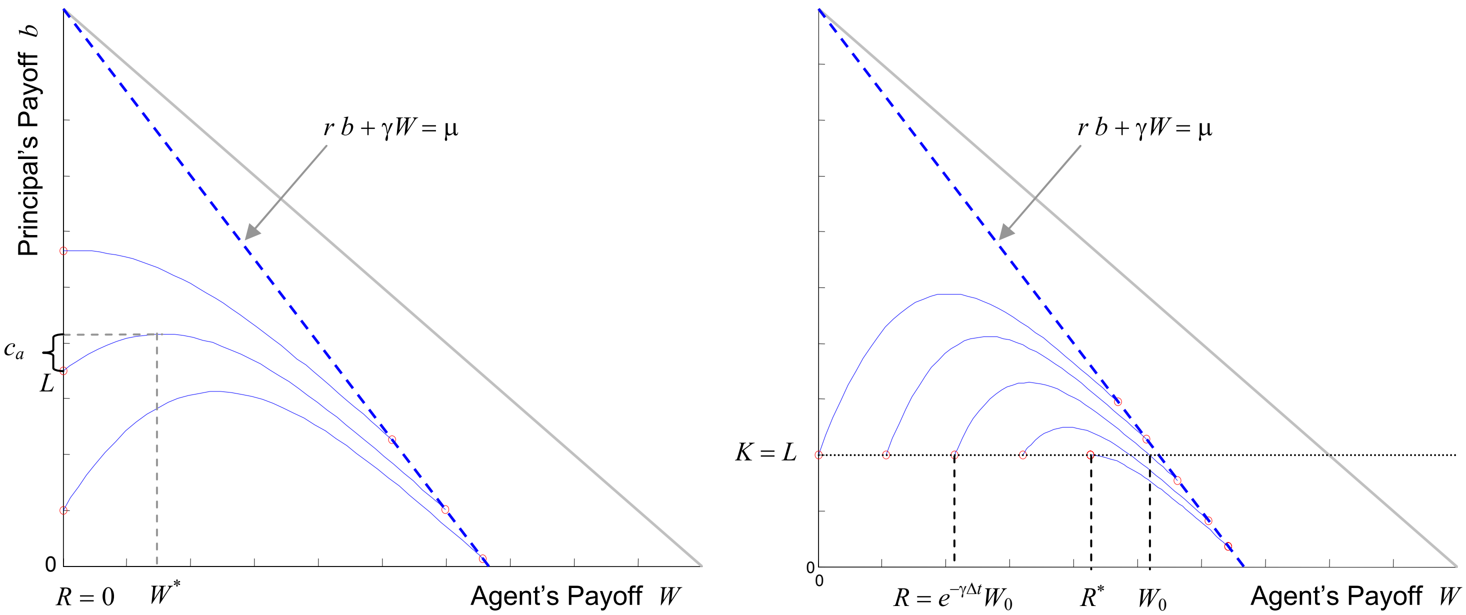


**Figure 7: Determining *L* or *R* endogenously.** The left panel considers the case in which the agent can be fired and replaced at cost $c_a$, so that $L = b(W*) - c_a$. The right panel considers the case in which the agent can quit and raise capital $K$ (in the example, $K = L$) to start a new firm with delay $\Delta t$, so that $R = e^{-\gamma\Delta t} W_0$.

## 5.2. Renegotiation-Proofness

Note that the optimal contracts in the basic model are generally not renegotiation-proof. When $b'(R) > 0$, then both the principal and the agent would like to renegotiate at termination time. Instead of termination, which gives the agent and the principal payoffs of $R$ and $L$, they could renegotiate by restarting the contract from with the agent's value $W > R$, which gives the principal profit $b(W) > L$.

To be renegotiation-proof, the principal's profit function $b(W)$ cannot have positive slope. To find this function we must solve the optimality equation from boundary conditions $b'(W^1) = -1$ and $rb(W^1) + \gamma W^1 = \mu$ for an appropriate choice of $W^1$, such that the maximum of the resulting function is $b(R^*) = L$. This is equivalent to the case in Section 5.1 of a unique agent that can restart the firm immediately ($\Delta t = 0$). Let us set $b(W) = L$ on the interval $[R, R^*]$.

A renegotiation-proof contract, under which the principal breaks even, exists only if the required external capital $K \le L$. In that case, the agent's continuation value $W_0$ is such that $b(W_0) = K$. Until termination the agent's continuation value $W_0$ evolves in the interval $[R^*, W^1]$ as

$$dW_t = \gamma W_t\, dt + \lambda(d\hat{Y}_t - \mu\, dt) - dI_t + dP_t,$$

where processes $I$ and $P$ reflect $W_t$ at endpoints $W^1$ and $R^*$ respectively. The project is terminated stochastically whenever $W_t$ is reflected at $R^*$. The probability that the project continues at time $t$ is

$$\Pr(\tau \geq t) = \exp\left(\frac{-P_t}{R^* - R}\right).$$

Then $W_t$ is the agent's true expected future payoff. Indeed, whenever $W_t$ hits $R^*$ and $dP_t$ is added to the agent's continuation value, the project is terminated with probability $dP_t / (R^* - R)$ to account for this increment to the agent's value.

The implementation of a renegotiation-proof contract involves a credit line and debt as in the optimal contract of Section 2.3 with $R^*$ in place of $R$. Since $R^* > R$, both the credit line and debt decrease. This is intuitive, because renegotiation-proofness reduces the profitability of the project.[22]

## 5.3. Private Benefits and Differing Opinions

Suppose the agent receives private benefits of control from running the project. Specifically, suppose that prior to termination the agent earns additional utility at rate $\gamma\omega$. In this case, the agent's continuation value should evolve according to

$$dW_t = \gamma(W_t - \omega)\,dt - dI_t + \lambda(d\hat{Y}_t - \mu\,dt)$$

How does this alter the form of the optimal contract? Interestingly, as the following result shows, this is equivalent to simply reducing the agent's outside opportunity by $\omega$.

**Proposition 8.** *Suppose the agent earns private benefits at rate $\gamma\omega$ while running the project. Then the optimal contract is the same as the optimal contract without private benefits and termination payoff for the agent of $R - \omega$. That is, while the project is running, the principal accounts for the agent's payoff through state variable $\hat{W}_t$ that evolves as*

$$d\hat{W}_t = \gamma\hat{W}_t\,dt - dI_t + \lambda(d\hat{Y}_t - \mu\,dt)$$

*in the interval $[R - \omega, W^1]$. Under this contract, given a value of the state variable $\hat{W}_t$, the agent's total payoff including private benefits is $W_t = \hat{W}_t + \omega$.*

**Proof:** See Appendix. ◆

Thus, using our comparative statics results for $R$ from Section 3.2, increasing the agent's private benefits increases the credit limit and amount of debt in the optimal capital structure. Intuitively, the potential threat of losing the private benefits in termination enhances the agent's incentives and hence increases the debt capacity of the firm. Moreover, $\hat{W}_0$ rises, so that the agent's total payoff rises by more than a dollar for each dollar of private benefits, all else equal.

[22] Gromb (1999) also considers renegotiation-proofness in a related discrete-time model. While not providing a complete characterization, he does show that in an infinite-horizon stationary setting the maximum external capital the firm can raise is the liquidation value $L$.

A similar result follows if the agent and the investor have different beliefs about the mean of the cash flows, μ. For example, suppose the agent believes the mean is μ + δ. Holding these beliefs fixed, the agent's continuation payoff should evolve according to

$$\begin{aligned} dW_t &= \gamma W_t dt - dI_t + \lambda(d\hat{Y}_t - (\mu+\delta)\, dt) \\ &= \gamma(W_t - \lambda\delta/\gamma)\, dt - dI_t + \lambda(d\hat{Y}_t - \mu\, dt) \end{aligned}$$

Thus, a discrepancy δ between the agent's and investor's beliefs is therefore equivalent to a private benefit of magnitude ω = λδ/γ.

## 5.4. Incorrect γ

What happens if an agent whose subjective discount factor is $\gamma' \neq \gamma$ receives contract $(I,\tau)$, which is optimal for the agent with discount factor $\gamma$? How will the agent behave? It depends on whether γ′ is greater or less than γ. We focus on the case when λ = 1.

If γ′ < γ, then when $W_t < W^1$ then the agent will deposit all cash flows from the project onto the credit line, and have balance zero in his savings account. The agent does not steal because a dollar on the credit line earns a higher rate of interest than the agent's subjective discount rate. Also the agent pays the credit line before saving because the credit line has a higher interest rate, and the agent is free to draw from the credit line at any time.

What happens when the balance on the credit line is paid off, i.e. $W_t = W^1$? It can be shown that the agent will choose to save at interest rate $\rho$ as long as $S_t < S^1$ for some critical value $S^l$. When $S_t = S^1$, the agent consumes all excess cash flows. Intuitively, for $S_t < S^1$, a dollar saved gives the agent more than a dollar of utility because it makes it less necessary to draw on the high-interest credit line to cover potential future losses. When $S_t = S^1$, the agent is on the margin indifferent between saving and consuming because the savings account earns an interest rate lower than the agent's subjective discount factor. We show in the following proposition that the principal earns the same profit as if the agent had discount factor γ, i.e. $b_\gamma(W_0)$. When γ is close to γ′, then $b_\gamma(W_0)$ is close to $b_{\gamma'}(W_0)$, so the mistake of overestimating the agent's discount factor is not too costly for the principal.

The following result summarizes this, and shows that the agent's behavior is drastically different if γ′ > γ :

**PROPOSITION 9.** *Suppose that the principal offers a contract designed for the agent with discount rate γ to an agent whose true discount rate is γ′ < γ. Then this agent would derive utility greater than $W_0$, and the principal would receive profit of exactly $b_\gamma(W_0)$.*

*If the agent's true discount factor γ′ is greater than γ for which the contract is designed, then the agent will draw the entire credit line and default immediately. The agent earns utility $W_0$, whereas the principal earns $L - (W_0 - R)$.*

**PROOF:** See Appendix. ◆

We conclude that underestimating the agent's subjective discount factor is a very costly mistake for the principal. This drastic difference raises the issue of robustness: what contract is optimal if the principal is uncertain about the agent's rate of time preference?

# 6. Conclusion

We analyzed a situation in which an agent or entrepreneur needs to raise external capital to (i) start-up a profitable project, (ii) cover future operating losses that may occur, and (iii) consume. In our setting, the agent can divert cash flows from the project for personal consumption without the investor's knowledge. To enforce payments, the investors can threaten to withhold future funding and terminate the project. We analyze an optimal contract between the investors and the agent in this setting.

An optimal contract takes a similar form both in a discrete-time setting of DeMarzo and Fishman (2003) and in our continuous-time setting. The contract involves a credit line, debt and equity. Debt, outside equity, and possibly the credit line provide the funds for start-up capital and initial consumption for the agent. For the duration of the project, the credit line provides the flexibility to cover possible operating losses. The agent has incentives to pay interest and not consume from the credit line because in case of default he has to surrender the project to investors. The agent holds an equity stake and has discretion over the payment of dividends. The agent's equity stake is sufficiently large that he does not divert excess cash flows for personal consumption, but pays them out as dividends appropriately.

The continuous-time setting of our paper has several advantages. First, the features of an optimal contract are cleaner. Unlike in discrete time, an optimal contract in continuous time does not require stochastic termination. Second, some of the analysis is simplified. Because time is continuous, we do not have to consider problems associated with different points within each time period separately. Most importantly, a continuous time model provides a convenient characterization of the optimal contract, which involves an ordinary differential equation. With this characterization we can say a great deal about how the optimal capital structure is determined by the specific features of the project. Also, we are able to compute the values of securities that are involved in the implementation of an optimal contract. Finally, we can easily analyze extensions. For example, we show how our contract also solves a standard principal-agent setting with costly effort. Other extensions are considered; in many cases the solution only involves finding the appropriate boundary conditions for the differential equation that defines an optimal contract.

Our results open several thought-provoking questions for future research. Here are only a few examples. First, we discover that in our setting there is no asset substitution problem. That is, increased variance of cash flows does not benefit equity holders because it makes agent's incentive problem more difficult. Second, we find that it may be very important for the principal to assess the characteristics of the project correctly. If an investor makes a mistake, the agent may draw the entire credit line for personal consumption and default immediately. This raises the question of robust contract design. Finally, the simplicity of our characterization opens the possibility of embedding our model within other standard finance models. For example, we can consider extending our framework to allow for dynamic investment decisions or project choice, and determine how the dynamics of these other decisions relate to cash flows and optimal capital structure.

# 7. Appendix

**PROOF OF LEMMA A:** Consider any incentive-compatible contract (τ, I, C, Ŷ). To prove the proposition, we show that there is a new incentive-compatible contract, which gives the same payoff to the agent and the same or greater payoff to the principal, under which the agent reports cash flows truthfully and maintains zero savings. This contract is ($\tau'(Y) = \tau(\hat{Y}(Y)), I'(Y) = C(Y), C, Y$). Note that the agent's consumption is the same as under the old contract, so he earns the same payoff. Let us show that the agent's strategy is incentive-compatible and that the principal earns the same or greater payoff.

Under the new contract the agent cannot improve his payoff by a deviation ($C', \hat{Y}'$), because any feasible consumption $C'$ is feasible under the old contract as well. If $C'$ is feasible under the new contract, then the agent always has nonnegative savings if he reports $\hat{Y}(\hat{Y}'(Y))$ and consumes $C'$ under the old contract. Indeed,

$$S_t(C',\hat{Y}') = \int_0^t e^{r(t-s)}(dI_t(\hat{Y}(\hat{Y}')) - dC_t') =$$

$$\underbrace{\int_0^t e^{r(t-s)}(dI_t(\hat{Y}(\hat{Y}')) - dC(\hat{Y}'))}_{\text{savings under old contract if } \hat{Y}' \text{ are true cash flows}} + \underbrace{\int_0^t e^{r(t-s)}(dC(\hat{Y}') - dC_t')}_{\text{savings under new contract under deviation}} \geq 0.$$

To show that the principal is at least as well-off before, note that the new contract avoids the inefficiency due to stealing and due to inefficient savings (at rate $\rho<r$) by the agent. Therefore, the principal's profit improves by

$$E\left[\int_0^\tau e^{-rt}\left((1-\lambda)(dY_t - d\hat{Y}_t)^+ + (r-\rho)S_t dt\right)\right],$$

where $S$ denotes the agent's savings under the old contract. ◆

**PROOF OF PROPOSITION 1.** First, let us verify that function $b$ defined in the Proposition is concave. Note that $b'(W) \geq -1$ and $rb(W) < \mu - \gamma W$ imply that $b'' < 0$. Therefore, to the left of point $W^1$ with boundary conditions $b'(W^1)=-1$ and $rb(W^1) = \mu - \gamma W^1$ function b enters the region where it is concave. It stays concave, because a concave function can never exit this region (this can be seen geometrically).

Next, let us prove that $b$ represents the principal's optimal profit under, which is achieved by the contract outlined in the Proposition. Define

$$G_t \equiv \int_0^t e^{-rs}(dY_s - dI_s) + e^{-rt}b(W_t).$$

Under an arbitrary incentive-compatible contract, $W_t$ evolves according to (5). Then from Ito's lemma,

$$e^{rt}dG_t = \underbrace{\left(\mu + \gamma W_t b'(W_t) + \tfrac{1}{2}\beta_t^2 b''(W_t) - rb(W_t)\right)dt}_{\le 0} \underbrace{-(1 + b'(W_t))dI_t}_{\le 0} + (\sigma + \beta_t b'(W_t))dZ_t$$

From (14) and the fact that $b'(W_t) \ge -1$, $G_t$ is a supermartingale. It is a martingale if and only if $\beta_t = \lambda\sigma$, $W_t \le W^1$ for $t > 0$, and $I_t$ is increasing only when $W_t \ge W^1$.

We can now evaluate the principal's payoff for an arbitrary incentive compatible contract. Note that $b(W_\tau) = L$. For all $t < \infty$,

$$E\left[\int_0^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L\right] = E\left[G_{t\wedge\tau} + 1_{t\le\tau}\left(\int_t^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L - e^{-rt}b(W_t)\right)\right]$$

$$= \underbrace{E\left[G_{t\wedge\tau}\right]}_{\le G_0 = b(W_0)} + e^{-rt}E\left[1_{t\le\tau}\left(\underbrace{E_t\left[\int_t^\tau e^{-r(s-t)}(dY_s - dI_s) + e^{-r(\tau-t)}L\right]}_{\le \mu/r - W_t = \text{ first best}} - b(W_t)\right)\right]$$

Now, since $b'(W) \ge -1$, $\mu/r - W - b(W) \le \mu/r - R - L$. Therefore, letting $t \to \infty$,

$$E\left[\int_0^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L\right] \le b(W_0)$$

Finally, for a contract that satisfies the conditions of the proposition, $G_t$ is a martingale until time $\tau$ because $b'(W_t)$ stays bounded. Therefore, the payoff $b(W_0)$ is achieved with equality. ◆

**Remark.** It is easy to modify this proof to show that the principal cannot improve her profit by adding additional randomization. Such randomization would add an extra term to the expression for $dG_t$, but the process $G_t$ would still be a supermartingale since $b(W)$ is a concave function.

**Proof of Proposition 2:** Recall that the rate of return on savings is $\rho \le r$. We consider the case $\rho = r$ in which savings is most attractive without loss of generality. We also generalize the setting to allow the agent to save within the firm and on his own account (this will be useful in our implementation of the optimal contract). Savings within the firm are represented by $S_t^f$ and evolve according to

$$dS_t^f = rS_t^f\,dt + (dY_t - d\hat{Y}_t) - dQ_t$$

Here, $dQ_t$ represents the agent's diversion of the cash flows to his own account, which evolves according to

$$dS_t = rS_t\,dt + [dQ_t]^\lambda + dI_t - dC_t$$

Note that the agent bears the cost of diversion when moving funds outside the firm. Both accounts must maintain non-negative balances. We show that for an arbitrary feasible strategy $(C, \hat{Y})$ of the agent,

$$\hat{V}_t = \int_0^t e^{-\gamma s}\,dC_s + e^{-\gamma t}\left(S_t + \lambda S_t^f + W_t\right)$$

is a supermartingale. Now,

$$e^{\gamma t} d\hat{V}_t = dC_t + dS_t - \gamma S_t\, dt + \lambda(dS_t^f - \gamma S_t^f\, dt) + dW_t - \gamma W_t\, dt$$

Using (15) and the definitions of $dS_t$ and $dS_t^f$ ,

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= [dQ_t]^\lambda - \lambda dQ_t - (\gamma - r)(S_t + \lambda S_t^f)\, dt + \lambda(dY_t - \mu\, dt) \\ &= -(1-\lambda) dQ_t^- - (\gamma - r)(S_t + \lambda S_t^f)\, dt + \lambda \sigma\, dZ_t \end{aligned}$$

Because $\lambda \le 1$, $dQ_t^-$ is non-decreasing, $\gamma > r$, and the savings balances are non-negative, $\hat{V}$ is a supermartingale until time $\tau$ because $W_t$ is bounded below. If $W_t$ is bounded above and there is no savings, $S_t = S_t^f = 0$, and the agent reports truthfully so that $d\hat{Y}_t = dY_t$ and $dQ_t = 0$, then $\hat{V}$ is a martingale. Thus,

$$W_0 = \hat{V}_0 \ge E\left[\hat{V}_\tau\right] = E\left[\int_0^\tau e^{-\gamma s}\, dC_s + e^{-\gamma \tau}\left(S_\tau + \lambda S_\tau^f + R\right)\right]$$

with equality if the agent maintains zero savings and reports truthfully. ◆

**Proof of Proposition 3:** Let $Div_t$ be an increasing process representing the cumulative dividends paid by the firm. Then the credit line balance evolves according to

$$dM_t = \gamma M_t\, dt + x\, dt + dDiv_t - d\hat{Y}_t .$$

where we can assume $dDiv_t$ and $d\hat{Y}_t$ are such that $M_t \ge 0$. Defining $W_t$ from (17), and using, from (16), $\lambda x = \lambda r D = \lambda\mu - \gamma(R + \lambda C^L)$, we have

$$\begin{aligned} dW_t &= -\lambda\, dM_t = -\gamma\lambda M_t\, dt - \lambda x\, dt - \lambda\, dDiv_t + \lambda d\hat{Y}_t \\ &= \gamma(W_t - (R + \lambda C^L))dt - (\lambda\mu - \gamma(R + \lambda C^L))dt - \lambda\, dDiv_t + \lambda d\hat{Y}_t \\ &= \gamma W_t dt - \lambda\, dDiv_t + \lambda(d\hat{Y}_t - \mu\, dt) \end{aligned}$$

Letting $dI_t = \lambda\ dDiv_t$, the incentive compatibility result of Proposition 3 follows from Proposition 2. Optimality follows from (18), since then $M_t = 0$ implies $W_t = W^1$. ◆

**Proof of Proposition 4:** Let $b$ be the optimal continuation function for parameters (1, $R^\lambda$, $L^\lambda$) and define

$$b^\lambda(W) = \lambda b(W/\lambda) + (1-\lambda)(\mu/r)$$

We claim that $b^\lambda$ is the optimal continuation function with parameters ($\lambda$, $R$, $L$). To see this, we can easily check that $b^\lambda(R) = L$. Since

$$b^{\lambda}{}'(W) = b'(W/\lambda) \quad \text{and} \quad b^{\lambda}{}''(W) = \tfrac{1}{\lambda} b''(W/\lambda),$$

then $b^{\lambda}{}'(W^1) = -1 \Rightarrow b'(W^1/\lambda) = -1$ and $b''(W^1/\lambda) = 0 \Rightarrow b^{\lambda}{}''(W^1) = 0$ . Thus, $b^\lambda$ satisfies both boundary conditions at $W^1$. In addition, for $W \in [R, W^1]$,

$$\begin{aligned} rb^{\lambda}(W) &= \lambda rb(W/\lambda) + (1-\lambda)\mu \\ &= \lambda\Big[\mu + \gamma(W/\lambda)b'(W/\lambda) + \tfrac{1}{2}\sigma^2 b''(W/\lambda)\Big] + (1-\lambda)\mu \\ &= \mu + \gamma W b^{\lambda}{}'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b^{\lambda}{}''(W) \end{aligned}$$

so that $b^{\lambda}$ satisfies (11) and hence is the optimal continuation function. Thus, $W^1$ is the dividend boundary for parameters ($\lambda$, $R$, $L$) if and only if $W^1/\lambda$ is the dividend boundary for (1, $R^{\lambda}$, $L^{\lambda}$). Thus, from (16) and (18), the optimal debt structure is unchanged. ◆

## Market Values of Securities:

The following lemma is useful for computation of market values of securities and for comparative statics:

**LEMMA D.** *Suppose $W_t$ evolves as*

$$dW_t = \gamma W_t\, dt - dI_t + \lambda\,(d\hat{Y}_t - \mu\, dt)$$

*in the interval $[R, W^1]$ until time $\tau$ when $W_t$ hits $R$, where $I_t$ is a nondecreasing process that reflects $W_t$ at $W^1$. Let $k$ be a real number, and $g:[R, W^1] \to \Re$ a bounded function. Then the same function $G:[R, W^1] \to \Re$ both solves equation*

$$rG(W) = g(W) + \gamma W\, G'(W) + 1/2\, \lambda^2\sigma^2 G''(W) \tag{24}$$

*with boundary conditions $G(R) = L$ and $G'(W^1) = -k$ and satisfies*

$$G(W_0) = E\left[\int_0^{\tau} e^{-rt} g(W_t)dt - k\int_0^{\tau} e^{-rt} dI_t + e^{-r\tau} L\right] \tag{25}$$

**PROOF:** Suppose that $G$ solves (24), and let us show that it satisfies (25). Define

$$H_t = \int_0^{\tau} e^{-rt} g(W_t)dt - k\int_0^{\tau} e^{-rt} dI_t + e^{-r\tau} G(W_t)$$

Then using Ito's lemma,

$$e^{rt} dH_t = \left( g(W_t) + \gamma W_t G'(W_t) + \frac{1}{2}\lambda^2\sigma^2 G''(W_t) - rG(W_t) \right)dt - (k + G'(W_t))dI_t + G'(W_t)\lambda\sigma dZ_t$$

From equation (24), condition $G'(W^1) = -k$, and the fact that $I$ increases only when $W_t = W^1$, $H$ is a martingale. Because $G$ is bounded, $H$ is a martingale until time $\tau$, so

$$G(W_0) = H_0 = E[H_\tau] = E\left[\int_0^{\tau} e^{-rt} g(W_t)dt - k\int_0^{\tau} e^{-rt} dI_t + e^{-r\tau} L\right]$$

This completes the proof. ◆

The values of credit line, debt and equity can be expressed in terms the following functions, which can be computed by Lemma D:

$$G_\tau(W) = E\left[e^{-r\tau} \mid W_0 = W\right] \qquad \text{and} \qquad G_I(W) = E\left[\int_0^\tau e^{-rt} dI_t \mid W_0 = W\right]$$

By Lemma D, both of these functions solve differential equation

$$rG(W) = \gamma W\, G'(W) + 1/2\, \lambda^2 \sigma^2 G''(W) \tag{26}$$

with boundary conditions $G_\tau(R) = 1,\ G_\tau'(W^1)=0$ and $G_I(R) = 0,\ G_I'(W^1)=1.$ Functions $G_\tau$ and $G_I$ can be easily computed. To evaluate market values of securities, we also use the fact that

$$E\left[\int_0^\tau e^{-rt} dt \mid W_0 = W\right] = \frac{1 - G_\tau(W)}{r}$$

Then, expressed as functions of the agent's continuation value $W_t$, market values the credit line, debt and equity are

$$V_c(W) = E\left[\int_0^\tau e^{-rt}\left(dY_t - x dt - \frac{dI_t}{\lambda}\right) + e^{-r\tau} L_C \mid W_0 = W\right]$$

$$= \frac{\gamma W^1}{\lambda} \frac{1 - G_\tau(W)}{r} - \frac{G_I(W)}{\lambda} + L_C G_\tau(W)$$

$$V_D(W) = E\left[\int_0^\tau e^{-rt} x dt + e^{-r\tau} L_D \mid W_0 = W\right] = x\frac{1 - G_\tau(W)}{r} + L_D G_\tau(W)$$

$$V_E(W) = E\left[\int_0^\tau e^{-rt} \frac{dI_t}{\lambda} + e^{-r\tau} L_E \mid W_0 = W\right] = \frac{G_I(W)}{\lambda} + L_E G_\tau(W)$$

**PROOF OF PROPOSITION 5:** When $L < D$, then $L_E = 0$. Then, to demonstrate that equity holders prefer less volatility, we need to prove that $G_I$ is concave. From the stochastic representation, we see that $G_I$ is an increasing function. From (26),

$$1/2\, \lambda^2 \sigma^2 G_I''(R) = -\gamma R G_I'(R) < 0.$$

Suppose that $G_I$ were not concave everywhere on $[R, W^1]$, and let $V = \inf\{G_I''(W) > 0\}$. Then $V > R$ and $G_I''(V) = 0$ by continuity of $G_I''$. But then from (26)

$$1/2\, \lambda^2 \sigma^2 G'''(V) = (r-\gamma)\, G'(V) - \gamma V\, G''(V) = (r-\gamma)\, G'(V) < 0,$$

so $G_I''(V+\varepsilon) < 0$ for all sufficiently small $\varepsilon > 0$, contradiction. ◆

The following lemma tells us that when there are no outside equity holders, then no funds remain after debt and credit line holders are paid from the liquidation value.

**LEMMA E.** *If* $\lambda = 1$*, then in the optimal contract,* $L < D + C^L$.

**PROOF:** When $\lambda = 1$, $D + C^L = b(W^1) + W^1 - R$. Since $b'(W) > -1$ for $W \in (R, W^1)$, $b(W^1) + W^1 > b(R) + R = L + R$. Thus, $D + C^L > L$. ◆

## Comparative Statics Results:

**LEMMA F.** *Suppose* $\theta$ *is one of parameters* $L$, $\mu$, $\gamma$ *or* $\sigma^2$ *and denote by* $b_\theta(W)$ *the optimal continuation function for that parameter value. Then*

$$\frac{\partial b_\theta(W)}{\partial \theta} = E\left[\int_0^\tau e^{-rt}\left(\frac{\partial \mu}{\partial \theta} + \frac{\partial \gamma}{\partial \theta} W_t b'_\theta(W_t) + \frac{1}{2}\frac{\partial \sigma^2}{\partial \theta} b''_\theta(W_t)\right)dt + e^{-r\tau}\frac{\partial L}{\partial \theta} \mid W_0 = W\right]$$

**PROOF:** Consider a value of $W^1$ and a corresponding incentive-compatible contract of Proposition 5: one in which process $I$ reflects $W_t$ at $W^1$. Then the principal's profit under this contract is

$$b_{\theta,W^1}(W) = E\left[\int_0^\tau e^{-rt}\mu dt - \int_0^\tau e^{-rt} dI_t + e^{-r\tau} L \mid W_0 = W\right]$$

By Lemma D, $b_{\theta,W^1}(W)$ solves equation

$$rb_{\theta,W^1}(W) = \mu + \gamma W b'_{\theta,W^1}(W) + \tfrac{1}{2}\sigma^2 b''_{\theta,W^1}(W) \tag{27}$$

with boundary conditions $b_{\theta,W^1}(R) = L$ and $b'_{\theta,W^1}(W^1) = -1$. Denote by $W^1(\theta)$ the choice of $W^1$ that maximizes the principal's profit $b_{\theta,W^1}(W_0)$ for a given value of parameter $\theta$. Then $b_\theta(W) = b_{\theta,W^1(\theta)}(W)$. By the Envelope Theorem,

$$\frac{\partial b_\theta(W)}{\partial \theta} = \left.\frac{\partial b_{\theta,W^1}(W)}{\partial \theta}\right|_{W^1 = W^1(\theta)} \tag{28}$$

Differentiating (27) with respect to $\theta$ at $W^1 = W^1(\theta)$ and using (28) we find that $\frac{\partial b_\theta(W)}{\partial \theta}$ satisfies equation

$$r\frac{\partial b_\theta(W)}{\partial \theta} = \frac{\partial \mu}{\partial \theta} + \frac{\partial \gamma}{\partial \theta} W b'_\theta(W) + \gamma W \frac{\partial}{\partial W}\frac{\partial b_\theta(W)}{\partial \theta} + \frac{1}{2}\frac{\partial \sigma^2}{\partial \theta} b''_\theta(W) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial W^2}\frac{\partial b_\theta(W)}{\partial \theta}$$

with boundary conditions $\frac{\partial b_\theta(R)}{\partial \theta} = \frac{\partial L}{\partial \theta}$ and $\frac{\partial}{\partial W}\frac{\partial b_\theta(W^1)}{\partial \theta} = 0$. The conclusion of the lemma follows from Lemma D. ◆

**COROLLARY.** *From Lemma F and we obtain that*

$$\frac{\partial b(W)}{\partial L} = G_\tau(W), \qquad \frac{\partial b(W)}{\partial \gamma} = G_1(W), \qquad \frac{\partial b(W)}{\partial \mu} = \frac{1-G_\tau(W)}{r}, \ \ \textit{and}$$

$$\frac{\partial b(W)}{\partial \sigma^2} = G_2(W), \tag{29}$$

*where*

$$G_\tau(W) = E\left[e^{-r\tau} \mid W_0 = W\right], \qquad G_1(W) = E\left[\int_0^\tau e^{-rt} W_t b'(W_t)dt \mid W_0 = W\right],$$

$$\textit{and } G_2(W) = E\left[\int_0^\tau e^{-rt} b''(W_t)dt \mid W_0 = W\right]. \tag{30}$$

*Additionally, because the principal's profit remains the same if the agent's outside option increases by dR and liquidation value decreases by* $b'(R)dR$*, the effect of a change in R on the principal's profit is captured by*

$$\frac{\partial b(W)}{\partial R} = -b'(R)G_\tau(W).$$

To find how parameters affect the optimal choice of $W^1$, note that

$$b_\theta(W^1(\theta)) = \frac{\mu - \gamma W^1(\theta)}{r} \Rightarrow \frac{\partial W^1(\theta)}{\partial \theta} = \frac{r}{\gamma - r}\frac{\partial}{\partial \theta}\left(\frac{\mu - \gamma W^1}{r} - b_\theta(W^1)\right)\Bigg|_{W^1 = W^1(\theta)} \tag{31}$$

We can then compute the derivatives of $W^1$ with respect to parameters using the Corollary of Lemma F. Similarly, the derivative of $W_0$ with respect to the parameters can be found by differentiating $b_\theta(W_0(\theta)) = K$ with respect to $\theta$ and using the Corollary. We obtain comparative statics results summarized in Table 1 in the paper. In Table 1, we still need to sign the non-obvious entries in parentheses. The following Lemma allows us to compare the principal's profit for different γ's and to sign two entries that involve $G_1(W)$.

**Lemma G**. *Suppose that the principal offers a contract designed for the agent with discount rate γ to an agent whose true discount rate is γ′ < γ. Then this agent would derive utility greater than* $W_0$*, and the principal would receive profit of exactly* $b(W_0)$*.*

**Proof:** Let us investigate how an agent with discount rate $\gamma'$ responds to a contract created for an agent with discount rate $\gamma$. First, let us interpret the contract. The agent's value $W_t$ can be interpreted as the agent's balance on a high-interest savings account. It evolves as

$$dW_t = \gamma W_t\, dt + (d\hat{Y}_t - \mu\, dt)$$

where $d\hat{Y}_t - \mu\, dt$ is the flow of deposits. The high-interest account has a cap of $W^1$. The agent's consumption is

$$dC_t = dY_t \; -\mu \, dt - (d\hat{Y}_t - \mu \, dt) - dQ_t,$$

where $\mu \, dt$ is a tax and $dQ_t$ is the flow of deposits onto the low-interest savings account. The balance on that account is

$$dS_t = \rho S_t \, dt + dQ_t$$

When the agent manages these two accounts, it is optimal to never have positive balance on the low-interest savings account, unless the high-interest savings account is full (i.e. $W_t = W^1$). Also, it is optimal to deposit all cash flows onto the high-interest savings account and not consume when $W_t < W^1$, because those cash flows can earn a higher interest rate than the agent's own discount rate. The agent consumes only when $W_t = W^1$ and the balance on his own savings account is positive. This sort of strategy gives the agent of value higher than $W_t$, which he could get by simply drawing the credit line to the end and defaulting immediately.

Let us show that the principal still gets profit $b(W_t)$ when the agent follows any such strategy. When $W_t < W^1$, the agent deposits all cash flows onto the credit line, just like an agent with discount factor $\gamma$ would do. The only difficulty can come from the fact that when $W_t = W^1$, the agent may manage his own savings account with cash flows from the project, and keep the balance on the credit line at 0 by paying the principal a flow $\mu - \gamma W^1$ of coupon payments on long-term debt. This modification in the agent's strategy does not alter the principal's profit because $\mu - \gamma W^1 = rb(W^1)$, which is exactly what the principal needs to get to realize a profit of $b(W^1)$. ◆

Note that the contract in Lemma G is not optimal for agent $\gamma'$. An optimal contract would give the principal higher profit for the same value of the agent. Therefore, to every point $(W, b_\gamma(W))$ with $W \geq W^*(\gamma)$, there is a point $(W', b_{\gamma'}(W')) > (W, b_\gamma(W))$. We conclude that $b_\gamma(W)$ must be increasing as $\gamma$ falls for all $W \geq W^*(\gamma)$, so $G_1(W) < 0$.

**COROLLARY.** $-\dfrac{G_1(W_0)}{b'(W_0)} < 0 \;\; and \;\; G_1(W^*) < 0.$

For the remaining two entries of Table 1, we need to relate $b'(W)$ and $G_\tau(W)$.

**LEMMA H.** *The following inequality holds for all $W < W^1$:*

$$b'(W) < \frac{(\gamma - r)G_\tau(W)}{rG_\tau(W^1)} - \frac{\gamma}{r}. \tag{32}$$

**PROOF:** Differentiating equation (33) with respect to $W$ we find that $b'(W)$ satisfies

$$(r-\gamma)b'(W) = \gamma W b''(W) + \frac{\sigma^2}{2} b'''(W) \tag{34}$$

with boundary conditions $b'(W^1) = -1$ and $b''(W^1) = 0$. Denote the right hand side of (32) by $g(W) - \gamma/r$. From (30), we know that $g(W)$ satisfies

$$rg(W) = \gamma W g'(W) + \frac{\sigma^2}{2} g''(W) \Rightarrow$$
$$(r-\gamma)(g(W) - \frac{\gamma}{r}) + (r-\gamma)\frac{\gamma}{r} + \gamma g(W) = \gamma W g'(W) + \frac{\sigma^2}{2} g''(W) \tag{35}$$

with boundary conditions $g(W^1)=(\gamma-r)/r$ and $g'(W^1)=0$. Denote $f(W) = g(W)-\gamma/r - b'(W)$. To prove the lemma, we need to show that $f(W)>0$ for all $W<W^1$. Since $f(W^1)=0$, this property follows if we show that $f'(W)<0$ for all $W<W^1$. Subtracting (34) from (35), we find that

$$\frac{\sigma^2}{2} f''(W) = (r-\gamma) f(W) + (r-\gamma)\frac{\gamma}{r} + \gamma g(W) - \gamma W f'(W) \tag{36}$$

with boundary conditions $f(W^1)=0$ and $f'(W^1)=0$. From (36) we find that

$$\frac{\sigma^2}{2} f''(W^1) = (r-\gamma)\frac{\gamma}{r} + \gamma\frac{\gamma - r}{r} = 0$$
$$\frac{\sigma^2}{2} f'''(W^1) = (r-2\gamma) f'(W^1) + \gamma g'(W^1) + \gamma W^1 f''(W^1) = 0 \text{, and}$$
$$\frac{\sigma^2}{2} f^{(4)}(W^1) = (r-3\gamma) f''(W^1) + \gamma g''(W^1) + \gamma W^1 f'''(W^1) > 0$$

Therefore, $f'(W)<0$ for $W<W^1$ in a small neighborhood of $W^1$. If $f'(W)<0$ fails for some $W<W^1$, there has to be a largest point $V$ at which it fails. Then $f'(V)=0$ and $f(W)$ is positive and decreasing on $[V,W^1)$. But then from (36)

$$\frac{\sigma^2}{2} f''(V) = (r-\gamma) f(V) + (r-\gamma)\frac{\gamma}{r} + \gamma g(V) > 0, \text{ since } \; g(V) > \frac{\gamma - r}{r}.$$

We conclude that $f'(V+\varepsilon)>0$, which contradicts our definition of V as the largest point at which $f'(V)\geq 0$. We conclude that $f'(W)<0$ and $f(W)>0$ for $W<W^1$, so (32) holds. ◆

Now we can sign the remaining two fields in Table 1.

**COROLLARY.** *Applying (32) at W=R, we have*

$$\frac{rb'(R)G_\tau(W^1)}{\gamma - r} - 1 < -\frac{\gamma G_\tau(W^1)}{\gamma - r} < 0 \quad and \quad 1 - \frac{\gamma G_\tau(W^1)}{\gamma - r} > \frac{rb'(R)G_\tau(W^1)}{\gamma - r} > 0.$$

## Hidden Effort and Extensions:

**PROOF OF PROPOSITION 7:** Let $w^s = \lambda A/\gamma$ and $b^s = (\mu - A)/r$. We can rewrite (20) as $b^s \leq b(W) + \frac{\gamma}{r}\left(w^s - W\right) b'(W)$, and this must hold for all $W$, leading to the condition

$$b^s \leq f(w^s) = \min_W b(W) + \tfrac{\gamma}{r}\left(w^s - W\right) b'(W). \tag{37}$$

To prove that condition (21) of Proposition 7 guarantees (37), it is sufficient to show that for all $w$,

$$b\left(w^s\right)-\tfrac{\gamma-r}{r}\left(b(W^*)-b\left(w^s\right)\right)\le b\left(W\right)+\tfrac{\gamma}{r}(w^s-W)b'(W) \tag{38}$$

Note that since $b$ is concave and $\gamma > r$,

$$b(w^s)\le b(W)+\left(w^s-W\right)b'(W)\le b(W)+\tfrac{\gamma}{r}\left(w^s-W\right)b'(W)$$

if $(w^s - W)b'(W) > 0$, which implies (38) for $W$ not between $w^s$ and $W^*$. For $W$ between $w^s$ and $W^*$, note that

$$\begin{aligned}
b(w^s)-\tfrac{\gamma-r}{r}(b(W^*)-b(w^s)) &\le b(w^s)-\tfrac{\gamma-r}{r}(b(W)-b(w^s))\\
&\le b(w^s)-\tfrac{\gamma-r}{r}(W-w^s)b'(W)\\
&\le b(W)+(w^s-W)b'(W)-\tfrac{\gamma-r}{r}(W-w^s)b'(W)\\
&= b(W)+\tfrac{\gamma}{r}(w^s-W)b'(W)
\end{aligned}$$

so that (38) again holds, verifying the sufficiency of condition (21).

Note that $f'(w^s) = \gamma/r\ b'(W) \ge -\gamma/r$, whereas $\partial b^s/\partial w^s = -(\gamma/r)/\lambda$. Thus, both (37) and (21) imply a lower bound on $w^s$ (or equivalently $A$).

Finally, we note the following properties of $f$ described in the paper: Setting $W = w^s$ in (37) implies $f(w) \le b(w)$. Also, since $f$ is the lower envelope of linear functions it is concave. Finally, (21) implies that $f(W^*) = b(W^*)$. ◆

**PROOF OF PROPOSITION 8:** Let $b$ be the optimal continuation function given boundary condition $b(R-\omega) = L$. Then define $b^*(W) = b(W-\omega)$. Then $b^*(R) = L$ and

$$\begin{aligned}
rb^*(W) = rb(W-\omega) &= \mu+\gamma(W-\omega)b'(W-\omega)+\tfrac{1}{2}\lambda^2\sigma^2 b''(W-\omega)\\
&= \mu++\gamma(W-\omega)b^{*}{}'(W)+\tfrac{1}{2}\lambda^2\sigma^2 b^{*}{}''(W)
\end{aligned}$$

Finally, $b'(W^1) = -1$ implies $b^{*\prime}(W^1+\omega) = -1$ and $b^{*\prime\prime}(W^1+\omega) = 0$. Thus, by the same arguments as in the proof of Proposition 1, $b^*$ is the optimal continuation function for the setting with private benefits. ◆

**PROOF OF PROPOSITION 9:** The first result holds by Lemma G. Next, suppose the agent's true discount factor $\gamma'$ is greater than $\gamma$. The process

$$\hat{V}_t=\int_0^t e^{-\gamma' s}dC_s + e^{-\gamma' t}(S_t+W_t)$$

is a strict supermartingale. Indeed,

$$e^{\gamma t}d\hat{V}_t=-(1-\lambda)(dY_t-d\hat{Y}_t)^- -(\gamma'-\gamma)W_t dt-(\gamma'-\rho)S_t dt+\lambda\sigma dZ_t,$$

so $\hat{V}$ has a negative drift. Since $W_t$ and $S_t$ are bounded from below, $\hat{V}$ is strict supermartingale until time $\tau$. If the agent draws the entire credit line and defaults at time

0, then he gets a payoff of $W_0$. If he follows any other strategy, then $\tau > 0$ and the agent's payoff is

$$E\left[\int_0^\tau e^{-\gamma' s} dC_s + e^{-\gamma'\tau}(S_\tau + W_\tau)\right] = E\left[\hat{V}_\tau\right] < \hat{V}_0 = W_0$$

Therefore, the agent will draw the entire credit line immediately if $\gamma' > \gamma$. ◆

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