Security Design with Correlated Hidden Cash Flows: The Optimality of Performance Pricing*

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Abstract

This paper studies optimal security design in a dynamic setting with an agency problem that arises when an agent in charge of a project can divert cash flows for his own consumption at the expense of an outside investor. Cash flows are unobservable and unverifiable by the outside investor, who relies on the agent's reports, and has the right to liquidate the project. Unlike previous analyses, we allow cash flows to be correlated over time. We solve for the optimal contract and show that it can be implemented using a credit line with an interest rate that increases with the balance on the credit line. This finding is consistent with the fact that the majority of commercial loans are lines of credit with performance pricing. Thus, our model provides theoretical evidence that performance pricing is used to mitigate the agency cost. In addition, we develop a new recursive method to deal with a correlated privately observed variable in dynamic agency settings. It allows us to reduce the dimensionality of the problem and obtain a closed-form solution for the optimal contract.

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1 Introduction

More than 87% of all commercial and industrial loans made by large domestic banks are loans under commitment[1], otherwise known as lines of credit. A line of credit is a contract between a firm and a bank that lets the firm borrow from the bank during the life of the contract on terms specified in advance. Two main characteristics of a credit line are the credit limit, which stipulates the maximum amount of credit allowed, and the interest rate charged on the balance. Very often, instead of a fixed rate of interest, performance pricing schemes are used that connect the interest rate to some measure of the borrower's performance, such as the borrower's interest coverage ratio, debt-to-EBITDA ratio, leverage ratio, or current credit rating. Asquith, Beatty and Weber (2004) report that more than 50% of lending agreements[2] have performance pricing features. Moreover, most of the lending agreements require the borrower to pay a higher interest rate when the borrower performs poorly.

Manso, Strulovici and Tchistyi (2004) study different forms of performance pricing in a dynamic capital-structure model, in which default is endogenously triggered by equity holders when the firm's assets fall to a certain level. In a setting with a bankruptcy cost and tax benefits but without an agency problem, they prove that debt obligations that result in paying higher interest rates in times of poor performance and lower interest rates in times of high performance are not the most efficient way to raise money. Such debt obligations precipitate default and increase bankruptcy cost, since they impose a higher debt burden when the firm experiences financial strains. A loan with a fixed interest rate results in a higher value of the firm. This finding raises a question: why are such performance pricing schemes so widely used in practice?

The goal of this paper is to explain the existence of the very popular form of bank lending: a line of credit with an escalating interest rate. We develop a model of security design in which a line of credit with an escalating interest rate is a part of the optimal contract. In the friction-free world of Modigliani and Miller (1958), the market value of the firm is independent of its capital structure. As Manso, Strulovici and Tchistyi (2004) establish, the theory of trade-off between tax benefits and bankruptcy costs does not justify performance pricing. This paper introduces an agency problem into the model and shows that performance pricing is used to mitigate the agency cost.

Specifically, a risk-neutral agent with limited liability needs external financing for a prof-

[1] E.2 Survey of Terms of Business Lending, August 2-6, 2004 (Federal Reserve Statistical Release)

[2] Asquith, Beatty and Weber (2004) use the data on loan contracts in the Loan Pricing Corporation database.

itable business project. If funded, the project generates stochastic cash flows. An outside investor is unable to observe the cash flows, while the agent has an ability to divert the cash flows for his own consumption at the expense of the investor. Before initiating the project, the agent and the investor (or a group of investors) sign a contract that will govern their relationship after the project is initiated. In particular, the contract obligates the agent to report the cash flows to the investor, although the investor cannot verify the agent's reports. In addition, the contract specifies payments between the agent and the investor conditional on the history of the agent's reports, and the circumstances under which the control of the project's assets is transferred from the agent to the investor. In this latter case, the agent is fired and the investor makes optimal use of the assets. The transfer of control leads to inefficiencies, either due to a dead-weight cost associated with it, or because the investor is less capable than the agent of running the project and cannot find an equivalent replacement for the agent's managerial talent.

We assume that the cash flows are correlated over time and follow a two-state Markov process. The correlation is an important assumption, not only because it is a more realistic assumption than independent cash flows, but also because it creates an additional degree of informational asymmetry between the agent and the investor. With correlated cash flows, the agent has a superior knowledge, not only about the current cash flow realization that he observes directly, but also about the future cash flows, since their distribution is determined by the current cash flow realization.

We characterize the optimal contract in this setting and its implementation using standard securities. We find that the optimal contract can be implemented using a combination of equity, a coupon bond and a credit line with an interest rate that increases with the outstanding balance on the credit line. According to this implementation, the agent owns a fraction of the firm's equity, while being obligated to make coupon payments on the bond and interest payments on the credit line balance to the investor, who also owns the rest of the firm's equity. The initial draw on the credit line is determined by the amount of funds provided by the investor, as well as the bargaining power of the parties. The agent uses the cash flows generated by the project to make the current debt payments and to repay the balance on the credit line. When the cash flow is low, the agent is allowed to draw on the credit line to make the current debt payments, as long as he does not exceed the credit limit. The agent is in default if he is unable to fulfill his current debt obligations without exceeding the credit line limit. In this case, the investor may take control over the firm's assets and fire the agent.

In this combination of securities, the roles of the coupon bond and equity are straightforward. The coupon bond is used to extract verifiable cash flows, while dividends paid to the equity holders offer a reward to the agent for repaying the debt. The role of the credit line with an escalating interest rate is more sophisticated. The balance on the credit line can be considered as a memory device that summarizes all the relevant information regarding the agent's performance. The interest rate, along with the credit limit, determines the dynamics of the credit line balance and the timing of the default. When the agent reports a low cash flow and fails to make the interest payment, the balance on the credit line increases by the amount of the unmade interest payment. When the balance goes over the credit limit default occurs. The speed at which the balance grows is determined by the interest rate; and this speed is greater when the balance is higher. The agent has the incentive to use high cash flows to reduce the balance as well as the interest rate. The threat of losing control over the project induces the agent to pay the debt.

To see why the interest rate on the credit line should increase with the balance, consider what will happen when the agent keeps stealing the cash flows until the credit line is exhausted and default occurs. Because the cash flows are positively correlated, the average amount that can be stolen per period is negatively related to the length of the time interval during which the stealing takes place. Higher balance on the credit line means that the default will occur sooner if the agent keep stealing cash flows. Thus, to discourage the agent from stealing, a higher interest rate should be charged on the credit line when the balance is high.

An important aspect of this paper is the methodology, which has independent theoretical value. We develop a new dynamic programming approach for solving for an optimal contract (mechanism) in a setting with a correlated privately observed variable, where the standard dynamic programming technique does not work. The main advantage of our approach is that it allows us to reduce the dimensionality of the problem and obtain a closed-form solution for the optimal contract in our setting. We also believe that this approach is not only applicable to our setting, but can also be used in other dynamic principal-agent models with correlated hidden states.

1.1 Related Literature

A number of papers study optimal contracting in a setting in which an agent has an ability to divert cash flows. In a simple one-period model, Diamond (1984) demonstrates that the optimal contract is debt, where the agent's incentives to make payments to lenders are given in terms of non-pecuniary bankruptcy penalties. Bolton and Scharfstein (1990) consider a similar two-period model, in which the investor can threaten to cut off funding in the second period if the firm defaults in the first. This threat induces the firm to share the first period profit with the investor. In a dynamic setting with asymmetric information, Clementi and Hopenhayn (2004) demonstrate that borrowing constraints emerge as a feature of the optimal lending agreements.

The two studies that are most closely related to ours are DeMarzo and Fishman (2003), and DeMarzo and Sannikov (2004). Both of these papers study long-term financial contracting in a setting with privately observed independent cash flows. Unlike previous analyses, we allow cash flows to be correlated over time. It turns out that the correlation significantly changes the optimal contract between the agent and the investor. While in DeMarzo and Fishman (2003), and DeMarzo and Sannikov (2004) the optimal contract can be implemented using a credit line with a constant interest rate, we find that the implementation of the optimal contract in our setting requires a credit line from the investor with a variable interest rate. Since the presence of correlation of cash flows is overwhelming in practice, our model is a more realistic approximation of the reality, and should better fit the data.

Recent empirical studies support the theory that performance pricing and other covenants are used to mitigate agency costs. Analyzing a large database of commercial loans, Asquith, Beatty and Weber (2004) report that debt contracts are more likely to include performance pricing when re-contracting, adverse selection, and moral hazard costs are higher. They also estimate that more than 50% of debt contracts have performance pricing requirements. While using the same database, Bradley and Roberts (2004) come to the conclusion that debt covenants are used to reduce the agency cost of debt. Dichev and Skinner (2004) relate the existence of debt covenants to informational asymmetries between lenders and borrowers.

On the technical side, this paper develops a recursive method to solve for an optimal incentive-compatible contract in a setting in which a privately observed state variable is *correlated* over time. The vast majority of the literature on optimal contracting assumes, however, that privately observed economic variables are *independent* over time. In this literature, an optimal contract typically depends on a history of publicly observed outcomes, which is a multi-dimensional object. However, it is often possible to rewrite the problem recursively, summarizing all the relevant information in the history by a one-dimensional object - a continuation value. Green (1987), Abreu, Pearce, and Stacchetti (1990), Phelan and Townsend (1991), Korcherlakota (1996), DeMarzo and Fishman (2002), DeMarzo and Fishman (2003), among many others, utilize this approach.

Surprisingly, there are only a few papers that allow for correlation of privately observed variables. Fernandes and Phelan (2000) consider a dynamic model with a risk-averse agent whose endowment follows a first-order Markov process. Quadrini (2003) uses the methodology developed by Fernandes and Phelan (2000) to solve a dynamic principal-agent model with privately observed persistent shocks. Doepke and Townsend (2001) develop several new recursive methods to solve for optimal contracts in dynamic principal-agent models with hidden income and hidden actions. These papers resort to numerical simulations to characterize optimal contracts.

Our approach to solving an agency problem with history dependence is quite different from the methodology of Fernandes and Phelan (2000). Their method requires computation of the continuation functions that depend, not only on the state variable and the agent's continuation payoff, but also on the agent's deviation payoff. Although their method can be used in our setting for numerical simulations, the higher dimensionality of their method makes it virtually intractable analytically.

Battaglini (2004) obtains a closed form solution for the optimal contract between a monopolist and a consumer whose preferences follow a two-state Markov process and are unobservable by the monopolist. The consumer has unlimited wealth and the consumer's type at time zero is unknown to the monopolist. According to the optimal contract, the monopolist screens the consumer's type by distorting consumption of the low type. The contract instantly becomes efficient as soon as the consumer reveals his type is high. In a related paper, Battaglini and Coate (2004) study optimal income taxation of individuals whose income generating abilities evolve according to a two-state Markov process. The major distinction of our paper from Battaglini (2004), and Battaglini and Coate (2004) is that the agent has limited wealth. As a result, the first best cannot be implemented even when the cash flow is high. Consequently, the structure of the optimal contract in our setting is quite different from that in Battaglini (2004), and Battaglini and Coate (2004). In particular, in their papers, the optimal contract depends only on whether the agent has ever reported the high type, at which point the contract becomes first best. Here, the contract remains second best even after multiple reports of the high cash flows, and the optimal contract depends on history through a continuous variable, the balance on the credit line.

The paper is organized as follows. Section 2 introduces the dynamic contracting model with correlated privately-observed cash flows. Section 3 provides the derivation of the optimal contract. Section 4 discusses the initiation of the contract. Section 5 demonstrates that the optimal contract can be implemented by a combination of equity, a coupon bond and a credit line with an escalating interest rate. Section 6 concludes.

2 The Model

A risk-neutral agent evaluates consumption sequences $\{C_t\}$ according to $\sum_t \beta^t E[C_t]$, where β is the intertemporal discount factor. The agent has the managerial skills to run a project that generates stochastic cash flows $\{Y_t\}$. The agent's initial wealth W is not sufficient to initiate the project that requires an initial fixed investment $I > W$. To raise the lacking capital, the agent will have to enter into a contractual relationship with an investor who is also risk-neutral and has sufficient financial resources. The discount factor for the investor is also β, which corresponds to the risk-free interest rate $r = \frac{1}{\beta} - 1$.

The cash flows generated by the project are correlated over time. For simplicity, let the cash flows follow a two-state Markov chain: $Y_t \in \{y_L, y_H\}$ for all t, where $0 \leq y_L < y_H$. We will refer to y_L and y_H as the low and high cash flows respectively. Let $Q(y)$ denote the probability that $Y_t = y_H$ given the previous period cash flow realization $y \in \{y_L, y_H\}$:

$$Q(y) = \Pr\left(Y_t = y_H | Y_{t-1} = y\right).$$

The cash flows are assumed to be positively correlated, which in terms of transition probabilities means that $Q\left(y_H\right) > Q\left(y_L\right)$. One can verify that this implies that an expectation of a future cash flow is always higher in the high state:

$$E\left[Y_{t+k} | Y_t = y_H\right] > E\left[Y_{t+k} | Y_t = y_L\right], \tag{1}$$

where $k = 1, 2, ...$ For more on the properties of the cash flow process, see Lemma 4 in Appendix.

The agent privately observes realizations of the cash flows, while the investor must rely on the agent to report the cash flow realizations, without being able to verify the agent's reports. We assume that the low cash flow y_L is observable and collectible by the investor, but the agent can secretly divert the excess cash flow $y_H - y_L$ for his own consumption. Stealing may be costly for the agent. It is assumed that the agent is able to enjoy only a fraction $\lambda \in [0, 1]$ of the stolen amount. The fraction $(1 - \lambda)$ represents the cost of stealing, which can be attributed to different kinds of expenses and inefficiencies associated with the conspiracy. The agent can consume diverted cash flows immediately or save them at interest rate $\rho \leq r$ at his private bank account.

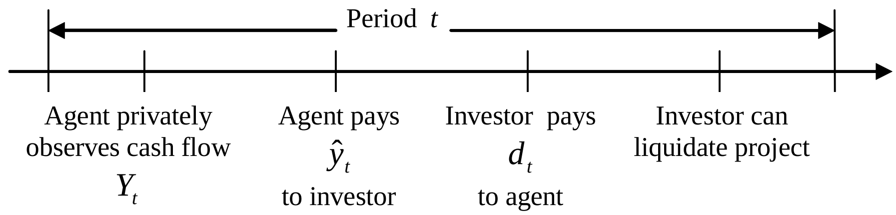

Figure 1: Sequence of Events

The agent has limited liability: he can quit at any time and get his reservation payoff R_t. For ease of presentation, we normalize the agent's reservation payoff to zero: $R_t = 0$, which means that the agent will never quit voluntarily. In addition, it is assumed that the project has a finite life, with T being the final date.

In exchange for the funding, the investor gains the right to take control over the project. We refer to it as the liquidation of the project. If the project is liquidated at date t, the investor sells the project's assets and collects the liquidation value L_t, while the agent is left with his reservation value R_t. All subsequent cash flows are permanently lost. We assume that the liquidation is inefficient, in the sense that the liquidation value is strictly below the expected value of the cash flows lost in the case of liquidation. The inefficiency may be caused by friction costs associated with a transfer of control, or by the fact that the investor does not have the agent's managerial talent to run the project efficiently on his own.

If the investor agrees to fund the project, at date 0 the agent and the investor sign a contract that will govern their relationship until the final date T. According to the contract, the agent must report realizations of the cash flows to the investor. Of course, the reported cash flow $\hat{y}_t$ can be different from the true realization y_t. Without loss of generality, it is assumed that the agent must pay the reported cash flow to the investor. The contract also specifies transfer payment d_t to the agent and the probability of liquidation p_t after any history of the agent's reports. Specifically, given the history $\hat{y}^t = \{\hat{y}_1, ..., \hat{y}_t\}$ of the agent's reports, the contract obligates the investor to make a payment of $d_t\left(\hat{y}^t\right) \geq 0$ to the agent, and liquidate the project with the probability $p_t\left(\hat{y}^t\right)$ at the end of period t. The sequence of the events is illustrated in Figure 1.

We will study contracts with full commitment. No renegotiation of the terms of the contract is allowed.

A contract is *optimal* if it maximizes the investor's continuation payoff subject to a certain payoff for the agent. The investor's income in each period is given by the difference between the reported cash flow $\hat{y}_t$ and the payment to the agent d_t and the proceeds L_t from the asset liquidation. Let $P_t\left(\hat{y}^t\right)$ be the probability that the project is active at the beginning of period t after the history $\hat{y}^t$ of reports, under the contract $\sigma = (d, p)$. One can verify that $P_t\left(\hat{y}^t\right) = \prod_{k=1}^{t-1}\left(1 - p_t\left(\hat{y}^k\right)\right)$. Given the initial state y_0 and the continuation payoff a_0 for the agent, the investor's problem is to choose an incentive compatible contract $\sigma = (d, p)$ that maximizes the investor's payoff:

$$b_0(y_0, a_0) = \max_{d,p} E_0\left[\sum_{t=1}^{T} \beta^t P_t\left(y^t\right)\left(Y_t - d_t(y^t) + p_t\left(y^t\right) L_t\right) |Y_0 = y_0\right], \tag{2}$$

subject to the incentive compatibility constraint

$$E\left[\sum_{t=1}^{T} \beta^t P_t\left(y^t\right) d_t\left(y^t\right) \mid Y_0 = y_0\right] \geq E\left[\sum_{t=1}^{T} \beta^t P_t\left(\hat{y}^t\right)\left(\lambda(Y_t - \hat{y}_t) + d_t\left(\hat{y}^t\right)\right) |Y_0 = y_0\right], \tag{3}$$

for all feasible reporting strategies $\hat{y}$, and the promise keeping constraint

$$a_0 = E_0\left[\sum_{t=1}^{T} \beta^t P_t\left(y^t\right) d_t\left(y^t\right) \mid Y_0\right]. \tag{4}$$

An optimal contract solves the investor's problem given by (2)-(4). Any other contract results in a payoff for the investor being equal to or lower than $b_0(y_0, a_0)$, given the agent's payoff a_0. Thus, function $b_0(y_0, a_0)$ represents the highest possible payoff attainable by the investor, given the payoff a_0 for the agent and the initial state y_0. We will refer to $b_0(y_0, a_0)$ as the continuation function at time zero.

In a similar manner, we can define continuation functions at any point of time in the future. In particular, let $b_t^y\left(y_{t-1}, a_t^y\right)$ denote the highest possible payoff attainable by the investor in period t before the cash flow Y_t is realized, given the agent's continuation payoffs a_t^y at the beginning of period t and the realization y_{t-1} of the cash flow Y_{t-1}. Let $b_t^d(y_t, a_t^d)$ denote the highest possible payoff attainable by the investor in period t after the cash flow Y_t is realized but before the liquidation decision is made, given the agent's continuation payoffs a_t^d at that time and the realization y_t of the cash flow Y_t. Let $b_t^e(y_t, a_t^e)$ denote the highest possible payoff attainable by the investor in period t after the liquidation decision is made, given the agent's continuation payoffs a_t^e at the end of period t and the realization y_t of the cash flow Y_t.

2.1 Optimality of a Truth-Telling Contract without Savings

In this section we demonstrate that the contracting problem can be rewritten recursively. We start our analysis of the model by showing that we can restrict our attention to the set of contracts in which the agent always tells the truth and does not save.

For any contract $\sigma = (d, p)$, the agent chooses an optimal strategy $\varphi = (\hat{y}, C, S)$ that, apart from the reporting strategy $\hat{y}$, also includes the agent's consumption C, and saving S, as functions of the history y^t of cash flow realizations. At every date t, the agent's consumption C_t, and the savings S_t must be non-negative. The agent's income

$$i_t = d_t + \zeta(y_t, \hat{y}_t)$$

consists of two components: the transfer d_t from the investor and the difference $\zeta(y_t, \hat{y}_t)$ between the reported cash flow $\hat{y}_t$ and the actual cash flow realization y_t:

$$\zeta(y_t, \hat{y}_t) = \lambda (y_t - \hat{y}_t)^+ - (y_t - \hat{y}_t)^- .$$

Note, that $\zeta(y_t, \hat{y}_t)$ can be negative, since the agent can use his savings to overreport cash flows.

One can use a Revelation-Principle type of logic to show that, for any contract $\sigma = (d, p)$, there exists a contract $\sigma' = (d', p')$ that results in the same payoff for the agent and equal or greater payoff for the investor, and for which the truth-telling is the optimal agent's strategy. Indeed, suppose the reporting strategy $\tilde{y}$ is optimal under the contract σ, then define $d'_t\left(\hat{y}^t\right) = d_t\left(\tilde{y}_t(\hat{y}^t)\right) + \zeta(\hat{y}_t, \tilde{y}_t(\hat{y}^t))$, and $p'_t\left(\hat{y}^t\right) = p_t\left(\tilde{y}_t(\hat{y}^t)\right)$, for the every history $\hat{y}^t$ of the agent's reports. One can see that if the agent tells the truth under σ', then his income in each period is equal to the income he receives under σ when he employs strategy $\tilde{y}$; and the investor's payoff under σ' is no less than under the old contract[3]. In addition, the truth-telling is optimal under σ', since $\tilde{y}$ is the optimal strategy under σ.

Savings are not necessary for the agent because the agent is risk-neutral. Since the agent finds it optimal to tell the truth under the contract σ', he never uses his savings to misrepresent cash flows. Thus, savings translate into delayed consumption. Since the agent is risk-neutral, he receives no benefits from consumption smoothing.

[3]If $\lambda < 1$, the investor's income can actually be higher under the contract σ' because of inefficiencies associated with the cash flow diversion under the contract σ.

This leads to the following result:

Proposition 1 *There exists an optimal contract that induces the agent to report cash flows truthfully and maintain zero savings.*

2.2 Temporary Incentive Compatibility Constraints

Now, we prove that the contract is incentive compatible if and only if at any point of time the agent's continuation payoffs satisfy temporary incentive compatibility constraints.

Given the result of Proposition 1, we will focus on direct-revelation contracts with no savings for the agent. To facilitate our analysis, we assume that the agent is not allowed to save, which implies that the agent cannot report $\hat{y}_H$ when y_L is realized. After finding an optimal contract with no saving, we will verify that the contract remains incentive compatible when the agent is allowed to save.

The agent's reporting strategy $\hat{y}$ is a sequence $\left\{\hat{y}_t\left(y^t\right)\right\}_{t=1}^{T}$ of the agent's reports regarding the cash flow realizations. If the agent reports $\hat{y}_t$, given the actual cash flow Y_t, his net income in period t is given by $Y_t - \hat{y}_t + d_t\left(\hat{y}^t\right)$. The reporting strategy $\hat{y}$ under the contract σ results in the following expected payoff for the agent:

$$a_0\left(y_0, \hat{y}, \sigma\right) = E\left[\sum_{t=1}^{T} \beta^t P_t\left(\hat{y}^t\right)\left(\lambda(Y_t - \widehat{y}_t) + d_t\left(\hat{y}^t\right)\right)|Y_0 = y_0\right]. \tag{5}$$

Note that the agent's payoff also depends on the initial state y_0.

We say that a contract $\sigma = (d, p)$ is *incentive compatible* if it induces the agent never to misreport the cash flows. That is, for any reporting strategies $\hat{y}$,

$$E\left[\sum_{t=1}^{T} \beta^t P_t\left(y^t\right) d_t\left(y^t\right)\ |Y_0 = y_0\right] \geq E\left[\sum_{t=1}^{T} \beta^t P_t\left(\hat{y}^t\right)\left(\lambda(Y_t - \hat{y}_t) + d_t\left(\hat{y}^t\right)\right)|Y_0 = y_0\right]. \tag{6}$$

Under an incentive compatible contract, the agent's best strategy is to truthfully reveal the cash flows in every period. Will the truth-telling strategy remain optimal on an off-equilibrium path? The next result shows that an incentive compatible contract remains incentive compatible even if the agent has deviated from truth-telling in the past.

Lemma 1 *Under any incentive compatible contract, at any point of time, the agent (weakly) prefers to tell the truth, even if he has lied in the past.*

Proof. See Appendix. ■

After a realization of cash flow y_H the agent has a dilemma. He can either tell the truth and report the high cash flow, or report the low cash flow and consume the stolen $\lambda(y_H - y_L)$. Reporting y_H truthfully means choosing a continuation contract that follows after the report y_H, while reporting y_L and diverting $y_H - y_L$ means choosing a continuation contract that follows after the report y_L. Let the continuation contract $\sigma_t(\hat{y}^t)$ correspond to the history $\hat{y}^t$ of agent's reports. The terms of this contract depend on the reported cash flows, but not on the actual realizations of the cash flows. However, the last cash flow realization matters for the agent's and the investor's continuation payoffs, because it determines the distribution of the future cash flows.

Let's consider a one-time deviation from the truth-telling strategy. Suppose $Y_t = y_H$. Given a history y^t of the cash flows realizations, such that the project can be active in period t, i.e. $P_t(y^t) > 0$, the agent's continuation payoff under the truth-telling strategy is given by

$$a_t\left(y^{t-1}, y_H\right) = E\left[\sum_{k=t}^{T} \beta^{k-t} \frac{P_k\left(y^t, y^{k-t}\right)}{P_t\left(y^t\right)} d_t\left(y^t, y^{k-t}\right) \mid Y_t = y_H\right].$$

If the agent truthfully reveals the cash flows in each period other than t, but cheats in period t, the agent's continuation payoff will be

$$\hat{a}_t\left(y^{t-1}, y_H\right) = E\left[\sum_{k=t}^{T} \beta^{k-t} \frac{P_k\left(y^{t-1}, y_L, y^{k-t}\right)}{P_t\left(y^{t-1}, y_L\right)} d_t\left(y^{t-1}, y_L, y^{k-t}\right) \mid Y_t = y_H\right].$$

We will call $\hat{a}_t\left(y^{t-1}, y_H\right)$ the *deviation* continuation payoff after history $\left(y^{t-1}, y_H\right)$.

The next theorem says that the contract is incentive compatible if and only if there is no one-time profitable deviation for the agent.

Theorem 1 *The contract $\sigma = (d, p)$ is incentive compatible if and only if for all time periods $t \leq T$, and all histories y^{t-1}, such that $P_k\left(y^{t-1}\right) > 0$,*

$$a_t\left(y^{t-1}, y_H\right) \geq \hat{a}_t\left(y^{t-1}, y_H\right) + \lambda(y_H - y_L). \tag{7}$$

Proof. See Appendix. ■

Equation (7) says that, under the contract σ, the agent does not find it optimal to deviate from the truth-telling strategy for one period. We will interpret equation (7) as the *temporary incentive compatibility* constraint. Theorem 1 establishes that a contract is incentive compatible if and only if at any point of time temporary incentive compatibility constraints are satisfied. We will use this property later when we formulate the contracting problem recursively.

3 The Optimal Contract

In this section, we solve for the optimal contract. Our methodology for solving for the optimal contract consists of the following main steps. First, we introduce a convenient way to represent the agent's continuation payoffs. Second, in order to rewrite the contracting problem recursively, we conjecture that the optimal contract is sequentially optimal. Third, given the conjecture, we solve for the optimal contract using a dynamic programming technique. A key ingredient of our technique is that, in order to write down temporary incentive compatibility constraints, we calculate the agent's continuation payoffs in the high state as a function of the agent's continuation payoffs in the low state under the optimal contract. Finally, we verify that the contract we derive under the sequential optimality conjecture is indeed the best possible contract.

3.1 Parametric Representation of Continuation Payoffs

In this subsection, we introduce a convenient way to represent the agent's continuation payoffs. Since the cash flows are correlated, the agent's continuation payoffs depend not only on the terms of the contact, but also on the last cash flow realization, which determines the distribution of the future cash flows. It turns out, as we will see later, this dependence can be conveniently incorporated into the agent's continuation payoffs, if these payoffs are represented in terms of the values of the cash flows that the agent can steal during a certain time interval.

For continuous time $\tau \in \Re$, let n_τ be the biggest integer, such that $n_\tau \leq \tau$, and $l_\tau = \tau - n_\tau$. We will use τ to denote a time interval of n_τ periods and the fraction l_τ of the next period. Given $Y_t = y$, the value of the cash flows that can be diverted by the agent during time τ starting from period $t+1$ is given by

$$V_\tau(y) \equiv E\left[\sum_{k=1}^{n_\tau} \beta^k \left(Y_{t+k} - y_L\right) + l_\tau \beta^{(n_\tau+1)} (Y_{t+n_\tau+1} - y_L) \middle| \, Y_t = y\right]. \tag{8}$$

Here, we assume that if the project operates a fraction l_τ of a period, then only the fraction l_τ of the cash flow realized in that period is counted. Alternatively, we can say that l_τ represents the probability that the agent will be allowed to run the project in the period $t+n_\tau+1$. Note that due to the Markov property of the cash flow process, the current date t is irrelevant for the value of V_τ.

Function V_τ has a number of good properties.

Lemma 2 *Function $V_\tau(Y_t)$ is continuous, strictly increasing in τ, and is piecewise linear, with the right-hand-side derivative*

$$\frac{\partial V_\tau(y)}{\partial \tau} = \beta^{(n_\tau+1)} E\left[\left(Y_{t+n_\tau+1} - y_L\right) \middle| Y_t = y\right]. \tag{9}$$

Moreover,

$$V_{\tau+1}(Y_{t-1}) = E\left[\beta\left(\left(Y_t - y_L\right) + V_\tau(Y_t)\right) | Y_{t-1}\right]. \tag{10}$$

Proof. See Appendix. ■

The equation (10) is self-evident. Its left-hand side is the value of the cash flows that can be stolen during time $\tau+1$, while the right-hand side represents the same value as a sum of the next period excess cash flow $(Y_t - y_L)$, and the value of the cash flows that can be stolen during time τ. The derivative (9) is obtained from (8), using the fact that $l_\tau = \tau - n_\tau$.

We will use $V_\tau(\cdot)$ to represent continuation payoffs for the agent. For example, if $Y_t = y_L$, and the agent's continuation payoff at the end of period t is $a > 0$, then there exist unique $\tau > 0$ such that $a = V_\tau(y_L)$. Thus, given state y_L, the agent's continuation payoff of a is equivalent to the expected value of the cash flows that can be stolen between time $t+1$ and time $t+1+\tau$.

We assume that the value of the liquidated firm in period t is given by

$$L_t(y_t) = \sum_{k=t+1}^{T} \beta^{k-t} y_L + \alpha V_{T-t}(y_t), \tag{11}$$

where $\alpha \leq 1$. According to (11), the investor is able to recover the full value of the future minimum cash flows and the fraction α of the value of the future excess cash flows. The liquidation is strictly inefficient when $\alpha < 1$. This assumption is made to simplicity and is not crucial for our subsequent results.

3.1.1 When the First Best is Implementable

The only case when a continuation contract does not ever require liquidation is when the agent's continuation payoff is no less than $\lambda V_{T-t}(y_t)$, which is the value of the cash flows the agent is able to steal running the project until the final date T. The optimal contract looks especially easy in this situation. The investor pays to the agent the difference between the agent's continuation payoff and $\lambda V_{T-t}(y_t)$ immediately and lets the agent consume the fraction λ of all subsequent excess cash flows. In this case, the continuation function at the end of period t is given by

$$b_t^e(y_t, a_t^e) = \sum_{s=t+1}^{T} \beta^{s-t} E[Y_s | Y_t = y_t] - a_t^e \text{ for } a_t^e \geq \lambda V_{T-t}(y_t). \tag{12}$$

When the agent continuation payoff a_t^e is below $\lambda V_{T-t}(y_t)$, the threat of liquidation must be real. Otherwise, the agent can steal all the subsequent excess cash flows and get the payoff $\lambda V_{T-t}(y_t)$.

3.2 The Optimal Contract with Restrictions on the Continuation Payoffs

In this section, we derive the optimal contract using a recursive procedure. This procedure requires, that at any point of time, all the relevant information contained in the history is summarized by the two state variables, the realization of the last cash flow and the agent's continuation payoff. To justify this procedure, we make use of the following conjecture.

3.2.1 Sequential Optimality Conjecture

Recall the definition of an optimal contract. We say that an incentive-compatible contract σ that implements payoffs $a_0(y_0|\sigma)$ and $b_0(y_0|\sigma)$ for the agent and the investor respectively is *optimal* at state y_0 if there is no other incentive-compatible contract $\tilde{\sigma}$ with the same payoff for the agent, but with a higher payoff for the investor: $a_0(y_0|\tilde{\sigma}) = a_0(y_0|\sigma)$ and $b_0(y_0|\tilde{\sigma}) > b_0(y_0|\sigma)$. This definition is based on the time zero payoffs.

In a similar manner, we can define an optimal continuation contract at any point of time:

Definition 1 An incentive-compatible continuation contract σ_t is *optimal* at state y_t if there is no other incentive-compatible continuation contract $\tilde{\sigma}_t$ such that $a_t(y_t|\tilde{\sigma}_t) = a_t(y_t|\sigma_t)$ and $b_t(y_t|\tilde{\sigma}_t) > b_t(y_t|\sigma_t)$.

Note that the definition of the optimality of a continuation contract at time t does not depend on what happened before this time. We may think about an optimal continuation contract as the best incentive-compatible contract for the investor that can be written at time t, given the agent's continuation payoff.

Definition 2 An incentive-compatible contract is *sequentially* optimal if at any point of time, after any history, its continuation contracts are optimal.

The definition of an optimal contract does not ensure its optimality in the future. We will show that a sequentially optimal contract exists. We will do it in two steps. First, we conjecture that the optimal contract is sequentially optimal. We will use this conjecture to justify our recursive algorithm to derive a contract. Second, we will verify that the derived contact is optimal by showing there is no other contract that can improve on the derived contract.

Conjecture 1 *There exists a sequentially optimal contract.*

In other words, we conjecture that the optimal contract not only maximizes the investor's expected payoff at time zero for a given initial payoff for the agent, but also maximizes the investor's continuation payoff at any point of time in the future, given the corresponding continuation payoff for the agent. Although the statement of Conjecture 1 is true in the case when cash flows are independent over time[4], there is no direct way to verify it in advance when cash flows are correlated. We will compute the optimal contract σ^*, assuming that Conjecture 1 is true, and then verify that no other contract can improve on σ^*.

3.2.2 Derivation of the Optimal Contract Using a Recursive Algorithm

Conjecture 1 allows us to calculate the optimal contract recursively. If a continuation contract σ_t is optimal at time t, then we can use σ_t to calculate a continuation contract σ_{t-1} that is optimal at time $t-1$. In particular, Conjecture 1 implies that under the optimal contract, the investor's continuation payoffs in period t – at the beginning, after the cash flow report, and in the end – are given by $b_t^y(y_{t-1}, a_t^y)$, $b_t^d\left(y_t, a_t^d\right)$, $b_t^e(y_t, a_t^e)$ correspondingly, where a_t^y, a_t^d, and a_t^e denote corresponding continuation payoffs for the agent. The continuation functions $b_t^y(y_{t-1}, a_t^y)$, $b_t^d\left(y_t, a_t^d\right)$, $b_t^e(y_t, a_t^e)$ can be computed recursively by solving a sequence of optimization problems. The solution of these problems will also give us the characterization of the optimal continuation contracts in terms of continuation payoffs and termination probabilities.

In each period $t < T$, the sequence of the following events takes place. First, the agent privately observes the realization of the cash flow Y_t. If $Y_t = y_H$, the agent decides whether to pay the cash flow to the investor, or divert it for his own consumption and report y_L.

[4] See for example, DeMarzo and Fishman (2003)

If $Y_t = y_L$, the agent has no other choice than to report the cash flow y_L. Given the agent's payment, the investor, in accordance with the contract, makes a decision regarding the termination of the project. In the case of termination, the agent's payoff is zero, and the investor's payoff is L_t. If the project is not terminated, it continues to operate in the next period.

For each period, we consider the start-of-period, intra-period (just prior to the termination decision), and end-of-period continuation functions, denoted by $b_t^y(Y_{t-1}, a_t^y)$, $b_t^d(Y_t, a_t^d)$, $b_t^e(Y_t, a_t^e)$ correspondingly, where a_t^y, a_t^d, and a_t^e denote the agent's continuation payoffs at the start, middle and end of period t respectively.

We start computing the continuation functions from the final period T. At the end of the final period, the project is liquidated, and no cash flows will be generated in the future. However, a transfer from the investor to the agent is allowed. Due to the limited liability of the agent, this transfer must be non-negative. Hence, the continuation function at the end of the last period is given by

$$b_T^e\left(y_T, a_T^e\right) = \begin{cases} -a_T^e \text{ for } a_T^e \geq 0 \\ -\infty \text{ for } a_T^e < 0 \end{cases},$$

where the agent's continuation payoff a_T^e is the final payment from the investor.

Given the continuation function in the last period, we can calculate the continuation functions in earlier periods by taking the following steps.

Step One: Liquidation Problem Consider the problem the investor faces after the cash flow announcement but before the liquidation decision. If the project is terminated in period t, the investor's payoff is L_t, while the agent gets nothing. Given the continuation function $b_t^e\left(y_t, \cdot\right)$ and the agent's continuation payoff a_t^d, the optimal probability of liquidation p_t solves

$$\begin{aligned} b_t^d\left(y_t, a_t^d\right) &= \max_{p_t,\ a_t^e} (1-p_t) b_t^e\left(y_t, a_t^e\right) + p_t L_t && (13) \\ s.t.\ a_t^d &= (1-p_t) a_t^e, && (14) \\ a_t^e &\geq \lambda V_1\left(y_t\right), && (15) \\ p_t &\in [0,1] && (16) \end{aligned}$$

where equation (14) ensures that the agent's continuation payoff a_t^d before the liquidation decision is consistent with the continuation payoff a_t^e after the liquidation decision. Constraint

(15) reflects the fact that once the project is allowed to continue into the next period, the agent's end-of-period payoff must be at least as high as the expected value of the cash flow $a^L(y_t) \equiv \lambda V_1(y_t)$ that the agent is capable of stealing in the next period. This also implies that if the agent's intra-period continuation payoff a_t^d is below $a_t^L(y_t)$, the investor must liquidate the project in period t with positive probability.

The next proposition states that the project is terminated with positive probability if and only if $a_t^d(Y_t)$ drops below $a^L(Y_t)$.

Proposition 2 *The probability of the termination in period t is given by*

$$p\left(Y_t, a_t^d\right) = \begin{cases} \frac{a^L(Y_t) - a_t^d}{a^L(Y_t)} \; for \; a_t^d \in [0, a^L(Y_t)) \\ 0 \quad for \quad a_t^d \geq a^L(Y_t) \end{cases} . \tag{17}$$

The intra-period continuation function $b_t^d(Y_t, a_t^d)$ is obtained from end-of-period continuation function $b_t^e(Y_t, \cdot)$ as follows:

$$b_t^d(Y_t, a_t^d) = \begin{cases} -\infty & for \quad a_t^d(Y_t) < 0 \\ \frac{a_t^d}{a^L(Y_t)} b_t^e(Y_t, a^L(Y_t)) + \left(1 - \frac{a_t^d}{a^L(Y_t)}\right) L_t \; for & a_t^d \in [0, a^L(Y_t)) \\ b_t^e(Y_t, a_t^d) & for \;\; a_t^d > a^L(Y_t) \end{cases} . \tag{18}$$

If the project is not terminated in period t the agent's continuation payoff evolves as follows:

$$a_t^e = \min\left(\lambda V_{T-t}(Y_t),\ \max\left(a_t^d,\ a^L(Y_t)\right)\right).$$

The proof of Proposition 2 is in the Appendix. It is strongly recommended to finish reading this section first before reading the proof, since the proof relies on other results of this section. We prove Proposition 2 and the other propositions in this section all together through a backward induction argument. Assuming that the statements of these propositions are true in the subsequent periods, we prove that they must be true in the current periods.

The intuition behind this result is simple. The termination of the project is inefficient. Therefore, the investor finds it optimal to refrain from early termination, unless she has exhausted all other means of providing proper incentives for the agent. Once the investor is in a position in which she has to resort to a liquidation, the liquidation probability is the smallest one that allows her to implement the agent's continuation payoff. The probability of the termination, given by (17) is proportional to the amount the agent's continuation payoff a_t^d is below the minimal implementable payoff $a^L(Y_t)$. If $a_t^d < a^L(Y_t)$, and the project was not terminated in period t, then the agent's continuation payoff at the end of period t is increased to $a^L(Y_t)$.

Step Two: Intra-Period agency Problem At every point of time, the optimal contract governing the relationship between the investor and the agent in the future is characterized by two state variables: the realization of the last cash flow that determines the distribution of the future cash flows, and the promised payoff for the agent. After observing the realization of y_t, the agent must report the earning to the investor. Let $a_t^d(y_t)$ denote the continuation payoff for the agent associated with the optimal contract if the agent truthfully announces the realization of y_t. If $y_t = y_L$, then the agent has no other choice but to tell the truth and get $a_t^d(y_L)$. If $y_t = y_H$, then the agent would get $a_t^d(y_H)$ if he tells the truth.

An important question is what the agent's continuation payoff would be if he reports y_L instead of y_H. The contract specifies future payments to the agent and termination probabilities as a function of the history of the agent's reports. In this sense, the terms of the future contract do not depend on whether or not the agent was honest. However, Y_t determines the distribution of future cash flows and, therefore, is relevant for the agent's continuation payoff under fixed terms of the contract. If the agent cheats on the investor, his continuation payoff, as we will see, would be higher than $a_t^d(y_L)$, since he faces better prospects regarding future cash flows.

For the contract σ^*, let $c_t\left(a_t^d(y_L)\right)$ denote the continuation payoff for the agent if he chooses to report y_L when y_H was actually realized in period t. We write it as a function of $a_t^d(y_L)$, since $a_t^d(y_L)$ fully determines the future terms of the optimal contract after y_L was reported. We will refer to c_t as the agent's *deviation* continuation payoff function.

Given the intra-period continuation function b_t^d, the start-of-period continuation function b_t^y is the solution of the following problem:

$$
\begin{aligned}
b_t^y(Y_{t-1}, a_t^y) &= \max_{a_t^d(\cdot)} E_t\left[Y_t + b_t^d\left(Y_t, a_t^d(Y_t)\right)|Y_{t-1}\right] && (19)\\
\text{s.t.}\quad (\text{IC})\; a_t^d(y_H) &\geq c_t\left(a_t^d(y_L)\right) + \lambda(y_H - y_L) && (20)\\
(\text{PK})\qquad a_t^y &= E_t\left[a_t^d(Y_t)|Y_{t-1}\right] && (21)\\
(\text{IR})\qquad a_t^d(\cdot) &\geq 0. && (22)
\end{aligned}
$$

The start-of-period continuation payoff for the investor is the conditional expectation of the sum of the cash flow Y_t and the intra-period continuation payoff, given by the function b_t^d. The intra-period continuation payoff for the agent conditional on realization of Y_t, $a_t^d(y_L)$ and $a_t^d(y_H)$ must satisfy the incentive compatibility constraint (IC), and the promise keeping constraint (PK), and the individual rationality constraint (IR). The first constraint insures that the agent has no incentive to divert the high cash flow. The second one says that the payoff a_t^y promised to the agent at the start of period t is equal to the expected intra-period continuation payoff. Since the agent has limited liability, his continuation payoff cannot be negative.

In the formulation of the problem (19), we explicitly use the conjecture that the contract σ^* results in the continuation payoffs lying on the upper frontiers of the payoff possibility sets, represented by the continuation functions $b_t^d(Y_t, \cdot)$.

Proposition 3 *The incentive compatibility constraint (IC) binds: for* $t = 1, ..., T$

$$a_t^d(y_H) = c_t\left(a_t^d(y_L)\right) + \lambda(y_H - y_L). \tag{23}$$

Proof. See Appendix. ■

The proof is interconnected with the proofs of the other propositions in this section.

Proposition 3 allows us to calculate function c_t. We will represent the agent's continuation payoffs using functions $V_\tau(\cdot)$. For any $a_t^d \leq \lambda V_{T-t}(y_t)$, there is a unique parameter $\tau \leq T-t$ such that $a_t^d = \lambda V_\tau(y_t)$.

Proposition 4 *Given contract* σ^**, if the agent's continuation payoff in the state* y_L *is* $\lambda V_\tau(y_L)$ *for some* $0 \leq \tau \leq T-t$ *, then the agent's continuation payoff in deviation is* $\lambda V_\tau(y_H)$*:*

$$c_t(\lambda V_\tau(y_L)) = \lambda V_\tau(y_H) \text{ for } 0 \leq \tau \leq T-t. \tag{24}$$

Proof. See Appendix. ■

The terms of a continuation contract depend only on the reported cash flows. In this respect, if the agent reports the low cash flow in the current period he faces the same continuation contract whether or not the true cash flow realization was low. However, the agent's continuation payoff depends on the true cash flow realization, since the distribution of future cash flows is a function of the current cash flow realization.

Let $\sigma_t^*(y_L)$ denote the continuation contract after the agent reports the low cash flow in period t. Given the probability of liquidation p_t in period t under $\sigma_t(y_L)$, we can represent the agent's continuation payoff in equilibrium as the expectation of the agent's continuation payoffs in the next period:

$$a_t^d(y_L) = E\left[\beta(1-p_t)\,a_{t+1}^d(Y_{t+1})\,|Y_t = y_L\right], \tag{25}$$

where $a_{t+1}^d(Y_{t+1})$ denotes the continuation payoff for the agent in period $t+1$ under the continuation contract $\sigma_t^*(y_L)$ conditional on the cash flow Y_{t+1}. Similarly, if the agent reports the low cash flow in period t, when, in fact, the high cash flow realized, the same contract results in the following continuation payoff after the deviation:

$$c_t\left(a_t^d(y_L)\right) = E\left[\beta(1-p_t)\,a_{t+1}^d(Y_{t+1})\,|Y_t = y_H\right]. \tag{26}$$

Given (25)-(26), one can calculate function c_t using backward induction.

Proposition **??** allows us to calculate the evolution of the agent's continuation payoffs. Since the continuation payoffs $a_t^d(y_L)$ and $a_t^d(y_H)$ satisfy the promise-keeping constraint (PK) and the incentive compatibility constraint (IC), which is binding, we can solve equations (PK) and (IC) for $a_t^d(y_L)$ and $a_t^d(y_H)$ as functions of a_t^y. This yields the following:

Proposition 5 *For the start-of-period continuation payoff given by $a_t^y = \frac{1}{\beta}\lambda V_{\tau+1}(Y_{t-1})$, with $0 \le \tau \le T-t$, the agent's intra-period continuation payoff is given by*

$$a_t^d(Y_t) = \lambda V_\tau(Y_t) + \lambda(Y_t - y_L)\ . \tag{27}$$

The start-of-period continuation function is obtained as follows:

$$b_t^y\left(Y_{t-1}, \frac{1}{\beta}\lambda V_{\tau+1}(Y_{t-1})\right) = E_t\left[Y_t + b_t^d\left(Y_t, \lambda V_\tau(Y_t) + \lambda(Y_t - y_L)\right)|Y_{t-1}\right].$$

Proof. According to Lemma 2,

$$\frac{1}{\beta}V_{\tau+1}(Y_{t-1}) = E\left[V_\tau(Y_t) + (Y_t - y_L)|Y_{t-1}\right].$$

Hence, the (PK) constraint is satisfied.

According to (27), $a_t^d(y_L) = \lambda V_\tau(y_L)$ and $a_t^d(y_H) = \lambda V_\tau(y_H) + \lambda(y_H - y_L)$. Proposition 4 says that $c_t\left(a_t^d(y_L)\right) = \lambda V_\tau(y_H)$. Putting all together:

$$a_t^d(y_H) = c_t\left(a_t^d(y_L)\right) + \lambda(y_H - y_L).$$

Thus, $a_t^d(y_L)$ and $a_t^d(y_H)$ satisfy the (IC) constraint as equality.

Substituting (27) into (19) gives $b_t^y\left(Y_{t-1}, \frac{1}{\beta}\lambda V_{\tau+1}(Y_{t-1})\right)$. ■

Step Three: Discounting between Periods So far, we have demonstrated that knowing the end-of-period continuation function b_t^e, one can obtain the intra-period continuation function b_t^d, and then the start-of-period continuation function b_t^y. To complete our recursive characterization of the optimal contract, we derive the continuation function b_{t-1}^e, given b_t^y.

With the exception of the payoff discounting, nothing takes place between the end of the prior period and the beginning of the next period. Hence, if the contract results in the payoff of a_{t-1}^e at the end of period $t-1$, then the agent's payoff at the start of period t must be a_{t-1}^e/β. The above argument, combined with the fact that the investor discounts future cash flows using the same discount factor β, yields the following:

Proposition 6 *Given the start-of-period continuation function b_t^y, the continuation function at the end of period t is given by*

$$b_{t-1}^e\left(y_{t-1}, a_{t-1}^e\right) = \beta b_t^y\left(y_{t-1}, \frac{a_{t-1}^e}{\beta}\right).$$

3.2.3 Summary of the Algorithm

Propositions 2, 5, 6, and 4, yield the algorithm of computing recursively the functions b_t^e, b_t^d, b_t^y, and c_t starting from the end of the last period T. In general, the functions b_t^e, b_t^d, b_t^y are too complicated to be written explicitly. However, the evolution of the agent's continuation payoff can be described compactly, as is done below:

$$a_{t-1}^e = \lambda V_\tau(Y_{t-1}) \tag{28}$$

$$\rightarrow \quad a_t^y = \lambda V_\tau(Y_{t-1})/\beta \tag{29}$$

$$\rightarrow \quad a_t^d(Y_t) = \lambda V_{\tau-1}(Y_t) + \lambda(Y_t - y_L) \tag{30}$$

$$\rightarrow \quad a_t^e = \min\left(\lambda V_{T-t}(Y_t),\ \max\left(a_t^d(Y_t),\ a^L(Y_t)\right)\right). \tag{31}$$

During the transition from the end of period $t-1$ to the beginning of period t, the agent's continuation payoff is adjusted by the discount factor β.

The transition from a_t^y to $a_t^d(Y_t)$ provides the agent with an incentive to report truthfully the realization of the cash flow Y_t. Comparing to a_t^y, the agent's continuation payoff is going to increase if $Y_t = y_H$, and decrease if $Y_t = y_L$:

$$a_t^d(y_L) < a_t^y < a_t^d(y_H).$$

The agent's continuation payoff is affected by the realization of Y_t in two ways. First, the current cash flow Y_t directly affects the continuation payoff through the term $\lambda(Y_t - y_L)$. This term represents the value of the realized cash flow without taking into account an impact of this cash flow realization on the subsequent cash flows. Second, the cash flow Y_t affects the agent's continuation payoff through its impact on the distribution of the future cash flows, which is reflected in the term $\lambda V_{\tau-1}(Y_t)$.

The project can be liquidated only if the agent's continuation payoff a_t^d becomes low enough that it cannot be implemented without a liquidation with a positive probability in period t. The probability of liquidation is given by

$$p_t\left(a_t^d(y_t)\right) = \max\left(\frac{a^L(y_t) - a_t^d(y_t)}{a^L(y_t)}, 0\right). \tag{32}$$

As an immediate consequence of this result, we have the following lemma:

Lemma 3 *The project is never liquidated in a period in which the high cash flow is reported.*

Indeed, at the beginning of a period t, the agent's continuation payoff must be at least $a^L(y_{t-1})/\beta = V_1(y_{t-1})/\beta$, which is the expected amount he will be able to steal in this period. After cash flow y_H is reported, his continuation payoff can only increase. Thus, the agent's continuation payoff never goes down into the liquidation zone $\left[0, a^L(y_t)\right]$, after a realization of the high cash flow.

The parameter τ can be interpreted as the earliest default time. Given the agent's continuation payoff $a_{t-1}^d = \lambda V_{\tau_{t-1}}(Y_{t-1})$ in period $t-1$, the liquidation will not occur during the next $n_{\tau_{t-1}}$ periods, no matter what cash flows the agent reports in these periods. However, if the agent reports the low cash flow $n_{\tau_{t-1}} + 1$ times in a row, the liquidation will occur in period $t + n_{\tau_{t-1}}$ with probability $l_{\tau_{t-1}}$. To see this, consider the evolution of the agent's continuation payoffs given by (28)-(31). If the agent reports the low cash flow in period t, his continuation payoff after the report will be $\lambda V_{\tau_{t-1}-1}(y_L)$ if the agent tells the truth and $\lambda V_{\tau_{t-1}-1}(y_H)$ if he lies, due to Proposition 4. Thus, the agent's continuation payoff will be $\lambda V_{\tau_t}(Y_t)$ in period t, where $\tau_t = \tau_{t-1} - 1$. The liquidation will not occur in period t as long as $\tau_t \geq 1$. Now, one can see that if the agent reports the low cash flow in period $t+1$, his continuation payoff will be $\lambda V_{\tau_{t+1}}(Y_t)$, where $\tau_{t+1} = \tau_t - 1$. Thus, the liquidation cannot occur before period $t + n_{\tau_{t-1}}$.

3.2.4 Outline of the proof

We are now in a position to prove that if there exists a sequentially optimal contract, then it must be the contract σ^* that has been derived in this section. In this subsection, we only provide an intuitive explanations for the form of the optimal contract, while the Appendix contains formal proofs of all the statements made in this section.

Our proof is by induction that starts from the last period T. Assuming that all the properties of the contract σ^* outlined above hold in periods $t+1$ through T, we prove that they must hold in period t to maximize the investor's expected payoff in this period. In each period, assuming we know the function b_t^e, we solve the liquidation problem (13)-(16), whose solution gives us the optimal liquidation probability p_t, and the continuation function b_t^d. The next step is to solve the intra-period agency problem (19)-(21) and obtain the continuation function b_t^y. The last step is the transition from the beginning of period t to the end of the period $t-1$, which involves discounting between the periods.

To prove Proposition 2, we need to solve the liquidation problem (13)-(16), in which we maximize the investor's continuation payoff $b_t^d\left(y_t, a_t^d\right)$ before the liquidation decision, given the continuation function b_t^e. A probabilistic liquidation means that we randomize between the payoff pair $(0, L_t)$ in the event of liquidation, and the pair $(a_t^e, b_t^e(y_t, a_t^e))$ in the event of continuation, where the agent's payoff in the event of continuation is given by $a_t^e = a_t^d/(1-p_t)$, and p_t is the probability of liquidation. Let $(\tilde{a}_t^L, b_t^e(y_t, \tilde{a}_t^e))$ denote the payoff point of tangency of a line originating from the point $(0, L_t)$ to the upper frontier of the payoff possibility set at the end of the period t, given by the function b_t^e. Since the function b_t^e is concave, the continuation function b_t^d is given by the line connecting the points $(0, L_t)$ and $(\tilde{a}_t^L, b_t^e(y_t, \tilde{a}_t^e))$ on the interval $[0, \tilde{a}_t^L]$ and coincides with b_t^e on $(\tilde{a}_t^L, \infty)$. We prove in the Appendix that $\tilde{a}_t^L = a_t^L(y_t)$. Figure 2 shows the function b_t^e (solid line), which is defined on $\left[a^L, \infty\right]$, and the function b_t^d, which is obtained from b_t^e by extending it over the liquidation region $\left[0, a^L\right]$ (dashed line).

Intuitively, we can think about the optimal contracting problem as minimizing the liquidation probabilities while keeping the contract incentive compatible. Due to the inefficiency of liquidation, the investor prefers to threaten the agent with the liquidation in the future rather than resorting to liquidation immediately. Only when no threat in the future can induce truth-telling in the current period does the investor liquidate the project.

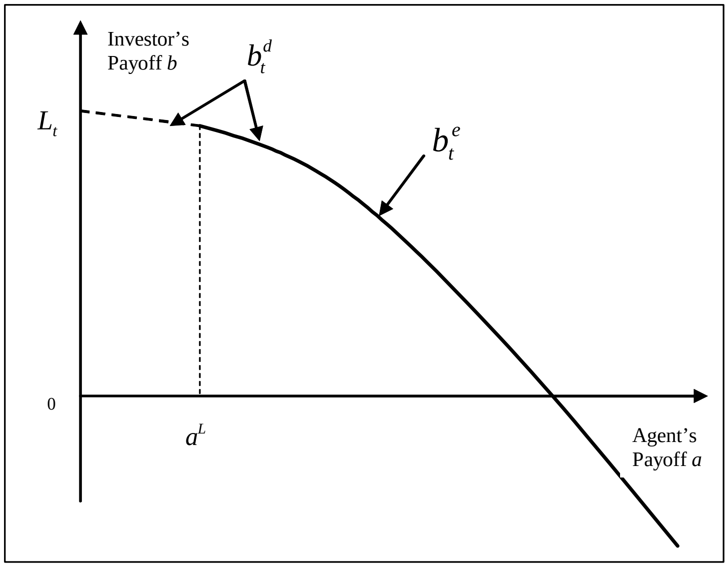

Figure 2: Continuation functions b_t^d and b_t^e

We now discuss the intra-period agency problem (19)-(22). In each period t, the agent's incentives to tell the truth are given through the continuation payoffs $a_t^d(y_t)$. The agent's continuation payoff increases $(a_t^d(y_H) > a_t^y)$ when the high cash flow is reported, and decreases $(a_t^d(y_L) < a_t^y)$ when the low cash flow is reported. The incentive compatibility constraint (20) is satisfied only when the difference between $a_t^d(y_H)$ and $a_t^d(y_L)$ is wide enough. According to Proposition 3, in order to maximize the investor's payoff the incentive compatibility constraints (20) must bind, meaning that the difference between $a_t^d(y_H)$ and $a_t^d(y_L)$ is minimized.

To see why the (IC) constraint should bind, consider how the liquidation probability would be different when the (IC) constraint does not bind. The liquidation is likely to occur sooner after the low cash flow realization than after the high cash flow realization, since the payoff $a_t^d(y_L)$ is closer than the payoff $a_t^d(y_H)$ to the liquidation region $\left(0, a^L\right)$. Since the promise-keeping constraint (21) insures that the mean of $a_t^d(y_H)$ and $a_t^d(y_L)$ is fixed and is equal to a_t^y, relaxing the (IC) constraint leads to a lower payoff $a_t^d(y_L)$ after the low cash flow realization and to a higher payoff $a_t^d(y_H)$ after the high cash flow realization, comparing to the case when the (IC) constraint binds. Thus, relaxing the (IC) constraint leads to even higher chances of liquidation when liquidation is more likely, and even lower chances of liquidation when it is less likely. As a result, the expected loss associated with the liquidation increases.

Thus, the (IC) constraint should bind.

3.3 The Optimal Contract without Restrictions

We assumed that Conjecture 1 is true when deriving the contract σ^*. The algorithm employed to derive the contract σ^* ensures that if there exists a sequentially optimal contract, it must be the contract σ^*. However, it may be possible that the optimal contract, i.e. a contract that maximizes the investor's payoff at time zero, given the agent's payoff, is not sequentially optimal, i.e. it results in suboptimal payoffs after some histories. In this case, the contract σ^* will remain incentive compatible, but will not maximize the investor's initial payoff.

In this section, we verify that the contract σ^* is the optimal contract. We relied on Conjecture 1 in the formulation of the intra-period agency problem (19)-(21). According to this problem, the investor maximizes his continuation payoff at the start of period t, by choosing the agent's continuation payoffs $a_t^d(y_L)$ and $a_t^d(y_H)$, such that they satisfy the promise-keeping constraint (PK) and the incentive compatibility constraint (IC). The (IC) constraint is written using the deviation payoff function c_t, which is associated with the continuation contract σ_t^* after y_L was reported in period t. We prove that the (IC) constraints bind for all $t \leq T$, and compute function c_t recursively.

According to the properties of σ^*, if the agent's continuation payoff at the start of period t is given by $a_t^y = \lambda V_\tau(Y_{t-1})/\beta$, then his continuation payoff will be $a_t^d(Y_t) = \lambda V_{\tau-1}(Y_t) + \lambda(Y_t - y_L)$, if he truthfully reports Y_t. Hence, the investor's continuation payoff will be $b_t^d\left(Y_t, a_t^d(Y_t)\right)$, if Y_t is realized. We, however, have not considered continuation contracts that result in payoffs below the upper frontiers of the payoff possibility sets, represented by $b_t^d(Y_t, \cdot)$. It is not obvious that such contracts are inferior to σ^*, since their incentive compatibility constraints can be different from those associated with the contract σ^*.

The next proposition states the main result of this section.

Proposition 7 σ^* *is the optimal contract.*

Proof. See Appendix. ■

Below, we discuss the main elements of the proof. The contract σ^* has two important properties. First, the incentive compatibility constraints always bind. Second, the project can be liquidated only if the agent's continuation payoff drops below $\lambda V_1(y_L)$, the lowest payoff value that can be implemented without liquidation. The first property says that the investor minimizes the distance between $a_t^d(y_H)$ and $a_t^d(y_L)$, subject to the (IC) constraint, while keeping the expectation of $a_t^d(Y_t)$ constant. Choosing continuation payoffs $\tilde{a}_t^d(y_H) > a_t^d(y_H)$ and $\tilde{a}_t^d(y_L) < a_t^d(y_L)$, so that the (PK) holds, would result in lower continuation payoff for the investor. Choosing continuation payoffs $\tilde{a}_t^d(y_H) < a_t^d(y_H)$ and $\tilde{a}_t^d(y_L) > a_t^d(y_L)$, so that the (PK) holds, would violate the (IC) constraint. According to Proposition 3, the incentive compatibility constraints bind for the contract σ^*, which means that if function b_t^d represents the upper frontier of the payoff possibility set (it does if σ^* is the optimal contract), then the investor's payoff at the beginning of period t is maximized when the difference between the agent's continuation payoffs $\tilde{a}_t^d(y_L)$ and $\tilde{a}_t^d(y_H)$ is minimized, subject to the incentive compatibility and the promise-keeping constraints.

To prove Proposition 7, we start by considering an arbitrary incentive compatible contract $\tilde{\sigma}$, whose incentive compatibility constraints do not necessarily bind. Let $\tilde{c}_t$ denote a deviation payoff function, associated with the contract $\tilde{\sigma}$. Since, by assumption, $\tilde{\sigma}$ is incentive compatible, the agent's continuation payoffs $\tilde{a}_t^d$, associated with $\tilde{\sigma}$, must satisfy

$$\tilde{a}_t^d(y_H) \geq \tilde{c}_t\left(\tilde{a}_t^d(y_L)\right) + \lambda(y_H - y_L) \text{ for all } t \leq T. \tag{33}$$

One can show recursively that equation (33) implies that $\tilde{c}_t(a) \geq c_t(a)$ for all $a \geq 0$, and $t \leq T$.

The function $\tilde{c}_t$ determines the difference between $\tilde{a}_t^d(y_H)$ and $\tilde{a}_t^d(y_L)$. The greater the value of the function $\tilde{c}_t$, the greater this difference is.

Since the continuation payoffs associated with the contract $\tilde{\sigma}$ lie below the upper frontier of the payoff possibility set, $\tilde{\sigma}$ can be an improvement on σ^*, only if $\tilde{\sigma}$ reduces the difference between the corresponding continuation payoffs for the agent. However, $\tilde{\sigma}$ never decreases this difference. Hence, $\tilde{\sigma}$ cannot be better than σ^*.

4 Initiating the Contract

The contract is initiated at time zero. The initial state Y_0 of the Markov cash flows and the continuation payoff $a_0^e(Y_0)$ for the agent uniquely determine the optimal contract between the agent and the investor. Thus, when Y_0 is commonly known, initiating the contract means choosing the payoffs $a_0^e(Y_0)$ and $b_0^e(y_L, a_0^e(y_L))$ for the agent and for the investor, respectively.

The situation is not much different when the initial state Y_0 is unknown but its distribution is the common knowledge. Suppose, Y_0 is unknown, but both the agent and the investor assess that $\Pr(Y_0 = y_H) = q_H$.

Proposition 8 *If the optimal contract σ^* implements payoffs $a_0^e(Y_0)$, and $b_0^e(Y_0, a_0^e(Y_0))$, when the state Y_0 is known, then, when Y_0 is unknown, σ^* is also optimal and implements payoffs $a_0 = E[a_0^e(Y_0)]$, and $b_0(a_0) = E[b_0^e(Y_0, a_0^e(Y_0))]$.*

Proof. See Appendix. ■

In the proof of Proposition 8, we use the fact that an optimal contract remains optimal in both states.

The contract sets initial continuation payoffs for the agent and the investor. Which pair of payoffs is chosen depends on the competitive environment. If the agent is a monopolist facing a competitive row of investors, he chooses the initial continuation payoff a_0^A, so that the investor will break-even:

$$a_0^A = \sup\{a : b_0(a) \geq I - W\}.$$

Proposition 9 *In the setting with a monopolistic agent and competitive investors, it is optimal for the agent to invest all his wealth in the project at date 0.*

Proof. See Appendix. ■

On the other hand, if the investor has all the market power, he chooses to maximize his payoff:

$$a_0^I = \arg\max_a b_0(a).$$

In general, the initial payoff a_0 for the agent can be anything between a_0^I and a_0^A depending on the market power of the agent and the investor, with a_0 increasing with the agent's market power and decreasing with the investor's market power.

5 The Implementation Result

In this section, we show that the optimal contract can be implemented using a combination of equity, a coupon bond, and a credit line with an escalating interest rate. We define these securities as follows:

Coupon Bond. A coupon bond represents the agent's commitment to make a coupon payments x to the investor at the end of each period $t \leq T$. If the agent is unable to make the coupon payment, the firm is in default.

Credit Line with Performance Pricing. A credit line is characterized by a credit limit C_t^L and an interest rate $r_t^C(M_{t-1})$ charged on the balance M_{t-1} at the end of the previous period. Note that we allow the interest rate to be a function of the credit line balance. The interest payment on the outstanding balance is due at the end of each period. The credit limit C_t^L determines the maximum amount of credit available for the agent in period t. Inability to make the current interest payment without exceeding the credit line limit leads to default. In addition, it is not allowed to borrow from the credit line to pay dividends.

Equity. The agent is allowed to use cash flows to pay dividends to the equity holders in proportion to their share of ownership.

Default occurs when the agent is unable to fulfil his financial obligations. Default leads to a probabilistic liquidation. Given the unmade payment z_t, the firm is liquidated with probability $p_t(z_t)$, while with probability $(1-p_t(z_t))$, the investor forgives the unmade payment z_t, and lets the agent operate the firm in the next period. In the even of liquidation, the investor sells the firm's assets and pockets the liquidation value L_t, while the agent gets nothing.

Although our definition of default is non-standard, it is consistent with the fact that creditors are often willing to write off a part of the debt instead of forcing bankruptcy. We can interpret the probabilistic liquidation as an uncertainty associated with the default procedure, which we do not model here. Our definition of the credit line imposes a restriction on the dividend policy, which is not unusual in practice.

Theorem 2 *The optimal contract is implemented by a combination of equity, a coupon bond, and a credit line with an escalating interest rate. The agent holds fraction λ of the equity, while the rest is held by the investor. The bond's coupon is equal to the verifiable cash flow:*

$$x = y_L.$$

The credit line has a credit limit given by

$$C_t^L = V_{T-t}(y_H) - V_1(y_H) \text{ for } t < T, \tag{34}$$

and an interest rate r_t^C that depends on the credit line balance as follows:
Let the balance M_{t-1} on the credit line at the end of period $t-1$ be represented as a function of parameter τ:

$$M_{t-1}(\tau) = V_{T-t}(y_H) - V_\tau(y_H)). \tag{35}$$

Then, the interest rate charged on this balance is given by

$$r_t^C\left(M_{t-1}\left(\tau\right)\right)=\frac{V_{T-t}\left(y_H\right)-V_{\tau-1}\left(y_H\right)}{V_{T-t+1}\left(y_H\right)-V_{\tau}\left(y_H\right)}-1. \tag{36}$$

The agent can draw on the credit line to make interest payments. However, he is not allowed to borrow from the credit line to pay dividends.

In the event of default, the unmade payment z_t results in the probability of liquidation

$$p_t\left(z_t\right)=\frac{z_t}{V_1\left(y_H\right)}. \tag{37}$$

With probability $(1-p_t\left(z_t\right))$, the project is not liquidated, and the unmade payment z_t is forgiven.

Proof. : See Appendix. ∎

In the optimal combination of securities, the roles of the coupon debt and equity are straightforward. The coupon debt is used to extract the verifiable cash flow y_L, while dividends paid to the equity holders represent a reward to the agent for repaying the credit line debt. Given his stake λ in the firm's equity, the agent is indifferent between stealing cash flows and issuing the dividends. The role of the credit line with an escalating interest rate is more sophisticated. The balance on the credit line can be considered as a memory device that summarizes all the relevant information regarding the past cash flow realizations. The interest rate along with the credit limit determines the dynamics of the credit line balance and the timing of the default. The threat of losing control over the project induces the agent to pay the credit line.

To prove Theorem 2, we show that the evolution of the balance on the credit line reflects the evolution of the agent's continuation payoffs induced by the contract σ^*. Specifically, the parameters of the credit line are chosen so that the continuation payoff for the agent $a_t^d\left(y_H\right)$ in high state under the optimal contract σ^* is always equal to the liquidation threshold $\lambda V_1\left(y_H\right)$ plus the amount of the unused credit on the credit line multiplied by λ:

$$a_t^d\left(y_H\right)=\lambda V_1\left(y_H\right)+\lambda(C_t^L-M_t).$$

As Figure 3 illustrates, zero balance on the credit line corresponds to the dividend threshold $\lambda V_{T-t}\left(y_H\right)$, while the balance equal to the credit limit corresponds to the liquidation threshold $\lambda V_1\left(y_H\right)$. An increase in the balance leads to a lower continuation payoff. Default occurs when the balance exceeds the credit limit.

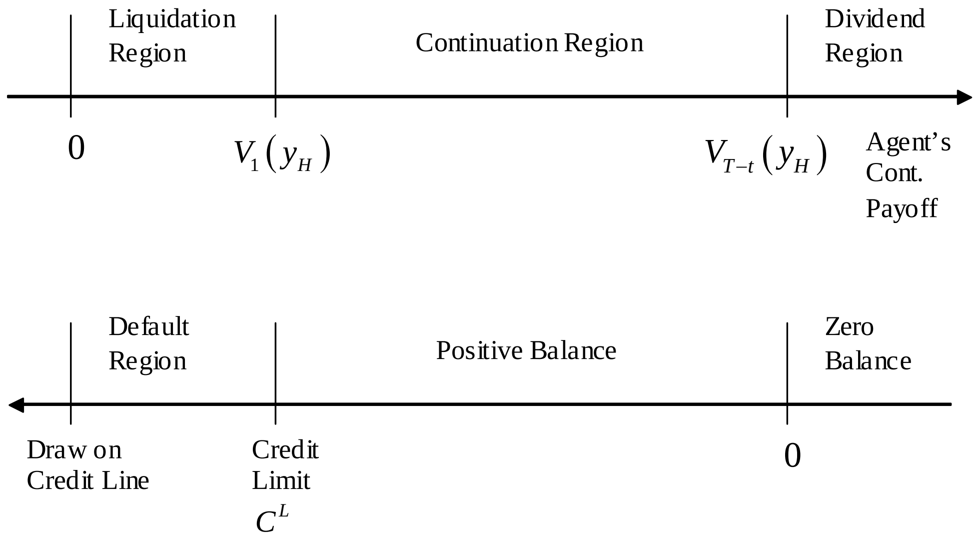

Figure 3: Agent's Continuation Payoff and Credit Line Balance

The agent uses all excess cash flows $(Y_t - y_L)$ to pay the credit line. Given the outstanding balance M_{t-1} in period $t-1$ and the payment $(Y_t - y_L)$ by the agent in period t, the new balance becomes

$$M_t = (1 + r_t^C(M_{t-1}))M_{t-1} - (Y_t - y_L).$$

When a cash flow is low, the agent has to draw on the credit line to make the interest payment, as long as his outstanding balance stays within the credit limit. On the other hand, a high cash flow leads to a reduction of the balance. The interest rate $r_t^C(M_{t-1})$ is chosen so that the evolution of the balance M_t is consistent with the evolution of the agent's continuation payoffs in the high state under the contract σ^*.

Since the balance on the credit line tracks the agent's continuation payoff in the high state, the agent has no incentive to divert excess cash flows. In the low state, there is no excess cash flow to divert. However, the credit limit in the low state is too generous compared to the agent's continuation payoff. If allowed, the agent would draw the credit line up to the limit, use all the borrowed cash to issue dividends and declare bankruptcy afterwards. To avoid this scenario, the credit line has the covenant that does not allow the agent to draw on the credit line to pay dividends.

5.1 The Optimal Interest Rate Structure.

We now examine properties of the optimal interest rate structure. The dependence of the interest rate r_t^C on the balance M_{t-1} is expressed through the parameter τ. Equations (35) and (36) should be read as follows: First, for a given balance M_{t-1}, we find τ that solves equation (35). Then, we substitute τ into equation (36) that gives us the interest rate $r_t^C(M_{t-1})$ charged on the balance.

It is more convenient to examine the interest rate structure in the stationary setting with $T \to \infty$. Let

$$V_\infty(y) \equiv E\left[\sum_{k=1}^{\infty} \beta^k (Y_k - y_L) \,|\, Y_0 = y\right].$$

We can omit the time index and simplify equations (35) and (36) that become

$$M_\tau = V_\infty(y_H) - V_\tau(y_H)), \tag{38}$$

and

$$r^C(M_\tau) = \frac{V_\tau(y_H) - V_{\tau-1}(y_H)}{M_\tau}. \tag{39}$$

We start our analysis of the optimal interest rate structure with two benchmark cases: independent cash flows and perfectly correlated cash flows.

Theorem 3 *When the cash flows are independent over time ($Q(y_L) = Q(y_H)$) or perfectly correlated ($Q(y_L) = 0$, $Q(y_H) = 1$), the optimal interest rate is equal to the risk free interest rate: $r^C(M) = r$.*

Proof. For ease of presentation, we prove the statement of Theorem 3 only for the case when the earliest default time τ is an integer. The reader can easily verify that the statement of Theorem 3 holds for any real τ.

For the earliest default times $\bar{\tau} = 1, 2, 3...$, equations (38) and (39) can be further simplified. In particular, the interest rate r^C can be represented as[5]

$$r^C(M_{\bar{\tau}}) = \beta^{\bar{\tau}} \frac{E[(Y_{\bar{\tau}} - y_L) \,|\, Y_0 = y_H]}{M_{\bar{\tau}}}, \tag{40}$$

[5] When the earliest default time τ is not an integer, the optimal interest rate is given by

$$r^C(M_\tau) = \beta^{n_\tau} \frac{(1 - l_\tau) E[(Y_{n_\tau} - y_L) \,|\, Y_0 = y_H] + l_\tau E[\beta(Y_{n_\tau+1} - y_L) \,|\, Y_0 = y_H]}{M_\tau}$$

where, the balance on the credit line $M_{\bar{\tau}}$ is equal to a time zero value of the excess cash flows that will be generated after date $\bar{\tau}$:

$$M_{\bar{\tau}} = \beta^{\bar{\tau}} \sum_{k=1}^{\infty} \beta^k E[(Y_{\bar{\tau}+k} - y_L) \,|\, Y_0 = y_H]. \tag{41}$$

When the cash flow are either independent or perfectly correlated, the conditional expectations of future excess cash flows are equal to the unconditional expectation:

$$E[(Y_{\bar{\tau}+k} - y_L) \,|\, Y_0 = y_H] \equiv Q\left(y_H\right)\left(y_H - y_L\right).$$

Let $\mu \equiv Q\left(y_H\right)\left(y_H - y_L\right)$. Then,

$$M_{\bar{\tau}} = \beta^{\bar{\tau}} \sum_{k=1}^{\infty} \beta^k \mu = \beta^{\bar{\tau}} \frac{\beta}{1-\beta} \mu. \tag{42}$$

Substituting, (42) into (40) gives

$$r^C\left(M_{\bar{\tau}}\right) = \frac{1-\beta}{\beta} = r.$$

■

Theorem 3 replicates the result by DeMarzo and Fishman (2003) who consider independent cash flows. The reason the optimal interest rate is constant when the cash flows are i.i.d. is that the expectation of the future cash flows is constant.

The correlation of the cash flows introduces a link between the current and future cash flows. This link becomes weaker as the time separating the current and future cash flows increases. This means, for example, that $E\left[(Y_{\bar{\tau}} - y_L) \,|\, Y_0 = y_H\right]$, the numerator in (40), is decreasing[6] with $\bar{\tau}$, when cash flows are positively correlated. Although $M_{\bar{\tau}}/\beta^{\bar{\tau}}$ also depends on $\bar{\tau}$, the overall effect on the interest rate r^C is that r^C is decreasing with $\bar{\tau}$, which means that r^C is increasing with the balance on the credit line:

Theorem 4 *The interest rate r^C charged on the credit line is increasing with the outstanding balance M. Moreover*

$$\lim_{M \to 0} r^C\left(M\right) = r,$$

[6]See Lemma 4 in Appendix.

and

$$r^C\left(C^L\right) \leq r\frac{Q\left(y_H\right)}{Q\left(y_L\right)}\left(1-\left(Q\left(y_H\right)-Q\left(y_L\right)\right)\right). \tag{43}$$

Proof. Consider the case when the earliest default time is an integer. According to (40) and (41), the interest rate on the credit line is given by

$$\begin{aligned} r^C\left(M_{\bar{\tau}}\right) &= \beta^{\bar{\tau}}\frac{E\left[\left(Y_{\bar{\tau}}-y_L\right)\mid Y_0=y_H\right]}{\beta^{\bar{\tau}}\sum\limits_{k=1}^{\infty}\beta^k E[\left(Y_{\bar{\tau}+k}-y_L\right)\mid Y_0=y_H]} \\ &= \left(\sum_{k=1}^{\infty}\beta^k\frac{E[\left(Y_{\bar{\tau}+k}-y_L\right)\mid Y_0=y_H]}{E\left[\left(Y_{\bar{\tau}}-y_L\right)\mid Y_0=y_H\right]}\right)^{-1}, \end{aligned} \tag{44}$$

for the earliest default time $\bar{\tau}=1,2,3...$

According to Lemma 4(iii) in Appendix,

$$\frac{E[\left(Y_{\bar{\tau}+k}-y_L\right)\mid Y_0=y_H]}{E\left[\left(Y_{\bar{\tau}}-y_L\right)\mid Y_0=y_H\right]} \leq \frac{E[\left(Y_{\bar{\tau}+1+k}-y_L\right)\mid Y_0=y_H]}{E\left[\left(Y_{\bar{\tau}+1}-y_L\right)\mid Y_0=y_H\right]}.$$

The last inequality means that if we increase the earliest default time by one, then every term in the sum in (44) will become bigger, and the interest rate will therefore become smaller. The higher the earliest default time $\bar{\tau}$, the smaller the balance $M_{\bar{\tau}}$ on the credit line is. Thus, the interest rate charged on the credit line increases with the balance on the credit line, for the earliest default time taking integer values. One can verify that this result holds for any earliest default time.

When $\bar{\tau}\to\infty$, $M_{\bar{\tau}}\to 0$. The cash flow process is asymptotically stationary. As a result,

$$\lim_{\bar{\tau}\to\infty}\frac{E[\left(Y_{\bar{\tau}+k}-y_L\right)\mid Y_0=y_H]}{E\left[\left(Y_{\bar{\tau}}-y_L\right)\mid Y_0=y_H\right]}=1.$$

Using (44),

$$\begin{aligned} \lim_{M\to 0} r^C\left(M\right) &= \left(\sum_{k=1}^{\infty}\beta^k\right)^{-1} \\ &= \frac{1-\beta}{\beta}=r. \end{aligned}$$

The interest rate charged on the credit line is the highest when the balance on the credit line reaches the credit limit, at which point the earliest default time $\tau=1$. Using (44) the fact[7] that

$$\begin{aligned} E[(Y_k - y_L) \,|\, Y_0 &= y_H] \geq \lim_{s \to \infty} E[(Y_s - y_L) \,|\, Y_0 = y_H] \\ &= \frac{Q(y_L)}{1 - (Q(y_H) - Q(y_L))}(y_H - y_L), \end{aligned}$$

we can write that

$$\begin{aligned} r^C(M_1) &\leq \frac{E[(Y_1 - y_L) \,|\, Y_0 = y_H]}{\sum_{k=1}^{\infty} \beta^k \frac{Q(y_L)}{1 - (Q(y_H) - Q(y_L))}(y_H - y_L)} \\ &= \frac{1-\beta}{\beta} \frac{Q(y_H)}{Q(y_L)} (1 - (Q(y_H) - Q(y_L))) \\ &= r \frac{Q(y_H)}{Q(y_L)} (1 - (Q(y_H) - Q(y_L))) . \end{aligned}$$

■

Theorem 4 says that the optimal interest rate schedule starts with the risk-free interest rate at the zero balance and monotonically increases with the balance. The interest rate, however, never goes above the boundary, given by (43). When cash flows are independent over time or perfectly correlated, the optimal interest rate is equal to the risk free interest rate. Thus, the correlation of the cash flows explains the fact that the optimal interest rate is increasing with the balance.

In order to understand the optimal interest rate structure, it is useful to reexamine the properties of the optimal contract. Under the optimal contract, the agent is always indifferent between using excess cash flows to pay down the credit line or diverting them for his own consumption. On the equilibrium path, the agent who holds fraction λ of the equity uses excess cash flows to pay off the credit line first and then to issue dividends. His continuation payoff in equilibrium is thus equal to the expected value of his share of the future dividends. Instead of paying down the credit line, the agent, however, can get the same continuation payoff by diverting all the future excess cash flows until he exhausts the credit line, at which point he defaults. The optimal interest rate minimizes expected costs of default, while discouraging cash flow diversion by the agent. Holding the credit limit constant, an interest rate schedule, which lies below the optimal schedule, would not be incentive competitive. On the other hand, an interest rate schedule, which lies above the optimal schedule, would result in unnecessary high chances of default on the equilibrium path.

[7] See Lemma 4(iv) in Appendix.

After diverting an excess cash flow, the agent is in a better position when the cash flows are positively correlated, than when they are independent over time. Indeed, subsequent cash flows are more likely to be high, given the high cash flow in the current period. Hence, given the same time interval, the agent is capable of stealing on average more cash when cash flows are positively correlated. Under the optimal contract, a higher interest rate charged on the credit line precipitates default and discourages the agent from stealing cash flows.

The optimal interest rate structure reflects an average rate at which the agent can divert cash flows. When the cash flows are positively correlated, the expectation of a future cash flow depends on the current cash flow as well as the time between the current date and the date of the future cash flow realization. Figure 4 presents an example of the expected amount of cash the agent can divert if he starts diverting cash flows in a period in which the high cash flow is realized. One can see that because the cash flows are positively correlated the expected value of the cash flows decreases as the times goes on. This fact has an important implication for the agent's willingness to divert cash flows. The higher balance on the credit line, the shorter time interval during which the agent can divert cash flows. However, as Figure 4 illustrates, the amount of cash per period that the agent can divert is higher, the shorter the time interval during which the stealing takes place. Thus, the agent has a stronger motive to divert cash flows when the balance is high. In order to prevent stealing, the time interval during which the agent is allowed to run the project when he reports the low cash flow in each period should be shorten. Under the optimal contract, this is done by charging higher interest rate at the end of the credit line.

When deriving the optimal contract, we assumed that the agent is not allowed to save and therefore cannot overreport cash flows. Given the result of Theorem 4, it is easy to see that optimal contract remains incentive compatible even when the agent is allowed to save. The interest rate r^C charged on the balance on the credit line is higher than the rate ρ, at which the agent's savings grow. Therefore, if the agent diverts the high cash flow today, saves it privately and and then uses the savings to pay down the credit line in the future by reporting higher cash flows, the resulting balance on the credit line will be higher than when he used all the cash flows to pay down the credit line immediately. Thus, the agent has no incentives to save under the optimal contract.

5.2 An Example

Now, we illustrate how the optimal interest rate schedule changes with the degree of the cash flow correlation. Consider a stationary case with the risk-free interest rate 5% ($r = 0.05$),

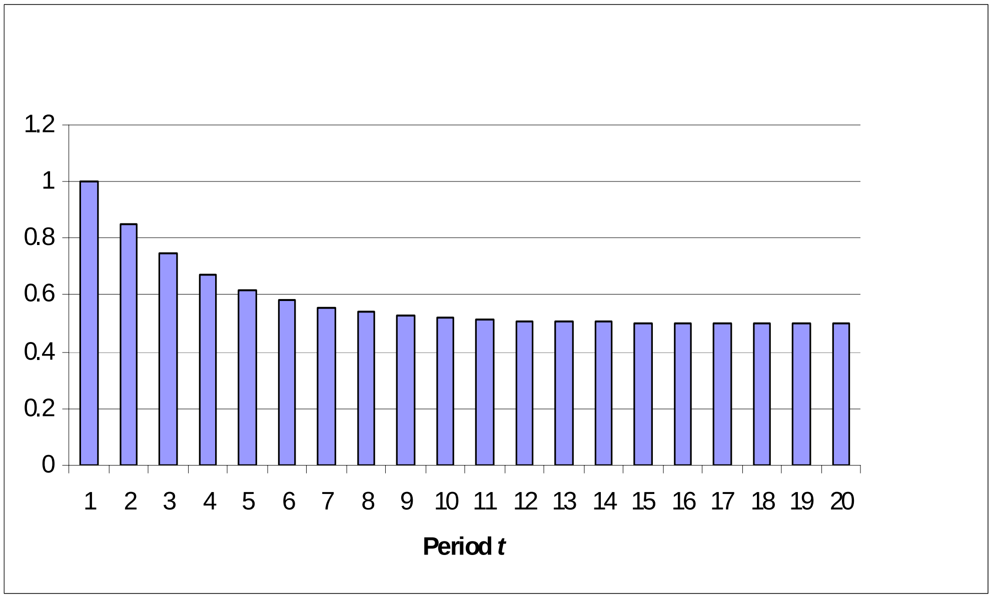

Figure 4: Expectation of future cash flows, given the high cash flow in period 1, $E[Y_t|Y_1 = y_H]$, for parameters $y_L = 0$, $y_H = 1$, $Q(y_L) = 0.15$, $Q(y_H) = 0.85$

and symmetric transition probabilities: $Q(y_H) = q$ and $Q(y_L) = 1 - q$, where $q \in [0.5, 1]$. Higher q means stronger correlation. When $q = 0.5$, the cash flows are independent over time. When $q = 1$, the cash flows are perfectly correlated.

As illustrated in Figure 5, the interest rate r^C on the credit line is equal to the risk-free interest rate when $q = 0.5$ for any size of the outstanding balance M. When q is increased to 0.6, the optimal interest rate r^C lies above the risk free rate r. However, this difference between r^C and r remains insignificant as long as the balance on the credit line stays below 86% of the credit limit. The interest rate grows sharply once the balance becomes greater than 86 percentage points of the credit limit, and almost reaches 6% when the credit line is completely exhausted. A greater degree of the cash flow correlation causes the interest rate to escalate earlier. For $q = 0.85$, the interest rate r^C becomes visibly greater than r when the balance crosses the 40% mark. Afterwards, r^C keeps growing and reaches 7.9% in the end. When the cash flows are extremely correlated ($q = 0.995$), the interest rate jumps sharply at the beginning and remains relatively flat afterwards, staying just below 0.055. In the deterministic limit ($q = 1$), the interest rate remains identically equal to the risk-free interest rate, as was the case with the independent cash flows.

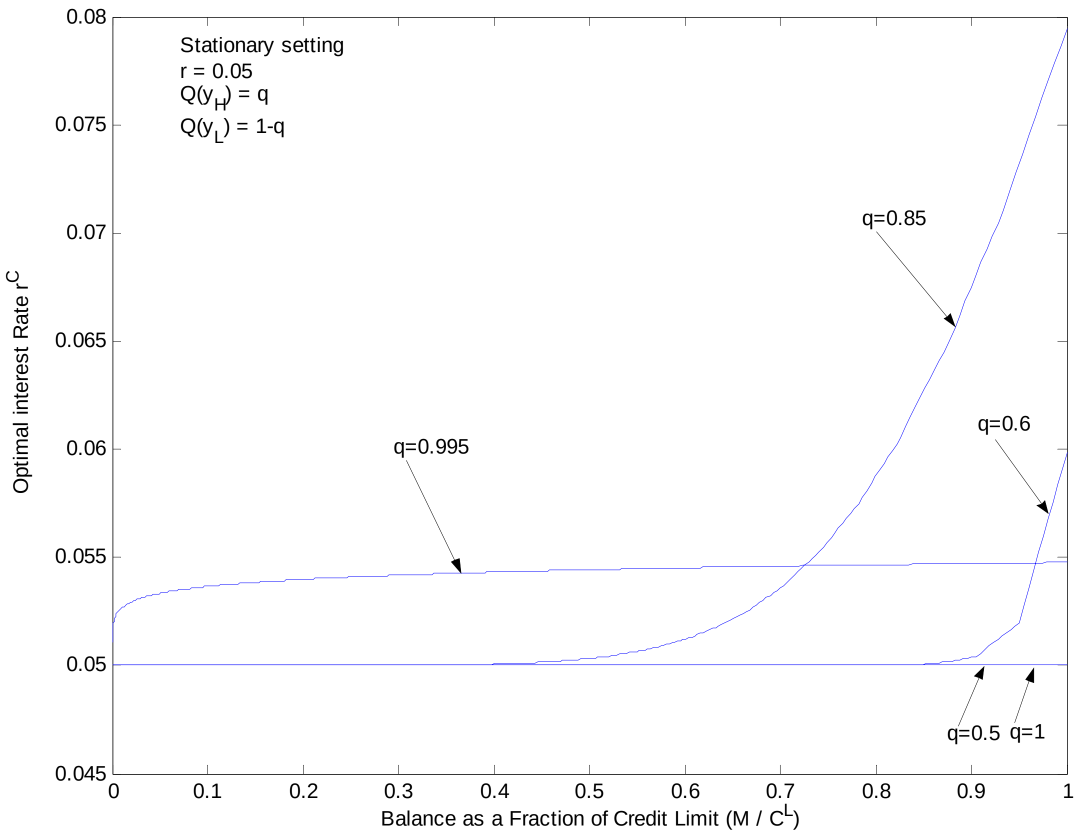

Figure 5: The optimal interest rate structure.

5.3 Alternative Implementations

Given the implementation of the optimal contract considered in this section, the interest rate charged on the credit line is a function of the balance. The balance on the credit line can be interpreted as a measure of the firm's performance. A lower balance indicates higher on average cash flows in the past, which can be seen as a better performance. In practice, however, other measures of performance are often used, such as the firm's leverage ratio and credit rating. In this subsection, we briefly discuss alternative implementations of the optimal contract using these measures.

In our settings, there is a one-to-one correspondence between the balance on the credit line and the leverage ratio, i.e. the ratio of debt value to the sum of the debt and equity values. The leverage ratio increases with the balance on the credit line. Thus, the leverage ratio can be used as a proxy for the balance on the credit line. Theorem 4 implies that the optimal interest rate increases with the leverage ratio.

In practice, the credit rating is one of the most often used measures in performance pricing. In our model, the firm's credit rating can be linked to the balance on the credit line, assuming the credit rating measures chances of default in the future. The probabilities of default by the agent depend on the balance on the credit line: higher balance on the credit line means higher probabilities of default in the future. Thus, the higher balance on the credit line on average corresponds to lower credit rating. Therefore, the interest rate on the credit line should go up with the credit rating deterioration.

6 Conclusion

This paper presents a model of security design in a setting with an agency problem. An agent seeks external financing for a project that, if initiated, generates Markov cash flows whose realizations are unobservable and unverifiable by an outside investor. The agent has an ability to secretly divert the cash flows for his own consumption. The investor who cannot detect stealing has the right to liquidate the project. We characterize an optimal contract between the investor and the agent in this setting.

We show that the optimal contract can be implemented using a credit line with an interest rate that increases with the outstanding balance on the credit line. The balance on the credit line can be considered as a performance measure that reflects all the relevant information regarding the past cash flow realizations. Alternative implementations of the optimal contract can be based on other performance measures, such as a leverage ratio, or the firm's credit rating. In particular, the optimal interest rate should increase with the leverage ratio, while an improvement in the firm's credit rating should lead to a lower interest rate.

In practice, a line of credit with performance pricing is the most prevalent form of banking lending. Our model demonstrates that a credit line with performance pricing is a part of the optimal contract in the setting with the agency problem. This finding is in line with the recent empirical studies by Asquith, Beatty and Weber (2002), Bradley and Roberts (2004), Dichev and Skinner (2004), which suggest that performance pricing, along with other debt covenants, is used to mitigate agency costs.

On the technical side, we develop a new recursive method to deal with correlated hidden states in dynamic settings. Unlike in the case with independent cash flows, the agent's preferences regarding continuation contracts are no longer the common knowledge in the setting with the correlated cash flows. A major challenge in reformulating the contracting problem recursively in this situation is to write incentive compatibility constraints. After reporting the low cash flow when the high cash flow is realized, the agent faces the same continuation contract that he would get after the low cash flow realized. However, the agent's continuation payoffs after the deviation is different from his continuation payoff after the low cash flow realization, due to the fact that the distribution of the future cash flows is determined by the current cash flow. A key element of our analysis is that we are able to find a one-to-one mapping between the agent's continuation payoffs in the states with the low and high cash flows. This allows us to formulate the contracting problem recursively using the agent's continuation payoff as a state variable and obtain a closed-form solution for the optimal contract. We also believe that this approach is not only applicable to our setting, but can also be used in other dynamic principal-agent models with correlated hidden states.

7 Appendix

7.0.1 Properties of the Markov Cash Flows

Lemma 4 *Given $Q(y_H) \geq Q(y_L)$, conditional expectations of future cash flows satisfy the following inequalities. For any $k, s > 0$,*

(i) $E[Y_{t+k}|Y_t = y_H] \geq E[Y_{t+k}|Y_t = y_L]$

(iia) $E[Y_{t+k}|Y_t = y_H] \geq E[Y_{t+k+s}|Y_t = y_H]$

(iib) $E[Y_{t+k}|Y_t = y_L] \leq E[Y_{t+k+s}|Y_t = y_L]$

(iii) $\dfrac{E[Y_{t+k}|Y_t = y_H]}{E[Y_{t+k+1}|Y_t = y_H]} \geq \dfrac{E[Y_{t+k+s}|Y_t = y_H]}{E[Y_{t+k+s+1}|Y_t = y_H]}$

(iv) $\lim_{k\to\infty} E\left[(Y_{t+k} - y_L)|Y_t = y_H\right] = \dfrac{Q\left(y_L\right)}{1 - \left(Q\left(y_H\right) - Q\left(y_L\right)\right)}(y_H - y_L)$

Proof. (Lemma 4)

(i) Inequality (i) is proven by induction. For $k = 1$, the inequality is true because $E\left[Y_{t+1}|Y_t\right] = y_L + Q\left(Y_t\right)\left(y_H - y_L\right)$ and $Q\left(y_H\right) \geq Q\left(y_L\right)$. Suppose that $E\left[Y_{t+k}|Y_t = y_H\right] \geq E\left[Y_{t+k}|Y_t = y_L\right]$, which implies that $E[Y_{t+k+1}|Y_{t+1} = y_H] \geq E[Y_{t+k+1}|Y_{t+1} = y_L]$. Now, consider $E\left[Y_{t+k+1}|Y_t\right]$. By the law of iterated expectations

$$\begin{aligned}
E\left[Y_{t+k+1}|Y_t\right] &= E\left[E[Y_{t+k+1}|Y_{t+1}]|Y_t\right] \\
&= Q(Y_t)E[Y_{t+k+1}|Y_{t+1} = y_H] + \left(1 - Q\left(Y_t\right)\right)E[Y_{t+k+1}|Y_{t+1} = y_L] \\
&= E[Y_{t+k+1}|Y_{t+1} = y_L] + Q(Y_t)(E[Y_{t+k+1}|Y_{t+1} = y_H] - E[Y_{t+k+1}|Y_{t+1} = y_L])
\end{aligned}$$

Since $Q\left(y_H\right) \geq Q\left(y_L\right)$, it follows that $E\left[Y_{t+k+1}|Y_t = y_H\right] \geq E\left[Y_{t+k+1}|Y_t = y_L\right]$.

(ii) To show that (iia) holds, we use the law of iterated expectations again.

$$\begin{aligned}
E\left[Y_{t+k+1}|Y_t = y_H\right] &= E\left[E[Y_{t+k+1}|Y_{t+1}]|Y_t = y_H\right] \\
&= Q(y_H)E[Y_{t+k+1}|Y_{t+1} = y_H] + \left(1 - Q\left(y_H\right)\right)E[Y_{t+k+1}|Y_{t+1} = y_L] \\
&= Q(y_H)E[Y_{t+k}|Y_t = y_H] + \left(1 - Q\left(y_H\right)\right)E[Y_{t+k}|Y_t = y_L] \\
&= E\left[Y_{t+k}|Y_t = y_H\right] - \left(1 - Q\left(y_H\right)\right)\left(E[Y_{t+k}|Y_t = y_H] - E[Y_t|Y_t = y_L]\right).
\end{aligned}$$

Using (i), we see that $E\left[Y_{t+k}|Y_t = y_H\right] \geq E\left[Y_{t+k+1}|Y_t = y_H\right]$. Repeating this argument yields (iia) for any s. Part (iib) is proven the same way.

(iii) By the law of iterated expectations,

$$\begin{aligned}
& E\left[Y_{t+k+s+1}|Y_t = y_H\right]E\left[Y_{t+k}|Y_t = y_H\right] - E\left[Y_{t+k+s}|Y_t = y_H\right]E\left[Y_{t+k+1}|Y_t = y_H\right] \\
= \; & \left(Q\left(y_H\right)E\left[Y_{t+k+s}|Y_t = y_H\right] + \left(1 - Q\left(y_H\right)\right)E\left[Y_{t+k+s}|Y_t = y_L\right]\right)E\left[Y_{t+k}|Y_t = y_H\right] \\
& -E\left[Y_{t+k+s}|Y_t = y_H\right]\left(Q\left(y_H\right)E\left[Y_{t+k}|Y_t = y_H\right] + \left(1 - Q\left(y_H\right)\right)E\left[Y_{t+k}|Y_t = y_L\right]\right) \\
= \; & \left(1 - Q\left(y_H\right)\right)\left(E\left[Y_{t+k+s}|Y_t = y_L\right]E\left[Y_{t+k}|Y_t = y_H\right] - E\left[Y_{t+k+s}|Y_t = y_H\right]E\left[Y_{t+k}|Y_t = y_L\right]\right) \\
\geq \; & 0
\end{aligned}$$

The last inequality follows from (iia) and (iib).

(iv) The cash flow process has a unique stationary distribution. One can verify that $\lim_{k\to\infty} P(Y_{t+k} = y_H|Y_t = y_H) = \dfrac{Q\left(y_L\right)}{1 - \left(Q\left(y_H\right) - Q\left(y_L\right)\right)}$. ■

7.0.2 Proof of Lemma 1

Let $\left\{\tilde{y}_k\left(y^k|y^{t-1},\hat{y}^{t-1}\right)\right\}_{k=t}^T$ denote a *continuation* reporting strategy after a history y^{t-1} of the realized cash flows, and a history $\hat{y}^{t-1}$ of the reported cash flows. We the contract σ is incentive compatible *after* $\left(y^{t-1},\hat{y}^{t-1}\right)$, provided the project is active at the beginning of period t, if for all $\left\{\tilde{y}_k\left(y^k|y^{t-1},\hat{y}^{t-1}\right)\right\}_{k=t}^T$,

$$
\begin{aligned}
& E\left[\sum_{k=t}^{T}\beta^{k-t}\frac{P_k\left(\hat{y}^{t-1},Y_t,y^{k-t}\right)}{P_t\left(\hat{y}^{t-1},Y_t\right)}d_t\left(\hat{y}^{t-1},Y_t,y^{k-t}\right)\mid Y_t\right] \\
\geq\ & E\left[\sum_{k=t}^{T}\beta^{k-t}\frac{P_k\left(\hat{y}^{t-1},\tilde{y}_k\left(y^k|y^{t-1},\hat{y}^{t-1}\right)\right)}{P_t\left(\hat{y}^{t-1},Y_t\right)}\left(\begin{array}{c}\lambda(Y_k-\tilde{y}_k\left(y^k|y^{t-1},\hat{y}^{t-1}\right)) \\ +d_t\left(\hat{y}^{t-1},\tilde{y}_k\left(y^k|y^{t-1},\hat{y}^{t-1}\right)\right)\end{array}\right)\mid Y_t\right].
\end{aligned}
$$

First, we consider the case in which the agent has been telling the truth until time t. If the contract is not incentive compatible after $\left(y^{t-1},y^{t-1}\right)$, then there is a continuation reporting strategy $\left\{\tilde{y}_k\left(y^k|y^{t-1},y^{t-1}\right)\right\}_{k=t}^T$ that results in a higher continuation payoff than the truth-telling. However, the existence of $\left\{\tilde{y}_k\left(y^k|y^{t-1},y^{t-1}\right)\right\}_{k=t}^T$ violates the time zero incentive compatibility constraint. A reporting strategy that consists of truth-telling until time t, truth telling after all histories other than y^{t-1} after time t, and $\left\{\tilde{y}_k\left(y^k|y^{t-1},y^{t-1}\right)\right\}_{k=t}^T$ after y^{t-1} is an improvement on the truth-telling. This contradicts the assumption that the contract is incentive compatible.

Suppose the contract is not incentive compatible after $\left(y^{t-1},\hat{y}^{t-1}\right)$, i.e. there exists a continuation reporting strategy $\left\{\bar{y}_k\left(y^k|y^{t-1},\hat{y}^{t-1}\right)\right\}_{k=t}^T$ that delivers higher continuation payoff for the agent than truth-telling. The terms of the contracts after date $t-1$ are determined by the history $\hat{y}^{t-1}$ of reported cash flows, while the distribution of the future cash flows depends on the realization of the cash flow in period t. Continuation strategy $\bar{y}$ must bring an improvement on truth-telling either after $y_t=y_L$, or $y_t=y_H$, or in the both cases. Let's assume that the agent's continuation payoff is increased after $y_t=y_L$. To show that this violates the time zero incentive compatibility constraint, we consider the following deviation from the truth-telling strategy. Let strategy y' consist of truth-telling until time t and after all histories of cash flows other than $\left(\hat{y}^{t-1},y_L\right)$, and follows $\bar{y}$ after $\left(\hat{y}^{t-1},y_L\right)$. Thus, the only difference between the truth-telling strategy and y' occurs after the cash flow history $\left(\hat{y}^{t-1},y_L\right)$ is realized. Since $\left(\hat{y}^{t-1},y_L\right)$ occurs with positive probability and $\bar{y}$ does better than the truth-telling after $\left(\hat{y}^{t-1},y_L\right)$, strategy y' results in an improvement on truth-telling, violating the time zero incentive compatibility constraint. The sane argument applies if the agent's continuation payoff is increased after $y_t=y_H$.

7.0.3 Proof of Theorem 1

We first argue that if for some t and (y^{t-1}, y_H), (7) is violated, then the contract is not incentive compatible. Indeed, the strategy in which the agent lies only at date t, given (y^{t-1}, y_H), will be an improvement on truth-telling, contradicting to the fact that (6) is satisfied.

Now, suppose that the contract $\sigma = (d, p)$ is such that (7) is always satisfied. To show that σ is incentive compatible, consider an arbitrary reporting strategy $\tilde{y}$. Let $k \leq T$ be the last period when $\tilde{y}$ can recommend lying. Let $\hat{y}$ be a reporting strategy that coincides with $\tilde{y}$ before time k and is truth-telling onward. Since both strategies $\tilde{y}$ and $\hat{y}$ result in the same income flows for the agent, as well as the same termination probabilities, before date k, and inequality (7) is satisfied at date k, then the date zero agent's expected payoff under the strategy $\hat{y}$ is equal to or greater than the corresponding payoff under the strategy $\tilde{y}$. Applying this argument inductively, we conclude $\tilde{y}$ cannot improve on truth-telling if (7) is satisfied. This proves that the contract is incentive compatible as long as (7) is satisfied for all time periods $t \leq T$, and all histories y^t.

7.0.4 Proof of Lemma 2

We use the definition of $V_\tau(Y_{t-1})$ and the law of iterated expectations to prove the lemma.

$$\begin{aligned}
V_{\tau+1}(Y_{t-1}) &= E\left[\sum_{k=1}^{n_\tau+1} \beta^k (Y_{t-1+k} - y_L) + l_\tau \beta^{(n_\tau+2)} (Y_{t+n_\tau+1} - y_L) |\, Y_{t-1}\right] \\
&= E\left[\beta\left((Y_t - y_L) + \sum_{k=1}^{n_\tau} \beta^k (Y_{t+k} - y_L) + l_\tau \beta^{(n_\tau+1)} (Y_{t+n_\tau+1} - y_L)\right) |\, Y_{t-1}\right] \\
&= E\left[\beta\left((Y_t - y_L) + E\left[\sum_{k=1}^{n_\tau} \beta^k (Y_{t+k} - y_L) + l_\tau \beta^{(n_\tau+1)} (Y_{t+n_\tau+1} - y_L) | Y_t\right]\right) |\, Y_{t-1}\right] \\
&= E\left[\beta\left((Y_t - y_L) + V_\tau(Y_t)\right) |\, Y_{t-1}\right]
\end{aligned}$$

7.1 Proof of Propositions 2-5.

We prove Propositions 2-5 by induction. Assuming that certain properties of the contract σ^* hold in periods $t+1$ through T, we verify that those properties must hold in the period t.

7.1.1 Proof of Proposition 2.

Let $\theta_t(y_t)$ denote the slope of the line connecting the points of liquidation payoffs $(0, L_t(y_t))$ and
$\left(a^L(y_t), b_t^e\left(y_t, a^L(y_t)\right)\right)$. By the definition,

$$\theta_t(y_t) = \frac{b_t^e\left(y_t, a^L(y_t)\right) - L_t(y_t)}{a^L(y_t)}. \tag{45}$$

To prove Proposition 2 we need the following Lemma.

Lemma 5 *The slope $\theta_t(y_t)$ does not depend on $y_t \in \{y_L, y_H\}$ and is given by*

$$\theta_t(y_t) = \frac{1}{\lambda(y_H - y_L)}\left[(1-\alpha)(y_H - y_L) + \left(b_{t+1}^d(y_H, \lambda(y_H - y_L)) - L_{t+1}(y_H)\right)\right] \tag{46}$$

Proof. (Lemma 5) If $a_t^e = a^L(y_t)$, then according to (28)-(30) the agent's continuation payoff in the next period after the cash flow realization will be $a_{t+1}^d(y_{t+1}) = \lambda(y_{t+1} - y_L)$. Using the fact that $b_{t+1}^d(y_L, 0) = L_{t+1}(y_L)$, we can express the investor's continuation payoff $b_t^d\left(y_t, a^L(y_t)\right)$ as follows:

$$b_t^d\left(y_t, a^L(y_t)\right) = \beta\left[Q(y_t)\left(y_H + b_{t+1}^d(y_H, \lambda(y_H - y_L))\right) + (1 - Q(y_t))(y_L + L_{t+1}(y_L))\right]. \tag{47}$$

Recall that

$$a^L(y_t) = \beta Q(y_t)\lambda(y_H - y_L), \tag{48}$$

and

$$L_t(y_t) = \sum_{k=t+1}^{T} \beta^{k-t} y_L + \alpha V_{T-t}(y_t).$$

Using the Law of iterated expectations we can represent $L_t(y_t)$ as follows:

$$L_t(y_t) = \beta\left[Q(y_t)(y_L + \alpha(y_H - y_L) + L_{t+1}(y_H)) + (1 - Q(y_t))(y_L + L_{t+1}(y_L))\right]. \tag{49}$$

Substituting (47), (48) and (49) into (45) we obtain formula (46). ■

Proof. (Proposition 2) The investor chooses the termination policy to maximize his continuation payoff, given the agent's continuation payoff. Since the agent and the investor get payoffs 0 and $L_t(y_t)$ in the case of termination, and the project can be terminated probabilistically, all payoffs are feasible within the convex hull of $(0, L_t(y_t))$ and the end-of-period payoff possibility set, whose upper frontier is given by b_t^e. Let $\left(\tilde{a}_t^L(y_t), b_t^e\left(y_t, \tilde{a}_t^L(y_t)\right)\right)$ be the point, where the line originating from $(0, L_t(y_t))$ is tangent to b_t^e. This line represents the highest possible continuation payoff for the investor before the liquidation decision has been made in period t, given the agent's continuation payoff belongs to the interval $\left(0, \tilde{a}_t^L(y_t)\right)$.

To prove the proposition we need to show that $\tilde{a}_t^L(y_t) = \lambda V_1(y_t)$. To do this we will show that the slope of the line connecting the liquidation payoff point $(0, L_t(y_t))$ with the point $(\lambda V_1(y_t), b_t^e(y_t, \lambda V_1(y_t)))$ is greater than the right-hand-side derivative of b_t^e at $\lambda V_1(y_t)$. The slope of the line is given by (46). Consider the right-hand-side derivative of b_t^e, taken at $a_t^L(y_t) \equiv \lambda V_1(y_t)$. Using (28)-(30), we can write that for $a_t^e = \lambda V_\tau(y_t)$ with $\tau \geq 1$

$$\begin{aligned} b_t^d(y_t, a_t^e) &= \beta(1 - Q(y_t))\left(y_L + b_{t+1}^d\left(y_L, a_{t+1}^d(y_L)\right)\right) \\ &\quad + \beta Q(y_t)\left(y_H + b_{t+1}^d\left(y_H, a_{t+1}^d(y_H)\right)\right), \end{aligned} \tag{50}$$

where

$$a_{t+1}^d(y_{t+1}) = \lambda V_{\tau-1}(y_t) + \lambda(y_t - y_L). \tag{51}$$

Differentiating (50) with respect to a_t^e gives

$$\begin{aligned} \frac{\partial b_t^d(y_t, a_t^e)}{\partial a_t^e} &= \nu_t\left(y_t, a_{t+1}^d(y_L)\right)\frac{\partial b_{t+1}^d\left(y_L, a_{t+1}^d(y_L)\right)}{\partial a_{t+1}^d(y_L)} \\ &\quad + \nu_t\left(y_t, a_{t+1}^d(y_H)\right)\frac{\partial b_{t+1}^d\left(y_H, a_{t+1}^d(y_H)\right)}{\partial a_{t+1}^d(y_H)}, \end{aligned} \tag{52}$$

where $\nu_t\left(y_t, a_{t+1}^d(y_H)\right) = \beta Q(y_t)\frac{\partial a_{t+1}^d(y_H)}{\partial a_t^e}$ and $\nu_t\left(y_t, a_{t+1}^d(y_L)\right) = \beta(1 - Q(y_t))\frac{\partial a_{t+1}^d(y_L)}{\partial a_t^e}$. Using (9), one can verify that

$$\nu_t\left(y_t, a_{t+1}^d(y_L)\right) = (1 - Q(y_t))\frac{E\left[(Y_{t+n_\tau+1} - y_L)\middle| Y_{t+1} = y_L\right]}{E\left[(Y_{t+n_\tau+1} - y_L)\middle| Y_t = y_t\right]}, \tag{53}$$

and that

$$\begin{aligned} \nu_t\left(y_t, a_{t+1}^d(y_H)\right) &= Q(y_t)\frac{E\left[(Y_{t+n_\tau+1} - y_L)\middle| Y_{t+1} = y_H\right]}{E\left[(Y_{t+n_\tau+1} - y_L)\middle| Y_t = y_t\right]} \\ &= 1 - \nu_t\left(y_t, a_{t+1}^d(y_L)\right). \end{aligned} \tag{54}$$

Thus, the right-hand-side derivative of b_t^e at the point $\lambda V_1(y_t)$ can be written as

$$\frac{\partial b_t^d(y_t, \lambda V_1(y_t)+)}{\partial a_t^d} = \frac{\partial b_{t+1}^d(y_L, 0+)}{\partial a_{t+1}^d(y_L)}\nu(y_t, 0+) + \frac{\partial b_{t+1}^d(y_H, \lambda(y_H - y_L)+)}{\partial a_{t+1}^d(y_H)}(1 - \nu(y_t, 0+)), \tag{55}$$

where

$$\nu(y_t, 0+) = (1 - Q(y_t))\frac{E[Y_{t+2}|Y_{t+1} = y_L]}{E[Y_{t+2}|Y_t = y_t]},$$

and "+" indicates the fact that we are considering the right neighborhood of a given point.

Subtracting (55) from (46) and rearranging terms gives

$$\begin{aligned}
&\theta(y_t) - \frac{\partial b_t^d(y_t, \lambda V_1(y_t)+)}{\partial a_t^d} \\
= &\left[\frac{(1-\alpha)}{\lambda} - \frac{\partial b_{t+1}^d(y_L, 0+)}{\partial a_{t+1}^d(y_L)}\nu(y_t, 0+)\right] \\
&+ \left[\frac{b_{t+1}^d(y_H, \lambda(y_H - y_L))}{\lambda(y_H - y_L)} - \frac{\partial b_{t+1}^d(y_H, \lambda(y_H - y_L)+)}{\partial a_{t+1}^d(y_H)}(1 - \nu(y_t, 0+)) - \frac{L_{t+1}(y_H)}{\lambda(y_H - y_L)}\right] \\
\geq &\left[\frac{(1-\alpha)}{\lambda} - \frac{\partial b_{t+1}^d(y_L, 0+)}{\partial a_{t+1}^d(y_L)}\nu(y_t, 0+)\right] \\
&+ \left[\frac{b_{t+1}^d(y_H, \lambda(y_H - y_L)\nu(y_t, 0+))}{\lambda(y_H - y_L)} - \frac{L_{t+1}(y_H)}{\lambda(y_H - y_L)}\right].
\end{aligned} \tag{56}$$

The inequality follows from concavity of the function b_{t+1}^d.

By the assumption of our inductive argument, Proposition 2 and Lemma 5 hold in period $t+1$. Therefore, b_{t+1}^d is affine on the interval $[0, \lambda V_1(y_t)]$, with $b_{t+1}^d(y_{t+1}, 0) = L_{t+1}(y_{t+1})$, and its derivative on this interval is equal to the slope $\theta_{t+1}(y_{t+1})$, which does not depend on y_{t+1}. Hence,

$$\begin{aligned}
\frac{\partial b_{t+1}^d(y_L, 0+)}{\partial a_{t+1}^d}\nu(y_t, 0+) &= \theta_{t+1}(y_L)\nu(y_t, 0+) \\
&= \theta_{t+1}(y_H)\nu(y_t, 0+) \\
&= \frac{\partial b_{t+1}^d(y_H, 0+)}{\partial a_{t+1}^d}\nu(y_t, 0+)
\end{aligned} \tag{57}$$

On the other hand,

$$\frac{b_{t+1}^{d}\left(y_{H},\lambda\left(y_{H}-y_{L}\right)\nu\left(y_{t},0+\right)\right)}{\lambda\left(y_{H}-y_{L}\right)}-\frac{L_{t+1}\left(y_{H}\right)}{\lambda\left(y_{H}-y_{L}\right)}=\frac{\partial b_{t+1}^{d}\left(y_{H},0+\right)}{\partial a_{t+1}^{d}}\nu\left(y_{t},0+\right) \tag{58}$$

Substituting (57) and (58) into (56) gives

$$\theta\left(y_{t}\right)-\frac{\partial b_{t}^{d}\left(y_{t},\lambda V_{1}\left(y_{t}\right)+\right)}{\partial a_{t}^{d}}\geq .\frac{\left(1-\alpha\right)}{\lambda}\geq 0$$

This proves that the tangent point is equal to $a_{t}^{L}\left(y_{t}\right)=\lambda V_{1}\left(y_{t}\right)$, which gives us the desired result. ■

7.1.2 Proof of Proposition 3

For $y\in\{y_{L},y_{H}\}$, let $\frac{\partial b_{t}^{d}\left(y,a_{t}^{d}\right)}{\partial a_{t}^{d}}$ denote the right-hand-side derivative of the continuation function b_{t}^{d}. The proof of Proposition 3 follows from the following lemma.

Lemma 6 *The continuation functions $b_{t}^{d}\left(Y_{t},a_{t}^{d}\right)$ are concave in the agent's continuation payoff a_{t}^{d}. Moreover,*

$$\frac{\partial b_{t}^{d}\left(y_{L},V_{\tau}\left(y_{L}\right)\right)}{\partial a_{t}^{d}}\geq\frac{\partial b_{t}^{d}\left(y_{H},V_{\tau}\left(y_{H}\right)\right)}{\partial a_{t}^{d}} \tag{59}$$

Proof. (Lemma 6). Since $b_{T}^{d}\left(y_{T},a_{T}^{d}\right)=-a_{T}^{d}$, the statement of the lemma is true in the final period. We assume that the statement of the Lemma is true in period $t+1$. If $\tau\leq 1$, then $\frac{\partial b_{t}^{d}\left(y_{L},V_{\tau}\left(y_{L}\right)\right)}{\partial a_{t}^{d}}=\frac{\partial b_{t}^{d}\left(y_{H},V_{\tau}\left(y_{H}\right)\right)}{\partial a_{t}^{d}}$, according to Lemma 5.

Let $\tau>1$. Using (52)-(54), we can represent a derivative the continuation function in period t as the weighted average of derivatives of the continuation function in period $t+1$:

$$\begin{aligned}\frac{\partial b_{t}^{d}\left(y_{t},V_{\tau}\left(y_{t}\right)\right)}{\partial a_{t}^{d}} &= \nu\left(y_{t},\tau\right)\frac{\partial b_{t+1}^{d}\left(y_{L},\lambda V_{\tau-1}\left(y_{L}\right)\right)}{\partial a_{t+1}^{d}\left(y_{L}\right)}\\ &\quad+\left(1-\nu\left(y_{t},\tau\right)\right)\frac{\partial b_{t+1}^{d}\left(y_{H},\lambda V_{\tau-1}\left(y_{H}\right)+\lambda\left(y_{H}-y_{L}\right)\right)}{\partial a_{t+1}^{d}\left(y_{H}\right)},\end{aligned}$$

where

$$\nu_{t}\left(y_{t},\tau\right)=\left(1-Q\left(y_{t}\right)\right)\frac{E\left[\left(Y_{t+n_{\tau}+1}-y_{L}\right)\middle|\,Y_{t+1}=y_{L}\right]}{E\left[\left(Y_{t+n_{\tau}+1}-y_{L}\right)\middle|\,Y_{t}=y_{t}\right]}.$$

Since by the assumption of induction, the statement of the lemma is true in period $t+1$,

$$\begin{aligned}\frac{\partial b_{t+1}^d\left(y_L,\lambda V_{\tau-1}\left(y_L\right)\right)}{\partial a_{t+1}^d\left(y_L\right)} &\geq \frac{\partial b_{t+1}^d\left(y_H,\lambda V_{\tau-1}\left(y_H\right)\right)}{\partial a_{t+1}^d\left(y_H\right)} \\ &\geq \frac{\partial b_{t+1}^d\left(y_H,\lambda V_{\tau-1}\left(y_H\right)+\lambda\left(y_H-y_L\right)\right)}{\partial a_{t+1}^d\left(y_H\right)},\end{aligned}$$

where the second inequality follows from the fact that $b_{t+1}^d\left(y_H,a_t^d\right)$ is concave in a_t^d.

Thus, to prove (59) it is enough to show that $\nu_t\left(y_L,\tau\right)>\nu_t\left(y_H,\tau\right)$. However, since $E\left[\left(Y_{t+n_\tau+1}-y_L\right)\middle| Y_t=y_L\right]<E\left[\left(Y_{t+n_\tau+1}-y_L\right)\middle| Y_t=y_H\right]$, it immediately follows that

$$\nu_t\left(y_L,\tau\right)>\nu_t\left(y_H,\tau\right),$$

which proves that the inequality (59) holds in period t. ■

Now it is easy to prove Proposition 3.

Proof. (Proposition 3). Consider Problem (19)-(22), in which the objective function is maximized by the pair of continuation payoffs $a_t^d\left(y_L\right)$ and $a_t^d\left(y_H\right)$ subject to the (IC), (PK), and (IR) constraints. Let the difference between them be denoted by $\xi\equiv a_t^d\left(y_H\right)-a_t^d\left(y_L\right)$. We will show that the objective function decreases in the difference ξ. Since the continuation payoffs satisfy the promise keeping constraint (PK) it must be the case that

$$\begin{aligned}a_t^d\left(y_L,\xi\right) &= a_t^y-Q\left(y_{t-1}\right)\xi \\ a_t^d\left(y_H,\xi\right) &= a_t^y+(1-Q\left(y_{t-1}\right))\xi.\end{aligned}$$

The investor's continuation payoff as a function of ξ is given by

$$\tilde{b}\left(\xi\right)=\left(1-Q\left(y_{t-1}\right)\right)\left(y_L+b_t^d\left(y_L,a_t^d\left(y_L,\xi\right)\right)+Q\left(y_{t-1}\right)\left(y_H+b_t^d\left(y_H,a_t^d\left(y_H,\xi\right)\right)\right).\quad (60)$$

Taking the derivative of $\tilde{b}$ with respect to ξ, we obtain that

$$\tilde{b}'\left(\xi\right)=-Q\left(y_{t-1}\right)\left(1-Q\left(y_{t-1}\right)\right)\left(\frac{\partial b_t^d\left(y_L,a_t^d\left(y_L,\xi\right)\right)}{\partial a_t^d}-\frac{\partial b_t^d\left(y_H,a_t^d\left(y_H,\xi\right)\right)}{\partial a_t^d}\right)\leq 0,$$

where the inequality follows from Lemma 6.

Since $\tilde{b}'\left(\xi\right)\leq 0$, the solution of Problem (19)-(22) is obtained by choosing the smallest ξ such that the (IC) constraint is satisfied, which corresponds to the case in which the (IC) constraint is satisfied with equality. ■

7.1.3 Proof of Proposition 4

Suppose the continuation contract $\sigma_t^*(y_L)$ results in the continuation payoff $a_t^d(y_L)$ for the agent after the low cash flow realization in period t. Lemma 1 says that since the contract σ^* is incentive compatible, it is always optimal for the agent to report cash flows truthfully in the future, even if the agent deviated in the past. Given the probability of liquidation p_t in period t, we can represent this payoff as the expectation of the agent's continuation payoffs in the next period:

$$a_t^d(y_L) = E\left[\beta(1-p_t)\, a_{t+1}^d(Y_{t+1})\, |Y_t = y_L\right], \tag{61}$$

where $a_{t+1}^d(Y_{t+1})$ denotes the continuation payoff for the agent in period $t+1$ under the continuation contract $\sigma_t^*(y_L)$ conditional on the cash flow Y_{t+1}.

Similarly, if the agent reports the low cash flow in period t, when, in fact, the high cash flow realized, the same contract results in the following continuation payoff after the deviation:

$$c_t\left(a_t^d(y_L)\right) = E\left[\beta(1-p_t)\, a_{t+1}^d(Y_{t+1})\, |Y_t = y_H\right]. \tag{62}$$

In the equations (62), the expectation is taken over the same continuation payoffs $a_{t+1}^d(Y_{t+1})$, but with respect to a different cash flow realization in period t.

We consider the cases with $a_t^d(y_L) < a^L(y_L)$ and $a_t^d(y_L) > a^L(y_L)$.

Case 1: $a_t^d(y_L) > a^L(y_L)$. Let

$$a_t^d(y_L) = \lambda V_\tau(y_L),$$

for some $\tau \geq 1$. According to Proposition 2 and equations (28)-(31), the probability of liquidation p_t is zero, and the continuation payoffs in the next period are given by

$$a_{t+1}^d(Y_{t+1}) = \lambda V_{\tau-1}(Y_{t+1}) + \lambda(Y_{t+1} - y_L). \tag{63}$$

Substituting this into (62) gives

$$\begin{aligned} c_t\left(a_t^d(y_L)\right) &= E\left[\beta(\lambda V_{\tau-1}(Y_{t+1}) + \lambda(Y_{t+1} - y_L))|Y_t = y_H\right] \\ &= \lambda V_\tau(y_H), \end{aligned}$$

where the second equality follows from Lemma 2. Thus,

$$c_t\left(\lambda V_\tau\left(y_L\right)\right)=\lambda V_\tau\left(y_H\right).$$

Case 2: $a_t^d\left(y_L\right)\leq a^L\left(y_L\right)$. Again, let

$$a_t^d\left(y_L\right)=\lambda V_\tau\left(y_L\right),$$

for some $\tau<1$. Since $\tau<1$, $V_\tau\left(y\right)=\tau E\left[\beta\left(Y_{t+1}-y_L\right)|Y_t=y\right]$. Then, the probability of liquidation is given by $p_t=1-\tau$. If the project is not liquidated in period t, the next period continuation payoffs are given by

$$a_{t+1}^d\left(Y_{t+1}\right)=\lambda(Y_{t+1}-y_L),$$

in accordance with Proposition 2 and equations (28)-(31). Then, the equation (63) gives

$$\begin{aligned}c_t\left(a_t^d\left(y_L\right)\right)&=E\left[\beta\tau\lambda(Y_{t+1}-y_L)|Y_t=y_H\right]\\&=\lambda V_\tau\left(y_H\right).\end{aligned}$$

Thus, we proved Proposition 4 for Case 2.

7.1.4 Proof of Proposition 7

Proof. In order to prove that σ^* is the optimal contract, we will demonstrate that for each date t and for any agent's continuation payoff a_t^y, there is no continuation contract that results in the investor's payoff greater than $b_t^y\left(y_{t-1},a_t^y\right)$.

Let $\tilde{\sigma}$ be an arbitrary incentive compatible contract, with the continuation payoffs after the cash flow realization in period t for the agent and the investor given by $\tilde{a}_t^d\left(y_t\right)$, and $\tilde{b}_t^d\left(y_t,\tilde{a}_t^d\left(y_t\right)\right)$. The continuation payoff for the investor at the beginning of period t is given by

$$\tilde{b}_t^y\left(Y_{t-1},a_t^y\right)=E_t\left[Y_t+\tilde{b}_t^d\left(Y_t,\tilde{a}_t^d\left(Y_t\right)\right)|Y_{t-1}\right].$$

Let $\tilde{c}_t$ denote a deviation payoff function, associated with the contract $\tilde{\sigma}$. Since, by assumption, $\tilde{\sigma}$ is incentive compatible, the agent's continuation payoffs $\tilde{a}_t^d$, associated with $\tilde{\sigma}$, must satisfy

$$\tilde{a}_t^d\left(y_H\right)\geq\tilde{c}_t\left(\tilde{a}_t^d\left(y_L\right)\right)+\lambda\left(y_H-y_L\right)\text{ for all }t\leq T. \tag{64}$$

In the last period, $\tilde{b}_T^d\left(Y_T, \tilde{a}_T^d(Y_T)\right) \leq b_T^d\left(Y_T, \tilde{a}_T^d(Y_T)\right)$. We assume by induction that $\tilde{b}_t^d\left(Y_t, \tilde{a}_t^d(Y_t)\right) \leq b_t^d\left(Y_t, \tilde{a}_t^d(Y_t)\right)$.

In the proof of Proposition 3, we demonstrate that the function $E_t\left[Y_t + b_t^d\left(Y_t, \tilde{a}_t^d(Y_t)\right)|Y_{t-1}\right]$ (see equation (60)) increases as the difference $\tilde{a}_t^d(y_H) - \tilde{a}_t^d(y_L)$ decreases. Since $\tilde{b}_t^d\left(Y_t, \tilde{a}_t^d(Y_t)\right) \leq b_t^d\left(Y_t, \tilde{a}_t^d(Y_t)\right)$, the only way that $\tilde{b}_t^y(Y_{t-1}, a_t^y)$ can be greater than $b_t^y(Y_{t-1}, a_t^y)$ is when $\tilde{a}_t^d(y_H) - \tilde{a}_t^d(y_L) < a_t^d(y_H) - a_t^d(y_L)$, where $a_t^d(y_H)$ and $a_t^d(y_L)$ the incentive compatibility constraint under the contract σ^*:

$$a_t^d(y_H) = c_t\left(a_t^d(y_L)\right) + \lambda(y_H - y_L).$$

To illustrate that this is impossible that $\tilde{a}_t^d(y_H) - \tilde{a}_t^d(y_L) < a_t^d(y_H) - a_t^d(y_L)$, we will show that $\tilde{c}_t(a) \geq c_t(a)$. We will show recursively that equation (64) implies that $\tilde{c}_t(a) \geq c_t(a)$, for all $a \geq 0$, and $t \leq T$. This must be true in the last period T for the contract $\tilde{\sigma}$ to be incentive compatible.

Suppose at time t,

$$\tilde{c}_t(a) \geq c_t(a), \text{ for all } a \geq 0. \tag{65}$$

Equations (64) and (65) imply that if $\tilde{a}_t^d(y_L) = \lambda V_\tau(y_L)$, then $\tilde{a}_t^d(y_H) = \lambda V_\tau(y_H) + \lambda(y_H - y_L) + \varphi$, with $\varphi \geq 0$, and $\varphi > 0$ if equation (65) holds with strict inequality.

Now, consider continuation payoffs in period $t-1$. Given the continuation payoffs $\tilde{a}_t^d(y_L)$ and $\tilde{a}_t^d(y_H)$ in period t, we can compute the continuation payoff in period $t-1$:

$$\begin{aligned}
\tilde{a}_{t-1}^d(y_L) &= E\left[\beta\tilde{a}_t^d(Y_t)|Y_{t-1} = y_L\right] \\
&= V_{\tau+1}(y_L) + Q(y_L)\beta\varphi \\
&= V_{\tau_1}(y_L),
\end{aligned} \tag{66}$$

The deviation payoff $\tilde{c}_{t-1}\left(\tilde{a}_{t-1}^d(y_L)\right)$ in period $t-1$ can be obtained as follows:

$$\begin{aligned}
\tilde{c}_{t-1}\left(\tilde{a}_{t-1}^d(y_L)\right) &= E\left[\beta\tilde{a}_t^d(Y_t)|Y_{t-1} = y_H\right] \\
&= V_{\tau+1}(y_H) + Q(y_H)\beta\varphi \\
&= V_{\tau_2}(y_H),
\end{aligned} \tag{67}$$

for where parameters τ_1 and τ_2 are chosen so that the corresponding equalities in (67) and (66) hold.

In particular, τ_1 solves

$$
\begin{aligned}
Q\left(y_L\right)\beta\varphi &= (1-l_\tau)\beta^{n_\tau+1}E\left[\left(Y_{n_\tau+1}-y_L\right)|Y_0=y_L\right] \\
&\quad+\sum_{k=\tau+2}^{n_{\tau_1}}\beta^k E\left[\left(Y_k-y_L\right)|Y_0=y_L\right]+l_{\tau_1}\beta^{n_{\tau_1}+1}E\left[\left(Y_{n_{\tau_1}+1}-y_L\right)|Y_0=y_L\right],
\end{aligned}
$$

where n_τ is the biggest integer, such that $n_\tau \leq \tau$, and $l_\tau = \tau - n_\tau$. Similarly, τ_2 solves

$$
\begin{aligned}
Q\left(y_H\right)\beta\varphi &= (1-l_\tau)\beta^{n_\tau+1}E\left[\left(Y_{n_\tau+1}-y_L\right)|Y_0=y_H\right] \\
&\quad+\sum_{k=\tau+2}^{n_{\tau_2}}\beta^k E\left[\left(Y_k-y_L\right)|Y_0=y_H\right]+l_{\tau_2}\beta^{n_{\tau_2}+1}E\left[\left(Y_{n_{\tau_2}+1}-y_L\right)|Y_0=y_H\right],
\end{aligned}
$$

Using the fact that $E\left[\left(Y_1-y_L\right)|Y_0=y\right]=Q\left(y\right)\left(y_H-y_L\right)$ and Lemma 4 (iia) and (iib),

$$
\frac{E\left[\left(Y_k-y_L\right)|Y_0=y_L\right]}{Q\left(y_L\right)\beta\varphi} \geq \frac{E\left[\left(Y_k-y_L\right)|Y_0=y_H\right]}{Q\left(y_H\right)\beta\varphi}.
$$

Therefore, $\tau_1 \leq \tau_2$.

Recall that $c_t\left(V_{\tau_1}\left(y_L\right)\right)=V_{\tau_1}\left(y_H\right)$. The fact that $\tau_1 \leq \tau_2$ gives us the desired result: $\tilde{c}_t\left(\tilde{a}_{t-1}^d\left(y_L\right)\right) \geq c_t\left(\tilde{a}_{t-1}^d\left(y_L\right)\right)$.

The function $\tilde{c}_t$ determines the difference between $\tilde{a}_t^d\left(y_H\right)$ and $\tilde{a}_t^d\left(y_L\right)$. The greater the value of the function $\tilde{c}_t$, the greater this difference is. Hence, $\tilde{b}_t^y\left(Y_{t-1},a_t^y\right)$ cannot be greater than $b_t^y\left(Y_{t-1},a_t^y\right)$. Consequently, $\tilde{\sigma}$ cannot be better than σ^*. ■

7.1.5 Proof of Proposition 8

The only parameter affected by the uncertainty about the initial state Y_0 is the distribution of date 1 earning Y_1. The probability of $Y_1 = y_H$ is given by $q_1 \equiv Q\left(y_H\right)q_H + Q\left(y_L\right)\left(1-q_H\right)$. Conditional on the realization of Y_1, the agent's and the investor's continuation payoffs are given by $a_1^d\left(Y_1\right)$ and $b_1^d\left(Y_1, a_1^d\left(Y_1\right)\right)$.

Given the agent's payoff a_0, the investor chooses the agent's continuation payoffs $a_1^d\left(\cdot\right)$ to solve the following optimization problem:

$$
\begin{aligned}
b_0\left(a_0\right) &\equiv \max_{a_1^d(\cdot)}\beta E\left[Y_1+b_1^d\left(Y_1,a_1^d\left(Y_1\right)\right)\right] \\
s.t.\ a_1^d\left(y_H\right) &\geq c_1\left(a_1^d\left(y_L\right)\right)+\lambda\left(y_H-y_L\right) \qquad (IC_1) \\
a_0 &= \beta E\left[a_1^d\left(Y_1\right)\right] \qquad (PK_1)
\end{aligned}
$$

Our previous analysis of a similar problem indicates that (IC$_1$) binds. Letting $a_0^e(Y_0) \equiv \beta E\left[a_1^d(Y_1)|Y_0\right]$, and $b_0^e(Y_0, a_0^e(Y_0)) \equiv \beta E\left[Y_1 + b_1^d\left(Y_1, a_1^d(Y_1)\right)|Y_0\right]$ gives the result.

7.1.6 Proof of Proposition 9

The proof is by showing that $b_0'(a_0) \geq -1$. Using the fact that $a_0 = E[a_0^e(Y_0)]$, $b_0(a_0) = E[b_0^e(Y_0, a_0^e(Y_0))]$ and $a_0^e(y_H) = c_0(a_0^e(y_L))$, we can write

$$b_0'(a_0) = \frac{\frac{db_0^e}{da_0^e(y_L)}}{\frac{a_0}{da_0^e(y_L)}} = \frac{(1-q_H)\, b_0^{e\prime}(y_L, a_0^e(y_L)) + q_H b_0^{e\prime}(y_H, c_0'(a_0^e(y_L)))\, c_0'(a_0^e(y_L))}{(1-q_H) + q_H c_0'(a_0^e(y_L))}.$$

Since $b_0^{e\prime}(Y_0, a_0^e(Y_0)) \geq -1$, $b_0'(a_0) \geq -1$.

7.1.7 Proof of Theorem 2.

Proof. The proof consists of two steps. First, given the parameters of the securities, we show that the agent's strategy of using all the cash flows to pay the debt replicates the agent's and the investor's continuation payoffs under the corresponding optimal contract σ^*. Then, we show that this strategy is incentive compatible.

Step 1. We assume that the agent adopts the "pay-debt-first" strategy: he uses cash flows generated by the project to pay the credit line balance until it is paid in full, and only after that he starts the consumption. Given this strategy, we will show that, in the credit line settings, for any history of the realizations of cash flows, the payments between the agent and the investor and the termination probabilities are identical to those under the optimal contract σ^*.

To see this, let $a_t^e(Y_t) = V_{\tau_t}(Y_t)$ be the agent's continuation payoffs under the optimal contract, which evolution is determined by the evolution of the earliest default time τ_t. Consider the following process:

$$g_t \equiv V_{\tau_t}(y_H).$$

One can see that $g_t = a_t^e(y_H)$, and $g_t > a_t^e(y_L)$.

Step 1.1. As an intermediate step, we show by induction that $M_t = V_{T-t}(y_H) - g_t$. Given $a_0^e(Y_0) = V_{\tau_0}(Y_0)$, the initial draw on the credit line is given by $M_0 = V_T(y_H) - g_0$, where $g_0 = V_{\tau_0}(y_H)$. Assume that $M_{t-1} = V_{T-t+1}(y_H) - g_{t-1} = V_{T-t+1}(y_H) - V_{\tau_{t-1}}(y_H)$. The interest rate corresponding to the balance M_{t-1} is given by

$$r_t^C = \frac{V_{T-t}(y_H) - V_{\tau_{t-1}-1}(y_H)}{V_{T-t+1}(y_H) - V_{\tau_{t-1}}(y_H)} - 1.$$

The agent pays $(Y_t - y_L)$ on the credit line in period t. The evolution of the balance is given by

$$M_t = \max\left\{0,\ \min\left\{\left(1+\hat{r}_t^C\right)M_{t-1} - (Y_t - y_L), C_t^L\right\}\right\}. \tag{68}$$

Substituting $\hat{r}_t^C$ and C_t^L into (68) gives

$$\begin{aligned} M_t &= \max\left\{0,\ \min\left\{V_{T-t}(y_H) - V_{\tau_{t-1}-1}(y_H) - (Y_t - y_L), V_{T-t}(y_H) - a^L(y_H)\right\}\right\} \\ &= \max\left\{0,\ (V_{T-t}(y_H) - \max\left\{V_{\tau_{t-1}-1}(y_H) + (Y_t - y_L), a^L(y_H))\right\}\right\} \\ &= V_{T-t}(y_H) - \min\left\{V_{T-t}(y_H),\ \max\left\{V_{\tau_{t-1}-1}(y_H) + (Y_t - y_L), a^L(y_H)\right\}\right\}. \end{aligned}$$

According to (31), $a_t^e(Y_t) = \min\left\{V_{T-t}(Y_t), \max\left\{V_{\tau_{t-1}-1}(Y_t) + (Y_t - y_L), a^L(Y_t)\right\}\right\}$, because $a_{t-1}^e(Y_{t-1}) = V_{\tau_{t-1}}(Y_{t-1})$. By the definition, if $Y_t = y_H$, then $g_t = a_t^e(y_H) = \min\left\{V_{T-t}(y_H), \max\left\{V_{\tau_{t-1}-1}(y_H) + y_H - y_L, V_1(y_H)\right\}\right\}$. If $Y_t = y_L$, then $a_t^e(y_L) = \min\left\{V_{T-t}(y_L), \max\left\{V_{\tau_{t-1}-1}(y_L), V_1(y_L)\right\}\right\}$ and, therefore, $g_t = \min\left\{V_{T-t}(y_H), \max\left\{V_{\tau_{t-1}-1}(y_H), V_1(y_H)\right\}\right\}$. Thus, $g_t = \min\left\{V_{T-t}(y_H), \max\left\{V_{\tau_{t-1}-1}(y_H) + (Y_t - y_L), V_1(y_H)\right\}\right\}$, which implies that

$$M_t = V_{T-t}(y_H) - g_t. \tag{69}$$

Step 1.2

From (69), one can see that, given the agent's strategy, the credit line replicates the optimal mechanism. A zero balance on the credit line corresponds to the continuation payoff of $a_t^e(Y_t) = \lambda V_{T-t}(Y_t)$. On the other hand, the liquidation under the credit line implementation happens with probability $\max\{0, (1-\tau)\}$ in period t when $g_t = V_\tau(y_H)$. This implies that the credit line, combined with the agent's strategy, leads to the same outcome in terms of payments between the agent and the investor and the same termination probabilities as those under the optimal contract.

Step 2.

To finish the proof we show that the agent's strategy is incentive compatible. Since the agent is not allowed to borrow from the credit line for his own consumption, the only possible deviation for him is to use a fraction of the earning for consumption before the credit line is paid off. This deviation is possible only when $Y_t = y_H$.

Given $Y_t = y_H$, consider a one-period deviation, i.e. the agent sticks to the "pay-debt-first" strategy in the subsequent periods. Then, $g_t = a_t^e(y_H)$, and, hence, $M_t = a_t^{FB}(y_H) - a_t^e(y_H)$. If $a_t^e(y_H) > a^L(y_H)$, then there is no liquidation. An increase in the credit line balance M_t is equivalent to a decrease in the continuation payoff $a_t^e(y_H)$ by the same amount. Therefore, it is optimal for the agent to pay the credit line first.

Now, consider termination. Let z_t be the unmade payment. The agent consumes z_t, and faces the probability of the termination $p_t = \frac{z_t}{a^L(y_H)}$. With the probability $(1-p_t)$ the unmade payment z_t is forgiven. In this case, $M_t = C_t^L = V_{T-t}(y_H) - V_1(y_H)$, which implies that agent's continuation payoff is equal to $\lambda V_1(y_H)$. The agent's continuation payoff in the middle of period t is, therefore, $(1-p_t)V_1(y_H) = V_1(y_H) - z_t$. One can see that the amount of the unmade payment is subtracted from the continuation payoff of the agent. Thus, the agent does not have any incentive to divert the earning, and the "pay-debt-first" strategy is incentive compatible. ■

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