# The Long and the Short of Asset Prices: Using long run consumption-return correlations to test asset pricing models

Jianfeng Yu
University of Pennsylvania
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**Abstract**

This paper examines a new set of implications of existing asset pricing models for the correlation between returns and consumption growth over the short and the long run. The findings suggest that models with external habit formation and time varying risk aversion are not consistent with two robust facts in the aggregate data. First, that stock market returns lead consumption growth, and second, that the correlation between returns and consumption growth is higher at low frequencies than it is at high frequencies. I show that in order to reconcile these facts with a consumption based model, one needs to focus on a class of models that are "forward looking", i.e. models that a) allow for both trend and cyclical fluctuations in consumption and b) link expected returns to the cyclical fluctuations in consumption. The models by Bansal and Yaron (2004) and Panageas and Yu (2006) provide examples of such models. The time series findings are re-confirmed by examining the same set of facts in the cross section.

# 1 Introduction

The standard consumption-based CAPM seems incompetent to reconcile the large equity premium, the low risk-free rate, and the cross-sectional differences across characteristics-based sorted portfolios[1]. Numerous generalizations based on the standard CCAPM have been proposed to address these asset market anomalies[2].

One of the most successful generalizations is the external habit-formation model. It has featured prominently in the recent asset pricing and business cycle literature[3]. In the habit-formation model, the usual assumptions being made are that the habit level is an exponentially weighted moving average of past consumption and that consumption growth is an *i.i.d.* process. Habit persistence generates time variation in investor preferences. The effective risk aversion coefficient is especially high after periods of unusually low consumption growth. As a result, the model can explain the large equity premium, the predictability of stock returns, and a counter-cyclical risk premium. Another type of successful models in this literature are the long-run risk model and the trend-cycle model, where the consumption consists of a small but persistent cycle component apart from the stochastic trend. The representative paper in this literature is Bansal and Yaron (2004). Both types of models can successfully match the first two moments of the aggregate data. More importantly, the main implcations of both types of models are crucially driven by their persistent state variables. In Campbell and Cochrane's habit formation model, the key variable is the slow-moving surplus ratio. In the long-run and trend-cycle models, the key state variable is the cyclical component. As a result, these models have clear low-frequency implications. This paper mainly focuses on the low-frequency features of different leading asset pricing models. The long-run correlation between consumption and asset returns is used to evaluate different models since these two types of models have different implications on the long-run correlation between consumption growth and asset returns.

Daniel and Marshall (1999) show that the performance of asset pricing models improves significantly at the two-year horizon. Parker and Julliard (2005) show that the standard consumption-based CAPM can explain the size and value premium much better in long horizons. Motivated by these papers, I first look at the relationship between consumption growth and asset returns at different frequencies. A few stylized facts are documented, then these stylized facts are used as an out-of-sample test for the existing asset pricing models, especially the external habit formation

[1]Notable papers on this issue include Hansen and Singleton (1982), Mehra and Prescott (1985), Weil (1989), and Hansen and Jagannathan (1991) on aggregate data, and Lettau and Ludvigson (2001) on cross-sectional data.

[2]A partial list of the papers on the generalization of the consumption CAPM consists of Abel (1990, 1999), Bansal and Yaron (2004), Barberis, Huang and Santos (2001), Campbell and Cochrane (1999), Constantinides (1990), and Constantinides and Duffie (1996).

[3]A partial list of related papers includes Buraschi and Jiltsov (2007), Menly, Santos and Veronesi (2004), Tallarini and Zhang (2005), Verdelhan (2007), and Watcher (2006).

model, given its popularity in the literature.

Specifically, the consumption and returns co-move more strongly over the long horizon than over the short horizon, and asset market returns lead consumption growth. Since many asset pricing models have implications on the relationship between asset prices and consumption over long horizons, it would be interesting to investigate whether these long horizon implications can match the data. Consumption CAPM is about how asset prices respond to shocks in consumption, and how small consumption shocks can result in big movement in asset prices. Here, I focus my analysis on the relation between consumption and returns. I could also analyze the relation between consumption and price dividend ratios. However, given the issue of measurements on dividends (Bansal and Yaron (2006)), I concentrate on the relation between consumption and returns.

In external habit-formation models, the habit level is an exponentially weighted average of past consumption, and the expected return is a decreasing function of the surplus ratio. Therefore, these models imply that past consumption growth predicts future returns. Although the models state that consumption leads returns, the data suggests the exact opposite. In habit formation models, the surplus ratio is persistent and the expected return is a decreasing function of the surplus ratio. Given that the surplus ratio is approximately a weighted past average of consumption growth, the model could generate a lower covariation between consumption growth and asset returns at low frequencies. In this paper, I show that as long as the external habit formation model produces a counter-cyclical equity premium, a pro-cyclical price dividend ratio, and an equity premium large enough, the model produces counterfactual predictions including an increasing cospectrum, and a negative low-frequency (long-horizon) correlation between consumption growth and asset returns.

For an asset pricing model to produce the desired lead-lag relation between consumption and returns, it is necessary for the expected return to depend on some forward-looking variable which can predict consumption growth in itself. If the log consumption is decomposed into a stochastic trend and a cycle, then the level of the cycle can predict future consumption growth. Hence, if an asset pricing model implies that the expected returns depend on the level of the cycle, then it can produce the correct lead-lag relation between consumption and asset returns. Since expected returns depend on the persistent cycle component, the co-movement between consumption and asset returns is tighter over longer horizons. In particular, the model in Panageas and Yu (2006) implies that expected stock market returns are high when the cycle is well below the trend. Bansal and Yaron (2004) also have the same implications when they incorporate stochastic volatility in consumption growth and the volatility is countercyclical.

Since expected asset returns depend on the level of the cycle, I test a conditional version of the CCAPM by using the filtered cyclical component from the log consumption as the conditional variable. The results indicate that these conditional models perform far better than unconditional models and roughly as well as the Fama-French three-factor models on portfolios sorted by size, book-to-market, and past realized returns. This conditional version of CCAPM increases the cross-sectional R-Squared from 24% to about 60%, as well as improving conditional CAPM R-Squared from 1% to about 60%.

**Related Literature:** Lettau and Wachter (2006) and Santos and Veronesi (2006a) argue that the external habit formation model generates counterfactual predictions in the cross section of stock returns. Santos and Veronesi (2006a) show that given the homogeneous cash flow risk for each firm, the external habit-formation model produces a growth premium rather than a value premium. Lettau and Wachter (2006) make a similar point. Instead of focusing on the cross-section of stocks which depends on how the heterogeneity of these stocks is modelled, I primarily focus on analyzing the aggregate market. For the conditional CAPM, Lettau and Ludvigson (2001) use *cay* and Santos and Veronesi (2006b) use labor income as conditional variables. They both show that conditional variables can improve the unconditional CCAPM and CAPM greatly. Panageas and Yu (2006) study the asset pricing implications of technological innovation. In the model, there is a delayed reaction of consumption to a large technological innovation, which helps to explain why short run correlations between returns and consumption growth are weaker than their long run counterparts. The delayed reaction of consumption also endogenously generates a cyclical component in consumption.

The remainder of the paper is organized as follows. In section 2, an external habit formation model with *i.i.d.* consumption growth is analyzed. Section 3 presents the long–run risk and trend-cycle models. In section 4, a general external habit formation model with predictable consumption growth is examined. Section 5 consists of a few robustness checks. Section 6 investigates the cross-sectional implications of the trend-cycle model and the habit formation model. Section 7 concludes the paper. All the technical derivations appear in the appendix.

# 2 External Habit Formation Model

There are two important features in the external habit formation model. One feature is that a raise in current consumption increases future effective risk aversion of the representative agent, the other is the slow-moving external habit level. Most of the key results for the external habit persistence model crucially depend on the slow-moving surplus ratio. In the meanwhile, this slow-moving feature of the model has clear implications for the long-run. Therefore, it is worthwhile to explore the low-frequency properties of the model.

I first set up a standard Campbell and Cochrane (1999) external habit-formation model with an *i.i.d.* consumption growth rate. The cointegration constraint between dividends and consumption is also incorporated into the model. Since the focus of this paper is the low-frequency implications of different models, this cointegration constraint could potentially play an important role.

Furthermore, a number of recent papers, including Bansal, Dittmar and Lundblad (2001), Hansen, Heaton and Li (2005), Bansal, Dittmar and Kiku (2006) and Bansal and Kiku (2007) suggest that dividends and consumption are stochastically cointegrated, and that this cointegration is important for understanding asset pricing. Then, a log-linear solution of the model is presented, and the long-run implications of the model is analytically derived under the log-linear approximation. I show that in order for the habit model to match the first two moments of the consumption and asset market data, the model will counterfactually produce bigger correlation between consumption growth and asset returns at high frequencies than at low frequencies and negative correlations at low frequencies (or long horizons). Furthermore, consumption leads asset returns in this external habit model. These implications contradict the data, as I will show later. As a robustness check, in section 4, I use a general $ARMA(2,2)$ process for consumption growth in the external habit formation model and the simulation results show that the conclusions in this section remain the same.

## 2.1 External Habit Formation Model with $I.I.D.$ Growth Rate

I now set up an external habit persistence model that closely follows the specification of Campbell and Cochrane (1999). The cointegration constraint between log consumption and log dividends is incorporated in the model. In this section, the consumption growth is an $i.i.d.$ process as in Campbell and Cochrane (1999). Let $c_t = \log(C_t)$ and $d_t = \log(D_t)$ denote log real per capita values of the consumption and the stock dividend. The consumption growth rate $g_{c,t} = c_t - c_{t-1}$ is generated as

$$g_{c,t} = \mu_c + \epsilon_{c,t}, \tag{2.1}$$

where $\epsilon_{c,t}$ is an $i.i.d.$ normal with standard error $\sigma_c$. The cointegrating constraint is that $d_t - c_t$ is a stationary process as follows

$$\begin{aligned} d_t &= \mu_{dc} + c_t + \delta_t \\ \delta_t &= \rho_\delta \delta_{t-1} + \epsilon_{\delta,t}, \end{aligned}$$

where $\epsilon_{\delta,t}$ is an $i.i.d.$ normal with standard error $\sigma_\delta$ and $\rho_{c\delta}$ is the correlation between $\epsilon_{c,t}$ and $\epsilon_{\delta,t}$. This model assumes that $0 \leq \rho_\delta \leq 1$. It follows that the dividend growth $g_{d,t}$ is generated as

$$\begin{aligned} g_{d,t} &= d_t - d_{t-1} = g_{c,t} + \delta_t - \delta_{t-1} \\ &= \mu_c + \epsilon_{c,t} + (\rho_\delta - 1)\,\delta_{t-1} + \epsilon_{\delta,t} \end{aligned}$$

This setup of the dynamics of consumption and dividends is a direct extension of Campbell and Cochrane (1999). Here, $c_t$ and $d_t$ are each $I(1)$, and these two series are cointegrated except for the case of $\rho_\delta = 1$, in which the model reduces to that of Campbell and Cochrane (1999) and the dividends can wander arbitrarily far from consumption as time passes. The agent is assumed to maximize the life time utility

$$E_t \sum_{k=0}^{\infty} \delta^k \frac{\left(C_{t+k} - X_{t+k}\right)^{1-\gamma} - 1}{1-\gamma}$$

where $C_t$ is the real consumption, $X_t$ is the agent's habit level at time $t$, $\gamma$ is the risk aversion coefficient and $\delta$ is the time preference of the agent. The surplus ratio is defined as $S_t = \frac{C_t - X_t}{C_t}$ and $s_t = \log\left(S_t\right)$. The dynamics of the log surplus ratio $s_t$ is given by

$$s_{t+1} = (1-\phi)\,\bar{s} + \phi s_t + \lambda\left(s_t\right)\epsilon_{c,t+1}, \tag{2.2}$$

where $\bar{s}$ is the steady state of the log surplus ratio, $\phi$ determines the persistence of the surplus ratio (which also largely determines the persistence of the price dividend ratio), and the sensitivity function $\lambda\left(s\right)$ is given by

$$\lambda\left(s_t\right) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1 - 2\left(s_t - \bar{s}\right)} - 1, & s_t \le s_{\max} \\ 0, & s_t \ge s_{\max} \end{cases}$$

with

$$s_{\max} = \bar{s} + \frac{1}{2}\left(1 - \bar{S}^2\right), \quad \bar{S} = \sigma_c \sqrt{\frac{\gamma}{1-\phi}}$$

In the continuous time limit, $s_{\max}$ is the upper bound on $s_t$. The implication of the above specification is that the risk-free rate is a constant and habit moves non-negatively with consumption. Under the assumption of external habit, the pricing kernel $M_t$ satisfies

$$\begin{aligned} M_{t+1} &= \delta\left(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma} \\ &= \delta \exp\left\{-\gamma\left[(\phi - 1)\left(s_t - \bar{s}\right) + \left[1 + \lambda\left(s_t\right)\right]\epsilon_{c,t+1} + \mu_c\right]\right\}. \end{aligned}$$

Hence, by the Euler equation, the functional equation for the price dividend ratio $Z_t = P_{d,t}/D_t$ for the asset that pays the dividend $D_t$ is

$$Z_t = E_t\left[M_{t+1}\left(Z_{t+1} + 1\right)\frac{D_{t+1}}{D_t}\right]$$

Therefore, the price dividend ratio $Z_t$ is a function of the state variable $(s_t, \delta_t)$, and can be obtained as the solution to the following functional equation,

$$Z\left(s_t, \delta_t\right) = \delta E_t\left[\begin{array}{c} \exp\left\{-\gamma\left[(\phi - 1)\left(s_t - \bar{s}\right) + \left[1 + \lambda\left(s_t\right)\right]\epsilon_{c,t+1} + \mu_c\right]\right\} \\ \cdot\left(Z\left(s_{t+1}, \delta_{t+1}\right) + 1\right) \cdot \exp\left(\mu_c + \epsilon_{c,t+1} + \left(\rho_\delta - 1\right)\delta_t + \epsilon_{\delta,t+1}\right) \end{array}\right]. \tag{2.3}$$

The above setup is the standard external habit-formation model except the cointegration constraint. To further explore long-run implications of the model, in the following section, a log-linear approximation of the model is provided and some qualitative features of the model in the long-run are analytically derived.

## 2.2 Log-linear Solution of the Model

Before solving the functional equation (2.3) numerically, it is worthwhile to work on the log-linear approximation of the log price dividend ratio to gain intuitions of the model. Although the first order approximation is not numerically accurate given the highly nonlinear nature of the model, it provides right intuition. Assume that the log price dividend ratio $z_t = \log(Z_t)$ can be approximated by a linear function of the state variables

$$z_t \approx a_0 + a_1 s_t + a_2 \delta_t,$$

where the constant coefficients $a_0$, $a_1$, and $a_2$ are to be determined. Furthermore, I approximate the nonlinear sensitivity function $\lambda(s)$ by a linear function[4],

$$\lambda(s) \approx -a_\lambda (s - s_{\max}),$$

where $\alpha_\lambda$ is a proper constant to closely approximate the sensitivity function. The results that will be obtained in the following manner are not sensitive to the choice of $a_\lambda$. In the appendix, the coefficients $a_0$, $a_1$, and $a_2$ are solved in closed-form. Hence, a linear approximation of the log price dividend ratio can be obtained. Now, plugging this linear approximation of the log price dividend ratio back into the Campbell-Shiller log-linear approximation on returns gives[5]

$$\begin{aligned} r_{t+1} &\approx \kappa_0 + g_{d,t+1} + \rho z_{t+1} - z_t \\ &\approx \alpha + \beta_S S_t + \left[1 + a_1 \rho \frac{1-\bar{S}}{\bar{S}}\right] \epsilon_{c,t+1} + [1 + a_2 \rho]\, \epsilon_{\delta,t+1}, \end{aligned} \tag{2.4}$$

where $\beta_S = \frac{a_1(\rho\phi-1)}{\bar{S}}$ and the constants $\alpha$ is given by equation (8.4) in the appendix. $\beta_S$ is negative if and only if $a_1$ is positive. Hence, as long as the price dividend ratio is procyclical, $\beta_S$ is negative, and hence, the risk premium is countercyclical. Notice that the parameters $\rho$ and $\kappa_0$ satisfy

$$\begin{aligned} \rho &= \frac{\exp(E[z_t])}{1 + \exp(E[z_t])} \\ \kappa_0 &= -\log \rho - (1-\rho) \log\left(\frac{1}{\rho} - 1\right). \end{aligned}$$

Hence, $\rho$ and $\kappa_0$ are determined endogenously. This is quite easy to implement numerically.

The habit level $X_t$ can be further approximated as an exponentially weighted average of past consumption

$$X_t \approx \sum_{k=1}^{\infty} \frac{1-\phi}{\phi} \phi^k C_{t-k}, \tag{2.5}$$

[4] Another linear approximation around the steady state $\bar{s}$, $\lambda(s) \approx \frac{1}{\bar{S}} - 1 + \frac{1}{\bar{S}}(s - \bar{s})$ is also used and the results are almost identical.

[5] Since the riskfree rate is a constant in this model, the returns are equivalent with the excess returns.

where $\phi$ is the measure of habit persistence. Equation (2.5) implies that the habit level $X_t$ and the consumption level are cointegrated. Substitute equation (2.5) back into the definition of the surplus ratio, approximate to the first order and simplify to obtain

$$S_t \approx \tilde{S}_t \equiv \sum_{j=1}^{\infty} \phi^{j-1} g_{t+1-j}. \tag{2.6}$$

Hence, the asset returns can be approximated by

$$r_{t+1} \approx \alpha + \beta_S \sum_{j=1}^{\infty} \phi^{j-1} g_{t+1-j} + \left[1 + a_1 \rho \frac{1-\bar{S}}{\bar{S}}\right] \epsilon_{c,t+1} + \left[1 + a_2 \rho\right] \epsilon_{\delta,t+1}. \tag{2.7}$$

With the above approximation on returns, some long-run properties of the model can be analytically derived now. The $K$-horizon covariance between asset returns and consumption is (see the appendix for the detailed calculations)

$$\begin{aligned} cov\left(\sum_{j=1}^{K} r_{t+j}, \sum_{j=1}^{K} g_{c,t+j}\right) \quad = \quad & -\frac{\beta_S \sigma_c^2}{1-\phi} - \frac{\phi\left(1-\phi^{K-1}\right)\beta_S \sigma_c^2}{(1-\phi)^2} \\ & + \left[\left(1 - a_1\rho - \frac{a_1(1-\rho)}{\bar{S}(1-\phi)}\right)\sigma_c^2 + (1+a_2\rho)\,\sigma_{c\delta}\right] \cdot K. \end{aligned}$$

When horizon $K$ is sufficiently large, the sign of the correlation at very long horizons will be determined by the coefficient in front of $K$ in the above equation. Hence, the model implies a negative long-horizon correlation if and only if

$$1 - a_1\rho - \frac{a_1(1-\rho)}{\bar{S}(1-\phi)} + (1+a_2\rho)\frac{\sigma_{c\delta}}{\sigma_c^2} < 0. \tag{2.8}$$

Furthermore, the correlation between consumption growth and asset returns is decreasing as the horizon increases. To see this, first write down the long-horizon asset returns

$$\sum_{j=1}^{K} r_{t+j} \approx \alpha K + \beta_S \sum_{j=1}^{K} \tilde{S}_{t+j-1} + \left(1 + a_1\rho\frac{1-\bar{S}}{\bar{S}}\right) \sum_{j=1}^{K} \epsilon_{c,t+j} + (1+a_2\rho) \sum_{j=1}^{K} \epsilon_{\delta,t+j}. \tag{2.9}$$

The long horizon correlation between asset returns and growth rate comes from the last three terms in the above equation. Notice that the surplus ratio $\tilde{S}_{t+j-1}$ is a smoothed average of the past consumption growth rate. As the horizon $K$ increases, more negative correlation results from the second term since $\beta_S < 0$ while the correlation from the last two terms stays constant. Hence, the correlation between consumption growth and asset returns decreases as horizon $K$ increases.

The above approximation analysis provides good intuition on how the model works and the qualitative features of the model in the long-run. To obtain the quantitative implications of the model, I further solve this model numerically by assuming that the log price dividend ratio is a quadratic function of the state variables[6]. Using the linear approximation as the initial value, the algorithm converges very fast. The parameter values are chosen close to Campbell and Cochrane's (1999) as in table 1. Since the cointegration is incorporated into the model, the persistence parameter $\rho_\delta$ for the difference of log dividends and log consumption need to be chosen. That parameter is taken from Bansal, Gallant and Tauchen (2007)[7]. $48,000$ quarters of artificial data are simulated to calculate population values for a variety of statistics. Table 2 shows the summary statistics of the equity premium, riskfree rate, and price dividend ratio from the simulated model. To facilitate the comparison with Campbell and Cochrane (1999), I report the simulated moments of the consumption and asset returns together with that of both the post-war sample and the long sample from table 2 of Campbell and Cochrane (1999). As in Campbell and Cochrane (1999), the external habit formation model matches these moments well.

The long-run feature of the model is demonstrated in table 3, which lists the correlation between consumption growth and asset returns at different horizons. For the data, this correlation is increasing as the horizon increases until 6 quarters, then slowly declines. However, for the habit formation model, the correlation is monotonically decreasing with horizon[8], and the correlations are negative at very long horizons. When $\rho_\delta$ is set to 1, consumption and dividends are not cointegrated as in Campbell and Cochrane (1999), the correlation between consumption and dividends is indeed lower as shown in the last column of table 3. The correlation is also monotonically decreasing, and the correlations are more negative at very long horizons. Here, the focus of the analysis is the dynamics of correlations over different horizons, not the level of the correlations. The correlation between consumption growth and asset returns is too large in the model, which is a common drawback for most asset pricing models. Furthermore, the level of correlation can be lowered when the parameter values are changed to other combinations. However, the decreasing pattern in the correlation over long-horizon remains.

A formal way to address the long-run implications of the model is the cross-spectral analysis of consumption growth and asset returns. Moreover, the spectral analysis (i.e., the phase spectrum) can provide information on the lead-lag relation between consumption and asset returns. Since spectral analysis is not a standard tool in finance, a brief explanation of coherence, cospectrum and phase spectrum is now provided below. The coherence of the consumption growth rate and stock market returns at frequency $\lambda$ measures the correlation between the consumption growth and returns at frequency $\lambda$. Essentially, the coherency analysis splits each of the two series into a set

[6] As in Tallarini and Zhang (2004), Bansal, Gallant and Tauchen (2007) , a quadratic polynomial approximation works well enough.

[7] In Lettau and Wachter (2007), they use $\rho_\delta = 0.91$ for annual frequency, or equivalently, $\rho_\delta = 0.9922$ for monthly frequency. The results will remain the same if $\rho_\delta$ is set to be 0.9922

[8] If the model is simulated at monthly frequency, and time-averaged to quarterly frequency, the correlation could increase from first quarter to second quarter, then it decreases monotonically as the horizon increases.

of Fourier components at different frequencies, then determines the correlation of a set of Fourier components for the two series around each frequency. When the frequency is $\lambda$, the corresponding length of the cycle is $1/\lambda$ quarters. Hence, when $\lambda = 0.5$, the corresponding cycle is 2 quarters. Since the coherency is always positive, the sign of the correlation at different frequencies can't be told from the coherency spectrum. To identify the sign of the correlation, the cospectrum needs to be examined. The cospectrum at frequency $\lambda$ can be interpreted as the portion of the covariance between consumption growth and asset returns that is attributable to cycles with frequency $\lambda$. Since the covariance can be positive or negative, the cospectrum can also be positive or negative. The slope of the phase spectrum at any frequency $\lambda$ is the *group delay* at frequency $\lambda$, and precisely measures the number of leads or lags between consumption growth and asset returns. When this slope is positive, consumption leads the market return. On the other hand, when this slope is negative, asset market returns lead consumption growth.

In the appendix, it is shown that the cross-spectrum between consumption growth and asset returns can be given by

$$f_{12}(\lambda) = \frac{1}{2\pi}\left(\beta_S \frac{e^{i\cdot\lambda} - \phi}{1 + \phi^2 - 2\phi\cos(\lambda)} + 1 + a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)\sigma_c^2 + \frac{1}{2\pi}\left[1 + a_2\rho\right]\sigma_{c\delta}. \tag{2.10}$$

Hence, the cospectrum $C_{sp}(\lambda)$ (the real part the the cross-spectrum $f_{12}(\lambda)$) can be given by

$$C_{sp}(\lambda) = \left(\beta_S \frac{\cos(\lambda) - \phi}{1 + \phi^2 - 2\phi\cos(\lambda)} + 1 + a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)\frac{\sigma_c^2}{2\pi} + (1 + a_2\rho)\frac{\sigma_{c\delta}}{2\pi}.$$

Taking the derivative of the above equation yields

$$C'_{sp}(\lambda) = \frac{-\beta_S\sin(\lambda)}{2\pi\left(1 + \phi^2 - 2\phi\cos(\lambda)\right)^2}\left(1 - \phi^2\right),$$

which is positive as long as $\beta_S < 0$. Hence, the portion of the covariance contributed by component at frequency $\lambda$ is increasing as the frequency $\lambda$ is increased when $\beta_S < 0$. This partially confirms the early result that the correlation between consumption growth and asset returns decreases as the horizon increases.

Another way to show the negative correlations at long horizons is to examine the sign of the cross-spectrum between consumption growth and asset returns at the frequency $\lambda = 0$. The cross-spectrum at frequency zero is

$$f_{12}(0) = \frac{1}{2\pi}\left(-\frac{a_1(1-\rho)}{\bar{S}(1-\phi)} + 1 - a_1\rho\right)\sigma_c^2 + \frac{1}{2\pi}(1 + a_2\rho)\sigma_{c\delta}.$$

Later it will be shown that equation (2.8) will typically be satisfied in the models that can match the first two moments of the aggregate data. When equation (2.8) holds, the low-frequency correlations between consumption growth rate and asset returns are negative (since the function $f_{12}(\lambda)$ is continuous in $\lambda$), which is in contradiction with the real data. Therefore, the sign of the correlation of at frequency $\lambda = 0$ is the same with the sign of the long-horizon correlation, which is not unexpected.

From the expression for the cross-spectrum in equation (2.10), the phase spectrum $\phi(\lambda)$ can be calculated as follows

$$\tan(\phi(\lambda)) = \frac{\beta_S \sin(\lambda)\sigma_c^2}{\beta_S(\cos(\lambda)-\phi)\sigma_c^2 + \left[\left(1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)\sigma_c^2 + (1+a_2\rho)\sigma_{c\delta}\right](1+\phi^2-2\phi\cos(\lambda))}. \tag{2.11}$$

To investigate the lead-lag relation between consumption growth and asset returns, I need to examine the sign of the slope of the phase spectrum by differentiating equation (2.11). Indeed, if the correlation between consumption innovation and return innovation is positively correlated, then it follows that

$$\begin{aligned}
\phi'(\lambda) \quad &\propto \quad -\frac{a_1(\rho\phi-1)}{\bar{S}} + 2\phi\left(1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right) + 2\phi\left[1+a_2\rho\right]\frac{\sigma_{c\delta}}{\sigma_c^2} \\
&\quad -\left\{1+a_1\rho\frac{1-\bar{S}}{\bar{S}} + (1+a_2\rho)\frac{\sigma_{c\delta}}{\sigma_c^2} + \phi^2 - a_1\rho\phi^2 + \frac{a_1\phi}{\bar{S}} + (1+a_2\rho)\phi^2\frac{\sigma_{c\delta}}{\sigma_c^2}\right\}\cos(\lambda) \\
&\geq \quad -\left[1-a_1\rho-\frac{a_1(1-\rho)}{\bar{S}(1-\phi)} + (1+a_2\rho)\frac{\sigma_{c\delta}}{\sigma_c^2}\right](1-\phi)^2,
\end{aligned}$$

where "$\propto$" denotes that the signs on the left and right sides of "$\propto$" are the same and the last inequality requires the following assumption

$$(1-a_1\rho)\left(1+\phi^2\right) + \frac{a_1(\rho+\phi)}{\bar{S}} + (1+a_2\rho)\left(1+\phi^2\right)\frac{\sigma_{c\delta}}{\sigma_c^2} \geq 0,$$

which is true if the correlation between the innovation in consumption and innovation in returns is positive, that is

$$\left(1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)\sigma_c^2 + (1+a_2\rho)\sigma_{c\delta} \geq 0. \tag{2.12}$$

Note that a positive slope at frequency $\lambda$ ($\phi'(\lambda) > 0$) implies that consumption growth leads asset returns at frequency $\lambda$. Hence, when equation (2.8) holds and the correlation between the innovation in consumption and innovation in returns is positive, consumption growth leads asset market returns in the external habit formation model. The above discussions lead to the following two propositions. I relegate all proofs to the appendix.

**Proposition 1:**

*If equation (2.8) holds,*

$$1-a_1\rho-a_1\frac{(1-\rho)}{\bar{S}(1-\phi)} + (1+a_2\rho)\frac{\sigma_{c\delta}}{\sigma_c^2} < 0,$$

*then there exist a frequency $\lambda^*$ such that, for $\lambda < \lambda^*$, the correlation between the consumption growth rate and asset returns at frequency $\lambda$ is negative. If, in addition, equation (2.12) holds,* *the slope of the phase spectrum between consumption growth and asset returns is positive. Hence, consumption growth leads asset returns.*

**Proposition 2:**

*Under the external habit-formation model, the analytical approximation shows that when*

$$\beta_S = \frac{a_1(\rho\phi - 1)}{\bar{S}} < 0,$$

*the cospectrum between consumption growth and asset returns is an increasing function of the frequency. The portion of the covariance between consumption growth and asset returns that is attributable to cycles with frequency $\lambda$ is increasing with the frequency $\lambda$. Hence, the high frequency cycles contribute more to the covariance between consumption growth and asset returns.*

It is very natural for consumption to lead returns in this model since the expected returns depend on the surplus ratio which is a smoothed average of the *past* consumption innovations. Now, I want to see when equation (2.8) can be satisfied, so the low-frequency correlation between consumption and asset returns is negative. Notice that $\delta_t = d_t - c_t$, hence, it is reasonable to assume that $\sigma_{c\delta} \leq 0$. Notice that $-1 \leq a_2\rho = \frac{\rho_\delta - 1}{1-\rho\rho_\delta}\rho \leq 0$, hence, $(1 + a_2\rho)\frac{\sigma_{c\delta}}{\sigma_c^2} \leq 0$. Therefore, for equation (2.8) to hold, only need the condition $1 - a_1\rho - a_1\frac{(1-\rho)}{\bar{S}(1-\phi)} < 0$. Furthermore, since $a_1$ can be found as the positive root of a quadratic equation, which usually ranges from 0.5 to 1.5, and $\bar{S}$ is usually less than 0.1 to produce a high equity premium, the condition $1 - a_1\rho - a_1\frac{(1-\rho)}{\bar{S}(1-\phi)} < 0$ can be easily satisfied. Therefore, equation 2.8 typically holds. Notice that equation 2.8 holds as long as $a_1$ is not too small. Since $a_1$ is the exposure of price dividend ratio to surplus ratio, if $a_1$ is too small, the model can't produce quantitative results for the first two moments of the aggregate data. Hence, for the model to make quantitative sense, $a_1$ can't be too small, and the condition in proposition 1 is typically satisisfied.

If $\beta_S < 0$, then the expected asset returns are high when the surplus ratio is low. Hence, the equity premium is countercyclical. Therefore, a negative $\beta_S$ is a very reasonable assumption. Indeed, as I show in the appendix, under very mild conditions, $\beta_S$ is negative. For example, when the correlation between consumption growth and dividend growth is positive, $\beta_S$ is negative. Also notice that $\beta_S < 0$ if and only if $a_1 > 0$. A positive $a_1$ implies a procyclical price dividend ratio. Therefore, as long as the external habit persistence model produces a procyclical price dividend ratio, the cospectrum between consumption growth and asset returns is an increasing function of the frequency $\lambda$, which contradicts the data.

Proposition 1 implies that the low-frequency correlation between consumption growth and asset returns is typically negative for a external habit formation model. At first glance, this seems contradictory to the cointegration constraint between dividends and consumption. However, the low-frequency correlation between consumption growth and asset returns is not necessarily positive. To see this, it follows from the Campbell-Shiller decomposition of the returns, the cumulative returns can be written as

$$\begin{aligned}\sum_{j=1}^{K} r_{t+j} &\approx K\kappa_0 + \sum_{j=1}^{K} g_{d,t+j} + \rho \sum_{j=1}^{K} z_{t+j} - \sum_{j=1}^{K} z_{t+j-1} \\ &= K\kappa_0 + \sum_{j=1}^{K} g_{d,t+j} + (\rho - 1) \sum_{j=1}^{K-1} z_{t+j} + z_{t+K} - z_t.\end{aligned}$$

Since the log price dividend ratio $z_t$ is stationary, the correlation between long-run returns and long-run consumption resulting from the term $z_{t+K} - z_t$ is negligible. In the long run, $\sum_{j=1}^{K} g_{d,t+j}$ and $\sum_{j=1}^{K} g_{c,t+j}$ are perfectly correlated. However, the term $(\rho - 1) \sum_{j=1}^{K-1} z_{t+j}$ is negatively correlated with $\sum_{j=1}^{K} g_{c,t+j}$ because $\rho - 1$ is negative and the price dividend ratio is positively correlated with the surplus ratio ($z_t \approx a_0 + a_1 s_t + a_2 \delta_t$). To see why price dividend ratio is positively correlated with the surplus ratio, I argue as follows. When the realized consumption growth is high, the surplus ratio is also high. Hence, the effective risk aversion is low. Therefore, the impled discount rate is lower and the price dividend ratio is higher. That is, the price dividend ratio is positively correlated with the consumption growth rate. Since each $z_{t+j}$ includes a smoothed average of past consumption growth, the covariance between $(1 - \rho) \sum_{j=1}^{K-1} z_{t+j}$ and $\sum_{j=1}^{K} g_{c,t+j}$ could be higher than the covariance between $\sum_{j=1}^{K-1} g_{d,t+j}$ and $\sum_{j=1}^{K} g_{c,t+j}$ if the horizon $K$ is big enough.When the negative effect between $(\rho - 1) \sum_{j=1}^{K-1} z_{t+j}$ and $\sum_{j=1}^{K} g_{c,t+j}$ dominates, the long-run correlation between consumption growth and asset returns could be negative. The following simple example can also provide some intuition. Suppose that the consumption realizations are very low over many periods, then the cumulative consumption growth rate is also low. Furthermore, low consumption realizations result in low surplus ratios during these periods, and hence, a high expected return in each of these periods. As a result, the realized asset returns are very likely to be large during these periods. Consequently, the long-horizon correlation between consumption growth and asset returns could be negative in this model.

Proposition 1 and proposition 2 provide the qualitative features of the cross-spectral between consumption and asset returns by a log-linear approximation. The exact cross-spectral can be obtain based on $48,000$ quarters of artificial data simulated from the model with the parameter values given by table 1. The top panel of figure 1 plots the coherency between consumption growth and asset returns from the simulation of the model, and the middle panel plots the cospectrum. It can be seen that in the simulated model, the cospectrum is increasing as shown by the solid line. The dotted line is the cospectrum from the analytical approximation. The approximation is quite accurate in general. Given the highly nonlinearity of the model, the difference between the linear approximation and the exact solution is not negligible for some region. However, the shape of the spectrum is very similar. The bottom panel is the phase spectrum which is increasing. It can be seen that the exact solution and the analytic approximation are extremely close for the phase spectrum. Since the phase spectrum determines the sign of the cospectrum, the claim about the sign of correlations based on the analytical approximation is also valid under the exact solution.

For the real data, the top panel of figure 2 confirms Daniel and Marshall's (1999) finding that the coherency between the quarterly consumption growth and the quarterly market excess return is much higher at low frequencies (around 0.5) than at high frequencies (around 0.1). Therefore, most of the correlation between the consumption growth and asset market returns comes from the co-movement at low frequencies. The middle panel also shows that most of the covariance comes from the low frequency covariation. The 95% confidence interval is also given by the dotted line, and the confidence interval for cospectrum is above 0 at frequency 0, while the cospectrum at frequency 0 is negative for the simulated model. The high frequency cospectrum is close to zero. The bottom panel of figure 2 shows that the phase spectrum is nearly monotonically decreasing. For most frequencies, in this phase spectrum, the slope is negative. Hence, it is the market returns that lead consumption growth. Figure 3 plot the coherency, cospectrum, and phase spectrum for both the model and the data together. From this graph, it can be seen that*, the coherency, cospectrum and the phase spectrum are all declining in the data, while they are all increasing in the external habit formation model.* In the simulated model, the correlation between consumption innovation and return innovation is very large. Therefore, it is not surprising that there is a very high coherency between consumption growth rate and asset returns as in figure 1. This excessively high correlation between consumption and asset returns is a common problem for most asset pricing models.

Instead of simulating the model for $48,000$ quarters in one shot, I run 1000 Monte Carlo experiments, each with 100 years of observations. Band-pass filter is used to calculate the low-frequency (with cycle longer than 5 years) and high-frequency (with cycle between 0.5 and 5 years) correlations between consumption and asset returns in each Monte Carlo experiment. Then, the difference between the low-frequency correlation and high-frequency correlation is obtained for each experiment. The Monte Carlo result shows that the 90% quantile of the differences is negative. Hence, we can reject the hypothesis at 10% level that the model can produce a larger low-frequency correlation than high-frequency correlation. Furthermore, in the data, the difference between low-frequency correlation and high frequency correlation is about 15%˜35%. None of the 1000 Monte Carlo experiments can produce such a big difference. Hence, it can be safely claimed that the model can't produce the same long-horizon feature as that in the data.

I have shown that the external habit formation model with difference utility form can't match the long-run features of the data. Abel (1990) proposes a ratio form of external habit formation model (Abel calls this catching up with the Joneses). Under Abel's model, it can be shown that both coherency and cospectrum between consumption growth and gross equity returns are increasing as those in the difference form of external habit formation models[9]. Even with predictable consumption

[9]Notice that under *i.i.d.* consumption growth case, the coherence and cospectrum between consumption and excess

growth, the above results are still true if the risk aversion coefficient is large enough to produce a reasonable equity premium.

## 3 Long-Run Risk and Trend-Cycle Models

Section 2 has shown that the standard external habit formation model has difficulty matching coherency, cospectrum and phase spectrum between consumption growth and asset market returns. Hence, the question is what kind of model can produce the correct long-run correlation and lead-lag relation between consumption growth and asset returns. In the standard Lucas tree model, where *i.i.d.* consumption growth and CRRA preferences are assumed, the coherency, cospectrum and phase spectrum are all flat. To obtain a decreasing coherency, cospectrum, and phase spectrum, it is necessary to modify either the preferences or the consumption dynamics. It is difficult to match the first two moments of the equity premium and the riskfree rate by modifying the consumption dynamics alone[10]. The external habit formation model is a representative model with generalized preferences which are proposed to resolve asset pricing puzzles. As an out-of-sample test, it has been shown in last section that this type of model can not generate the same shape of the cross-spectrum as that in the data.

If in a model, the expected return depends on a forward-looking variable, which can predict consumption growth in itself, then the model could potentially produce the desired lead-lag relation between consumption and asset returns. When the log consumption is decomposed to a stochastic trend and a cycle, the level of the cycle can predict the future consumption growth. Hence, if an asset pricing model (for example, Panageas and Yu (2006)) implies that expected returns depend on the level of the cycle, then the model could produce the correct lead-lag relation between consumption and asset returns. Since expected returns depend on the persistent cycle component, the co-movement could be tighter between consumption and asset returns over longer horizons. As a result, this type of model could potentially produce the right low-frequency property as that in the data. In the following, I give an sketch of a structural trend-cycle model to provide the motivation for the consupmtion dynamics and expected return dynamics. Then through a reduced-form model to show the intuition on how this type of model can produce the right patterns in the cross-spectra. At last, two structrual models are simulated to show that these models can generate the desired

returns is constant. While in the data, the coherence and cospectrum between consumption growth and gross returns (and excess returns) are all decreasing.

[10] If the CRRA preferences are maintained, but the consumption growth is a predictable process (for example, AR(1)), and a large risk aversion coefficient is assumed to generate enough equity premium, then the model could generate decreasing coherency and cospectrum. However, the phase spectrum would be increasing in this case since the expected return depends on the state variable, past consumption growth. This dependence is especially strong when risk aversion coefficient is large.

long-run features.

### 3.1 Structural Trend-Cycle Model in Panageas and Yu (2006)

In this section, I give a sketch of the trend-cycle model in Panageas and Yu (2006) to motivate the dynamics for consumption and the expected returns. There exists a continuum of firms indexed by $j \in [0,1]$. Each firm owns a collection of trees that have been planted in different technological epochs, and its total earnings is just the sum of the earnings produced by the trees it owns. Each tree in turn produces earnings that are the product of three components: a) a vintage specific component that is common across all trees of the same technological epoch, b) a time invariant tree specific component and c) an aggregate productivity shock. To introduce notation, let $Y_{N,i,t}$ denote the earnings stream of tree $i$ at time $t$, which was planted in the technological epoch $N \in (-\infty .. -1, 0, 1, .. +\infty)$. In particular, assume the following functional form for $Y_{N,i,t}$:

$$Y_{N,i,t} = \left(\overline{A}\right)^N \zeta(i)\theta_t \tag{3.1}$$

$\left(\overline{A}\right)^N$ captures the vintage effect. $\overline{A} > 1$ is a constant. $\zeta(\cdot)$ is a positive strictly decreasing function on $[0,1]$, so that $\zeta(i)$ captures a tree specific effect. $\theta_t$ is the common productivity shock and evolves as a geometric Brownian motion. Technological epochs arrive at the Poisson rate $\lambda > 0$. Once a new epoch arrives, the index $N$ becomes $N+1$, and every firm gains the option to plant a single tree of the new vintage at a time of its choosing.

Firm heterogeneity is introduced as follows: Once epoch $N$ arrives, firm $j$ draws a random number $i_{j,N}$ from a uniform distribution on $[0,1]$. This number informs the firm of the type of tree that it can plant in the new epoch. In particular a firm that drew the number $i_{j,N}$ can plant a tree with tree specific productivity $\zeta(i_{j,N})$. These numbers are drawn in an i.i.d fashion across epochs.

Any given firm determines the time at which it plants a tree in an optimal manner. Planting a tree requires a fixed cost which is the same for all trees of a given epoch. Let $K_{N,t} \in [0,1]$ denote the mass of firms that have updated their technology in technological epoch $N$ up to time $t$. It is formally shown that $K_{N,t}$ will coincide with the index of the most profitable tree that has not been planted yet (in the current epoch). Hence, the aggregate output is given as

$$Y_t = \left[\sum_{n=-\infty..N-1} \overline{A}^{(n-N)} \left(\int_0^{K_{n,\overline{\tau_n}}} \zeta(i)di\right) + \int_0^{K_{N,t}} \zeta(i)di\right] \overline{A}^N \theta_t$$

where $\overline{\tau_n} = \tau_{n+1}$ denotes the time at which epoch $n$ ended (and epoch $n+1$ started). Further define $F(x) = \int_0^x \zeta(i)di$. Then, the total consumption $c_t = \log(C_t) = \log(Y_t)$ can be rewritten as

$$c_t = \log(\theta_t) + N\log(\overline{A}) + x_t \tag{3.2}$$

where

$$x_t = \log\left[\sum_{n=-\infty..N-1} \overline{A}^{(n-N)} F\left(K_{n,\overline{\tau_n}}\right) + F(K_{N,t})\right], \tag{3.3}$$

and $x_t$ is a geometrically declining average of the random terms $F(K_{n,\overline{\tau_n}})$. This means that $x_t$ would behave exactly as an autoregressive process (across epochs). Hence, the model is able to produce endogenous cycles, on top of the pure random walk stochastic trend $\log(\theta_t) + N\log(\overline{A})$ that we assumed at the outset. Notice that the expected excess return on the market is a weighted averge of the returns on asset in place, and the returns on the options to adopt the new technologies and the expected return on options are higher than that of asset in place. When the current level of consumption is below its stochastic trend, this implies that there is a large number of unexploited investment opportunities for firms. Accordingly, the relative weight of growth options will be substantial. Hence, up to first order approximation, the expected excess return can be written as

$$\mu_t - r \approx \alpha + \beta x_t. \tag{3.4}$$

In a nutshell, this model implies that the consumption consists of a random walk and an autoregressive cycle and the expected excess return is approximately a linear function of the cyclical component in consumption. To see how the trend-cycle models can produce the right pattern of cross-spectrum, it is easiest to first work on a reduced-form model. Then I simulate two structural models: Bansal and Yaron (2004) and Panageas and Yu (2006). Previous literature usually assumes that the consumption growth rate follows an *i.i.d.* process. However, a predictable consumption growth rate is key for trend-cycle models. Hence, before turning to the reduced-form model, an $ARMA$ process is fitted for the quarterly data on consumption growth rate.

## 3.2 The Estimation of Consumption Dynamics

The estimation results indeed show that a good description for log consumption is a stochastic trend plus an $AR(2)$ cycle, which is equivalent to an $ARIMA\,(2,1,2)$ process[11]. For an $ARIMA\,(2,1,2)$ log consumption $c_t$, the consumption growth rate $g_{c,t}$ has the following dynamics

$$g_{c,t} - \mu_c = \rho_{c,1}\left(g_{c,t-1} - \mu_c\right) + \rho_{c,2}\left(g_{c,t-2} - \mu_c\right) + \epsilon_{c,t} + \theta_{c,1}\epsilon_{c,t-1} + \theta_{c,2}\epsilon_{c,t-2} \tag{3.5}$$

[11] As in Morley, Nelson and Zivot (2003), there is a one-to-one correspondence between ARIMA(2,1,2) and a trend-cycle decomposition with an AR(2) cycle component for the log consumption level. Furthermore, the AR(2) cyclic component is the simplest cycle dynamics such that all the parameters in the trend-cycle model are identifiable. In later analysis, we will assume that log consumption follows a trend-cycle process which is equivalent to the current ARIMA(2,1,2).

where $\epsilon_{c,t} \tilde{} WN\left(0, \sigma_c^2\right)$. This $ARIMA\left(2,1,2\right)$ process has the following equivalent trend-cycle representation for log consumption,

$$\begin{aligned} c_t &= T_t + x_t \\ T_t &= T_{t-1} + \mu_c + \xi_t \\ x_t &= \rho_{x,1} x_{t-1} + \rho_{x,2} x_{t-2} + \epsilon_{x,t} \end{aligned} \tag{3.6}$$

where $T_t$ is the stochastic trend, $x_t$ is the cyclical component in the log consumption, $\epsilon_{x,t} \tilde{} WN\left(0, \sigma_{\epsilon_x}^2\right)$, $\xi_t \tilde{} WN\left(0, \sigma_\xi^2\right)$ and $corr\left(\xi_t, \epsilon_{x,t}\right) = \rho_{\xi,\epsilon_x}$. Table 4 gives the estimates for the consumption process. All coefficients of the $ARIMA\left(2,1,2\right)$ are significant at 5% level. Moreover, the implied correlation between the trend innovation and the cycle innovation is highly negative with $\rho_{\xi,\epsilon_x} = -0.9569$. This negative correlation is consistent with the implication of Panageas and Yu (2006), in which the investment and consumption experience a delay when a new round of technological advancement arrives. Morley, Nelson and Zivot (2003) also find a large negative correlation coefficient between the innovations in the trend and cycle components in the GDP. A positive productivity shock (i.e., the invention of the internet) will immediately shift the long run path of output upwards, leaving actual output below the trend until it catches up. This yields a negative contemporaneous correlation since this positive trend shock is associated with a negative shock to the transitory component.

### 3.3 A Reduced-Form Trend-Cycle Model

In this section, a reduced-form trend-cycle risk model is analyzed to provide intuition on why this type of model can produce the desired pattern in the cross-spectrum. I assume that the log consumption $c_t$ consists of a stochastic trend component $T_t$ plus an $AR(2)$ cycle component $x_t$ as in equation (3.6), which is equivalent to the $ARIMA(2,1,2)$ process for log consumption. Hence, the consumption growth rate is given by $g_{c,t} = x_t - x_{t-1} + \xi_t$. The expected return is further assumed to be negatively correlated with current cycle component $x_t$ in the following way[12],

$$E_t\left(r_{t+1}\right) = \alpha_0 + \beta x_t, \quad \text{where } \beta < 0.$$

In a reduced-form model without cointegration constraint, the realized return can be written as

$$r_t = \alpha_0 + \beta x_{t-1} + u_t,$$

where the innovation $u_t$ is normally distributed with mean 0 and standard error $\sigma_u$. If the dividends (in logs) are assumed to be cointegrated with the consumption (in logs)

[12]Since the cycle component $x_t$ is assumed to be an $AR\left(2\right)$ process, it is more reasonable to assume that expected returns also depend on the lagged cycle component. Here, for simplicity, I ignore the lagged cyclical component. The results are robust if the lagged cyclical component is included in the expected returns.

$$\delta_t \equiv d_t - c_t = \mu_{dc} + \sum_{k=0}^{\infty} \psi_k \epsilon_{\delta,t-k},$$

where $\sum_{k=1}^{\infty} |\psi_k| < \infty$. Then, as shown in the appendix, by plugging the constraint on the conditional expected returns into the Campbell-Shiller's log-linear approximation on returns, it follows that

$$r_t \approx \alpha_0 + \epsilon_{\delta,t}\bar{\psi} + \epsilon_{x,t} \cdot (\rho^* - \beta\rho\bar{\rho}) + \xi_t + \beta x_{t-1}, \tag{3.7}$$

where $\bar{\psi}$, $\rho^*$, and $\bar{\rho}$ are proper constants defined by equation (8.6) in the appendix. Therefore, the cointegration constraint simply adds a restriction on the innovations in returns

$$u_t \equiv \epsilon_{\delta,t}\bar{\psi} + \epsilon_{x,t} \cdot (\rho^* - \beta\rho\bar{\rho}) + \xi_t.$$

To determine the cross-spectrum between consumption growth and asset returns, we only need to know the correlation between the innovation in returns $u_t$ and innovation in growth rate $(\xi_t, \epsilon_{x,t})$. Instead of estimating the parameters $\psi_k$, $\rho_{\delta,\xi} \equiv corr\,(\epsilon_{\delta,t}, \xi_t)$, $\rho_{\delta,\epsilon_x} \equiv corr\,(\epsilon_{\delta,t}, \epsilon_{x,t})$ (which might have substantial errors), and taking care of the internal link between the parameter $\rho$, the price dividend ratio and returns, here I just fix the correlations $\rho_{u,\xi} \equiv corr\,(u_t, \xi_t)$ and $\rho_{u,\epsilon_x} \equiv corr\,(u_t, \epsilon_{x,t})$ at different values and plot the cross-spectrum under different scenarios. This approach allows me to examine the sensitivity of the cross-spectrum to the underlying parameters.

Figure 4 through figure 7 plot the coherency, cospectrum, and phase spectrum under different parameter values. I fix the values for the parameters on the consumption dynamics and change the values of $\beta$, $\rho_{u,\xi}$, and $\rho_{u,\epsilon_x}$. Moreover, $\sigma_u$ is fixed at 0.08 to match the market volatility. In figure 4, the parameter values are $\beta = -2$, $\rho_{u,\xi} = 0$, and $\rho_{u,\epsilon_x} = 0$. In the data, these correlations are indeed very small. It can be seen that all the of them are downward sloping. When the values on the correlation are changed to $\rho_{u,\xi} = 0.2$, $\rho_{u,\epsilon_x} = -0.2$, and $\rho_{u,\xi} = 0.5$, $\rho_{u,\epsilon_x} = -0.5$, the coherency increases and the slope is steeper. However, this decreasing pattern remains. As the predictability of returns is increased to $\beta = -5$, the results are similar. To see why this reduced-form model can produce desired pattern of the spectrum, first examine the long horizon correlation between consumption growth rate and asset returns. For simplicity, assume $x_t$ is an AR(1) process, then

$$\sum_{j=1}^{K} g_{t+j} = \sum_{j=1}^{K} \xi_{t+j} + \sum_{j=1}^{K} \epsilon_{x,t+j} + (\rho_{x,1} - 1) \sum_{j=1}^{K} x_{t+j-1} \tag{3.8}$$

$$\sum_{j=1}^{K} r_{t+j} \approx K\alpha_0 + \sum_{j=1}^{K} u_{t+j} + \beta \sum_{j=1}^{K} x_{t+j-1}. \tag{3.9}$$

The correlation cumulative consumption and cumulative returns is a weighted average of the correlations between the three summations in equation 3.8 and the two summation in equation 3.9. Since the correlation between $(\rho_{x,1} - 1) \sum_{j=1}^{K} x_{t+j-1}$ and $\beta \sum_{j=1}^{K} x_{t+j-1}$ is just 1. Hence, if the weight on this correlation is high, the correlation between consumption and return will be high. The weight depends on the variance of the individual summation terms. Since $\sum_{j=1}^{K} \xi_{t+j}$, $\sum_{j=1}^{K} \epsilon_{x,t+j}$, and $\sum_{j=1}^{K} u_{t+j}$ are just summations of *i.i.d.* shocks, their variances just increase linearly with horizon. However, $\sum_{j=1}^{K} x_{t+j-1}$ is a summation of persistent terms, its variance will increase faster than linearly because of the cross-covariances. Hence, as horizon increases, the weight on the correlation between $(\rho_{x,1} - 1) \sum_{j=1}^{K} x_{t+j-1}$ and $\beta \sum_{j=1}^{K} x_{t+j-1}$ will increase. Since this correlation is 1, the correlation between cumulative consumption growth and cumulative returns also increases with horizon. To see the lead-lag relation, assume that the current cycle $x_{t-1}$ is relatively low, then the expected future consumption growth is high. Moreover, the expected asset returns are also high. Hence, given a high realized asset return, it is likely that the past cycle level $x_{t-1}$ is low. Since the cycle component $x_t$ is mean-reverting, then the future consumption growth is expected to be high. Hence, high asset returns predict a high consumption growth rate.

## 3.4 Simulation From Structural Models: Bansal and Yaron (2004) and Panageas and Yu (2006)

I have shown through a reduced-form trend-cycle model that this type of models can produce the desired relationship between growth and asset returns in long horizons. Now, based on a calibrated structural model in Panageas and Yu (2006)[13], $48,000$ quarters of excess returns and the consumption growth rate are simulated and the cross-spectra of these two simulated series are plotted. Figure 8 plots the coherency, cospectrum, and the phase spectrum for the simulated data. These spectra are indeed all decreasing. The magnitude of the cospectrum is higher than that in the data because the calibrated model has a higher consumption volatility[14], return volatility, and correlation between consumption and return than those values in the data. Moreover, the correlation between consumption growth and asset returns increases with horizons first, then slowly decreases, which is the same with the pattern in the data.

Since Bansal and Yaron (2004) is a representative long-run risk model which has attracted a lot of attention in the current literature, I explore its long-run implications in the following. Following Bansal and Yaron (2004), the dynamics of consumption and dividends are assumed to

[13]In Panageas and Yu (2006), their model features two types of shocks: "small" , frequent and disembodied shocks to productivity and "large" technological innovations, which are embodied into new vintages of the capital stock. The latter types shocks affect the economy with lags, since firms need to invest before they can take advantage of the new technology. This delayed reaction of consumption to large technological innovation helps them explain why the short-run correlation between consumption and asset returns are weaker than their long-run counterparts.

[14]The model is calibrated to match the long sample of consumption data which has a much higher volatility.

be

$$\begin{aligned}
x_{t+1} &= \rho_x x_t + \varphi_e \sigma_t e_{t+1} \\
g_{c,t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\
g_{d,t+1} &= \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \\
\sigma_{t+1}^2 &= \sigma^2 + \nu_1 \left(\sigma_t^2 - \sigma^2\right) + \sigma_w w_{t+1}.
\end{aligned}$$

where the innovations $e_t$, $\eta_t$, $u_t$, and $w_t$ are *i.i.d.* $N(0,1)$. It follows from equation $(A12)$ and equation $(A14)$ of Bansal and Yaron (2004), the excess returns can be approximated by

$$\begin{aligned}
r_{t+1} &\approx \beta_{m,w}\lambda_{m,w}\sigma_w^2 - 0.5\beta_{m,w}^2\sigma_w^2 + \left(\beta_{m,e}\lambda_{m,e} - 0.5\beta_{m,e}^2 - 0.5\varphi_d^2\right)\sigma_t^2 \\
&\quad + \kappa_{1,m}A_{1,m}\varphi_e\sigma_t e_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} + \varphi_d\sigma_t u_{t+1}, \qquad (3.10)
\end{aligned}$$

where all the constants are defined in appendix of Bansal and Yaron (2004). Taking the parameter values from the calibrated model of Bansal and Yaron (2004), the model can match the first two moments of equity premium, risk free rate and consumption growth. With the above return and consumption dynamics, $48,000$ quarters of artificial data are simulated. The resulting coherency, cospectrum and phase spectrum are all decreasing as shown in figure 9. Furthermore, the correlation between asset returns and consumption growth increases with horizons first, then slowly decreases, which is the same with the pattern in the data.

I also run 1000 Monte Carlo experiments, each with 100 years of observations, as in section 2. Band-pass filter is used to obtain the difference between the low-frequency correlation and high-frequency correlation for each experiment. The Monte Carlo result shows that the 90% quantile of the differences is 0.3903, while this difference in the data is about 15%~35%. Hence, the 90% confidence interval from the model includes the corresponding value from the data. The same is true for the model in Panageas and Yu (2006), with a 90% quantile of the differences equal 0.4844.

# 4 Habit Formation Model With Predictable Cash Flow

It has been shown that the external habit model with *i.i.d.* consumption growth specification can not produce a consistent cross-spectrum between consumption and asset returns as seen in the data. It has also been shown that when consumption is assumed to have a cyclical component, the long-run risk and trend cycle models could produce the desired relation between consumption growth and asset returns. As a robustness check, I use the same $ARMA(2,2)$ consumption growth for the external habit formation model (as estimated in section 5) and assume that consumption is cointegrated with dividends as follows,

$$\begin{aligned}
g_{c,t} &= \mu_c\left(1 - \rho_{c,1} - \rho_{c,2}\right) + \rho_{c,1}g_{c,t-1} + \rho_{c,2}g_{c,t-2} + \epsilon_{c,t} + \theta_{c,1}\epsilon_{c,t-1} + \theta_{c,2}\epsilon_{c,t-2} \\
\delta_t &\equiv d_t - c_t - \mu_{dc} = \rho_\delta\delta_{t-1} + \epsilon_{\delta,t}.
\end{aligned}$$

Hence, the dividend growth is given by

$$\begin{aligned} g_{d,t} &= \mu_c(1-\rho_{c1}-\rho_{c2})+\rho_{c,1}g_{c,t-1}+\rho_{c,2}g_{c,t-2} \\ &\quad +\epsilon_{c,t}+\theta_{c,1}\epsilon_{c,t-1}+\theta_{c,2}\epsilon_{c,t-2}+(\rho_\delta-1)\,\delta_{t-1}+\epsilon_{\delta,t}. \end{aligned}$$

Here, the same dynamics for the log surplus ratio $s_t$ as in equation (2.2) is assumed. Therefore, the state variables in this economy are $(\delta_t, g_t, g_{t-1}, s_t, \epsilon_{c,t}, \epsilon_{c,t-1})$. In this model, the riskfree rate is no longer a constant. Instead, it depends on the state variables. However, its variation is still very small. Notice that

$$\begin{aligned} E_t M_{t+1} &= \delta E_t \exp\left\{-\gamma\left[(\phi-1)(s_t-\bar{s})+\lambda(s_t)\,\epsilon_{c,t+1}+g_{c,t+1}\right]\right\} \\ &= \delta \exp\left\{0.5\gamma^2\left[1+\lambda(s_t)\right]^2\sigma_c^2-\gamma\left[\begin{array}{c}(\phi-1)(s_t-\bar{s})+\mu_c(1-\rho_{c,1}-\rho_{c,2}) \\ +\rho_{c,1}g_{c,t}+\rho_{c,2}g_{c,t-1}+\theta_{c,1}\epsilon_{c,t}+\theta_{c,2}\epsilon_{c,t-1}\end{array}\right]\right\}. \end{aligned}$$

Hence, the risk-free rate follows

$$\begin{aligned} r_t^f &= -\log(E_t M_{t+1}) \\ &= -\log(\delta)+\gamma\left[\begin{array}{c}(\phi-1)(s_t-\bar{s})+\mu_c(1-\rho_{c,1}-\rho_{c,2}) \\ +\rho_{c,1}g_{c,t}+\rho_{c,2}g_{c,t-1}+\theta_{c,1}\epsilon_{c,t}+\theta_{c,2}\epsilon_{c,t-1}\end{array}\right]-0.5\gamma^2\left[1+\lambda(s_t)\right]^2\sigma_c^2. \end{aligned}$$

To solve for the price dividend ratio, a log-linear approximation on the log P/D ratio is derived the same way as in the *i.i.d.* case. Then, this linear approximated function is used as the initial point to numerically solve for the exact price dividend ratio. This approach stabilizes the numerical solution. Table 6 lists the parameter values used in the simulation. The parameters for consumption dynamics are taken from the estimation results in section 2, table 4. Table 7 reports the summary statistics of the equity premium, risk-free rate, and price dividend ratio from the simulated data. As in the *i.i.d.* case, the model can match both the equity premium and risk-free rate.

Table 8 shows the correlation between consumption and asset returns at different horizons. The correlation is decreasing for the simulated model as the horizon increases. Figure 10 plots the coherency, cospectrum, and phase spectrum in this generalized model. It can be seen that the long-run correlation between consumption growth and asset returns are still negative as that in the last section. Furthermore, from the phase spectrum, consumption still leads asset returns, same as in the last section. Although there is a hump-shaped cospectrum as that in the data, the cospectrum in the model is very large at high frequencies compared with these quantities in the data. Hence, the main message in the *i.i.d.* case remains true even the consumption growth rate is assumed to be an $ARMA(2,2)$ process in the external habit formation model.

To understand why the results still hold in the case with predictable consumption growth, a simplified model with $AR(1)$ consumption growth is considered in the following. That is, let $\rho_{c,2}=0$, $\theta_{c,1}=0$, and $\theta_{c,2}=0$. The previous analysis shows that the cointegration constraint doesn't play a significant role. Therefore, to simplify the model further, assume $\rho_\delta = 1$. Under this set of simplified assumptions, it is shown in the appendix that

$$\begin{aligned} r_{t+1} \quad &\approx \quad \alpha_1 + \left(\rho a_3 + 1 + \rho a_1 \frac{1-\bar{S}}{\bar{S}}\right) \epsilon_{c,t+1} + \epsilon_{\delta,t+1} \\ &+\gamma \rho_{c,1} g_{c,t} + \frac{(\phi\rho - 1)\, a_1}{\bar{S}} \tilde{S}_t \end{aligned} \tag{4.1}$$

where $\alpha_1$ is some proper constant. Notice that $\frac{(\phi\rho-1)a_1}{\bar{S}}$ is usually around 1, and $\rho_{c,1}$ is around 0.5. It follows from the same argument below equation (2.9) in section 2 that the correlations between consumption growth and asset returns are decreasing as horizons increase, as long as the risk aversion coefficient $\gamma$ is not too big. If we allow a large risk aversion $\gamma$, then equation (4.1) implies that consumption predict return positively, which is also in contradiction with data since it is the return that predicts consumption growth. Accordingly, the external habit persistence model with a predictable consumption growth can't produce decreasing coherency, cospectrum, and phase spectrum.

# 5 Robustness Checks

At the end of section 2, a cross-spectral analysis shows that the coherency, cospectrum, and phase spectrum between quarterly consumption growth and quarterly asset returns are decreasing. The purpose of this section is to show that these features in the data are robust across different data samples and econometric methods. Furthermore, a band-pass filter analysis and Granger's causality test are applied to the simulated data from different models. The results corroborate the previous cross-spectral analysis.

## 5.1 Band-Pass Filter and Granger's Causality Test

In this section, band-pass filter analysis and Granger's causality test are performed on the real data and artificial data simulated from different models. The band-pass filter (see Baxter and King (1999)) is used to extract the low frequency and high frequency components of consumption and asset returns. The resulting correlations between the consumption growth rate and the market return at different frequencies are then calculated. The correlation is 0.114 for higher frequencies (with cycle among 2˜20 quarters) and 0.342 for lower frequencies (with cycle longer than 20 quarters). Table 9 lists the low-frequency and high-frequency correlations for different models and the data. I can be seen that the external habit formation models produce a higher correlations at high frequencies, while the long–run risk model, trend-cycle models and the real data generate a higher correlations at lower frequencies. This confirms the earlier from the coherency and cospectrum.

The phase spectrum analysis in section 2 shows that stock market returns lead consumption growth. Now, I conduct a formal Granger's causality test. To implement this test, I assume an autoregressive lag length of 2 and estimate the following equation by OLS

$$r_t = c_1 + \alpha_1 r_{t-1} + \alpha_2 r_{t-2} + \beta_1 g_{c,t-1} + \beta_2 g_{c,t-2} + u_{r,t},$$

where $r_t$ is the quarterly market excess return, and $g_{c,t}$ is the quarterly consumption growth rate. Then an $F$ test of the following null hypothesis is conducted

$$H_0 : \beta_1 = \beta_2 = 0.$$

Similarly, I can estimate the following OLS

$$g_{c,t} = c_2 + \gamma_1 g_{c,t-1} + \gamma_2 g_{c,t-2} + \eta_1 r_{t-1} + \eta_2 r_{t-2} + u_{c,t},$$

then conduct an $F$ test of the null hypothesis

$$H_0 : \eta_1 = \eta_2 = 0.$$

The $p$-value of Granger's causality test of consumption Granger-causing return is 0.4482, while the $p$-value of Granger's causality test of return Granger-causing consumption is $4.3770 \times 10^{-4}$. Hence, the statistical test indicates that stock market returns do Granger-cause consumption, while consumption does not Granger-cause stock market returns. Therefore, Granger's causality test confirms our phase spectrum result. For the annual data, the results are stronger.

The table 10 and 11 report the Granger causality test for the Fama-French 25 portfolios and the consumption growth rate. Table 10 gives the $p$-value for the test of consumption growth Granger-causes asset returns. All these $p$-values are large, so consumption growth does not Granger-cause asset returns. Table 11 gives the $p$-value for the test of asset returns Granger-cause consumption growth. All of these $p$-values are very small. Hence, the Fama-French 25 portfolio returns do Granger-cause the consumption growth. This confirms our results for the aggregate market data.

Table 12 presents the p-values from Granger's causality tests for different models and real data. The results show that for the external habit formation models, the consumption growth Granger causes asset returns. However, it is the asset returns that Granger cause consumption for the long-run risk model, trend-cycle model, and the real data. Hence, the band-pass filter analysis and the Granger's causality tests reconfirm the earlier results from cross-spectral analysis.

## 5.2 Parametric Estimation for Cross-Spectrum and Spectral Analysis for Annual Data

To reconfirm the results from the cross-spectral analysis for the real data in section 2, a more detailed cross-spectral analysis is performed for the real data in this section. Section 2 has shown that the coherency, cospectrum and the phase spectrum between consumption and asset returns are all decreasing for the quarterly data. When the whole sample is chopped into two sub-samples, the resulting graphs looks nearly identical as the graphs in figure 2.

A parametric method is also be used to estimate the cross-spectrum. First, I estimate a $VAR(2)$ for consumption growth and market excess returns. Then, by using the estimated parameter values, the cross-spectrum between consumption growth and asset returns can be obtained analytically as plotted in figure 11 (detailed calculations are given in the appendix). It can be seen that the decreasing pattern remains. The phase spectrum is increasing for very high frequencies. However, it is decreasing for horizons longer than 1 year. It is worth noting that the phase spectrum at very high frequencies are sensitive to different estimation methods and sub-samples. In particular, the confidence intervals at high frequencies are very wide. However, all of the other decreasing patterns are very robust to different estimation approaches and sub-samples. Since there might be serious measurement errors in the pre-war consumption data, I will mainly focus on the post-war quarterly consumption data. Furthermore, the quarterly data has more observations, so the power of the statistical inference is larger. However, as a robustness check, I also plot the cross-spectrum for annual data in figure 12. The observed patterns are the same with those in the quarterly data.

The higher correlation at long horizons could result from frictions such as delayed consumption. However, it is hard to believe that these frictions can affect the correlation at horizons longer than 1 year. Therefore, the higher correlation between consumption growth and asset return must originate from more fundamental economic reasons. This paper does not investigate the origin of these forces.

## 6 Cross-Sectional Analysis (Very Preliminary)

I have shown that the low-frequency features in the aggregate data can be used to evaluate asset pricing models. The same spectral analysis can be applied to individual portfolios. Since value premium is a long-standing puzzle in the literature, the same procedure is applied to evaluate different models which are proposed to resolve the value premium puzzle. Figure 13 plots the cross-spectra between consumption and growth portfolio and value portfolio. It can be seen that the slope of the coherency and cospectrum is steeper for value portfolio than growth portfolio. Actually, the correlation between the average returns for 10 book-to-market portfolios and the slopes for the coherency between 10 book-to-market portfolios and consumption is about 0.9. In the following, I use this property to evaluate different asset pricing models which can produce value premium, especially those models with slow-moving features. To show how the procedure works, two recent models are chosen to investigate. One is the duration model proposed by Lettau and Watcher (2006) which generates a value through the duration effect of cash flow. First, calculate the correlation between the average returns for 10 book-to-market portfolios and the slopes for the coherency between 10 book-to-market portfolios and consumption for the calibrated model in Lettau and Watcher (2006). This correlation is about $-0.95$, while this number is about 0.9 in the data (EXPLAIN WHY HERE). Further, figure 14 plots the cross-spectra for between consumption and value stocks, and the growth stocks from Bansal and Yaron' model. Here, the value stocks are those with a high value of $\phi$. The pattern is the same with the data as shown in figure 13.

Furthermore, cross-sectional implications of the tend-cycle model is tested. The objective is to see whether the trend-cycle decomposition can help to explain cross-sectional asset returns. Since the conditional expected return is related the cyclical component, I first use Kalman filter to extract that component from the consumption data, then use this cyclical component as a conditional variable (the same way as *cay* in Letteu and Ludvigson (2001)). The following cross-sectional results show that the cyclical component can help the conditional CCAPM and CAPM explain cross-sectional differences in asset returns. From table 13 and table 14, it can be seen that the conditional CCAPM and CAPM can explain the 10 size, book-to-market, and momentum portfolios with a R-squared of about 60%. The R-squared is improved significantly over the unconditional models. The Fama-French three-factor model can only explain about 30% of the cross-sectional difference in these 30 portfolios. However, the Fama-French 3 factor model can explain 77% of the cross-sectional variations in the Fama-French 25 portfolios. Our conditional CCAPM and CAPM can also explain about 60% of the variations for Fama-French 25 portfolios, which is slightly below that of the Fama-French three-factor model. When I use the Kalman filter to extract the cyclical component, I can fix the correlation between trend innovation and cycle innovation at 0. If I use this cyclical variable as the conditional variable, the R-squared in the cross-sectional regression becomes much lower. Therefore, the negative correlation between trend innovation and cycle innovation is crucial for our empirical cross-sectional analysis. Since the surplus ratio is also a state variable in the external habit-formation model, I also test the conditional consumption-based CAPM with the surplus ratio as the conditional variable. Figure 15 plots the fitted and the realized quarterly return under the unconditional CAPM and consumption CAPM. As expected, the R-squared is very small for both of the unconditional models. Figure 16 graphs the pricing errors of the two versions of the conditional consumption CAPM. The top panel uses cyclical component in consumption as the conditional variable, and the pricing error is relatively small for the 25 Fama-French portfolios. The bottom panel uses the approximate surplus ratio ($S_t \approx \sum_{k=1}^{20} \phi^k g_{t-k}$) as the conditional variable, the pricing error is still relatively large, although there is a significant improvement when compared to the unconditional model.

# 7 Conclusions

In this paper, I argue that the standard external habit formation model has a difficult time generating the same coherency, cospectrum and phase spectrum between consumption and the market returns as in the data. However, when the log consumption is decomposed to a stochastic trend and an $AR(2)$ cycle, and the expected return is a decreasing function of the cyclical component, the model can generate the same pattern as that found in the data. Instead of matching the first two moments of the aggregate data as most of the current literature does, I analyze the asset pricing model from a different perspective, by especially focusing on the low-frequency implications. I conclude that forward-looking behavior in the model is important for the model to be consistent with the observed data.

# 8 Appendix

**Parametric Estimation of the Cross-spectrum:**

I first estimate the following $VAR(2)$ for consumption growth and asset returns,

$$\begin{aligned} r_t &= c_1 + \alpha_1 r_{t-1} + \alpha_2 r_{t-2} + \beta_1 g_{t-1} + \beta_2 g_{t-2} + u_t \\ g_{c,t} &= c_2 + \gamma_1 g_{c,t-1} + \gamma_2 g_{c,t-2} + \eta_1 r_{t-1} + \eta_2 r_{t-2} + \epsilon_{c,t}, \end{aligned}$$

After the parameters are estimated, the cross-spectrum can be found in closed-form by the following argument. First, write down the equations for the orthogonal increment processes $Z_{g_c}$, $Z_r$, $Z_{\epsilon_c}$, and $Z_u$ in the spectral representations of $\{g_{c,t}\}$, $\{r_t\}$, $\{\epsilon_{c,t}\}$ , and $\{u_t\}$,

$$\begin{aligned} dZ_r &= \left(\alpha_1 e^{-i\lambda} + \alpha_2 e^{-2i\lambda}\right) dZ_r + \left(\beta_1 e^{-i\lambda} + \beta_2 e^{-2i\lambda}\right) dZ_g + dZ_u \\ dZ_{g_c} &= \left(\gamma_1 e^{-i\lambda} + \gamma_2 e^{-2i\lambda}\right) dZ_{g_c} + \left(\eta_1 e^{-i\lambda} + \eta_2 e^{-2i\lambda}\right) dZ_r + dZ_{\epsilon_c} \end{aligned}$$

Rearrange to obtain

$$\begin{aligned} dZ_{g_c} &= \frac{\bar{\eta}}{D_e} dZ_u + \frac{(1-\bar{\alpha})}{D_e} dZ_{\epsilon_c} \\ dZ_r &= \frac{1-\bar{\gamma}}{D_e} dZ_u + \frac{\bar{\beta}}{D_e} dZ_{\epsilon_c}, \end{aligned}$$

where

$$\bar{X} = X_1 e^{-i\lambda} + X_2 e^{-2i\lambda} \text{ for } X = \alpha, \beta, \gamma, \text{ and } \eta,$$

and

$$D_e = (1-\bar{\alpha})(1-\bar{\gamma}) - \bar{\beta}\bar{\eta}.$$

Hence, the cross-spectrum can be obtained as

$$\begin{aligned} 2\pi f_{11} &= \left|\frac{\bar{\eta}}{D_e}\right|^2 \sigma_u^2 + \left|\frac{(1-\bar{\alpha})}{D_e}\right|^2 \sigma_{\epsilon_c}^2 + 2real\left(\left[\frac{\bar{\eta}}{D_e}\right]\left[\frac{(1-\bar{\alpha})}{D_e}\right]'\right)\sigma_{\epsilon_c,u} \\ 2\pi f_{22} &= \left|\frac{1-\bar{\gamma}}{D_e}\right|^2 \sigma_u^2 + \left|\frac{\bar{\beta}}{D_e}\right|^2 \sigma_{\epsilon_c}^2 + 2real\left(\left[\frac{(1-\bar{\gamma})}{D_e}\right]\left[\frac{\bar{\beta}}{D_e}\right]'\right)\sigma_{\epsilon_c,u} \\ 2\pi f_{12} &= \frac{\bar{\eta}}{D_e}\left(\frac{1-\bar{\gamma}}{D_e}\right)'\sigma_u^2 + \frac{(1-\bar{\alpha})}{D_e}\left(\frac{\bar{\beta}}{D_e}\right)'\sigma_{\epsilon_c}^2 + \left[\frac{\bar{\eta}}{D_e}\left(\frac{\bar{\beta}}{D_e}\right)' + \frac{(1-\bar{\alpha})}{D_e}\left(\frac{1-\bar{\gamma}}{D_e}\right)'\right]\sigma_{\epsilon_c,u}. \end{aligned}$$

**Log-Linear Approximation to Price Dividend Ratio and Returns:**

To derive the log-linear approximation to the price dividend ratio and asset returns, let $m_t = \log(M_t)$ be the log IMRS. Plugging the log-linear approximation to returns $r_{t+1} \approx k_0 + g_{d,t+1} + \rho z_{t+1} - z_t$, and the linear approximation to log price dividend ratio $z_t \approx a_0 + a_1 s_t + a_2 \delta_t$ into the

Euler equation, I obtain

$$\begin{aligned} 1 &= E_t\left(\exp\left(m_{t+1}+r_{t+1}\right)\right) \\ &= \exp\left(\begin{array}{c} \left[\begin{array}{c} \gamma\left(\phi-1\right)\bar{s}+\log\left(\delta\right)+k_0+\mu_c\left(1-\rho_{c1}\right)\left(1-\gamma\right) \\ +a_0\rho-a_0+a_1\rho\left(1-\phi\right)\bar{s}+0.5\left[1+a_2\rho\right]^2\sigma_\delta^2 \end{array}\right] \\ +\left[-\gamma\left(\phi-1\right)-a_1+a_1\rho\phi\right]s_t \\ +\left[a_2\rho\rho_\delta-a_2+\left(\rho_\delta-1\right)\right]\delta_t \\ +0.5\left[1+a_1\rho\lambda\left(s_t\right)-\gamma\left[1+\lambda\left(s_t\right)\right]\right]^2\sigma_c^2 \\ +\left[1+a_1\rho\lambda\left(s_t\right)-\gamma\left[1+\lambda\left(s_t\right)\right]\right]\left[1+a_2\rho\right]\sigma_{c\delta} \end{array}\right) \end{aligned}$$

Replacing the sensitivity function $\lambda(s)$ with its linear approximation $\lambda(s)\approx -a_\lambda(s-s_{\max})$ in the above equation, then, setting the coefficients in front of the state variables to be zero, it follows that

$$a_2=\frac{\rho_\delta-1}{1-\rho\rho_\delta}, \tag{8.1}$$

$a_1$ can be determined as the unique positive root of the following quadratic equation if consumption growth is positively correlated with dividend growth[15],

$$\begin{aligned} 0 &= \left[2a_\lambda\rho^2-\frac{2\rho^2}{\bar{S}^2}\right]a_1^2+\left[\frac{\rho\phi-1}{0.5\sigma_c^2}-\frac{1}{0.5\sigma_c^2}a_\lambda\left[1+a_2\rho\right]\sigma_{c,\delta}\rho-2a_\lambda\rho\left(1+\gamma\right)+4\gamma\rho\frac{1}{\bar{S}^2}\right]a_1 \\ &\quad+\left[\frac{1}{0.5\sigma_c^2}a_\lambda\left[1+a_2\rho\right]\sigma_{c,\delta}\gamma+2a_\lambda\gamma-\frac{\gamma\left(\phi-1\right)}{0.5\sigma_c^2}-\frac{2\gamma^2}{\bar{S}^2}\right] \end{aligned} \tag{8.2}$$

and $a_0$ can be determined by the following equation,

$$a_0=\frac{1}{1-\rho}\left[\begin{array}{c} \gamma\left(\phi-1\right)\bar{s}+\log\left(\delta\right)+k_0+\mu_c\left(1-\gamma\right) \\ +a_1\rho\left(1-\phi\right)\bar{s}+0.5\left[1+a_2\rho\right]^2\sigma_\delta^2 \\ +0.5\sigma_c^2\left[\begin{array}{c} \left(a_1\rho-\gamma\right)^2\frac{1}{\bar{S}^2}\left(1+2\bar{s}\right)+2\left(1-a_1\rho\right)\left(a_1\rho-\gamma\right)\left(1+a_\lambda s_{\max}\right) \\ +\left(1-a_1\rho\right)^2+\left[1-\gamma+\left(a_1\rho-\gamma\right)a_\lambda s_{\max}\right]\cdot\left[1+a_2\rho\right]\frac{\sigma_{c,\delta}}{0.5\sigma_c^2} \end{array}\right] \end{array}\right] \tag{8.3}$$

Plug this linear approximation on price dividend ratio back into the Campbell-Shiller log-linear

[15] Notice that $\lambda'(s)\le\frac{1}{\bar{S}^2}$, hence, it is natural to choose $0<a_\lambda<\frac{1}{\bar{S}^2}$ to approximate the sensitivity function. Hence, the coefficient $2a_\lambda\rho^2-\frac{2\rho^2}{\bar{S}^2}$ in the quadratic equation (8.2) is negative. Further, the constant term in the quadratic equation satisfies

$$\begin{aligned} &\frac{1}{0.5\sigma_c^2}a_\lambda\left[1+a_2\rho\right]\sigma_{c,\delta}\gamma+2a_\lambda\gamma-\frac{\gamma\left(\phi-1\right)}{0.5\sigma_c^2}-\frac{2\gamma^2}{\bar{S}^2} \\ =&\left[\left(\frac{1-\rho}{1-\rho\rho_\delta}\right)\frac{\sigma_{c,\delta}}{\sigma_c^2}+1\right]2a_\lambda\gamma \end{aligned}$$

If we assume $\frac{\sigma_{c,\delta}}{\sigma_c^2}\ge -1$, then the constant term in the quadratic equation is positive. Hence, there is a unique positive root for equation (8.2). Notice that $\epsilon_{c,t}$ is the innovation in $c_t$, and $\epsilon_{\delta,t}$ is the innovation in $d_t$ minus the innovation in $c_t$. As long as the innovations in $c_t$ and the innovations $d_t$ are positively correlated (i.e. the consumption growth and dividend growth are positively correlated), $\frac{\sigma_{c,\delta}}{\sigma_c^2}>-1$ holds.

approximation on returns to obtain

$$\begin{aligned}
r_{t+1} &\approx \kappa_0 + g_{d,t+1} + \rho z_{t+1} - z_t \\
&= [\kappa_0 + \mu_c - a_0 + a_0\rho + a_1\rho(1-\phi)\bar{s}] + [a_1\rho\phi - a_1] s_t + [(\rho_\delta - 1) - a_2 + a_2\rho\rho_\delta]\delta_t \\
&\quad + [1 + a_1\rho\lambda(s_t)]\epsilon_{c,t+1} + [1 + a_2\rho]\epsilon_{\delta,t+1} \\
&\approx [\kappa_0 + \mu_c - a_0 + a_0\rho + a_1\rho(1-\phi)\bar{s}] + [a_1\rho\phi - a_1]\left(\frac{S_t}{\bar{S}} + \log\left(\bar{S}\right) - 1\right) \\
&\quad + [(\rho_\delta - 1) - a_2 + a_2\rho\rho_\delta]\delta_t + [1 + a_1\rho\lambda(s_t)]\epsilon_{c,t+1} + [1 + a_2\rho]\epsilon_{\delta,t+1} \\
&\approx \left[\kappa_0 + \mu_c - a_0 + a_0\rho + a_1\rho(1-\phi)\bar{s} + (a_1\rho\phi - a_1)\left(\log\left(\bar{S}\right) - 1\right)\right] + \frac{a_1\rho\phi - a_1}{\bar{S}} S_t \\
&\quad + [1 + a_1\rho\lambda(s_t)]\epsilon_{c,t+1} + [1 + a_2\rho]\epsilon_{\delta,t+1}
\end{aligned}$$

Letting

$$\alpha = \kappa_0 + \mu_c - a_0 + a_0\rho + a_1\rho(1-\phi)\bar{s} + (a_1\rho\phi - a_1)\left(\log\left(\bar{S}\right) - 1\right) \tag{8.4}$$

$$\beta_S \approx \frac{a_1(\rho\phi - 1)}{\bar{S}}, \tag{8.5}$$

and further approximating $\lambda(s_t)$ with $\lambda(\bar{s}) = \frac{1-\bar{S}}{\bar{S}}$, it follows that

$$r_{t+1} = \alpha + \beta_S S_t + \left[1 + a_1\rho\frac{1-\bar{S}}{\bar{S}}\right]\epsilon_{c,t+1} + [1 + a_2\rho]\epsilon_{\delta,t+1}.$$

**Proof of Proposition 1 & 2:**

First, surplus ratio $S_t$ can be approximated by a smoothed average of past consumption innovations as equation (2.6). Notice that the habit level can be approximated by

$$X_t \approx \sum_{k=1}^{\infty} \rho_k C_{t-k} = \sum_{k=1}^{\infty} \frac{1-\phi}{\phi}\phi^k C_{t-k}$$

where the weight $\rho_k = \frac{1-\phi}{\phi}\phi^k$. Plug the above equation back into the definition of surplus ratio $S_t$ to obtain

$$\begin{aligned}
S_t &= 1 - \sum_{k=1}^{\infty} \rho_k \frac{C_{t-k}}{C_t} \\
&= 1 - \sum_{k=1}^{\infty} \rho_k \exp\left(-\sum_{j=t-k+1}^{t} g_j\right) \\
&\approx 1 - \sum_{k=1}^{\infty} \rho_k \left(1 - \sum_{j=t-k+1}^{t} g_j\right) \\
&= \sum_{j=1}^{\infty}\left(\sum_{k=j}^{\infty} \rho_k\right) g_{t+1-j} \\
&= \sum_{j=1}^{\infty} \phi^{j-1} g_{t+1-j}.
\end{aligned}$$

Hence, equation (2.6) follows. Now, substituting equation (2.6) back into equation (2.4), then I replace each term in equation (2.4) and equation (2.1) by its spectral representation. Noting that the resulting equations are valid for all $t$, I obtain the following equations for the orthogonal increment processes $Z_{g_c}$, $Z_r, Z_{\epsilon_c}$ and $Z_{\epsilon_\delta}$ in the spectral representations of $\{g_t\}$, $\{r_t\}$, $\{\epsilon_{c,t}\}$ and $\{\epsilon_{\delta,t}\}$:

$$\begin{aligned} dZ_{g_c}(\lambda) &= dZ_{\epsilon_c}(\lambda) \\ dZ_r(\lambda) &= \beta_S \sum_{j=1}^{\infty} \phi^{j-1} e^{-ij\lambda} \cdot dZ_{g_c}(\lambda) + \left[1 + a_1\rho \frac{1-\bar{S}}{\bar{S}}\right] dZ_{\epsilon_c}(\lambda) + [1 + a_2\rho]\, dZ_{\epsilon_\delta}(\lambda) \end{aligned}$$

Notice that $\sum_{j=1}^{\infty} \phi^{j-1} e^{-ij\lambda} = \frac{e^{-i\cdot\lambda}}{1-\phi e^{-i\cdot\lambda}}$ and solve for $dZ_{g_c}(\lambda)$ and $dZ_r(\lambda)$ to obtain

$$\begin{aligned} dZ_{g_c}(\lambda) &= dZ_{\epsilon_c}(\lambda) \\ dZ_r(\lambda) &= \left[\beta_S \frac{e^{-i\cdot\lambda}}{1-\phi e^{-i\cdot\lambda}} + 1 + a_1\rho \frac{1-\bar{S}}{\bar{S}}\right] dZ_{\epsilon_c}(\lambda) + [1 + a_2\rho]\, dZ_{\epsilon_\delta}(\lambda). \end{aligned}$$

It follows that the multivariate spectrum is given by

$$\begin{aligned} 2\pi f_{11}(\lambda) &= \sigma_c^2 \\ 2\pi f_{22}(\lambda) &= \left|\beta_S \frac{e^{-i\cdot\lambda}}{1-\phi e^{-i\cdot\lambda}} + 1 + a_1\rho \frac{1-\bar{S}}{\bar{S}}\right|^2 \sigma_c^2 + [1 + a_2\rho]^2 \sigma_\delta^2 \\ &\quad + 2Re\left(\beta \frac{e^{-i\cdot\lambda}}{1-\phi e^{-i\cdot\lambda}} + 1 + a_1\rho \frac{1-\bar{S}}{\bar{S}}\right) [1 + a_2\rho]\, \sigma_{c\delta} \\ 2\pi f_{12}(\lambda) &= \left(\beta_S \frac{e^{-i\cdot\lambda}}{1-\phi e^{-i\cdot\lambda}} + 1 + a_1\rho \frac{1-\bar{S}}{\bar{S}}\right)' \sigma_c^2 + [1 + a_2\rho]\, \sigma_{c\delta} \\ &= \left(\beta_S \frac{e^{i\cdot\lambda} - \phi}{1 + \phi^2 - 2\phi\cos(\lambda)} + 1 + a_1\rho \frac{1-\bar{S}}{\bar{S}}\right) \sigma_c^2 + [1 + a_2\rho]\, \sigma_{c\delta} \end{aligned}$$

For the cospectrum $C_{sp}(\lambda)$, which measure the portion of the covariance between consumption growth and the asset returns attributable to cycles with frequency $\lambda$, it is the real part of the cross-spectrum $f_{12}(\lambda)$, then

$$C_{sp}(\lambda) = \left(\beta_S \frac{\cos(\lambda) - \phi}{1 + \phi^2 - 2\phi\cos(\lambda)} + 1 + a_1\rho \frac{1-\bar{S}}{\bar{S}}\right) \sigma_c^2 + [1 + a_2\rho]\, \sigma_{c\delta}$$

Therefore, the derivative of the cospectrum is

$$\begin{aligned} C'_{sp}(\lambda) &= \beta_S \frac{-\sin(\lambda)\left(1 + \phi^2 - 2\phi\cos(\lambda)\right) - 2\phi\sin(\lambda)(\cos(\lambda) - \phi)}{(1 + \phi^2 - 2\phi\cos(\lambda))^2} \\ &= \frac{-\beta_S \sin(\lambda)}{(1 + \phi^2 - 2\phi\cos(\lambda))^2} \left[1 - \phi^2\right] \geq 0 \end{aligned}$$

Hence, the portion of covariance contributed by components at frequency $\lambda$ is increasing as I increasing the frequency $\lambda$. By definition, the coherency and the phase are

$$\begin{aligned} h(\lambda) &= \frac{|f_{12}|}{\sqrt{f_{11} f_{22}}} \\ \tan(\phi(\lambda)) &= \frac{\beta \frac{-\sin(\lambda)}{1+\phi^2-2\phi\cos(\lambda)} \sigma_\epsilon^2}{\left(\beta \frac{\cos(\lambda)-\phi}{1+\phi^2-2\phi\cos(\lambda)} + 1 + a_1 \rho \frac{1-\bar{S}}{\bar{S}}\right) \sigma_c^2 + [1 + a_2 \rho] \sigma_{cd}}. \end{aligned}$$

At the frequency $\lambda = 0$, the cross-spectrum is

$$\begin{aligned} f_{12}(0) &= \left(\beta_S \frac{1-\phi}{1+\phi^2-2\phi} + 1 + a_1 \rho \frac{1-\bar{S}}{\bar{S}}\right) \sigma_c^2 + [1 + a_2 \rho] \sigma_{c\delta} \\ &= \left(-\frac{a_1 (1-\rho)}{\bar{S}(1-\phi)} + 1 - a_1 \rho\right) \sigma_c^2 + [1 + a_2 \rho] \sigma_{c\delta} \end{aligned}$$

Therefore, if and only if $\left(-\frac{a_1(1-\rho)}{\bar{S}(1-\phi)} + 1 - a_1\rho\right) \sigma_c^2 + [1 + a_2\rho] \sigma_{c\delta} < 0$, the low frequency correlation between consumption growth and asset returns is negative. To find the conditions for a negative correlation at long horizons, first write down the long horizon returns and the long horizon consumption growth rate

$$\begin{aligned} \sum_{j=1}^{K} r_{t+j} &\approx \sum_{j=1}^{K} \alpha + \beta_S \tilde{S}_{t+j-1} + \left(1 + a_1 \rho \frac{1-\bar{S}}{\bar{S}}\right) \epsilon_{c,t+j} + (1 + a_2 \rho) \epsilon_{\delta,t+j} \\ &\approx K\alpha + \beta_S \sum_{j=1}^{K} \sum_{k=1}^{\infty} \phi^{k-1} \epsilon_{c,t+j-k} + \left(1 + a_1 \rho \frac{1-\bar{S}}{\bar{S}}\right) \sum_{j=1}^{K} \epsilon_{c,t+j} + (1 + a_2 \rho) \sum_{j=1}^{K} \epsilon_{\delta,t+j} \\ \sum_{j=1}^{K} g_{c,t+j} &= K\mu_c + \sum_{j=1}^{K} \epsilon_{c,t+j}. \end{aligned}$$

Then, the long horizon covariances between return and consumption are

$$\begin{aligned} & cov\left(\sum_{j=1}^{K} r_{t+j}, \sum_{j=1}^{K} g_{c,t+j}\right) \\ = \; & \beta_S \sum_{i=1}^{K-1} \frac{1-\phi^{K-i}}{1-\phi} \sigma_c^2 + K\left(\left[1 + a_1 \rho \frac{1-\bar{S}}{\bar{S}}\right] \sigma_c^2 + [1 + a_2 \rho] \sigma_{c\delta}\right) \\ = \; & -\beta_S \sigma_c^2 \frac{1}{1-\phi} - \beta_S \sigma_c^2 \frac{\phi}{1-\phi} \frac{1-\phi^{K-1}}{1-\phi} + K\left(\sigma_c^2 \frac{\beta_S}{1-\phi} + \left[1 + a_1 \rho \frac{1-\bar{S}}{\bar{S}}\right] \sigma_c^2 + [1 + a_2 \rho] \sigma_{c\delta}\right). \end{aligned}$$

Notice that the long run variances are

$$
\begin{aligned}
var\left(\sum_{j=1}^{K} r_{t+j}\right) &= var\left(\begin{array}{c} \sum_{i=1}^{K-1}\left[\beta_S \frac{1-\phi^{K-i}}{1-\phi}+1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right]\epsilon_{c,t+i}+\beta_S\sum_{i=0}^{\infty}\left[\phi^i\frac{1-\phi^K}{1-\phi}\right]\epsilon_{c,t-i} \\ +\left[1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right]\epsilon_{c,t+K}+\left[1+a_2\rho\right]\sum_{j=1}^{K}\epsilon_{\delta,t+j} \end{array}\right) \\
&= \left[\begin{array}{c} (K-1)\left(\frac{\beta_S}{1-\phi}+1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)^2+\left(\frac{\beta_S}{1-\phi}\right)^2\phi^2\frac{1-\phi^{2(K-1)}}{1-\phi^2} \\ -2\left(\frac{\beta_S}{1-\phi}+1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)\frac{\beta_S}{1-\phi}\phi\frac{1-\phi^{K-1}}{1-\phi} \end{array}\right]\sigma_c^2 \\
&\quad+\left[\frac{1-\phi^K}{1-\phi}\right]^2\sigma_c^2\frac{\beta_S^2}{1-\phi^2}+\left(1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)^2\sigma_c^2+K\left(1+a_2\rho\right)^2\sigma_\delta^2 \\
&\quad+\left[(K-1)\frac{\beta}{1-\phi}+K\left(1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)-\frac{\beta\phi}{1-\phi}\frac{1-\phi^{K-1}}{1-\phi}\right]\left(1+a_2\rho\right)\sigma_{c\delta}
\end{aligned}
$$

Hence, the correlation at horizon $K$ is just $\sqrt{\frac{cov\left(\sum_{j=1}^{K} r_{t+j},\sum_{j=1}^{K} g_{c,t+j}\right)}{var\left(\sum_{j=1}^{K} r_{t+j}\right)var\left(\sum_{j=1}^{K} g_{c,t+j}\right)}}$. When the horizon $K$ is sufficiently large, the following quantity determines the sign of the correlation at the very long horizon

$$
\begin{aligned}
&\sigma_c^2\frac{\beta_S}{1-\phi}+\left[1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right]\sigma_c^2+\left[1+a_2\rho\right]\sigma_{c\delta} \\
=\;&\frac{a_1\left(\rho\phi-1\right)}{\bar{S}\left(1-\phi\right)}\sigma_c^2+\left[1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right]\sigma_c^2+\left[1+a_2\rho\right]\sigma_{c\delta} \\
=\;&\left[1-a_1\rho-\frac{a_1\left(1-\rho\right)}{\bar{S}\left(1-\phi\right)}\right]\sigma_c^2+\left(1+a_2\rho\right)\sigma_{c\delta}.
\end{aligned}
$$

It can be seen that the sign of long-run correlation is the same with the sign of the correlation of at frequency $\lambda=0$.

By differentiating equation (2.11), the sign of the slope of the phase spectrum can be examined. Indeed,

$$
\begin{aligned}
\phi'\left(\lambda\right) \;\propto\; &\left\{\beta_S\left(\cos\left(\lambda\right)-\phi\right)\sigma_c^2+\left[\left(1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)\sigma_c^2+\left[1+a_2\rho\right]\sigma_{c\delta}\right]\left(1+\phi^2-2\phi\cos\left(\lambda\right)\right)\right\} \\
&\cdot\beta_S\cos\left(\lambda\right)-\beta_S\sin\left(\lambda\right)\left\{-\beta_S\sin\left(\lambda\right)\sigma_c^2+2\phi\left[\left(1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)\sigma_c^2+\left[1+a_2\rho\right]\sigma_{c\delta}\right]\sin\left(\lambda\right)\right\}
\end{aligned}
$$

Rearrange and simplify to obtain

$$
\begin{aligned}
\phi'\left(\lambda\right) \;\propto\;& -\frac{a_1\left(\rho\phi-1\right)}{\bar{S}}+2\phi\left(1+a_1\rho\frac{1-\bar{S}}{\bar{S}}\right)+2\phi\left[1+a_2\rho\right]\frac{\sigma_{c\delta}}{\sigma_c^2} \\
&-\left\{1+a_1\rho\frac{1-\bar{S}}{\bar{S}}+\left(1+a_2\rho\right)\frac{\sigma_{c\delta}}{\sigma_c^2}+\phi^2+\frac{-a_1\rho\bar{S}\phi^2}{\bar{S}}+\frac{a_1\phi}{\bar{S}}+\left(1+a_2\rho\right)\phi^2\frac{\sigma_{c\delta}}{\sigma_c^2}\right\}\cos\left(\lambda\right) \\
\geq\;& a_1\frac{\left(1-\rho\right)\left(1-\phi\right)+\rho\bar{S}\left(1-\phi\right)^2}{\bar{S}}-\left(1-\phi\right)^2-\left(1+a_2\rho\right)\frac{\sigma_{c\delta}}{\sigma_c^2}\left(1-\phi\right)^2 \\
=\;& -\left[1-a_1\frac{1-\rho+\rho\bar{S}\left(1-\phi\right)}{\bar{S}\left(1-\phi\right)}+\left(1+a_2\rho\right)\frac{\sigma_{c\delta}}{\sigma_c^2}\right]\left(1-\phi\right)^2,
\end{aligned}
$$

where the inequality requires the following assumption

$$
\left(1-a_1\rho\right)\left(1+\phi^2\right)+\frac{a_1\rho}{\bar{S}}+\frac{a_1\phi}{\bar{S}}+\left(1+a_2\rho\right)\left(1+\phi^2\right)\frac{\sigma_{c\delta}}{\sigma_c^2}>0,
$$

which is true if the correlation between the innovations of return and consumption is positive.

**Log-linear Approximation for the Habit Model with $ARMA\,(2,2)$ Consumption Growth:**

First, I assume a linear approximation of the P/D ratio,

$$z_t = a_0 + a_1 s_t + a_2 \delta_t + a_3 g_{c,t} + a_4 g_{c,t-1} + a_5 \epsilon_{c,t} + a_6 \epsilon_{c,t-1}$$

Plugging back into the Euler equation, we have

$$1 = \exp\left(\begin{array}{c} \left[\begin{array}{c} \gamma\left(\phi-1\right)\bar{s}+\log\left(\delta\right)+k_0+\mu_c\left(1-\rho_{c1}-\rho_{c2}\right)\left(1-\gamma\right) \\ +a_0\rho-a_0+a_3\rho\mu_c\left(1-\rho_{c,1}-\rho_{c,2}\right)+a_1\rho\left(1-\phi\right)\bar{s} \end{array}\right] \\ \left[-\gamma\left(\phi-1\right)-a_1+a_1\rho\phi\right]\cdot s_t \\ +\left[a_2\rho\rho_\delta+\left(\rho_\delta-1\right)-a_2\right]\cdot\delta_t \\ +\left[\rho_{c,1}-a_3+\lambda_{\delta c}-\gamma\rho_{c,1}+a_3\rho\rho_{c,1}+a_4\rho\right]\cdot g_{c,t} \\ +\left[-\gamma\rho_{c,2}+a_3\rho\rho_{c,2}-a_4+\rho_{c,2}\right]\cdot g_{c,t-1} \\ +\left[\theta_{c,1}-\gamma\theta_{c,1}+a_6\rho+a_3\rho\theta_{c,1}-a_5\right]\cdot\epsilon_{c,t} \\ +\left[\theta_{c,2}-\gamma\theta_{c,2}+a_3\rho\theta_{c,2}-a_6\right]\cdot\epsilon_{c,t-1} \\ +0.5\left[1-\gamma\left(1+\lambda\left(s_t\right)\right)+a_1\rho\lambda\left(s_t\right)+a_5\rho+a_3\rho\right]^2\sigma_c^2+0.5\left[1+a_2\rho\right]^2\sigma_\delta^2 \\ +\left[1-\gamma\left(1+\lambda\left(s_t\right)\right)+a_1\rho\lambda\left(s_t\right)+a_5\rho+a_3\rho\right]\left[1+a_2\rho\right]\sigma_{c\delta} \end{array}\right)$$

Again, the sensitivity function $\lambda\left(s\right)$ can be replaced with its linear approximation. Then, by matching the coefficients, I obtain

$$a_2 = \frac{\rho_\delta - 1}{1-\rho\rho_\delta}$$

and,

$$\begin{aligned} a_3 &= -\frac{\left(\rho_{c,1}-\gamma\rho_{c,1}\right)+\left(\rho_{c,2}-\gamma\rho_{c,2}\right)\rho}{\left(\rho\rho_{c,1}-1\right)+\rho^2\rho_{c,2}} \\ a_4 &= -\frac{\left(\rho_{c,1}-\gamma\rho_{c,1}\right)\rho\rho_{c,2}-\left(\rho_{c,2}-\gamma\rho_{c,2}\right)\left(\rho\rho_{c,1}-1\right)}{\left(\rho\rho_{c,1}-1\right)+\rho^2\rho_{c,2}} \end{aligned}$$

and

$$\begin{aligned} a_6 &= \theta_{c,2}-\gamma\theta_{c,2}+a_3\rho\theta_{c,2} \\ a_5 &= \theta_{c,1}-\gamma\theta_{c,1}+a_6\rho+a_3\rho\theta_{c,1} \end{aligned}$$

Furthermore, $a_1$ can be found as the positive root of the following quadratic equation,

$$\begin{aligned} 0 &= \left[a_\lambda\sigma_c^2\rho^2-\sigma_c^2\frac{\rho^2}{\bar{S}^2}\right]a_1^2 \\ &+\left[\sigma_c^2\frac{2\rho\gamma}{\bar{S}^2}-a_\lambda\sigma_c^2\rho\left(1+a_3\rho+a_5\rho+\gamma\right)+\left(\rho\phi-1\right)-a_\lambda\left[1+a_2\rho\right]\sigma_{c\delta}\rho\right]a_1 \\ &+a_\lambda\sigma_c^2\left(1+a_3\rho+a_5\rho\right)\gamma-\sigma_c^2\frac{\gamma^2}{\bar{S}^2}-\gamma\left(\phi-1\right)+a_\lambda\left[1+a_2\rho\right]\gamma\sigma_{c\delta} \end{aligned}$$

At last, $a_0$ can be found as follows,

$$a_0 = \frac{1}{(1-\rho)} \left[ \begin{array}{c} \gamma(\phi-1)\bar{s} + \log(\delta) + k_0 + \mu_c(1-\rho_{c1}-\rho_{c2})(1-\gamma) \\ +a_3\rho\mu_c(1-\rho_{c,1}-\rho_{c,2}) + a_1\rho(1-\phi)\bar{s} + 0.5[1+a_2\rho]^2\sigma_\delta^2 \\ +0.5\sigma_c^2 \left[ \begin{array}{c} (a_1\rho-\gamma)^2 \frac{1}{\bar{S}^2}(1+2\bar{s}) + (1-a_1\rho+a_3\rho+a_5\rho)^2 \\ +2(a_1\rho-\gamma)(1+a_\lambda s_{\max})(1-a_1\rho+a_3\rho+a_5\rho) \\ +[1-\gamma+a_3\rho+a_5\rho+a_\lambda(a_1\rho-\gamma)s_{\max}][1+a_2\rho]\frac{\sigma_{c\delta}}{0.5\sigma_c^2} \end{array} \right] \end{array} \right]$$

Hence, a linear approximation of the log P/D ratio is obtained. Then the approximated return is

$$\begin{aligned} r_{t+1} \approx\ & \alpha + \beta_S S_t + \beta_\delta \delta_t + \beta_{g,1} g_{c,t} + \beta_{g,2} g_{c,t-1} + \beta_{\epsilon,1}\epsilon_{c,t} + \beta_{\epsilon,2}\epsilon_{c,t-1} \\ & + (1 + a_5\rho + a_3\rho + a_1\rho\lambda(s_t))\epsilon_{c,t+1} + (a_2\rho+1)\epsilon_{\delta,t+1} \end{aligned}$$

where

$$\begin{aligned} \alpha &= \kappa_0 + \mu_c(1-\rho_{c1}-\rho_{c2}) - a_0 + \rho a_0 + a_1\rho(1-\phi)\bar{s} + a_3\rho\mu_c(1-\rho_{c,1}-\rho_{c,2}) \\ \beta_S &\approx \frac{a_1(\rho\phi-1)}{\bar{S}} \\ \beta_\delta &= \rho_\delta - 1 + a_2\rho\rho_\delta - a_2 \equiv 0 \\ \beta_{g,1} &= \rho_{c,1} + a_3\rho\rho_{c,1} + a_4\rho - a_3 \\ \beta_{g,2} &= \rho_{c,2} + a_3\rho\rho_{c,2} - a_4 \\ \beta_{\epsilon,1} &= \theta_{c,1} + a_3\rho\theta_{c,1} + a_6\rho - a_5 \\ \beta_{\epsilon,2} &= \theta_{c,2} + a_3\rho\theta_{c,2} - a_6 \end{aligned}$$

**The Reduced Form Forward-Looking Risk Model with Cointegration:**

In the following, I will derive equation (3.7) in section (3). First, notice that

$$\begin{aligned} \left(1-\rho_{x,1}L-\rho_{x,2}L^2\right)^{-1} &= (1-\hat{\rho}_1 L)^{-1}(1-\hat{\rho}_2 L)^{-1} \\ &= \sum_{k=0}^{\infty}(\hat{\rho}_1 L)^k \sum_{k=0}^{\infty}(\hat{\rho}_2 L)^k = \sum_{k=0}^{\infty}\tilde{\rho}_k L^k \end{aligned}$$

where

$$\hat{\rho}_i = \frac{-2\rho_{x,2}}{\rho_{x,1} \pm \sqrt{\rho_{x,1}^2 + 4\rho_{x,2}}}, \text{ and } \tilde{\rho}_k = \sum_{j=0}^{k} \hat{\rho}_1^j \hat{\rho}_2^{k-j}$$

then, $x_t = \sum_{k=0}^{\infty}\tilde{\rho}_k \epsilon_{x,t-k}$. I can rewrite the dynamics of consumption growth and dividend growth as follows

$$\begin{aligned} g_{c,t} &= \sum_{k=0}^{\infty}\tilde{\rho}_k\epsilon_{x,t-k} - \sum_{k=0}^{\infty}\tilde{\rho}_k\epsilon_{x,t-1-k} + \xi_t \equiv \sum_{k=0}^{\infty}\tilde{\rho}_k^*\epsilon_{x,t-k} + \xi_t \\ \Delta d_t &= \sum_{k=0}^{\infty}\tilde{\rho}_k^*\epsilon_{x,t-k} + \xi_t + \Delta\delta_t \end{aligned}$$

where

$$\tilde{\rho}_k^* = \tilde{\rho}_k + \tilde{\rho}_{k-1}, \text{ and } \tilde{\rho}_{-1} \equiv 0.$$

Substituting the above equations into the log-linearized equation in asset returns $r_t$, it follows

$$\begin{aligned}
r_t - E_{t-1} r_t &= (E_t - E_{t-1}) \left[ \sum_{j=0}^{\infty} \rho^j \Delta d_{t+j} - \sum_{j=1}^{\infty} \rho^j r_{t+j} \right] \\
&= -\left\{ \sum_{j=0}^{\infty} \rho^j E_{t-1} [\Delta \delta_{t+j}] - \sum_{j=0}^{\infty} \rho^j E_t [\Delta \delta_{t+j}] \right\} + \xi_t \\
&\quad + \left\{ \sum_{j=0}^{\infty} \rho^j E_t \left[ \sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] - \sum_{j=0}^{\infty} \rho^j E_{t-1} \left[ \sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] \right\} \\
&\quad + \left\{ \sum_{j=1}^{\infty} \rho^j E_{t-1} \left( \beta \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right) - \sum_{j=1}^{\infty} \rho^j E_t \left( \beta \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right) \right\}.
\end{aligned}$$

Each of the three terms in the curly bracket will be simplified in order. For the first term,

$$\begin{aligned}
&\sum_{j=0}^{\infty} \rho^j E_{t-1} [\Delta \delta_{t+j}] - \sum_{j=0}^{\infty} \rho^j E_t [\Delta \delta_{t+j}] \\
=\; &\sum_{j=0}^{\infty} \rho^j E_{t-1} \left[ \sum_{k=0}^{\infty} \psi_k \epsilon_{\delta,t+j-k} - \sum_{k=0}^{\infty} \psi_k \epsilon_{\delta,t+j-1-k} \right] - \sum_{j=0}^{\infty} \rho^j E_t \left[ \sum_{k=0}^{\infty} \psi_k \epsilon_{\delta,t+j-k} - \sum_{k=0}^{\infty} \psi_k \epsilon_{\delta,t+j-1-k} \right] \\
=\; &\sum_{j=0}^{\infty} \rho^j \left[ \sum_{k=j+1}^{\infty} \psi_k \epsilon_{\delta,t+j-k} - \sum_{k=j}^{\infty} \psi_k \epsilon_{\delta,t+j-1-k} \right] - \sum_{j=0}^{\infty} \rho^j \left[ \sum_{k=j}^{\infty} \psi_k \epsilon_{\delta,t+j-k} - \sum_{k=\max(0,j-1)}^{\infty} \psi_k \epsilon_{\delta,t+j-1-k} \right] \\
=\; &\epsilon_{\delta,t} (\rho - 1) \sum_{j=0}^{\infty} \rho^j \psi_j
\end{aligned}$$

where $\psi_{-1}$ is defined to be 0. For the second term,

$$\begin{aligned}
&\sum_{j=0}^{\infty} \rho^j E_t \left[ \sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] - \sum_{j=0}^{\infty} \rho^j E_{t-1} \left[ \sum_{k=0}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] \\
=\; &\sum_{j=0}^{\infty} \rho^j E_t \left[ \sum_{k=j}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] - \sum_{j=0}^{\infty} \rho^j E_{t-1} \left[ \sum_{k=j+1}^{\infty} \tilde{\rho}_k^* \epsilon_{x,t+j-k} \right] = \epsilon_{x,t} \sum_{j=0}^{\infty} \rho^j \tilde{\rho}_j^*
\end{aligned}$$

and for the last term,

$$\begin{aligned}
&\sum_{j=1}^{\infty} \rho^j E_{t-1} \left( \beta \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right) - \sum_{j=1}^{\infty} \rho^j E_t \left( \beta \sum_{k=0}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right) \\
=\; &\beta \sum_{j=1}^{\infty} \rho^j \left[ \sum_{k=j}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} - \sum_{k=j-1}^{\infty} \tilde{\rho}_k \epsilon_{x,t+j-1-k} \right] = -\beta \epsilon_{x,t} \sum_{j=1}^{\infty} \rho^j \tilde{\rho}_{j-1}
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
r_t &= a_0 + \epsilon_{\delta,t}\bar{\psi} + \epsilon_{x,t}\sum_{j=0}^{\infty}\rho^j\tilde{\rho}_j^* - \beta\epsilon_{x,t}\sum_{j=1}^{\infty}\rho^j\tilde{\rho}_{j-1} + \xi_t + \beta x_{t-1} \\
&= a_0 + \epsilon_{\delta,t}\bar{\psi} + \epsilon_{x,t}\cdot(\rho^* - \beta\rho\bar{\rho}) + \xi_t + \beta x_{t-1}
\end{aligned}$$

where

$$\bar{\psi} = (1-\rho)\sum_{j=0}^{\infty}\rho^j\psi_j; \qquad \rho^* = \sum_{j=0}^{\infty}\rho^j\tilde{\rho}_j^*, \qquad \bar{\rho} = \sum_{j=0}^{\infty}\rho^j\tilde{\rho}_j \tag{8.6}$$

Based on the derived dynamics on consumption and asset returns, the equations for the orthogonal increment processes in the spectral representations can be derived as before, and then the expressions for the coherency, cospectrum and the phase can be analytically obtained.

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Table 1: Parameter choices for the external habit formation model with *i.i.d.* consumption: All the parameter values are annualized.

| Statistics | Variable | Value |
|---|---|---|
| Mean consumption growth (%) | $g_c$ | 1.89 |
| Standard deviation of consumption growth (%) | $\sigma_c$ | 1.22 |
| Log risk-free rate (%) | $r^f$ | 0.094 |
| Persistence coefficient in habit | $\phi$ | 0.87 |
| Persistence coefficient in $\delta_t$ | $\rho_\delta$ | 0.89 |
| Standard deviation of the innovation in $\delta_t$ | $\sigma_\delta$ | 0.112 |
| Risk aversion coefficient | $\gamma$ | 2 |
| Correlation between innovation in consumption and $\delta_t$ | $\rho_{c,\delta}$ | -0.1 |
| Subjective discount factor | $\delta$ | 0.89 |

Table 2: Summary statistics of simulated data for external habit formation model with *i.i.d.* consumption growth, and cointegrated consumption and dividends. All the quantities are annualized.

| Statistics | Equity | Postwar Sample | Long Sample |
|---|---|---|---|
| $E(g_c)$ | 1.90 | 1.89 | 1.72 |
| $\sigma(g_c)$ | 1.22 | 1.22 | 3.32 |
| $E(r^f)$ | 0.094 | 0.094 | 2.92 |
| $E(r - r^f)$ | 6.71 | 6.69 | 3.90 |
| $\sigma(r - r^f)$ | 15.34 | 15.7 | 18.0 |
| $exp[E(p - d)]$ | 18.2987 | 24.7 | 21.1 |
| $\sigma(p - d)$ | 0.3136 | 0.26 | 0.27 |
| $AC_1(p - d)$ | 0.8432 | 0.87 | 0.78 |

Table 3: Long-horizon correlations for external habit-formation model with *i.i.d.* consumption growth. The calculations are based on quarterly frequency data. $\rho_\delta = 1$ is the case where consumption and dividends are not cointegrated.

| Horizon (in quarters) | Data | Habit-Formation($\rho_\delta = 0.89$) | Habit-Formation($\rho_\delta = 1$) |
|---|---|---|---|
| 1 | 0.1561 | 0.8731 | 0.7604 |
| 2 | 0.2041 | 0.8568 | 0.7353 |
| 3 | 0.2466 | 0.8400 | 0.7101 |
| 4 | 0.2702 | 0.8234 | 0.6854 |
| 5 | 0.2779 | 0.8074 | 0.6612 |
| 6 | 0.2844 | 0.7916 | 0.6377 |
| 7 | 0.2812 | 0.7761 | 0.6151 |
| 8 | 0.2769 | 0.7612 | 0.5938 |
| 9 | 0.2551 | 0.7465 | 0.5728 |
| 10 | 0.2259 | 0.7322 | 0.5529 |

Table 4: Estimation for consumption dynamics based on quarterly consumption data.

| $ARIMA(2,1,2)$ | $\mu_c$ | $\rho_{c,1}$ | $\rho_{c,2}$ | $\theta_{c,1}$ | $\theta_{c,2}$ | $\sigma_c$ |
|---|---|---|---|---|---|---|
| $Estimate$ | 0.0055 | 1.3040 | -0.5535 | -1.0288 | 0.4359 | 0.0042 |
| $Standard\ Error$ | 0.0005 | 0.3756 | 0.2388 | 0.3661 | 0.1489 | 0.0002 |
| $Trend + AR(2)$ | $\mu_c$ | $\rho_{x,1}$ | $\rho_{x,2}$ | $\sigma_x$ | $\sigma_\xi$ | $\rho_{\xi,\epsilon_x}$ |
| $Implied\ Value$ | 0.0055 | 1.3040 | -0.5535 | 0.0050 | 0.0068 | -0.9569 |

The following consumption dynamics is estimated

$$g_{c,t} - \mu_c = \rho_{c,1}\left(g_{c,t-1} - \mu_c\right) + \rho_{c,2}\left(g_{c,t-2} - \mu_c\right) + \epsilon_{c,t} + \theta_{c,1}\epsilon_{c,t-1} + \theta_{c,2}\epsilon_{c,t-2} \tag{8.7}$$

where $\epsilon_{c,t}\tilde{\ }WN\left(0,\sigma_c^2\right)$. This $ARIMA\left(2,1,2\right)$ process has the following equivalent trend-cycle representation for log consumption,

$$\begin{aligned} c_t &= T_t + x_t \\ T_t &= T_{t-1} + \mu_c + \xi_t \\ x_t &= \rho_{x,1}x_{t-1} + \rho_{x,2}x_{t-2} + \epsilon_{x,t} \end{aligned} \tag{8.8}$$

where $T_t$ is the stochastic trend, $x_t$ is the cyclical component in the log consumption, $\epsilon_{x,t}\tilde{\ }WN\left(0,\sigma_{\epsilon_x}^2\right)$, $\xi_t\tilde{\ }WN\left(0,\sigma_\xi^2\right)$ and $corr\left(\xi_t,\epsilon_{x,t}\right) = \rho_{\xi,\epsilon_x}$.

Table 5: Long-horizon correlations under the parameter values $\beta = -2$, $\rho_{\xi,\epsilon_x} = -0.9569$, $\rho_{u,\xi,} = 0$ and $\rho_{u,\epsilon_x} = 0$ for the reduced-form of forward-looking risk model.

| horizon | data | Forward-Looking |
|---|---|---|
| 1 | 0.1561 | 0.0969 |
| 2 | 0.2041 | 0.1943 |
| 3 | 0.2466 | 0.2652 |
| 4 | 0.2702 | 0.3133 |
| 5 | 0.2779 | 0.3464 |
| 6 | 0.2844 | 0.3674 |
| 7 | 0.2812 | 0.3795 |
| 8 | 0.2769 | 0.3869 |
| 9 | 0.2551 | 0.3919 |
| 10 | 0.2259 | 0.3909 |

Table 6: Parameter choices for the external habit formation model with $ARMA(2,2)$ consumption growth : All parameter values are in quarterly frequency.

| Statistics | Variable | Value |
|---|---|---|
| Mean consumption growth (%) | $g_c$ | 0.5458 |
| Standard deviation of the innovation in consumption (%) | $\sigma_c$ | 0.4158 |
| Persistence coefficient in habit | $\phi$ | 0.9658 |
| $AR(1)$ Coefficient of Consumption Growth | $\rho_{c1}$ | 1.3034 |
| $AR(2)$ Coefficient of Consumption Growth | $\rho_{c2}$ | -0.5535 |
| $MA(1)$ Coefficient of Consumption Growth | $\theta_{c1}$ | -1.0288 |
| $MA(2)$ Coefficient of Consumption Growth | $\theta_{c2}$ | 0.4359 |
| Persistence coefficient in $\delta_t$ | $\rho_\delta$ | 0.9719 |
| Standard deviation of the innovation in $\delta_t$ | $\sigma_\delta$ | 0.056 |
| Risk aversion coefficient | $\gamma$ | 2 |
| Correlation between innovation in consumption and $\delta_t$ | $\rho_{c,\delta}$ | -0.1 |
| Subjective discount factor | $\delta$ | 0.9740 |

Table 7: Summary statistics of simulated data for the external habit-formation model with $ARMA(2,2)$ consumption growth. All the quantities are annualized.

| Statistics | Equity |
|---|---|
| $E(g_c)$ | 2.18 |
| $\sigma(g_c)$ | 0.90 |
| $E(r^f)$ | 1.21 |
| $\sigma(r^f)$ | 0.68 |
| $E(r-r^f)$ | 6.26 |
| $\sigma(r-r^f)$ | 17.73 |
| $exp[E(p-d)]$ | 16.7159 |
| $\sigma(p-d)$ | 0.3754 |
| $AC_1(p-d)$ | 0.8414 |

Table 8: Long-horizon correlations for the external habit-formation model with ARMA(2,2) consumption growth. The calculations are based on quarterly data.

| horizon (in quarters) | data | Habit-Formation |
|---|---|---|
| 1 | 0.1561 | 0.7680 |
| 2 | 0.2041 | 0.7214 |
| 3 | 0.2466 | 0.6987 |
| 4 | 0.2702 | 0.6886 |
| 5 | 0.2779 | 0.6813 |
| 6 | 0.2844 | 0.6727 |
| 7 | 0.2812 | 0.6619 |
| 8 | 0.2769 | 0.6486 |
| 9 | 0.2551 | 0.6335 |
| 10 | 0.2259 | 0.6172 |

Table 9: Band-Pass Filter Analysis: Band-pass filter analysis for the real data and artificial data simulated from different models. Here, C-C is the Campbell and Cochrane (1999) model, IID is the external habit formation model with IID consumption growth and cointegrated consumption and dividends. ARMA is the external habit formation model with ARMA(2,2) consumption growth and cointegrated consumption and dividends. B-Y is the calibrated model from Bansal and Yaron (2004), P-Y is the calibrated model from Panageas and Yu (2006). High frequency includes components with cycle less than 8 years. Low frequency includes components with cycle more than 8 years, which are the medium and long run components.

| | C-C | IID | ARMA | B-Y | P-Y | data |
|---|---|---|---|---|---|---|
| Low-Frequency Correlation | 0.4551 | 0.4666 | 0.6032 | 0.2809 | 0.7040 | 0.3417 |
| High-Frequency Correlation | 0.7833 | 0.8876 | 0.7901 | -0.0291 | 0.3811 | 0.1138 |

Table 10: Granger's Causality Test: p-value of the test of consumption growth Granger-causes asset returns based on quarterly data.

| | $BM1$ | $MB2$ | $BM3$ | $BM4$ | $BM5$ |
|---|---|---|---|---|---|
| $S1$ | 0.2061 | 0.2255 | 0.2815 | 0.4396 | 0.5542 |
| $S2$ | 0.1138 | 0.2249 | 0.5238 | 0.3315 | 0.6353 |
| $S3$ | 0.0849 | 0.2774 | 0.4562 | 0.5686 | 0.3665 |
| $S4$ | 0.1155 | 0.2585 | 0.3853 | 0.8124 | 0.8279 |
| $S5$ | 0.3041 | 0.3664 | 0.6938 | 0.7115 | 0.6400 |

Table 11: Granger's Causality Test: p-value of the test of asset returns Granger-cause consumption growth based on quarterly data.

| | $BM1$ | $MB2$ | $BM3$ | $BM4$ | $BM5$ |
|---|---|---|---|---|---|
| $S1$ | 0.0016 | 0.0045 | 0.0035 | 0.0081 | 0.0032 |
| $S2$ | 0.0034 | 0.0071 | 0.0044 | 0.0048 | 0.0026 |
| $S3$ | 0.0024 | 0.0013 | 0.0017 | 0.0014 | 0.0026 |
| $S4$ | 0.0051 | 0.0137 | 0.0018 | 0.0004 | 0.0174 |
| $S5$ | 0.0008 | 0.0137 | 0.0443 | 0.0027 | 0.0136 |

Table 12: Granger's Causality Test: The p-values of Granger's causality test for real data and artificial data simulated from different models.C-C is the Campbell and Cochrane (1999) model, IID is the external habit formation model with IID consumption growth and cointegrated consumption and dividends. ARMA is the external habit formation model with ARMA(2,2) consumption growth and cointegrated consumption and dividends. B-Y is the calibrated model from Bansal and Yaron (2004), P-Y is the calibrated model from Panageas and Yu (2006). All the data are quarterly frequency.

| | C-C | IID | ARMA | B-Y | P-Y | data |
|---|---|---|---|---|---|---|
| Consumption Causes Returns | 0 | 0 | 0 | 0.7740 | 0.7900 | 0.4482 |
| Returns Cause Consumption | 0.2361 | 0.2249 | 0.2015 | 0 | 0 | $4.3770 \times 10^{-4}$ |

Table 13: Conditional CAPM: : 30 portfolios with 10 size, 10 BM and 10 momentum portfolios. The conditional variable is the cyclical component in the log consumption. The portfolio data is downloaded from Kenneth French's website, and the consumption data is take from FED at Saint Louis. (To compare our result with *cay*, we only use data from 1952Q1-2005Q4)

| Row | Constant | $R_m$ | $Cycle$ | $Cycle \cdot R_m$ | $R^2$ |
|---|---|---|---|---|---|
| 1 | 3.674 | -0.328 | | | 0.0042 |
| | (3.6758) | (-0.3424) | | | |
| 2 | 2.294 | 1.232 | -3.9312 | -15.1479 | 0.6086 |
| | (2.5589) | (1.3831) | (-4.6591) | (-2.8213) | |
| 3 | 1.037 | 2.467 | | -24.6311 | 0.4384 |
| | (1.0828) | (2.5871) | | (-4.5858) | |

Table 14: Conditional CCAPM: : 30 portfolios with 10 size, 10 BM and 10 momentum portfolios. The conditional variable is the cyclical component in the log consumption.

| Row | Constant | $g_c$ | $Cycle$ | $Cycle \cdot g_c$ | $R^2$ |
|---|---|---|---|---|---|
| 1 | 2.019 | 4.0814 | | | 0.2411 |
| | (4.4048) | (2.9822) | | | |
| 2 | 2.924 | 4.0151 | -3.5451 | -0.0296 | 0.6247 |
| | (7.7356) | (2.0745) | (-5.3980) | (-5.0146) | |
| 3 | 2.827 | 1.6138 | | -0.0316 | 0.5913 |
| | (7.4106) | (1.4106) | | (-5.3752) | |

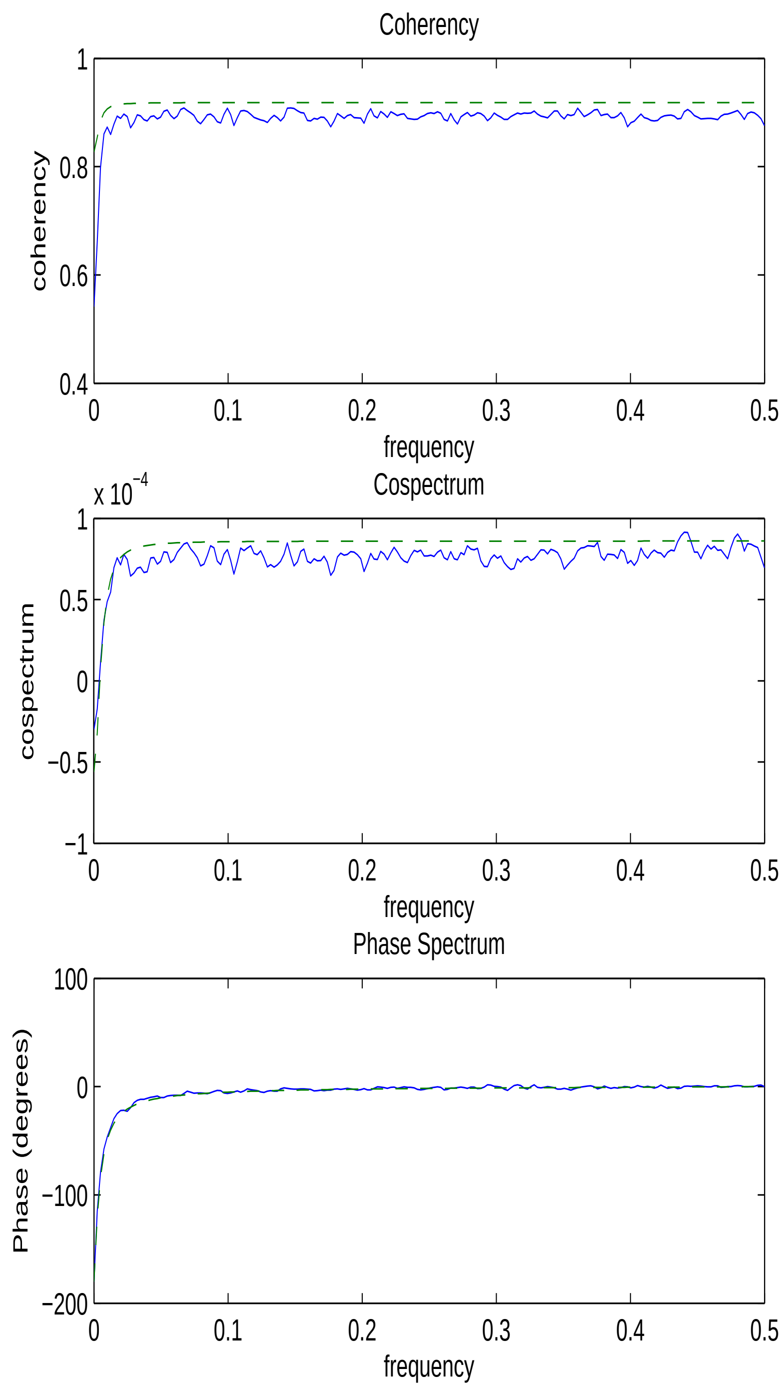


Figure 1: The coherency, cospectrum, and phase spectrum between consumption growth rate and the stock market excess return in the simulated model: The solid line is calculated from the analytical approximation, and the dotted line is calculated from the 40,000 quarters of simulation. A 95% confidence band is also plotted.

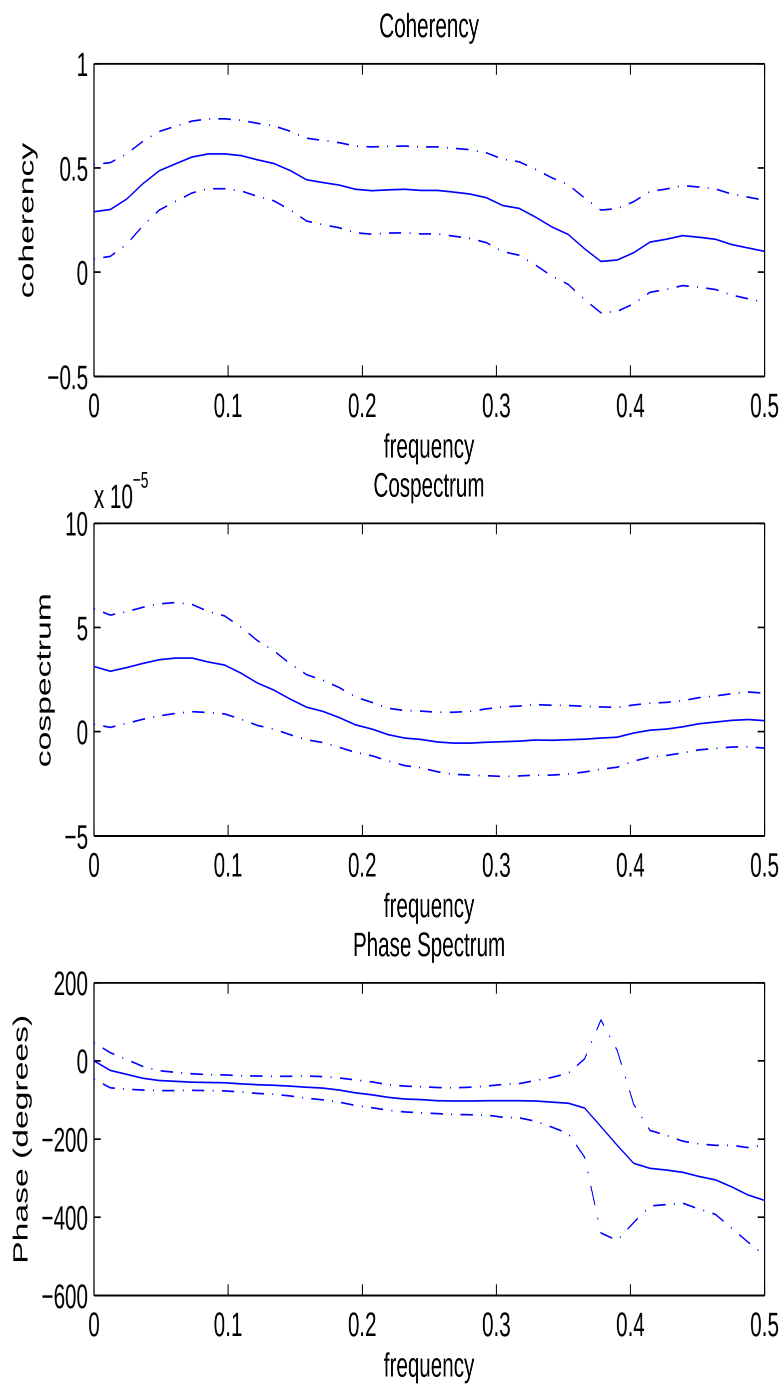


Figure 2: The nonparametric estimation of the coherency, cospectrum, and phase spectrum between the quarterly consumption growth rate and quarterly stock market excess returns in the data: The quarterly consumption data and population data over the period 1952Q1-2006Q4 are taken from Fed St Louis, and the quarterly excess market return is taken from CRSP VW index. A modified Bartlett estimate of the multivariate spectrum is used with $lag = 20$ quarters.

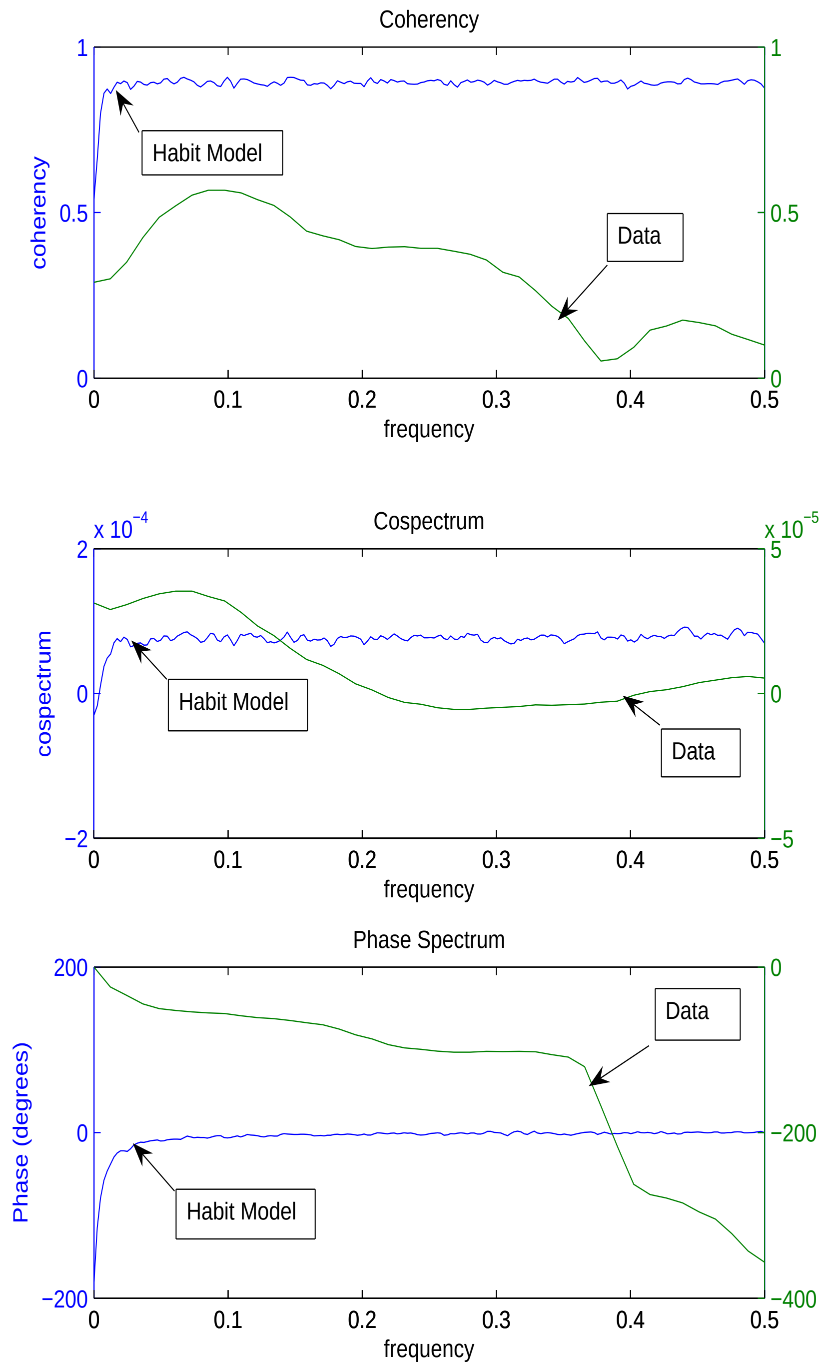


Figure 3: The smoothed lines are the coherency, cospectrum, and phase spectrum between the quarterly consumption growth rate and quarterly stock market excess returns in the data: The rough lines are the coherency, cospectrum, and phase spectrum between consumption and asset returns for the simulated model with *i.i.d.* consumption growth.

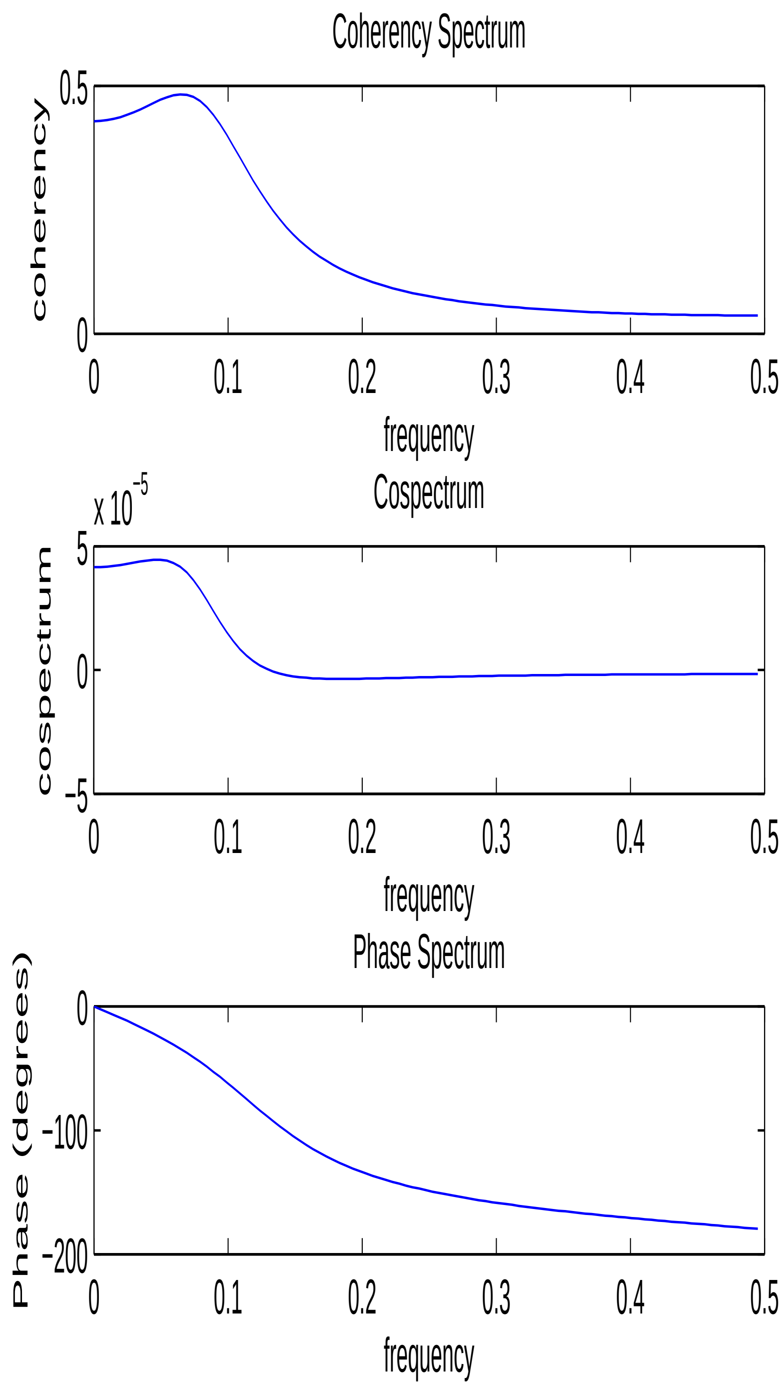


Figure 4: The three panels are the coherency, cospectrum, and phase spectrum between consumption growth rate and the stock market excess returns under the parameter values $\beta = -2$, $\rho_{u,\xi,} = 0$ and $\rho_{u,\epsilon_x} = 0$ for the reduced-form of forward-looking risk models.

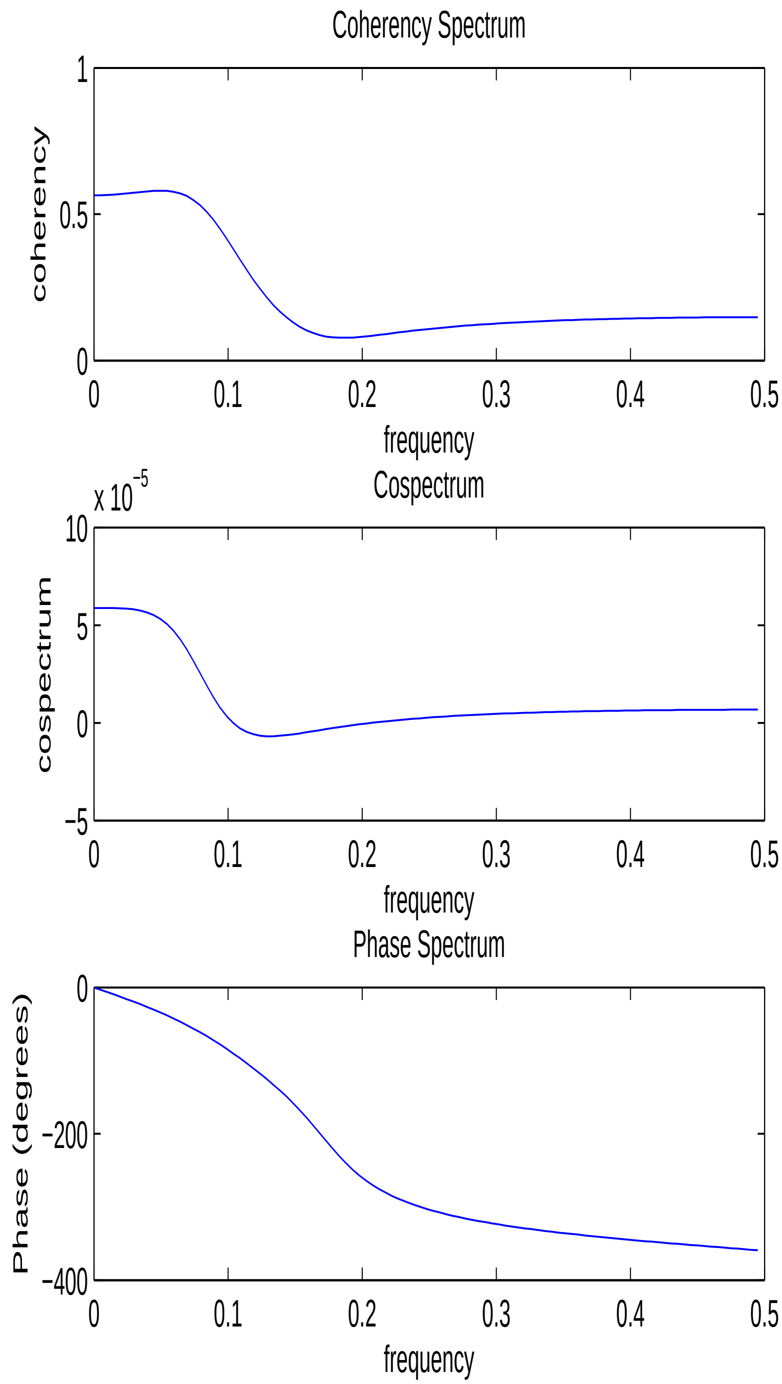


Figure 5: The three panels are the coherency, cospectrum, and phase spectrum between consumption growth rate and the stock market excess returns under the parameter values $\beta = -2$, $\rho_{u,\xi,} = 0.2$ and $\rho_{u,\epsilon_x} = -0.2$ for the reduced-form of forward-looking risk models.

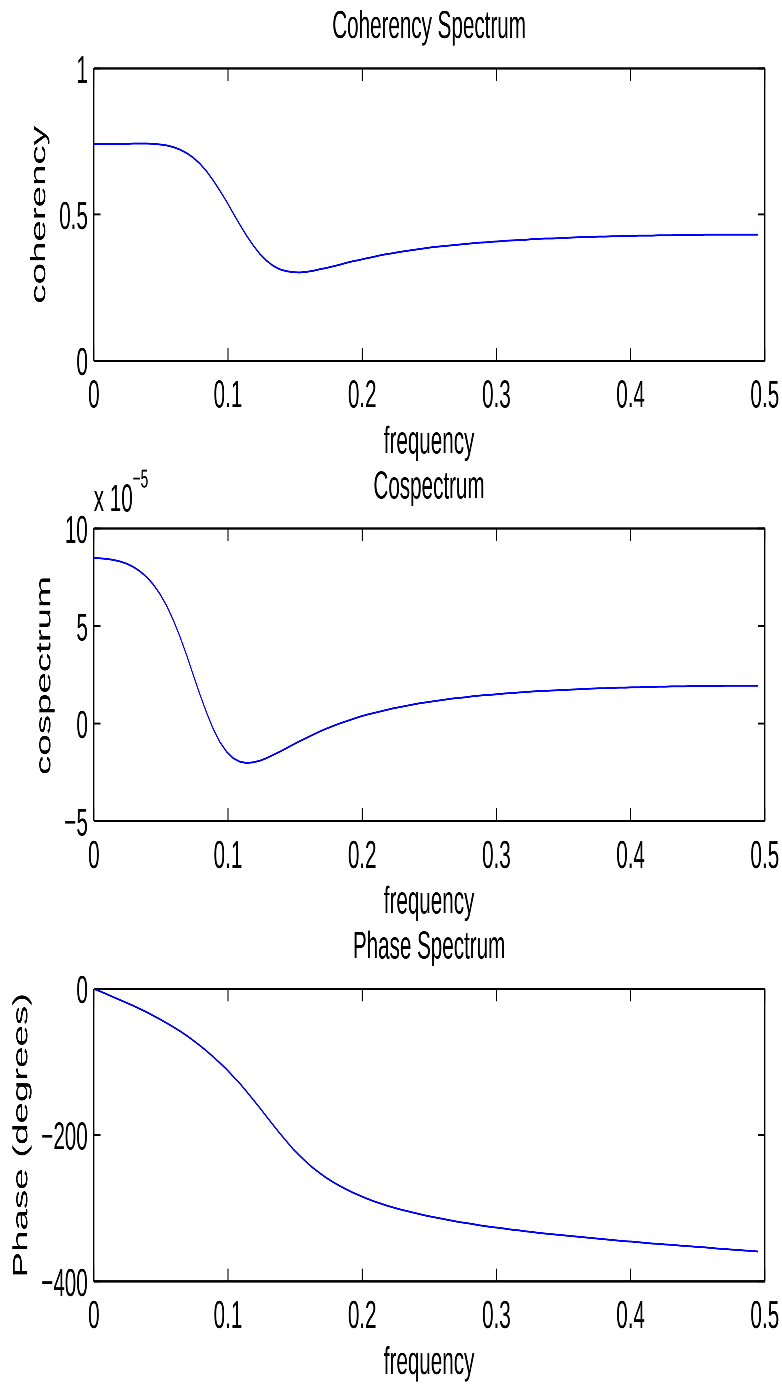


Figure 6: The three panels are the coherency, cospectrum, and phase spectrum between consumption growth rate and the stock market excess returns under the parameter values $\beta = -2$, $\rho_{u,\xi,} = 0.5$ and $\rho_{u,\epsilon_x} = -0.5$ for the reduced-form of forward-looking risk models.

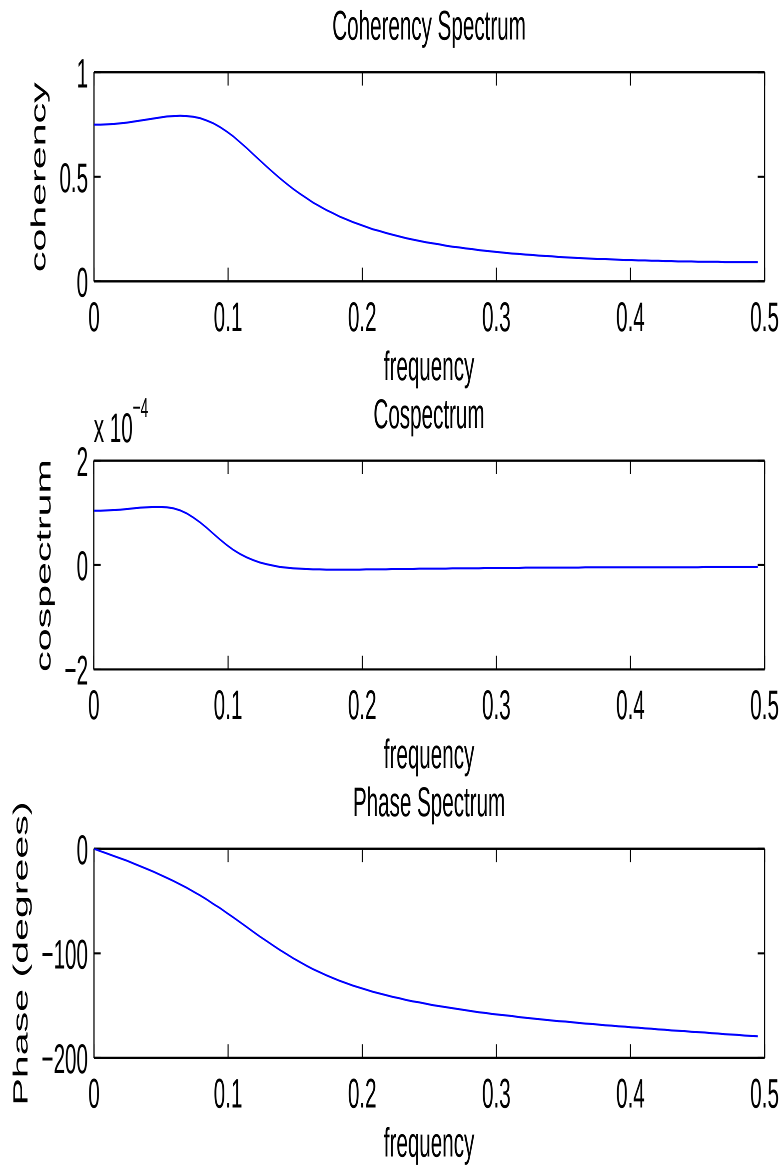


Figure 7: The three panels are the coherency, cospectrum, and phase spectrum between consumption growth rate and the stock market excess returns under the parameter values $\beta = -5$, $\rho_{u,\xi,} = 0$ and $\rho_{u,\epsilon_x} = 0$ for the reduced-form of forward-looking risk models.

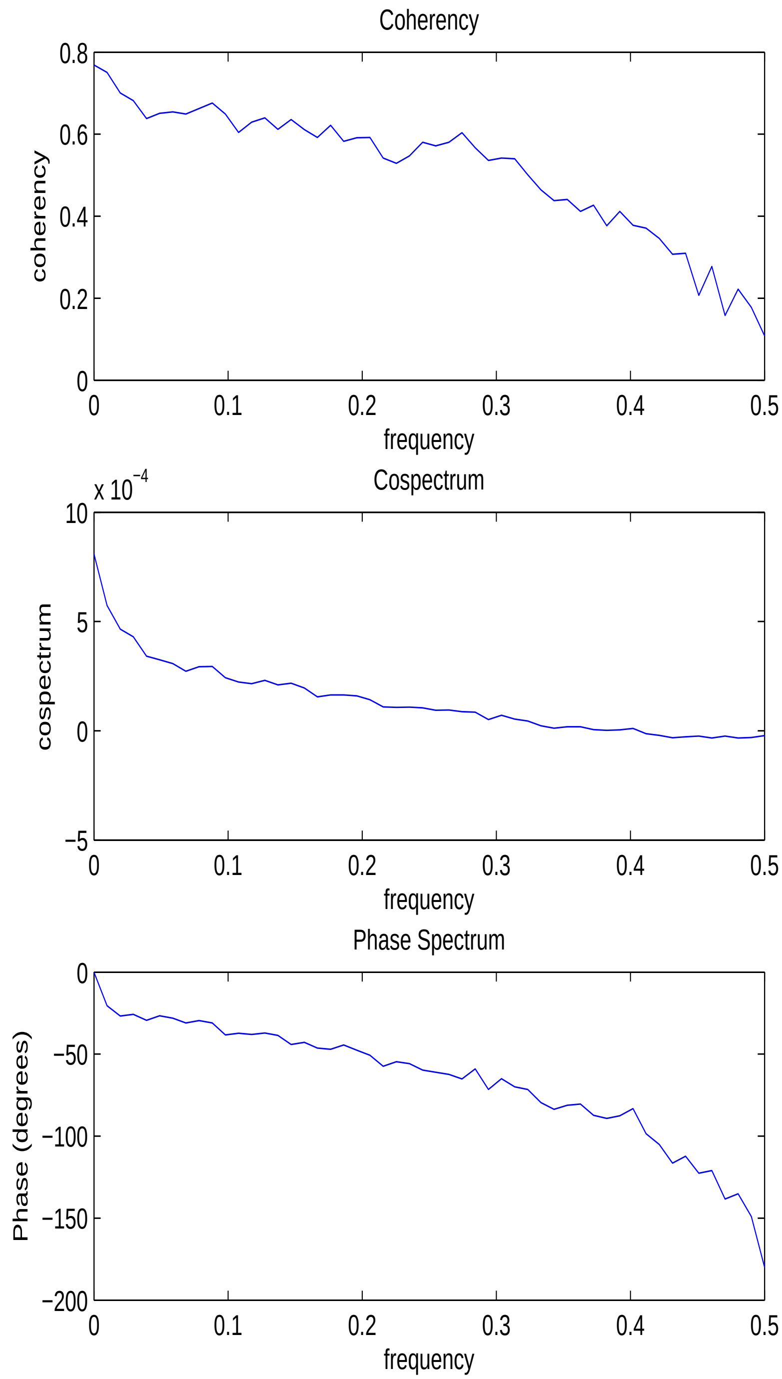


Figure 8: The three panels are the coherency, cospectrum, and phase spectrum between the quarterly consumption growth rate and the quarterly stock market excess returns for the simulated data in a calibrated model of Panageas and Yu (2006)

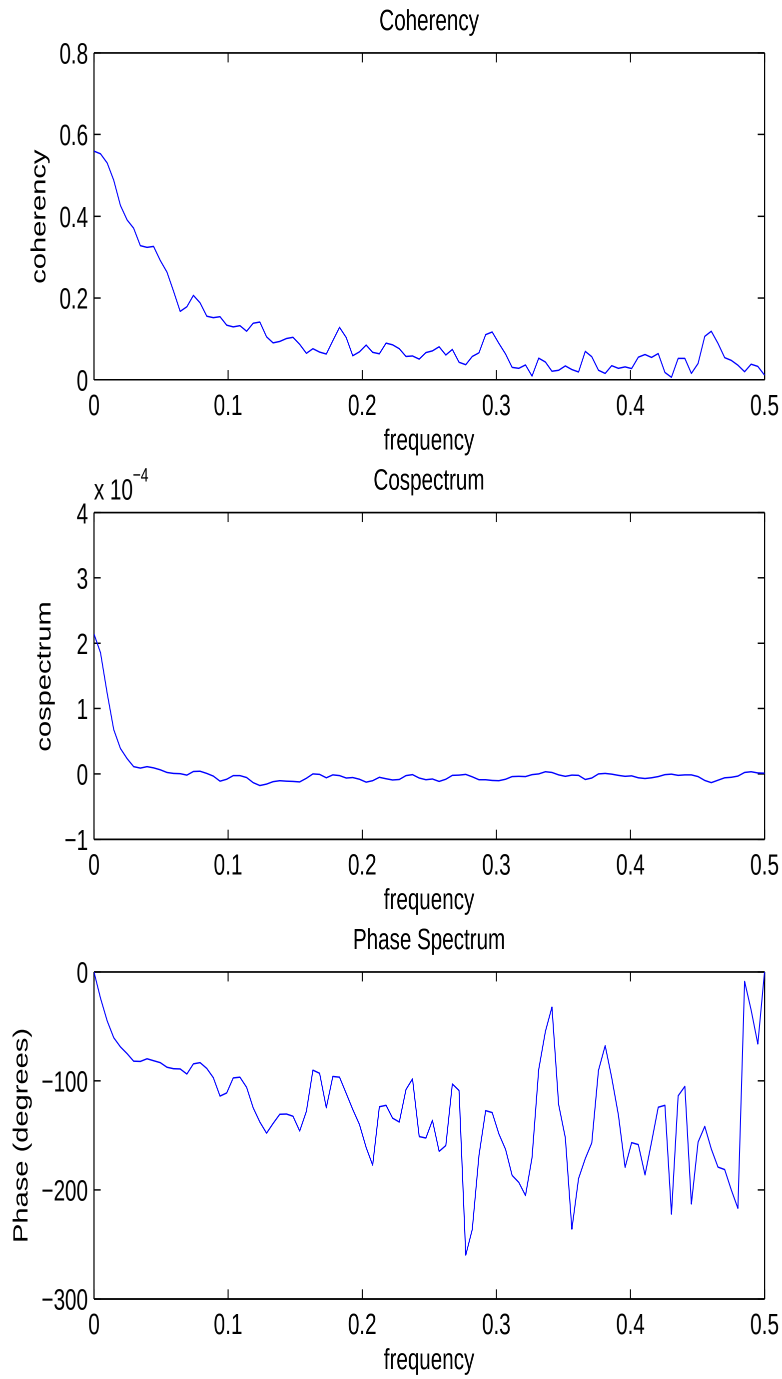


Figure 9: The three panels are the coherency, cospectrum, and phase spectrum between the quarterly consumption growth rate and the quarterly stock market excess returns for the simulated data in a calibrated model of Bansal and Yaron (2004)

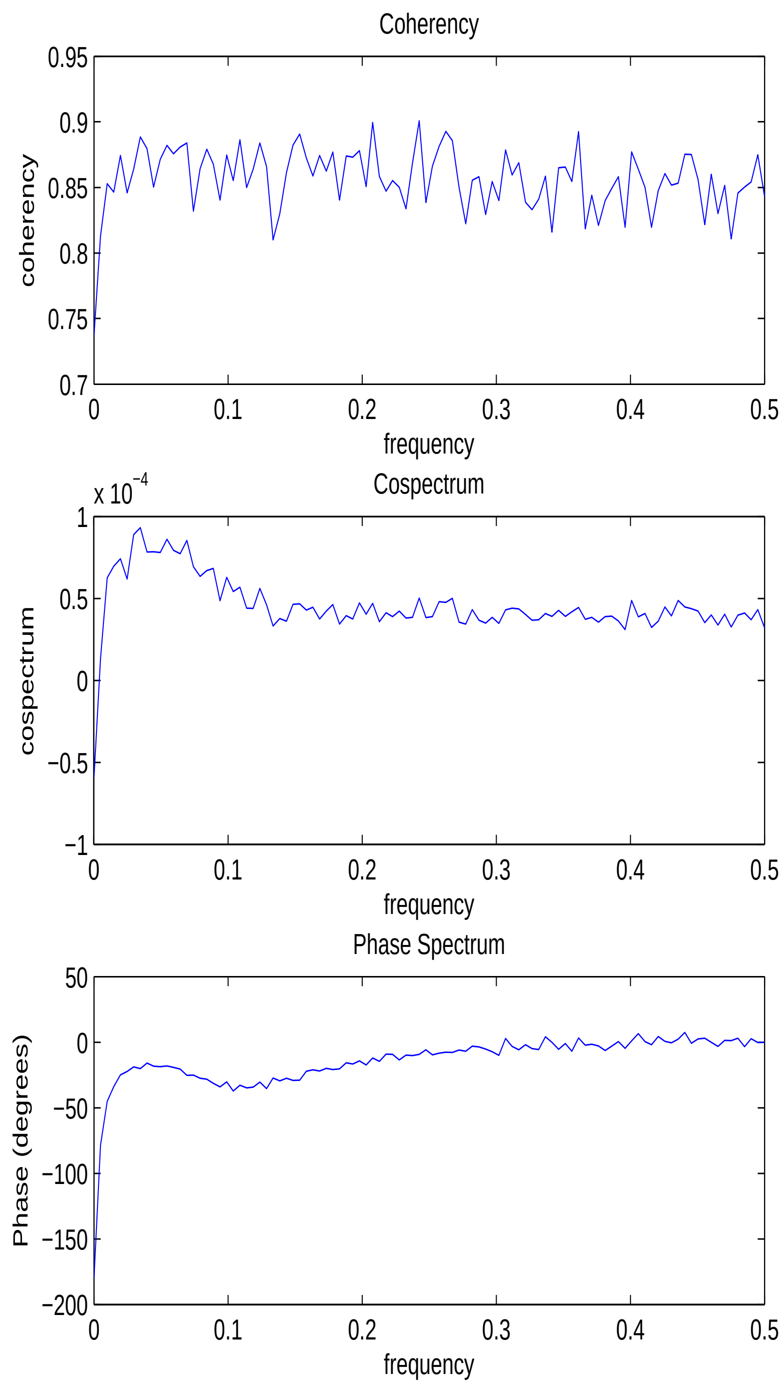


Figure 10: The coherency and phase spectrum between consumption growth rate and the stock market excess return in the simulated external habit model with $ARMA(2,2)$ consumption growth rate.

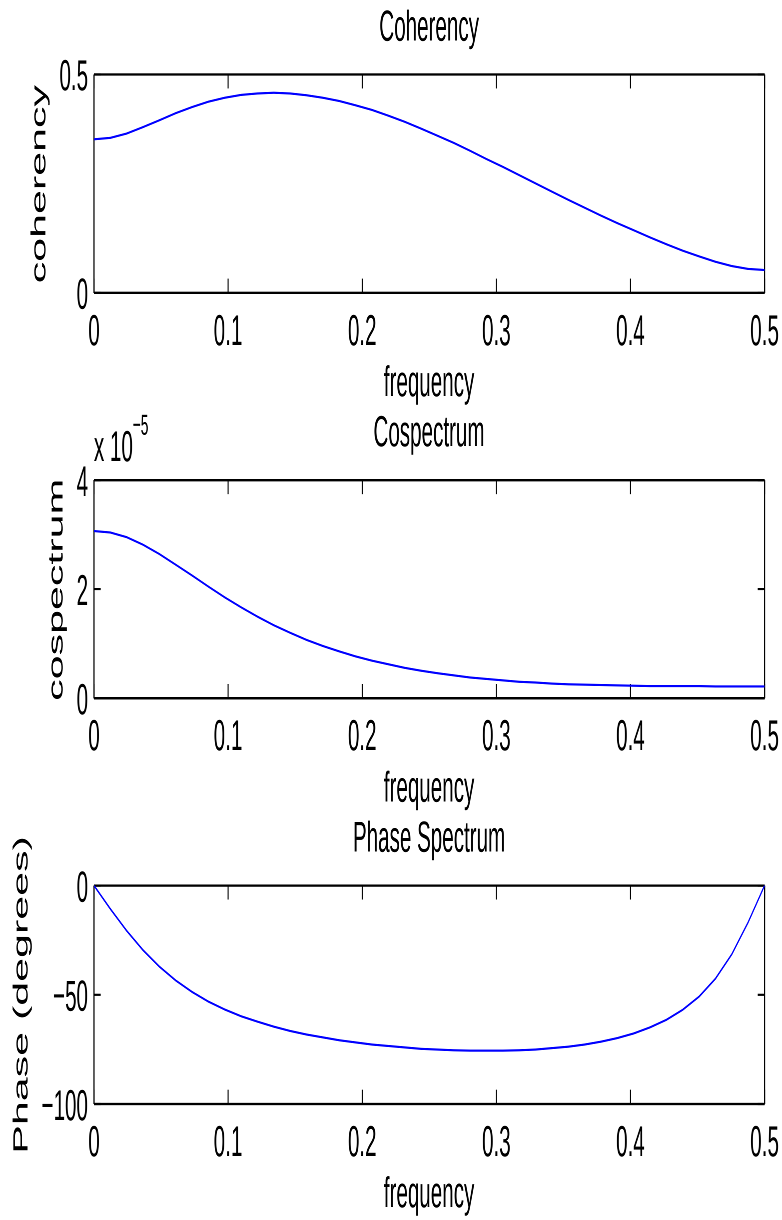


Figure 11: The parametric estimation of the coherency, cospectrum, and phase spectrum between consumption growth rate and the stock market excess return in the data: The quarterly consumption data and population data over the period 1952Q1-2006Q4 are taken from BEA, and the quarterly excess market return is taken from CRSP VW index. We first fit an VAR(2) on the consumption and returns, then obtain the analytical cross-spectrum by plugging in the estimated parameter values.

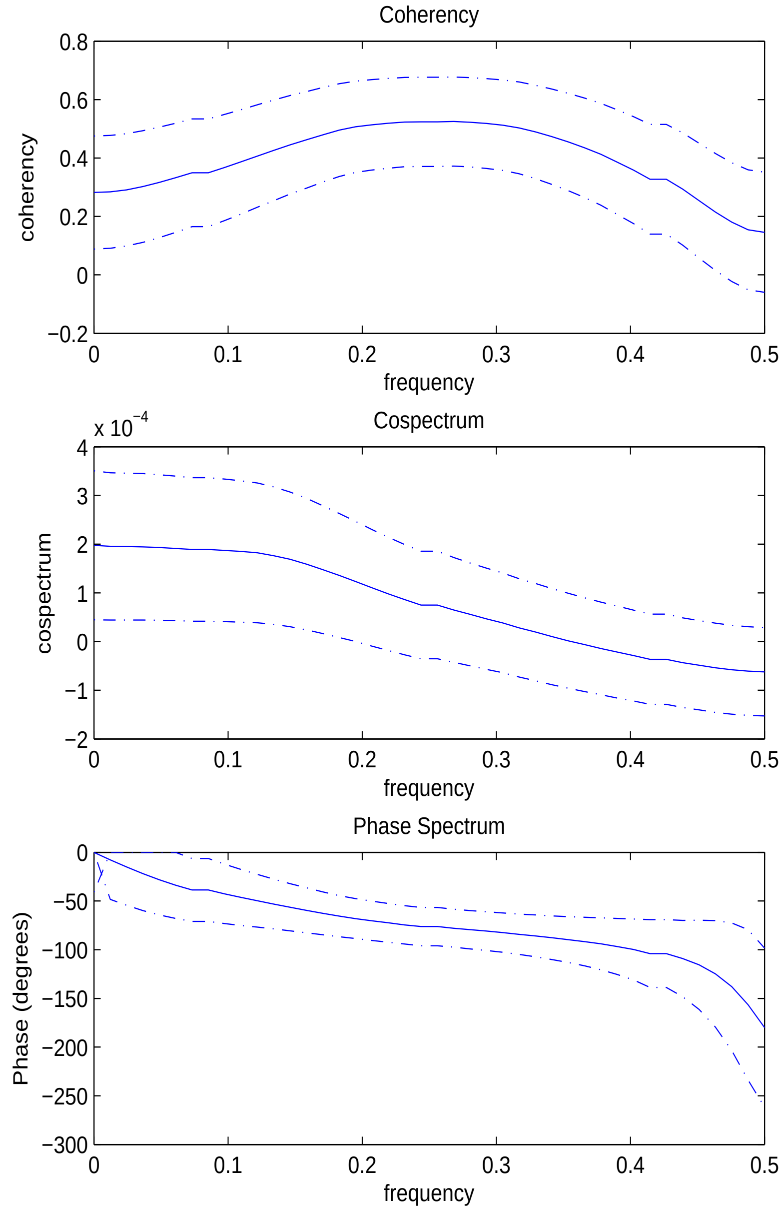


Figure 12: The nonparametric estimation of the coherency, cospectrum, and phase spectrum between annual consumption growth rate and the annual stock market excess return in the data: The annual consumption data and population data over the period 1930-2006 are taken from BEA, and the annual excess market return is calculated from CRSP VW index. A modified Bartlett estimate of the multivariate spectrum is used with $lag = 28$ years.

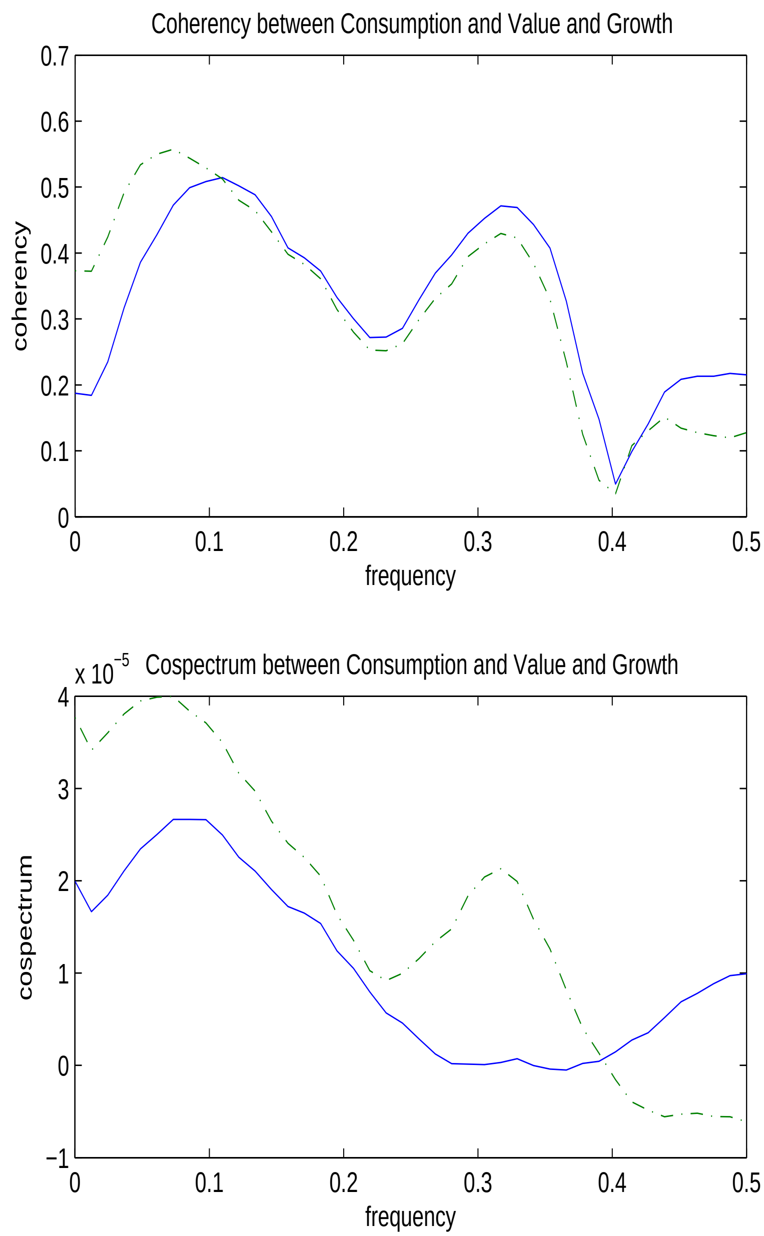


Figure 13: The coherency and cospectrum between consumption growth and growth stocks, and the coherency and cospectrum between consumption and value stocks for real data. The solid line is the spectra between consumption and growth stocks, while the dotted line is for the spectra between consumption and value stocks.

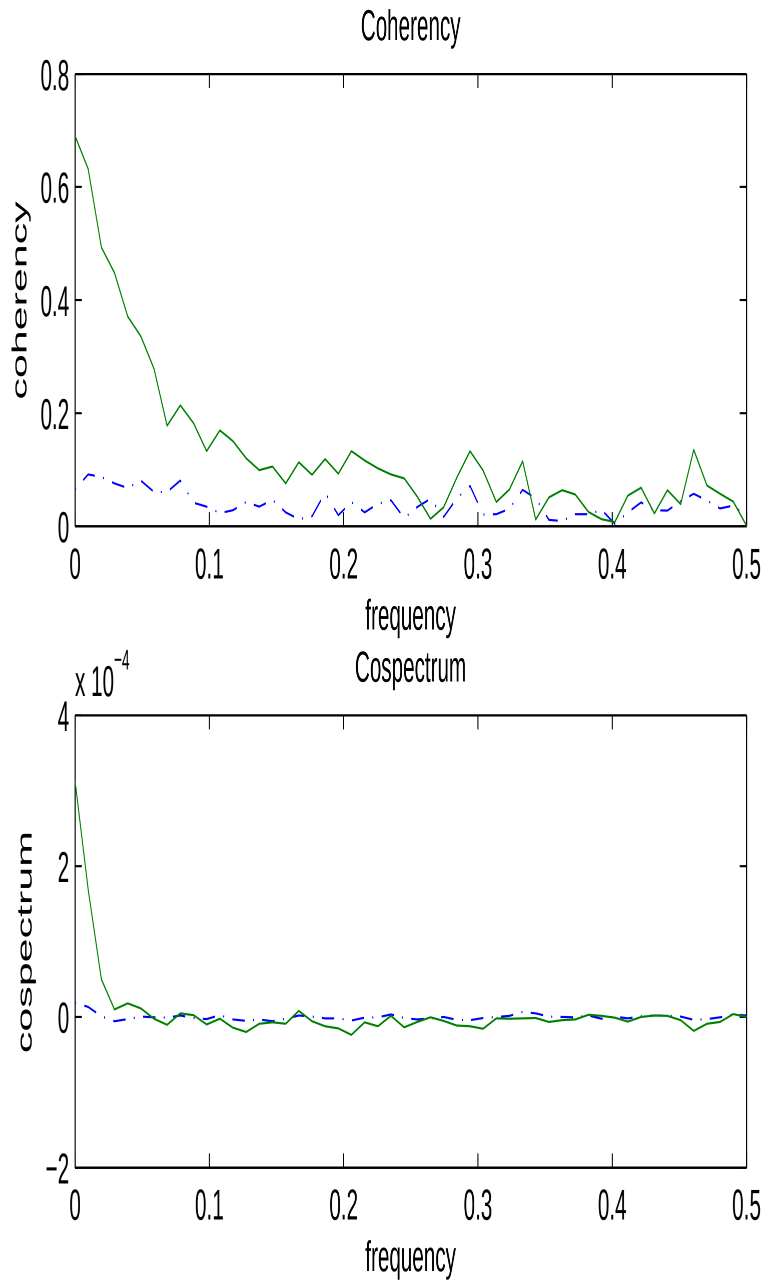


Figure 14: The coherency and cospectrum between consumption growth and growth stocks, and value stocks for the model of Bansal and Yaron (2004), where the solid line is the spectra between consumption and value stocks ($\phi = 4$), while the dotted line is for the spectra between consumption and growth stocks ($\phi = 1$).

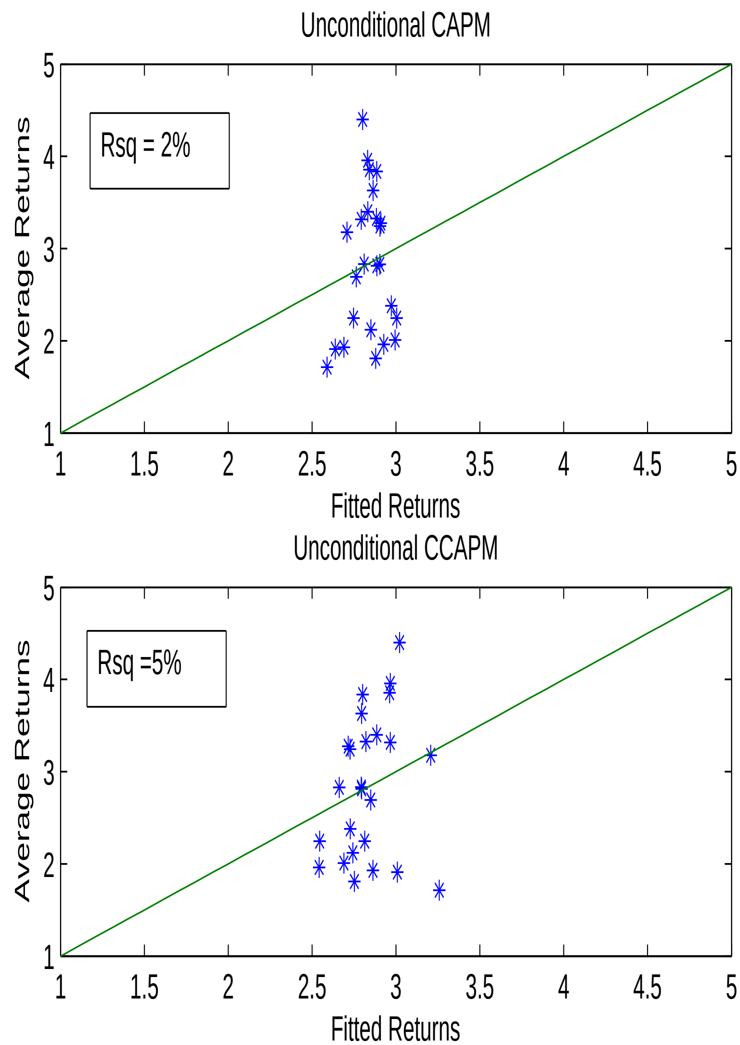


Figure 15: Realized vs. fitted returns: 25 Fama-French portfolios. The upper panel is the unconditional CAPM, the bottom panel is the unconditional consumption CAPM.

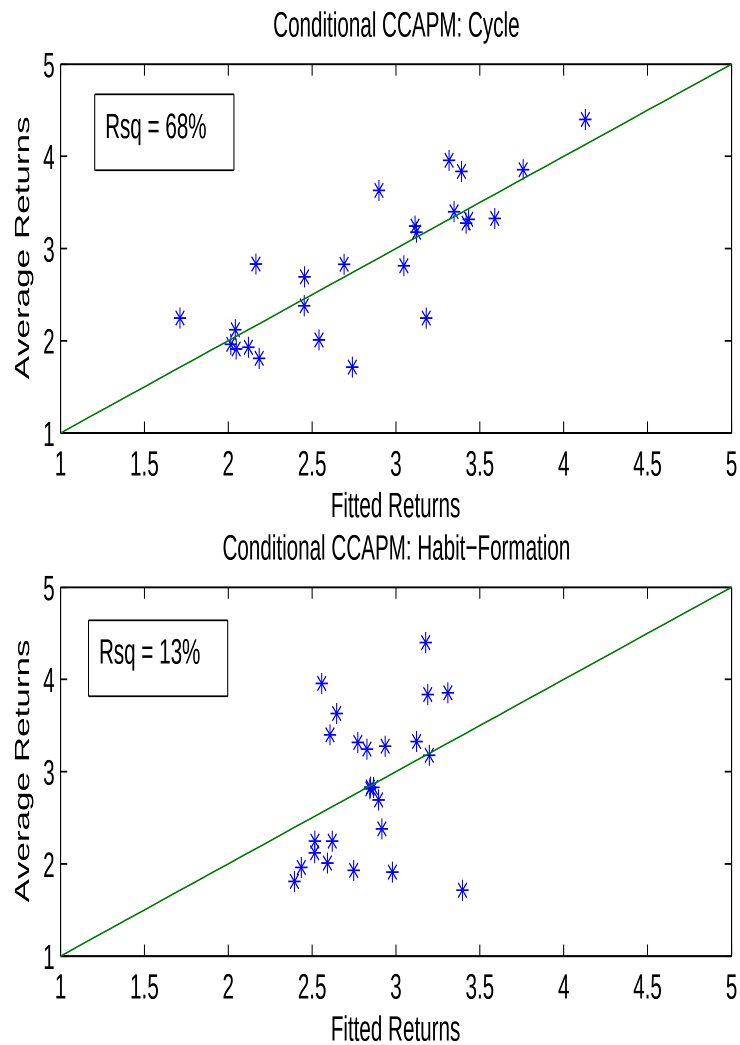


Figure 16: Realized vs. fitted returns: 25 Fama-French portfolios. The upper panel is the conditional consumption CAPM where the conditional variable is the cycle, the bottom panel is the conditional consumption CAPM where the conditional variable is the approximate surplus ratio.