Abstract

We propose a general equilibrium model of defaultable debt where investors hire fund managers to invest their capital either in risky bonds or in a riskless asset. There is only a small fraction of informed managers who can perfectly predict if there is going to be default. Looking at the past performance, investors update their beliefs on the information of their managers and make hiring and firing decisions. This leads to career concerns which affect the investment decision of uninformed managers, generating a “reputational premium”. When the default probability is high enough, uninformed managers prefer to invest in the riskless asset to reduce the probability of being fired. On the contrary, if the probability of default is low enough, investing in the risky bonds has a reputational advantage and the premium is negative. As the economic and financial conditions change, the reputational premium can switch sign, amplifying the reaction of prices and capital flows. We also propose an extended version of the model where the default decision is endogenous, generating a feedback effect from asset pricing to the real economy, which further magnifies the reaction of prices to shocks.
1 Introduction

In the last few years, before the subprime turmoil in August 2007, market observers seemed to be concerned about a growing “overenthusiasm” for risky investments, including high-yield corporate bonds, mortgage-backed assets and emerging market bonds. One observer notices:

Bonds issued by Ecuador, which is politically very unstable, are among the riskiest bets in the emerging markets. It is hard to predict what will happen there next month, let alone in 10 years time. Yet buyers appear to be ready and willing to line up for a sale by the government of up to Dollars 750m in 10-years bonds, the first international bond offer since the country defaulted in 1999. The issue, [...] is the latest example that the prolonged love affair with emerging market debt is far from over. (December 9, 2005, Financial Times).

A similar observation related to leveraged buy-out deals follows:

The head of one of the biggest commercial lenders in the US describes the amount of leverage on some buy-out deals as “nutty”. Much of the wildest lending is being done by hedge funds awash with cash, he says. “Some funds believe they have to invest the money even if it’s not a smart investment. They think the people that gave them the money expect them to invest it. But it’s madness.” (March 14, 2005, Financial Times).

Figure 1 shows the pattern of the yield spreads of some emerging market bonds, the AAA and the B-graded corporate bonds, and the BBB graded commercial mortgage-backed assets, between October 1994 and February 2008. The figure shows at least two periods in which all spreads shrunk to very low levels, close to the AAA corporate spreads: in 1996-1997 and then again from 2005 to the summer of 2007.\(^1\) Observers describe these periods as periods of overenthusiasm which typically occur right before the emergence of a crisis (e.g. Kamin and von Kleist, 1999, IMF, 1999b, Duffie et al., 2003). The figure also shows three episodes of high turbulence in which the spreads of many high-risk bonds jump up and capital tends to flow out of these markets, a phenomenon dubbed as flight-to-liquidity or flight-to-quality.

We propose a stylized dynamic general equilibrium model able to rationalize both types of episodes. In our paper, investors rationally allocate their capital to fund managers, who may

---

\(^1\) As a columnist of the Wall Street Journal observes, the 5-year credit default swap spreads for Brazil, Peru, Columbia were at the record-tight levels of 0.70, 0.65 and 0.80 percentage point at the time when, for example, the Boston Scientific Corp, an investment grade company traded at 0.78 percentage point. (April 24, 2007, Tight spreads are emerging, WSJ).
have different degrees of information about risky investments. The core of our model builds on the career concerns of the uninformed managers, which affect their investment decisions. This leads to rational “overinvestment” in risky investments when the expected default probability is low enough, and “underinvestment” when it is high enough, generating “excess volatility” of prices and capital flows.

Our economy is populated by three types of agents: investors, fund managers, and entrepreneurs. Investors delegate their portfolio decision to risk-neutral fund managers. Fund managers can invest either in risk-less assets or in risky bonds issued by entrepreneurs running a risky project. After observing the realization of the project’s productivity, the entrepreneurs can choose to default on their debt. As shown in Figure 2, the model is structured on two sets of interactions: investors/managers and managers/borrowers.

On the one hand, the interaction between investors and managers shapes the managers’ career concerns. There is a small portion of informed managers who have private information about the productivity of the risky project. Using this information, they can formulate a more
precise estimate of the default probability of the risky bond than the uninformed managers. At the end of each period, based on the managers’ performance, investors update their beliefs and decide whether to keep their manager or to hire a new one. The firing decision of the investors distorts the investment decision of uninformed managers who would like to be perceived as informed managers.

On the other hand, the interaction between managers and borrowers determines the price of the risky bond, the probability of default and the value of debt in the economy. The investment choice of the managers determines the required bond price for a given probability of default. The entrepreneurs issue bonds to cover their consumption and the fixed cost of the risky project. At the end of the period, they observe the productivity of the project and decide whether to pay back the outstanding debt or to default and suffer a loss. For a given price, their default rule determines the ex-ante probability of default on the bond. Hence, the equilibrium bond price and default probability are jointly determined by the conditions of both the financial market and the fundamentals of the risky project.

The focus of our paper is to study the effect of the agency problem between investors and managers, coming from the first interaction, on the equilibrium bond price and default frequency, coming from the second interaction.

Our main result is that managers’ career concerns amplify the effect of both financial and fundamental shocks on the bond price, the default probability, and the level of borrowing. This amplification effect arises in general equilibrium as the outcome of two reinforcing mechanisms. First, on the real side, when borrowing is more expensive, entrepreneurs respond by borrowing less, but, given their smoothing desire, the value that they have to repay is higher and, hence, they default with larger probability. Second, on the financial side, career concerns impose a reputational premium on the spread of risky bonds that depends on the default probability.

Figure 2: The structure of the model
Uninformed fund managers try to time the market in order to behave as if they were informed and knew in advance if there would be default or not. Default will hurt the reputation of uninformed managers who invest in the risky bond, and no default will hurt the reputation of uninformed managers who invest in the risk-less bond. Thus, when the probability of default is high, the reputational premium is positive to compensate for the foregone reputation. When instead the default probability is low, the risky bond will trade with a negative reputational premium. The real side of the model implies that a larger return on bond leads to a larger probability of default. The financial side implies that a larger probability of default leads to a larger return on bond, also because of a larger reputational premium. These two mechanisms reinforce each other in equilibrium and generate excess volatility in bond prices: bond spreads are particularly low in good times and high in bad times.

A natural application of our model is to think of the entrepreneurs as an emerging economy. In this context, our results are in line with the empirical evidence of excess volatility in emerging market bond spreads and capital flows in Neumeyer and Perri (2005) and Uribe and Yue (2006). However, our result more generally applies to any type of credit market characterized by substantial fluctuations in the fundamentals of the underlying risk and by a crucial role of delegated portfolio management.

On the empirical side, our results are also broadly consistent with the puzzle that a large proportion of the variation in prices of both corporate and emerging market bonds cannot be explained by the variation of fundamentals and that a large part of this unexplained component is common across bonds (see Collin-Dufresne at al., 2000, Gruber et al., 2001, Westphalen, 2001). Furthermore, the recent papers of Singleton and Pan (2007) and Longstaff et al (2007) show that US financial market conditions have a large role in explaining the variation of emerging market spreads compared to emerging market fundamentals. Our model argues that fund managers’ career concerns generate an important channel through which financial markets can affect the pricing of debt.

**Literature review.** To our knowledge, this is the first paper to address the asset pricing consequences of the interaction between the real economy and the agency problem between fund managers and investors. Our work is related to several areas of macroeconomics and finance.
First, our paper is related to reputational herding models, where, as in our paper, decision makers with career concerns make inefficient decisions to convince their clients that they are informed. However, there are two main points of departure from our work. On the one hand, this literature traditionally concentrates on partial equilibrium models while our focus is on the interaction of career concerns and asset prices. On the other hand, these papers present mechanisms in which each decision maker herds on others’ decision because going against the average action is a bad signal about his ability. In our model, at the equilibrium prices, fund managers choose the inefficient action regardless of other managers’ decision. That is, there are no strategic complementarities. The closest paper to ours is Rajan (1994), who shows that reputational herding might motivate bank executives to overextend credit in good times by amplifying real shocks. In contrast to our model, Rajan (1994) predicts that in bad times banks provide the right amount of credit while we argue that in bad times managers underinvest in the risky bonds.

Second, there is a growing literature which analyzes the effect of delegated portfolio management on asset prices, including Shleifer and Vishny (1997), Allen and Gorton (1993), Cuoco and Kaniel (2007), Vayanos (2003), Dasgupta and Prat (2005, 2007), Dasgupta, Prat and Verardo (2008) He and Krishnamurthy (2007). This literature is silent about the real effects of the agency frictions in financial markets. Unlike our work, most of this literature takes managers’ distorted incentives as given. A notable exception is Dasgupta and Prat (2006, 2007) and Prat and Verardo (2008) who introduce reputational herding into a Glosten-Milgrom type of sequential trading model. They show that reputational concerns can lead to excessive trading, slow revelation of information and (if the market maker has market power) biased prices. This series of work is a predecessor to our model in the sense that they are the first to use the term reputational premium and to point out that the potential trade-off between reputation and trading profits might lead managers to choose bets with negative net present value. However, our context is different as we are interested in the way reputational effects amplify the price response of financial and real shocks depending on the state of the economy. We also emphasize that reputational effects have systematic price effects even in a standard, competitive, asset pricing model.

Our paper is also related to a large literature on the propagation and amplification of fundamental shocks due to the interaction between asset values and collateralized lending.

---

Seminal papers in this area are Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) on the macro-side, and Gromb and Vayanos (2002) on the finance side. The main difference with our mechanism is that these papers have typically an asymmetric distortion, given that collateral constraints build into the model an external finance premium, usually generating underinvestment. In our model, instead, we microfound the financial distortion and we generate a premium that can be either positive or negative.

Finally, our application on emerging markets is also related to the vast literature on sovereign debt, reversal of capital flows and financial crisis in emerging economies. However, this literature abstracts away from the effects of intermediation in financial markets.

The rest of the paper is organized as follows. In Section 2, we describe the model. In Section 3, we define and characterize an equilibrium. In Section 4, we analyze the limit equilibrium when the mass of informed managers is infinitesimal. In Section 5, we analyze an economy where productivity is persistent and we propose some numerical exercises. Finally, Section 6 concludes. The appendix includes all the proofs which are not in the text.

2 An Example

In this section, we introduce a simple example to show the main mechanism of the model: incentive schemes can distort fund managers’ investment decisions and, hence, affect equilibrium prices and capital flows.

Assume that a large group of risk-neutral fund managers have to decide whether to invest a unit of capital in a risky asset or in a riskless asset. The risky asset has price \( p \) and defaults with probability \( q \). It pays 1 if there is no default and 0 otherwise. The riskless asset pays the safe return \( R < 1/p \). Moreover, assume that a manager obtains a bonus \( W \) if he succeeds in his investment, that is, if he invests in the risky bond when there is no default or in the riskless asset otherwise. The riskless asset is in infinite supply, while the supply of the risky bond is fixed and smaller than the total capital invested by the managers.

It is straightforward to see that the bond market clears if and only if managers are indifferent between investing in the risky bond and in the riskless asset. Hence, the equilibrium price of

---


the risky bond has to satisfy the following indifference condition
\[(1 - q) \left( \frac{1}{p} + W \right) = R + qW. \tag{1} \]

The left-hand side of equation (1) represents the expected payoff of a manager who invests in the risky bond. With probability \(1 - q\) there is no default and the manager gets a return \(1/p\) and the bonus \(W\). If instead there is default, the manager gets zero revenues and no bonus. Similarly, the right-hand side of equation (1) represents the expected payoff of a manager who invests in the riskless asset. He gets always a return \(R\), but he obtains the bonus only if there is default.\(^5\)

In order to characterize the price distortion generated by the bonus \(W\), we define the _premium_ \(\Pi\) as the difference between the expected return and the risk free rate
\[\Pi \equiv \frac{1 - q}{p} - R.\]

The indifference condition (1) immediately implies that \(\Pi = 0\) when there is no bonus scheme, that is, \(W = 0\). In this case, fund managers care only about the expected returns of the bond and the premium is zero. When instead \(W > 0\), the premium can be negative or positive. In particular, if \(q > 1/2\), the payoff of the risky bond is skewed to the left as the probability of default is larger than the probability of no default. In this case, investing in the riskless asset has an advantage over the risky bond as this ensures the bonus payment with larger probability. If the expected return of the two assets were equal, all managers would prefer the riskless one, because of this advantage. Thus, in equilibrium there must be a positive premium on the risky bond to induce managers to hold it. Similarly, if \(q < 1/2\) the payoff of the risky bond is skewed to the right. In this case, the risky bond has an advantage and the premium is negative.

This simple example is suggestive, but it clearly calls for some microfundations behind the bonus scheme. Why investors should reward managers who make successful investment with an ex post bonus? The story we have in mind is a story of reputation, which needs both a dynamic environment and some form of heterogeneous information. In the rest of the paper we

\(^5\)The equilibrium price is consistent with the assumption that \(1/p > R\) if
\[
\frac{qR}{1 - q} + \left( \frac{q}{1 - q} - 1 \right) W > 0.
\]
This is always true if \(W\) is sufficiently small.
build a dynamic general equilibrium model of delegated portfolio management with informed and uninformed managers. Investors rationally learn about the type of the fund managers based on their past performance. Uninformed managers’ reputational concerns generate a reputational premium similar to the one described in this example, with $W$ becoming an equilibrium object. We show that small shocks to the financial market or to the fundamentals of the risky project may lead to large changes in asset prices and capital flows, due to the presence of the reputational premium. In Section 4, we also endogenize the default probability $q$ and the supply of the risky bond. We introduce entrepreneurs who issue the risky bond and can decide to default on their loans. This generates a feedback effect from asset pricing to the real economy, which further magnifies the reaction of prices to shocks.

3 Model

3.1 Set up

Consider an infinite horizon economy with discrete time. There is a mass $\Gamma$ of risk-neutral investors and two investment possibilities: a riskless asset, in infinite supply, which pays $R$ units of consumption per unit of capital invested, and a risky bond with fixed supply $b$ and price in terms of capital $p_t$, which pays 1 unit of consumption if there is no default and 0 otherwise. Let $\chi_{t+1}$ denote the default indicator such that $\chi_{t+1} = 1$ if there is default and $\chi_{t+1} = 0$ otherwise. Assume that the ex-ante probability of default is equal to $q$.

At any time $t$, each investor has one unit of capital and needs a fund manager to invest it. There are two types of risk-neutral fund managers: informed and uninformed. Informed managers know in advance $\chi_{t+1}$, that is, they can perfectly predict if there is going to be default. Uninformed managers, instead, expect the bond to default with probability $q$. There is a mass $M^I$ of informed managers and a larger continuum of uninformed managers. Fund managers do not have any capital, and need to be employed by an investor to make any investment decision. Each investor can employ only one fund manager and a fund manager can work only for a single investor. All the managers have to pay a cost $\kappa$ to search for an investor. Moreover, investors do not know the managers’ type.

At time $t$ there is a mass $\Gamma^I_t$ of employed managers of type $s$, and all investors have a manager working for them, so that $\Gamma^I_t + \Gamma^U_t = \Gamma$. Then, employed managers choose how to invest the unit of capital they manage. Next, investors observe the return of their manager’s
investment and decide whether to fire him or not. Moreover, they receive a signal that reveals the type of an uninformed manager with probability $1 - \omega$. Each manager has a probability $1 - \delta$ to die, in which case a new manager of the same type is born. Finally, all investors who do not have a manager, either because they have fired him or because he is dead, randomly search for a new one. At the same time, the unemployed managers decide whether to pay a cost $\kappa$ to search for a job. Let $N_t^s$ be the mass of managers of type $s$, for $s = I, U$, who decide to look for a job. The probability for a manager to find a job, denoted by $\mu_t$, is equal to the ratio of the mass of investors searching for a manager to the mass of managers searching for an investor.

We look for a stationary equilibrium where the mass of informed and uninformed employed managers, $\Gamma^I$ and $\Gamma^U$, and the matching probability, $\mu$, are constant over time. Hence, from now on we can drop the time dependence for these objects. Moreover, for simplicity, we fix the contract between investors and fund managers: fund managers keep a share $\gamma$ of the revenues and leave the rest to the investors. Both investors and managers fully consume their net revenues in each period.

At the beginning of time $t$, any employed manager selects a demand schedule $d^I_t (p_t, a_{t+1}) \in [0, 1]$ if he is informed and $d^U_t (p_t) \in [0, 1]$ if he is uninformed. At the same time, there is a mass $y_t$ of noise traders, with one dollar each, who demand the bond, where $y_t$ is a random variable uniformly distributed on the support $[0, \bar{y}]$. After collecting all the demand schedules, an auctioneer selects the equilibrium price $p_t$ and assigns the risky bonds to the managers and the noise traders. The selected price and bond allocation must be consistent with the submitted demand schedules and with the bond market clearing. Let us denote by $x^I_t$ and $x^U_t$ the equilibrium fraction of informed and uniformed managers who obtain the bond. The market clearing condition is then

$$\Gamma^I x^I_t + \Gamma^U x^U_t + y_t = p_t b.$$  

The right-hand side of the market clearing condition represents the value of the supply of bonds. The left-hand side, instead, represents the demand of bonds, which comes from three different sources: 1) a proportion $x^I_t$ of informed employed managers, 2) a proportion $x^U_t$ of uninformed employed managers, and 3) a mass $y_t$ of noise traders.

---

6 As we will discuss later on, this exogenous signal makes the analysis more tractable because it guarantees that we can focus on equilibria where a manager who does succeed in his investment is never fired.

7 This ensures that the mass of informed $M^I$ is constant over time.
At the beginning of time $t$, each investor has a manager $j$ working for him that he believes is informed with probability $\eta_{jt}$. Let $\theta_j^t$ be an indicator variable, which is equal to 1 if manager $j$ is allocated a unit of the risky bond, and to 0 otherwise. A law of large number ensures that $x_s^t$ represents also the probability for agent of type $s$ of receiving the risky bond. At the end of time $t$, the investor gets his share of the realized returns, observes $\theta_j^t$ and whether there has been default or not. Moreover, if his manager is uninformed, he discovers his type with probability $1 - \omega$. Then, according to the Bayes’ Rule, he updates his belief about the type of his manager to $\eta_{jt+1}^i$. Conditional on his posterior, he chooses his firing strategy $\phi_j^t$, that is, whether to keep his manager for next period ($\phi_j^t = 0$), or to fire him and randomly hire a new one ($\phi_j^t = 1$).\footnote{Notice that all the unemployed managers have the same probability of being informed, given that the ones who have a good history are never fired and the ones who have a bad history will never pay the cost $\kappa$ to search for a new investor.} Clearly, the investor’s firing decision is affected by the probability that a new hire is informed. The key feature of our model is that manager $j$ knows that his investment decision will affect the investors’ firing decision by changing his posterior belief. This generates career concerns affecting the investment strategy that are at the core of our model.

3.2 Equilibrium

Let us first introduce the definition of a stationary equilibrium for any $M^I > 0$. Next, we will propose the type of stationary equilibrium we are interested in, an interior equilibrium, we will verify that it is an equilibrium, and we will show under which conditions it exists.

Definition 1 For a given $M^I > 0$, a stationary equilibrium is a demand schedule for informed managers, $d^I(p_t, x_{t+1})$, a demand schedule for uninformed managers, $d^U(p_t)$, a firing strategy for investors, $\phi(\eta_{t+1}^i)$, bond allocations for the informed and uninformed managers, $x^I(\chi_{t+1}, y_t)$ and $x^U(\chi_{t+1}, y_t)$, a price $p(\chi_{t+1}, y_t)$, a constant mass of employed informed and uninformed managers, $I^I$ and $I^U$, a constant matching probability $\mu$, and an updating rule $\eta_{t+1}^j = \zeta(\eta_{t}^j, \theta_t^j, p_t, x_t)$, such that

1. investors maximize their expected utility, taking as given the equilibrium price, allocation and strategies of other agents;

2. fund managers maximize their expected utility, taking as given the equilibrium price, allocation and strategies of other agents;
3. the bond price and allocations are consistent with the managers’ demand schedules and with market clearing;

4. $\Gamma^I, \Gamma^U$ and $\mu$ are consistent with free entry in the labor market for fund managers;

5. investors’ beliefs are consistent with the Bayes’ rule.

### 3.3 Interior Equilibrium

When $M^I > 0$, employed uninformed managers face the risk of being fired and, hence, their investment decisions are affected by their expected future utility. Moreover, they can potentially extract some information on the strategy of the informed managers from the equilibrium price. We focus on equilibria where uninformed managers are typically the marginal traders, that is, are indifferent between investing in the bond and in the risk-free asset whenever prices are not fully revealing. We call this type of equilibrium, an “interior” equilibrium.

**Definition 2** Let $z_t \equiv \Gamma^I \left(1 - \chi_{t+1}\right) + y_t$. A stationary interior equilibrium is an equilibrium where

1. **prices are**
   
   $$p(z_t) = \begin{cases} 
   y_t/b & \text{if } z_t \in [0, \Gamma^I) \\
   p^* & \text{if } z_t \in [\Gamma^I, \bar{y}] \\
   1/R & \text{if } z_t \in (\bar{y}, \bar{y} + \Gamma^I] 
   \end{cases}$$  
   \hspace{1cm} (3)

   with $p^* \in (\bar{y}/b, 1/R)$;

2. **the managers’ demand schedules are**
   
   $$d^I(\chi_{t+1}, p_t) = \begin{cases} 
   \frac{1 - \chi_{t+1}}{b/R - y_t} & \text{if } p_t < 1/R \\
   0, 1 - \chi_{t+1} & \text{if } p_t = 1/R 
   \end{cases}$$

   and

   $$d^U(\chi_{t+1}, p_t) = \begin{cases} 
   0 & \text{if } p_t \leq \bar{y}/b \\
   \{0, 1 - \chi_{t+1}\} & \text{if } p_t \in (\bar{y}/b, 1/R] 
   \end{cases}$$

3. **the managers’ bond allocations are**
   
   $$x^I(\chi_{t+1}, y_t) = \begin{cases} 
   \frac{1 - \chi_{t+1}}{b/R - y_t} & \text{if } z_t \in [0, \bar{y}] \\
   0 & \text{if } z_t \in (\bar{y}, \bar{y} + \Gamma^I] 
   \end{cases}$$

   and

   $$x^U(\chi_{t+1}, y_t) = \begin{cases} 
   \frac{p*b-z_t}{\Gamma^U} & \text{if } z_t \in [0, \Gamma^I) \\
   0 & \text{if } z_t \in [\Gamma^I, \bar{y}] \\
   \frac{p*b-z_t}{1/R - p^*} & \text{if } z_t \in (\bar{y}, \bar{y} + \Gamma^I] 
   \end{cases}$$  
   \hspace{1cm} (4)

---

9Notice that $z_t$ represents the total potential demand of informed managers and noise traders.
(iv) the investors’ firing rule requires to fire manager \( j \) if the exogenous signal reveals that \( j \) is uninformed or \( \theta_t^j \neq 1 - \chi_t+1 \) and \( p_t < 1/R \), and keep him otherwise.

An interior equilibrium is characterized by three possible revelation regimes: \( p_t = 1/R \) reveals that there is going to be default, \( p_t = \bar{y}/b \) reveals that there is going to be no default, and, thanks to the uniform distribution of \( y \), \( p_t = p^* \) does not reveal any information. If \( p_t = 1/R \), the two assets are equivalent and all managers are indifferent between them. If \( p_t = \bar{y}/b \), there is revelation of default and all managers demand the riskless asset. Finally, when \( p_t = p^* \), informed managers know in advance if there is going to be default or not, and hence can time the market perfectly, by demanding the risky bond if and only if there is not going to be default. The uninformed managers, however, do not have this superior information and are indifferent between obtaining the bond or not. This is the notion of interior solution we refer to when we call the equilibrium an interior equilibrium. According to these demand schedules, the auctioneer determines the fraction of uninformed and informed managers who obtain the bond in order to guarantee that the bond market clears. Finally, investors fire their manager whenever they have an exogenous signal that he is uninformed and if \( p_t < 1/R \) and their manager does not time the market correctly.

In Appendix A, we verify that a stationary interior equilibrium as defined above is indeed an equilibrium. We first show that there are three possible revelation regimes and then we show that the equilibrium strategies described in definition 2 are optimal, taking as given the equilibrium price, allocation, and strategies of the other agents.

The following assumptions are sufficient to ensure that such an equilibrium exists:

\[
\frac{MI + \bar{y}}{b} < P(q) < \min \left\{ \frac{\Gamma - MI}{b}, \frac{1}{R} \right\}, \tag{A1}
\]

\[
MI \leq \bar{y}, \tag{A2}
\]

and

\[
\omega < \frac{1}{1 + \delta}. \tag{A3}
\]

Assumption A1 guarantees that there are always some uninformed managers investing in both the risky bond and the risk-less asset. Assumption A2 ensures that prices are not fully revealing with positive probability. Finally, assumption A3 ensures that the proportion of informed managers among those who are searching for a job is sufficiently small that if an uninformed
manager does not make a mistake, he is not fired. This last assumption is not crucial, but it makes the analysis simpler.

**Proposition 1** Suppose A1-A3 hold. Then an interior equilibrium exists.

### 3.4 Limit Equilibrium

Let us focus on a limit interior equilibrium where $M^I \to 0$. This limit case is very tractable and insightful at the same time. We show that as $M^I \to 0$, the sequence of interior equilibria converges to a limit where the bond price never reveals any information, and is constant over time. Intuitively, this can be the case because as the fraction of informed manager is infinitesimal, the uninformed managers will have to demand all the bonds supplied and hence won’t learn any information from the equilibrium price.

**Definition 3** When $M^I \to 0$, a limit interior equilibrium is an equilibrium where the price $p^* \in (\bar{y}/b, 1/R)$ is determined by the indifference condition of the uninformed managers; the informed managers’ demand schedule is $d^I(\chi_{t+1}) = 1 - \chi_{t+1}$ and their bond allocation is $x^I(\chi_{t+1}) = d^I(\chi_{t+1})$; the uninformed managers’ demand schedule is $d^U = \{0, 1\}$ and their bond allocation is $x^U(y_t) = (p^*b - y_t)/\Gamma$; the investors’ strategy is to fire manager $j$ if he receives a negative exogenous signal or if $\theta^j_t \neq 1 - \chi(a_{t+1})$, and keep him otherwise.

In a limit interior equilibrium, career concerns affect the bond price by generating a reputational premium. The equilibrium behaves in a similar way to the general one, in the case in which the price does not reveal any information. Investors fire the managers who reveal to be uninformed, while informed managers never make a mistake and are never fired. The market clearing condition for the risky bond determines the fraction $x^U(y_t)$ of uninformed managers who invest in the risky bond. Assumption A1 guarantees that $x^U(y_t) \in (0, 1)$ for any $y_t$, so that there are always some uninformed managers investing in the risky bond and some investing in the risk-free asset. Hence, it must be that the uninformed managers are indifferent between the two investment possibilities.

For a given default probability $q$, the equilibrium price $p$ is determined by the indifference condition for the uninformed managers, that is,

$$(1 - q) (\gamma/p + \delta \omega W) = \gamma R + q \delta \omega W,$$

(5)
where \( W \) is the expected continuation utility of an uninformed manager who keeps the job. This condition is the analogous to condition (1) in the example in Section 2. The left-hand side of equation (5) represents the expected payoff of a manager who invests in the risky bond. With probability \( 1 - q \), there is no default, the manager gets a return \( \gamma/p \). If there is no exogenous signal, he is not fired and gets expected continuation utility \( W \). If instead there is default, the manager gets zero revenues, is fired, and gets 0 continuation utility, given that there is free entry. Similarly, the right-hand side of equation (5) represents the expected payoff of a manager who invests in the risk-free bond. He gets always a return \( \gamma R \), but only if there is default and there is no exogenous signal, he is not fired and gets expected continuation utility \( W \). Otherwise, the investor learns that he was not informed and fires him. The continuation utility \( W \) is given by

\[
W = \frac{\gamma R}{1 - \delta q}.
\]

This expression is obtained by noticing that managers are always indifferent between investing in the risk-free asset and in the risky bond, and hence their expected utility can be calculated as the value of always investing in the risk-free asset.

Similarly to section 2, let the reputational premium \( \Pi \) be the difference between the expected repayment and the risk free rate \( R \), that is,

\[
\Pi \equiv \frac{1 - q}{p} - R.
\]

This premium characterizes the price distortion generated by the career concerns of the uninformed managers.

As a point of comparison, consider a model with \( M^I = 0 \). In this case, all managers are uninformed, so investors will be indifferent between keeping the manager they started with and hiring a new one. Then, there exists an equilibrium where managers are never fired and maximize their period by period profit and the bond price is determined by the standard no-arbitrage condition

\[
(1 - q) \frac{1}{p} = R.
\]

We call this equilibrium the benchmark equilibrium.

In the benchmark equilibrium, the standard arbitrage condition (8) immediately implies that \( \Pi \) is equal to zero. When instead there is a positive measure of informed managers, \( M^I > 0 \), the reputational premium can be negative or positive. Typically, it is positive when \( q \)
is sufficiently large and negative when \( q \) is sufficiently small. Betting on large probability events is especially attractive for an uninformed manager with career concerns, because it increases the chance that he will not make an unsuccessful decision and will not be fired. The equilibrium price reflects this preference for large probability events. Fund managers are willing to get a lower expected return in exchange for a large probability of not being fired. It is interesting to point out that there is a discontinuity at \( M^I = 0 \), given that the benchmark equilibrium disappears as \( M^I > 0 \), even if \( M^I \to 0 \).

Combining (5) and (6) implies that the limit equilibrium is characterized by

\[
p = P^L(q) = \frac{(1 - q)(1 - \delta \omega q)}{1 - \delta \omega (1 - q)} R.
\]

In the benchmark equilibrium the pricing rule is given by the standard no-arbitrage condition (8), that is, \( p^B = P^B(q) = (1 - q) / R \). Notice that the reputational premium is zero iff \( q = 1/2 \) and both \( P^L(q) \) and \( P^B(q) \) are decreasing in \( q \). This implies that \( \Pi > 0 \) iff \( q > 1/2 \) and \( \Pi < 0 \) otherwise.

Proposition 2 guarantees that under assumptions A1-A3, a limit interior equilibrium exists. The equilibrium regime is determined jointly by the fundamentals of the risky project and the state of the financial market. Figure 3 represents graphically both the limit equilibrium for an economy with career concerns (L) and the benchmark equilibrium (B). In this numerical example we assume \( q < 1/2 \) so that \( P^L(q) > P^B(q) \) and the reputational premium is positive. It is immediate to see that if \( q > 1/2 \) then \( P^L(q) < P^B(q) \) and the reputational premium is negative. The figure shows that in a limit interior equilibrium prices react more to changes in the fundamentals of the economy, both on the supply and on the demand side. This is simply due to the fact that the pricing rule \( P^L(q) \) is steeper than the benchmark pricing rule \( P^B(q) \).

For example if there is an exogenous shock either to the default probability \( q \) or to the return of the safe asset \( R \), it is easy to see that prices react more in a limit equilibrium than in the benchmark equilibrium. This shows the essence of the amplification result that we will discuss in more detail in Section 4.4, when we introduce the extended model with endogenous supply.

4 Endogenous Bond Supply

In this section, we extend the model in order to endogenize the default probability. In particular, we introduce entrepreneurs who issue the risky bond described so far in order to finance a risky project they have access to. We assume that they can decide to default ex post on their
loans at a given cost. In our baseline model we have established that the equilibrium bond price is affected by the default probability. This extended model generates a feedback effect from the bond price to the borrowing and default choice. We will see that this feedback effect magnifies the amplification generated by the presence of reputational concerns.

4.1 Entrepreneurs

The set up of the economy is identical to the one presented in Section 3.1 except that now $b$ and $q$ are endogenous objects. There are overlapping generations of entrepreneurs who live for two periods. A generation is represented by a continuum of measure 1 of entrepreneurs. In each period a new generation is born. Consider an entrepreneur born at time $t$. When she is young, she can choose to pay a cost $k > 0$ to invest in a risky project with return $a_{t+1}$, distributed according to the cumulative distribution function $F(a_{t+1})$ with $a_{t+1} \in [0, \infty)$, or to enjoy an outside option that gives her utility $V$. If she decides to undertake the risky project, she can borrow by issuing one-period discount bonds. Define $p_t$ the price of the bonds issued at time $t$. The entrepreneur chooses how much to borrow and how much to consume, taking
When she is young, her budget constraint is
\[ c_t + k \leq p_t b_{t+1}, \] (9)
where \( c_t \) represents consumption at time \( t \) and \( b_{t+1} \) represents the one-period discount bonds issued at time \( t \). There is an upper bound \( \bar{b} \) on how much entrepreneurs can borrow.

When she is old, she collects the project pay-off \( a_t \) and has the option to default on her debt \( b_{t+1} \). If she defaults she does not repay the debt, but she suffers an output loss of \((1 - \theta) a_t\), that is, she keeps only \( \theta a_t \) of the return on the project. If she does not default, she has to repay her debt and consume the rest. Her budget constraint when old is
\[ c_{t+1} \leq a_{t+1} - (1 - \chi_{t+1}) b_{t+1} - \chi_{t+1} (1 - \theta) a_{t+1}, \] (10)
where \( \chi_{t+1} : [p_t, 1/R] \times \mathbb{R}_+ \mapsto \{0, 1\} \) denotes the default decision that the agent is making at time \( t + 1 \) taking as given the price \( p_t \) and after observing the realization of \( a_{t+1} \). Hence, the default probability \( q \) which was exogenous in the previous section is now endogenous with \( q_t \equiv E_t[\chi_{t+1}] \).

The problem for an active entrepreneur born at \( t \) is to maximize her utility
\[ u(c_t) + \beta E[v(c_{t+1}) | p_t], \]
subject to (9) and (10), taking \( p_t \) as given. We assume that \( u(\cdot) \) and \( v(\cdot) \) are increasing and strictly concave and have continuous first and second derivative. Moreover,
\[ -cu''(c)/u'(c) \geq 1. \] (A4)

The problem can be rewritten as
\[ V(p_t) = \max_{b_{t+1} \leq \bar{b}, \chi_{t+1}} u(p_t b_{t+1} - k) + \beta E[v \{a_{t+1} - (1 - \chi_{t+1}) b_{t+1} - \chi_{t+1} (1 - \theta) a_{t+1}\} | p_t] \] (11)

Ex ante, an entrepreneur will choose to undertake the risky project if and only if \( V(p_t) \geq \nabla \).

We denote the aggregate supply of bonds issued by entrepreneurs at a given price by \( B(p_t) \), which corresponds to \( b \) in the previous section.

### 4.2 Interior Equilibrium

In this section we characterize the natural extension of a stationary interior equilibrium for our baseline model as defined in Definition 2. In the extended model, informed managers
have superior information about the project productivity $a_{t+1}$, from which they can infer the entrepreneurs’ default decision $\chi_{t+1} = \chi(p_t, a_{t+1})$. Again, there are going to be three regimes, two with full revelation with default and no default respectively, and one where no information is revealed. The last regime is the more interesting one, where uninformed managers are the marginal traders, that is, are indifferent between investing in the bond and in the risk-free asset. This is the sense in which this type of equilibrium is named “interior”. In the extended model, the equilibrium prices are the same as in (i) in Definition 2, except for the case of $z_t \in \{0, \Gamma^t\}$, when the price is $p$ instead of $y_t/b$, where $p$ is such that $V(p) = \bar{V}$. Moreover, now the entrepreneurs’ borrowing and default decisions are also equilibrium objects. In particular, the equilibrium borrowing strategy is

$$b(p_t, y_t) = \begin{cases} \bar{b} \text{ with pr. } y_t/(p \bar{b}) \text{ and } 0 \text{ otherwise} & \text{if } p_t = p \\ b^* & \text{if } p_t = \bar{p}^* \\ \bar{b} & \text{if } p_t = 1/R \end{cases},$$

(12)

where $b^* > \bar{b}$ in $[0, \bar{b}]$, and the equilibrium default rule is $\chi(p_t, a_{t+1}) = 1 \{a_{t+1} < \hat{a}(p_t)\}$ where $\hat{a}(p_t) = b(p_t) / (1 - \theta)$. Finally, the managers’ demand schedules, the bond allocation and the investors’ firing rule are exactly the same as in (ii)-(iv) in Definition 2, with the caveat that borrowing and default are now endogenous, that is, $b = B(p_t) = b(p_t)$ and $\chi_{t+1} = \chi(p_t, a_{t+1})$, and $p$ substitutes $\bar{y}/b$.

Entrepreneurs do not have superior information about $a_{t+1}$ and also learn information from the equilibrium price only when it is fully revealing. If $p_t = p$, entrepreneurs know that all the managers think that there is going to be default and that they are going to invest in the risk-free asset. Hence, they do not have any incentive not to default and if they are able to issue some bonds, they always choose to default and borrow as much as they can, that is, $\bar{b}$. However, in this case, the demand for bonds comes only from noise traders and is equal to $y_t$. Hence, in order to have an equilibrium it must be that the entrepreneurs are indifferent between borrowing $\bar{b}$ and default with probability 1 and get their outside option $\bar{V}$. This implies that $p$ must be exactly such that $V(p) = \bar{V}$ and a fraction $y_t/(p \bar{b})$ of entrepreneurs will become active and borrow from noise traders, while the others will stay out of the market and just get their outside option. If $p_t = p^*$, entrepreneurs do not have any additional information on $a_{t+1}$ and choose $b^*$ and $\hat{a}$ to maximize their expected utility. Finally, if $p_t = 1/R$, they know that all the managers believe that there is not going to be default and choose to borrow $\bar{b}$ and default whenever $a_{t+1} < \bar{b}/(1 - \theta)$, which is never the case. It turns out that $\bar{b} \geq b^*$,
given that there is no risk of default for the managers.

In order to ensure that such an equilibrium exists, we still need assumption A3 to be satisfied together with two modified versions of assumptions A1 and A2, that is,

\[ k \geq \bar{y} \geq 2M^I, \]  

(A1’)

and

\[ M^I + \bar{y} < \frac{pB(p)}{2} \text{ and } \Gamma - M^I > \frac{1}{R}b \left( \frac{1}{R} \right). \]  

(A2’)

The first inequality of assumption A1’ ensures that the supply of the risky bond is always big enough to cover the demand of the noise traders as long as the price is different from 0, while the second inequality ensures that prices are not fully revealing with positive probability. Assumption A2’ ensures that there are always some uninformed managers investing in both the risky bond and the risk-less asset.

In Appendix A we fully characterized an interior equilibrium for the extended model and show that it can be described by the equilibrium price and default probability, \( p^* \) and \( q^* \). These values can be found by solving the following fixed point problem: find a pair \((p^*, q^*)\) such that \( p^* = P(q^*) \) and \( q^* = Q(p^*) \), where \( P(\cdot) \) is the pricing function for a given probability of default and \( Q(\cdot) \) is the endogenous probability of default for a given bond price. Both \( P(\cdot) \) and \( Q(\cdot) \) are derived in the appendix.

In the next proposition, we find sufficient condition for the existence of a stationary equilibrium. Before stating the result, let us define

\[ q_1 \equiv F \left( \frac{B(1/R)}{1 - \theta} \right) \text{ and } q_2 \equiv F \left( \frac{\bar{b}}{1 - \theta} \right). \]

Proposition 2 For a given \( M^I > 0 \), such that assumptions A1’, A2’, A3, and A4 hold and

\[ P(q_1) < 1/R \text{ and } P(q_2) > p, \]  

(13)

an interior equilibrium exists.

4.3 Limit Equilibrium

As in the baseline case, we focus on a limit interior stationary equilibrium where \( M^I \to 0 \), which is the natural extension of the equilibrium described in section 3.4.
**Definition 4** A limit interior equilibrium with \( M^I \to 0 \), is an equilibrium where \( p^* \in (\hat{y}/b, 1/R) \) is determined by the indifference condition of the uninformed managers; the informed managers’ demand function is \( d^I(x_{t+1}) = 1 - \chi_{t+1} \) and their bond allocation is \( x^I(x_{t+1}) = d^I(x_{t+1}) \); the uninformed managers’ demand correspondence is \( d^U = \{0, 1\} \) and their bond allocation is \( x^U(y_t) = (p^*b - y_t)/\Gamma \); the investors’ strategy is to fire manager \( j \) if he receives a negative exogenous signal or \( \theta^I_t \neq 1 - \chi(a_{t+1}) \), and keep him otherwise; the borrowers’ strategy is \( b^* \in [0, b] \) and \( \chi(p_t, a_{t+1}) = 0 \) if \( a_{t+1} \geq \hat{a}^* \) and \( \chi(p_t, a_{t+1}) = 1 \) otherwise, with \( \hat{a}^* = b^*/(1 - \theta) \).

Along the same lines of an interior equilibrium with any \( M^I > 0 \), Definition 4 implies that the limit equilibrium can be characterized by a constant bond price and default probability, \( p^* \) and \( q^* \). These values can be determined by solving the following fixed point problem: find a pair \( (p^*, q^*) \) such that \( p^* = P^L(q^*) \) and \( q^* = Q(p^*) \). On the one hand, the pricing rule \( P^L(\cdot) \) is exactly the same as in the baseline model, that is,

\[
P^L(q) = \frac{(1 - q)(1 - \delta q)}{[1 - \delta (1 - q)]R},
\]

On the other hand, the repayment rule \( Q(\cdot) \) is derived in the appendix from the entrepreneurs’ optimal behavior,\(^{10}\) and is given by \( Q(p) \equiv F(B(p)/(1 - \theta)) \) with \( B(p) \) implicitly defined by

\[
p'u(pB(p) - k) - \beta \int_{B(p)}^{\infty} v'(a_{t+1} - B(p)) \, dF(a_{t+1}) = 0.
\]

Next lemma establishes some important properties of entrepreneurs’ optimal behavior.\(^{11}\)

**Lemma 1** In a limit equilibrium, as \( p \) increases, (i) the face value of debt \( B(p) \) decreases, (ii) the probability of default \( Q(p) \) decreases, (iii) the value of the bonds \( pB(p) \) increases, and (iv) the ex-ante value of becoming active entrepreneurs \( V(p) \) increases.

As borrowing becomes cheaper, entrepreneurs need to borrow less and, hence, there is higher chance that they can repay their debt. This implies that the probability of default decreases as a function of the bond price, that is, \( q \) is downward sloping in the space \( (p, q) \) as shown in Figure 3. Moreover, thanks to assumption A4, the value of the borrowing \( pb \) increases, because entrepreneurs want to smooth consumption between the two periods of their life and,

\(^{10}\) The repayment rule \( Q(\cdot) \) in a limit equilibrium coincides with the one derived in the appendix for a general interior equilibrium with any \( M^I > 0 \), under the regime of no revelation.

\(^{11}\) Note that the same Lemma applies to the general equilibrium defined in 2 when the economy is in a not revealing regime.
hence, they decrease $b$ less than proportionally with respect to the initial increase of $p$. Finally, as intuition suggests, the value of being an entrepreneur is increasing with the price, given that the revenues from running the risky project are higher if the cost of borrowing is lower.

Once again, as a point of comparison, consider the benchmark equilibrium, where entrepreneurs behave exactly as described above, but all managers are uninformed, $M_I = 0$, investors never fire any manager, and managers maximize their period by period profit. Such an equilibrium can also be characterized by a fixed point $(a^B, p^B)$, where $q^B = Q(p^B)$, but the pricing rule is given by the standard no-arbitrage condition (8), that is, $p^B = P^B(q^B) \equiv (1 - q^B) / R$.

Proposition 2 guarantees that a limit equilibrium exists. The equilibrium regime is determined jointly by the fundamentals of the risky project and the state of the financial market. Figure 3 represents graphically both the limit equilibrium for an economy with career concerns (L) and the benchmark equilibrium (B). The prices in the two equilibria, respectively, $p^*$ and $p^B$, correspond to the intersections of the repayment rule $Q(p)$ and the corresponding pricing rule, that is, $P^L(q)$ and $P^B(q)$, graphed in the space $(p, q)$.

Figure 4: The red line represents the repayment rule and the blue curve and the green curve represent the pricing rule in the economy with career concerns and in the benchmark economy, respectively. Points L and B denote respectively the limit interior equilibrium when $M_I \to 0$ and the benchmark equilibrium when $M_I = 0$. 

22
Notice that the reputational premium is zero iff \( q^* = q^B = 1/2 \) and \( p^* = p^B = 1/2R \). Moreover, both \( P^L (q) \) and \( P^B (q) \) are decreasing in \( q \). This proves the following proposition.

**Proposition 3** In equilibrium, one of the following regimes arises: (i) if \( q^* = 1/2 \), then \( \Pi^* = 0 \), (ii) if \( q^* < 1/2 \), then \( \Pi^* < 0 \); (iii) if \( q^* > 1/2 \), then \( \Pi^* > 0 \).

In the baseline numerical exercise we assume that \( u (c) = \log c \), \( v (c) = c \) and \( F (a) = 1 - a^\gamma a^{-\gamma} \). We work out this example in detail in Appendix C. In Figure 4, we choose \( a = 1.5 \), \( \gamma = .6 \), and \( k = .45 \). Such parameters implies that \( q^* < 1/2 \) and then \( p^* > p^B \), that is, that the reputational premium is positive. Changing the parameters, we can easily obtain the analogous figure where \( p^* < p^B \) and the reputational premium is negative. For example, this is the case if we decrease the lower bound for the productivity process \( a \) to 1.

### 4.4 Amplification

Next, we analyze some interesting properties of the limit equilibrium. In particular, we are interested in the reaction of the economy to shocks both to financial markets and to the fundamentals of the risky project. The first type of shocks affect the pricing rule and we refer to them as demand-side shocks; the second type affect the repayment rule and we label them supply-side shocks. Our main result is that there is an amplification effect that magnifies the reaction to both types of shocks of our equilibrium in comparison to the benchmark model. The mechanism behind this result is that both types of shocks can move the economy from one regime to the other, generating a natural amplification in the price and in the default probability.

Let us focus on the vector of parameters \( \sigma = \{ R, \alpha \} \), where \( R \) is the return on the risk-free asset and \( \alpha \) represents a parameter affecting the distribution of the productivity shock \( a \), such that if \( \alpha'' > \alpha' \), then \( F (a|\alpha'') < F (a|\alpha') \). A change in \( R \) represents a typical demand-side shock, that is, a change in the return of alternative investment opportunities. A change in \( \alpha \), instead, represents a first-order stochastic shift of the productivity distribution of the risky project, that is, a typical supply-side shock.

With a slight abuse of notation, let us denote by \((q^* (\sigma), p^* (\sigma))\) the default probability and price in the limit equilibrium when \( M^I \to 0 \) and the parameters are \( \sigma \), and \((q^B (\sigma), p^B (\sigma))\) the default probability and price in the benchmark equilibrium when \( M^I = 0 \) and the parameters
are $\sigma$. When there are multiple equilibria, let us focus on the equilibrium with the highest bond price.

Next proposition states our main amplification result.

**Proposition 4** Suppose there exists a pair $(\sigma', \sigma'')$ such that $q^B(\sigma') < 1/2$, $q^B(\sigma'') > 1/2$, and $q^*(\sigma'') > 1/2$. Then, there is amplification, that is, $q^*(\sigma'') - q^*(\sigma') > q^B(\sigma'') - q^B(\sigma')$ and $p^*(\sigma'') - p^*(\sigma') > p^B(\sigma'') - p^B(\sigma')$.

Proposition 4 shows that if there is a change in the parameters such that the equilibrium switches regime from a positive premium to a negative premium, then both prices and default probabilities respond more than in the benchmark model. Suppose, for example, we start from a regime where the reputational premium is negative. As the outside investment opportunities improve, that is, $R$ increases, the bond price decreases making borrowing more expensive and default happening more often. If the shock is big enough, it can generates a shift in the sign of the premium and a switch of regime. Alternatively, the economy can move from a regime to another because of a change in the parameters on the supply-side of the model. For example, a big enough decrease in $\alpha$ can increase the default probability enough to make the reputational premium negative. The effect on both prices and quantities is amplified in comparison to the benchmark model.

Assume that $\lim_{\alpha \to -\infty} F(a_t|\alpha) = 1$. Next proposition shows that when there exists a unique interior equilibrium for a given set of parameters, it is possible to change $R$ or $\alpha$ enough that the regime shifts.

**Proposition 5** Suppose that $\lim_{\alpha \to -\infty} F(a_t|\alpha) = 1$ and that there exists a unique interior equilibrium with $\sigma' = \{R',\alpha'\}$ such that $Q(p^*|\alpha') < 1/2$, where $\alpha' \in [\underline{a}, \overline{a}]$ for some $\underline{a}$ and $\overline{a}$. Then,

1. if $Q(p|\alpha') > 1/2$, there is an $\tilde{R} > R'$ such that for any $R'' > \tilde{R}$, $Q(p^*|\alpha') > 1/2$;
2. there is an $\hat{\alpha}$ with $\underline{a} < \hat{\alpha} < \alpha'$ such that for any $\alpha''$ with $\underline{a} < \alpha'' < \hat{\alpha}$, $Q(p^*|\alpha'') > 1/2$.

In the example illustrated in figure 3 and worked out in Appendix C, there is a unique equilibrium.$^{12}$

---

$^{12}$In particular, it is possible to show that if $u(c) = \log c$, $v(c) = c$ and $F(a) = 1 - \frac{a^\gamma}{a^{-\gamma}}$, there exists a unique equilibrium if $k$ is small enough.
Propositions 4 and 5 show that as the financial environment or the fundamentals of the risky project change, the economy can switch from a regime with low bond spreads (high $p$) and high level of capital invested in the risky bond market (high $pb$) to a regime with high bond spreads (low $p$) and low level of capital invested (low $pb$). The first type of regimes are frequently described as regimes of abundant liquidity or with traders reaching for yield. To describe phenomena where the economy switch to the second type of regime, common terms are flight-to-quality, flight-to-liquidity, disappeared liquidity, or drop in risk appetite. In our model, phenomena of this type can arise even if fund managers are risk-neutral and their aggregate funds are constant. We argue that abrupt changes in prices can be caused by managers’ career concerns. In good times, when the default probability of credit instruments is low, it is very attractive for uninformed fund managers to invest in these instruments because if they prefer less risky investment opportunities, they are likely to produce lower returns, lose reputation, and, hence, funds. If suddenly a negative shock hits either the demand or the supply side of the market, the probability of default increases, and investing in the risk-free asset increases the probability of losing their reputation. Hence, prices increase not only because of the higher probability of default, but also because of an additional premium coming from reputational concerns. This generates the amplification result we have discussed.

Our main result is that the impact of shocks can be amplified by the sign change in the reputational premium, leading to excess volatility of the bond price, default probability, and capital flows. This is consistent with the empirical evidence that shows that emerging market bond prices fluctuate more than what is accounted for by changes in probability of default. On the one hand, Broner, Lorenzoni and Schmukler (2007) argue that the premium over the expected repayment on emerging market bonds is especially high during crises times. On the other hand, Duffie et al. (2003) document that the implied short spread of Russian bonds was very low during the first 10 months of 1997. Moreover, their estimation shows that in one short interval in 1997, bond prices were so high that the implied default adjusted short spread was negative. Although this observation is model specific, it is still interesting to point out that this is inconsistent with most risk-aversion based explanation, but consistent with our model. Note also that the result that demand-side shocks can be important determinants of bond prices is broadly consistent with the empirical evidence that a large proportion of the variation in prices of both corporate bonds and emerging market bonds cannot be explained by the variation of fundamentals, and that a large part of this unexplained component is common

5 Persistent Productivity Shock

In this section, we introduce persistency in the productivity process of the risky project. In particular, assume that $a_{t+1}$ is distributed according to a first-order Markov process with cumulative density function $G(a_{t+1}|a_t)$. The environment is a natural generalization of the one with i.i.d. shock, where $a_t$ represents an additional state variable. We look for Markovian equilibria.

5.1 Equilibrium characterization

The equilibrium we focus on is a natural generalization of the interior limit equilibrium described in definition 3.

**Definition 5** A Markovian interior equilibrium with $M^I \rightarrow 0$, is an equilibrium where $p^*(a_t) \in (0,1/R)$ is determined by the indifference condition of the uninformed managers; the borrowers’ strategy is $b^*(a_t) > 0$ and $\chi(y_t,a_{t+1}) = 0$ if either $a_{t+1} \geq \hat{a}^*(a_t)$, and $\chi(y_t,a_{t+1}) = 1$, otherwise; the informed managers’ allocation is $x^I(y_t,a_{t+1}) = 1 - \chi(y_t,a_{t+1})$; the uninformed managers’ allocation is $x^U(y_t,a_t) = (p^*(a_t) b^*(a_t) - y_t) / \Gamma$; the investors’ strategy is to fire manager $j$ if $\theta_j \neq 1 - \chi(y_t,a_{t+1})$ and keep him otherwise.

When the process for $a_t$ is not i.i.d., the expected utility of the uninformed managers at time $t$ depends on the realization of $a_t$, because they can use the past information to update the distribution of $a_{t+1}$, that is,

$$W(a_t) = \gamma R + \delta \int_0^{\hat{a}(a_t)} W(a_{t+1}) dF(a_{t+1}|a_t).$$

(14)

Moreover, their indifference condition becomes

$$(1 - q(a_t)) \frac{\gamma}{p(a_t)} + \delta \int_{\hat{a}(a_t)}^{\infty} W(a_{t+1}) dF(a_{t+1}|a_t) = \gamma R + \delta \int_0^{\hat{a}(a_t)} W(a_{t+1}) dF(a_{t+1}|a_t).$$

This condition implicitly defines the equilibrium price as a function of the state $a_t$, $P(a_t, q(a_t)) = p(a_t)$, and the default rule $\hat{a}(a_t)$ where $\hat{a}(a_t) = F^{-1}(q(a_t))$.

Also the borrowers update their expectation of the distribution of $a_{t+1}$, conditional on $a_t$. Their default rule is $Q(a_t,p(a_t)) = F(B(a_t,p(a_t)) / (1 - \theta) | a_t)$, where $B(a_t,p(a_t)) = b(a_t)$.
is implicitly defined by
\[
p(a_t) u'(p(a_t) b(a_t) - k) - \beta \int_{b(a_t)}^{\infty} v'(a_{t+1} - b(a_t)) dF(a_{t+1}|a_t) = 0.
\]

Hence, a Markovian interior equilibrium is characterized by a fixed point such that \( p^*(a_t) = P(a_t, q^*(a_t)) \) and \( q^*(a_t) = Q(a_t, p^*(a_t)) \). When \( M^I = 0 \) and the shock \( a_t \) is persistent, the benchmark equilibrium is defined as a fixed point such that \( q^B(a_t) = Q(a_t, p^B(a_t)) \) and
\[
p^B(a_t) = \frac{1 - q^B(a_t)}{R}.
\]

### 5.2 Numerical example

Here we present some numerical exercises to illustrate the dynamic properties of our equilibrium when productivity shocks are persistent. In particular, we show how career concerns can magnify the reaction of the economy to shocks, hence, increasing the volatility of prices.

First, we show how the default probability, the bond price, the amount of capital borrowed by entrepreneurs, and the reputational premium vary with the realization of the productivity shock. Let us start with the equilibrium behavior in the benchmark economy. As a bad shock hits, the financial market will realize that, even for a given default rule, the probability of default will be higher and will require a lower bond price. As borrowing becomes more expensive, borrowers will then increase their default cut-off, magnifying the reduction in the bond price. A lower bond price also decreases the amount of capital entrepreneurs will borrow, so capital flows out from the market of risky bonds. Hence, for low realizations of productivity, the default cut-off will be higher and the bond price and the dollar value of outstanding bonds lower. However, the change in the bond prices is limited by the fact that the expected pay-off from holding the bonds will remain constant. Now, consider the economy with career concerns. Suppose the default probability is high enough that the reputational premium is positive. In this case, the financial market will require a bond price even lower than the benchmark economy because of the reputational premium. Given that productivity is persistent, a bad realization of the shock will further increase the probability of default, increasing the fear of the uninformed managers of being fired and pushing the bond price further down. This implies that the reputational premium itself is higher after bad shocks. Moreover, if the economy starts from a regime where the reputational premium is negative, a bad shock not only increases the
premium, but can even make it switch sign. Thus, the effect of the productivity shock on the bond price, the probability of default and the capital flows is amplified by the career concerns of managers.

We report the numerical results for an example similar to the one illustrated on Figure 3 with \( u(c) = \log c \) and \( v(c) = c \). However, now \( \log (a_t) \) follows an AR(1) process.\(^{13}\) Figures (5) and (6) show how the reputational premium, the bond price, and the default probability vary in equilibrium with the different realizations of the productivity shocks.

![Figure 5](image_url)

Figure 5: The figure shows the reputational premium as a function of the realization of \( \log(a) \). The blue line is the premium with career concerns and the green line shows the premium in the benchmark case.

Now, consider an economy that at time zero is hit by a shock. Figure 7 shows how the equilibrium prices react in expected terms to a bad and to a good shock, both with and without career concerns. The figure shows our amplification result: the economy with career concerns reacts more to the shocks than the benchmark economy. Moreover, notice that in the economy considered, the reputational premium would be negative in expected terms and a bad shock can actually make the economy shift regime.

\(^{13}\)Figures (5), (6) and (7) use the following process for \( a_t \): \( \log (a_{t+1}) = (1 - \rho) \mu + \rho \log (a_t) + \varepsilon_t \) with \( \rho = .7 \), \( \mu = 2.8 \) and \( \varepsilon_t \sim N(0, 2) \). Moreover, \( \beta = .75 \), \( \delta = .5 \), \( \gamma = 1 \), \( k = .4 \), and \( \theta = .1 \).

28
6 Conclusion

In this paper, we have proposed a general equilibrium model of delegated portfolio management with endogenous default. Investors hire fund managers to invest their capital either in a defaultable bond or in a riskless one and only a small fraction of managers have precise information about the default risk. Looking at the past performance, investors update their beliefs on the information of their fund managers. This leads to career concerns that affect the managers’ investment decisions, generating a “reputational premium”. When the probability of default is sufficiently high, fund managers prefer to invest in safe bonds even at a lower expected return to reduce the probability of being fired. On the contrary, if the probability of default is low enough, investing in the risky bond has a reputational advantage. The reputational premium can switch sign in response to shocks, both to the financial market and to the fundamentals of the risky project that requires financing. This can generate an overreaction of the market leading to excess volatility of spreads and capital flows.

For future research, it would be interesting to introduce alternative risky assets in the portfolio choice of the managers. In this case, our mechanism would generate contagion.
Figure 7: The two panels show the reaction of the equilibrium prices to a bad and a good shock, respectively. The blue line represents the price in the benchmark economy, and the green line the price in the economy with career concerns. At time zero productivity drops to the lowest possible realization in the first case and rises to the highest possible one in the second case.

Imagine that there are two risky bonds and a risk-less asset. The reputational cost of investing in the risk-less asset depends on the default probability of both the risky bonds. If none of them defaults, the manager who invests in the risk-less bond will lose his reputation. Thus, if the probability of default of any of the risky bonds decreases, the risk-less asset will be less attractive, and the prices of both bonds will have to increase in order to make uninformed managers indifferent between different investment opportunities.

Finally, an interesting application of our model is to emerging market bonds. A large literature on business cycle characteristics of emerging markets\footnote{See Neumeyer and Perri (2005), Uribe and Yue (2006), Arellano (2006), Aguiar and Gopinath (2006), Longstaff et al (2007)} highlights that emerging market bond spreads are very volatile. Also, capital flows are more volatile in small emerging market economies than in developed economies of comparable size. In particular, the magnitude of volatility of interest rates is hard to reconcile with models where bond prices are determined by the standard no-arbitrage condition. Our model provides an appealing framework to think about this excess volatility. It would be interesting to calibrate our model to quantify how...
much of the volatility of specific emerging markets bonds can be explained with our mechanism.

Appendix A

Baseline model

Here we want to show that an allocation as the one described in definition 2 is actually an equilibrium.

Revelation regimes. First of all, let us show that in an interior equilibrium, each period there are three possible regimes: the bond price is fully revealing and the entrepreneurs default for sure, the price is fully revealing and the entrepreneurs do not default, or the price does not reveal any information. Given that $y_t$ is uniformly distributed on $[0, \bar{y}]$, $z_t$ must be in $[0, \bar{y} + \Gamma']$. From the equilibrium pricing condition (3), if $p_t \leq \bar{y}/b$, then $z_t \in [0, \Gamma']$ and the uninformed managers discover that $\chi_{t+1} = 1$. If, instead, $p_t = 1/R$, then $z_t \in (\bar{y}, \bar{y} + \Gamma']$ and they know that $\chi_{t+1} = 0$. In both these cases, the price is fully revealing. When instead $p_t = p^*$, then $z_t \in [\Gamma', \bar{y}]$ and the uninformed managers update their beliefs about the probability of default as follows:

$$
\Pr (\chi_{t+1} = 1 | p_t = p^*) = \frac{\Pr (\chi_{t+1} = 1, z_t \in [\Gamma', \bar{y}])}{\Pr (\chi_{t+1} = 1, z_t \in [\Gamma', \bar{y}]) + \Pr (\chi_{t+1} = 0, z_t \in [\Gamma', \bar{y}])}.
$$

Notice that $z_t$ can be in $[\Gamma', \bar{y}]$ in two cases: when $\chi_{t+1} = 1$ and $y_t \in [\Gamma', \bar{y}]$ and when $\chi_{t+1} = 0$ and $y_t \in [0, \bar{y} - \Gamma']$. Given that $y_t$ is uniformly distributed, the first case arises with probability $q (\bar{y} - \Gamma') / \bar{y}$ and the second with probability $(1 - q) (\bar{y} - \Gamma') / \bar{y}$. It follows that $\Pr [\chi_{t+1} = 1 | p = p^*] = q$. This shows that, thanks to the uniform distribution of $y_t$, the price $p^*$ does not reveal any information. Let us define $\pi = \Gamma' / \bar{y}$ the probability that in equilibrium the price is fully revealing.

Managers’ flows. In an interior equilibrium, thanks to free-entry, the uninformed managers make zero profits ex-ante, that is,

$$
\mu W - \kappa = 0,
$$

where $W$ represents the expected utility of an employed uninformed manager. On the contrary, the informed managers make positive profits, given that they never make a mistake and hence
they are never fired. It follows that all of them will choose to pay the cost $\kappa$ at each point in time, so that $N_I^t = M^I - \delta \Gamma^I_t$.

Moreover, in an interior equilibrium, $\Gamma^I$, $\Gamma^U$ and $\mu$ are constant over time. In particular, $\Gamma^I$ must be such that the mass of employed informed managers stay constant after $\delta$ of them die and $\mu$ of the unemployed ones are hired, that is,

$$\Gamma^I = \delta \Gamma^I + \mu (M^I - \delta \Gamma^I).$$

(16)

Also, $\Gamma^U$ must be such that the mass of employed uninformed managers stay constant after $1 - \delta$ of them die, $1 - \omega \xi_t$ of them are fired and $\mu$ of the unemployed uninformed are hired, that is,

$$\Gamma^U = \delta \Gamma^U + (1 - \omega \xi_t) \delta \Gamma^U + \mu N_U^t,$$

(17)

where $\xi_t$ denotes the proportion of uninformed managers who do not make mistakes, that is,

$$\xi_t = \left\{ \begin{array}{ll}
(1 - x_U^t) \chi_{t+1} + x_U^t (1 - \chi_{t+1}) & \text{if } z_t \in [\Gamma^I, \bar{y}] \\
1 & \text{if } z_t \notin [\Gamma^I, \bar{y}]
\end{array} \right..$$

(18)

Condition (17) shows that in order to obtain a constant equilibrium value for $\Gamma^U$, the mass of uninformed managers who are unemployed $N_U^t$ must change over time together with $\xi_t$, in order to guarantee that there is a stationary equilibrium.

**Bond’s demand and allocation.** Here, we show that the demand schedules of the hired managers, who maximize their expected utility taking as given the equilibrium regime, price, bond allocation, and strategies of other agents, are consistent with an interior equilibrium.

When $p_t = 1/R$, there is full revelation that there is no default, so the risky bond pays for sure $1/R$ and is equivalent to the risk-free asset. It follows that both the informed and the uninformed managers are indifferent between the two bonds and $d^I(p_t, \chi_{t+1}) = d^U(p_t) = \{0, 1\}$ if $p_t = 1/R$. In this case, the auctioneer is free to decide whether to allocate or not the risky bond to any manager. Market clearing requires that he allocates the bond to a faction $(b/R - y_t)/\Gamma$ of both types of managers. Hence, in equilibrium $x^I(\chi_{t+1}, y_t) = x^U(\chi_{t+1}, y_t) = (b/R - y_t)/\Gamma$.

When instead $p_t < 1/R$, informed managers will always choose to invest in the risk-free asset if there is going to be default, and in the bond if there is not going to be default, that is, $d^I(p_t, \chi_{t+1}) = 1 - \chi_{t+1}$. This is optimal for them given that investors fire managers who do not do that. Also, they do not have any incentive to deviate, given that the bond price is smaller
than $1/R$. In this case the auctioneer has to satisfy their demand and assign them the bond iff they demand it, so that $x^I(\chi_{t+1}, y_t) = 1 - \chi_{t+1}$. However, the uninformed managers cannot follow the same strategy because they do not know $\chi_{t+1}$. When $p_t \leq \bar{y}/b$, there is full revelation and they discover that there is going to be default. Hence, they demand 0 bonds in order to avoid to be fired and $d^U(p_t) = 0$. If, instead, $p_t = p^*$, there is no information revelation and assumption A1 together with market clearing requires that some of them invest in the risky asset and some of them in the risk-less one, that is, that $x^U(\chi_{t+1}, y_t) \in (0, 1)$. Given that the auctioneer’s allocation must be consistent with the managers’ demand schedule, when $p_t = p^*$ the uninformed managers must be indifferent between the two assets, that is, $d^U(p_t) = \{0, 1\}$.

Notice that nobody would demand some of both assets, since he would reveal immediately to be uninformed. It follows that in order to have an equilibrium the price $p^*$ must guarantee this indifference condition. This is the notion of interior solution we refer to when we call the equilibrium an interior equilibrium. In particular, the market clearing condition requires $x^U(\chi_{t+1}, y_t) = (bp^* - z_t)/\Gamma^U$.

**Equilibrium prices.** Given the equilibrium conditions (ii)-(iv) in Definition 2, it is easy to show that the equilibrium price satisfies (3).

If $z_t \in [0, \Gamma^I)$, neither informed or uninformed managers demand the risky bond. Hence, market clearing requires that $p(y_t, \chi_{t+1}) = y_t/b$ and there is full revelation of default. If $z_t \in (\bar{y}, \bar{y} + \Gamma^I]$, all the managers are indifferent between the two assets. This requires that $p(y_t, \chi_{t+1}) = 1/R$ and there is full revelation of no default. If $z_t \in [\Gamma^I, \bar{y}]$, there is no information revelation and the price $p_t(y_t, \chi_{t+1}) = p^* \in (\bar{y}/b, 1/R)$ must be such that the uninformed managers are indifferent between demanding the risky asset and the risk-free one. That is, it is implicitly defined by

$$ (1 - q) \left( \frac{\gamma}{p^*} + \delta \omega W \right) = \gamma R + q \delta \omega W, \tag{19} $$

where $W$ is the expected continuation utility of an uninformed manager who keeps the job and is given by

$$ W = \gamma R + \delta \omega \left[ \pi + q (1 - \pi) \right] W. \tag{20} $$

Recall that $\pi = \Gamma^I/\bar{y}$ is the probability of full revelation, where $\Gamma^I$ defined by the combination of (15), (16), and (20). When the price is fully revealing, with probability $\pi$, uninformed managers always gain $\gamma R$, given that if they learn that there is default they invest in the risk-free asset, while if they learn that there is no default, the bond price will be equal to
1/R. Moreover, they are never fired and get the continuation utility W, as long as there is no exogenous signal. When instead the price does not reveal any information, with probability 1 – π, uninformed managers are indifferent in each point in time between investing in the risk-free asset or in the risky bond, and hence their expected utility can be calculated as the value of always investing in the risk-free asset. Again, they will always gain γR, but, in this case, they will get the continuation utility W only if there is default and no exogenous signal.

By combining equations (19) and (20) we immediately obtain the not revealing price as a function of the default probability q, that is, we obtain the following pricing rule:

\[ p^* = P(q) \equiv \frac{(1-q)(1-\delta[\pi + q(1-\pi)])}{R[1-\delta(1-q)(1+\pi)]}. \]  

(21)

It is immediate to see that \( p^* \in (\bar{y}/b, 1/R) \) as we wanted to show.

**Firing rule.** Finally, we show that the investors choose the equilibrium firing rule, once they maximize their expected utility, taking all the other equilibrium objects as given.

First of all, notice that the investors are indifferent between an informed and an uninformed manager when \( p_t = 1/R \), given that in this case all the managers choose the same demand schedule. However, when \( p_t < 1/R \) they prefer to have informed managers investing their capital. Hence, in order to make their firing decisions, investors need to assess the probability that employed managers are informed. At the end of time \( t \), each investor observes the investment realization of his manager \( \theta^j_t \) and the default realization \( \chi_{t+1} \). Then if \( p_t < 1/R \) and either \( \chi_{t+1} = 0 \) and \( \theta^j_t = 1 \), or \( \chi_{t+1} = 1 \) and \( \theta^j_t = 0 \), he realizes that his manager is not informed, that is, \( \eta_{t+1}^j = 0 \). In this case, the investor fires him, since there is always a positive probability that a new manager is informed and the probability of finding a new manager is always equal to 1. For the same reason, the investor also fires an uninformed manager whose type is revealed by an exogenous signal with probability \( 1 - \omega \). On the other hand, if the manager does not make a mistake, that is, if \( \theta^j_t = \chi_{t+1} \) and/or \( p_t = 1/R \), so that he does not reveal to be uninformed, then he is not fired if and only if his updated belief \( \eta_{t+1}^j \) is higher than the probability that a new hire is informed, denoted by \( \varepsilon_{t+1} \). By definition \( \varepsilon_{t+1} \) satisfies

\[ \varepsilon_{t+1} = \frac{M^I - \delta \Gamma^I}{M^I - \delta \Gamma^I + N^U_t} > 0. \]

When manager \( j \) realizes \( \theta^I_t = 1 - \chi_{t+1} \) and/or \( p_t = 1/R \), the investor’s belief is updated as follows:

\[ \eta_{t+1}^j = \frac{\eta^j_t}{\eta^j_t + \omega \xi_t(1 - \eta^j_t)}, \]
where $\xi_t$, defined in (18), represents the proportion of uninformed managers who make the same investment decision of the informed managers. Next, we show that assumption A3 is sufficient to make sure that in equilibrium $\eta^j_{t+1} \geq \varepsilon_{t+1}$ for any $\varepsilon_t$ and $\eta^j_I > 0$.

First, consider an investor who has just hired manager $j$ and hence, by definition, has prior belief $\eta^j_t = \varepsilon_t$. In this case, if $\theta^j_t = 1 - \chi_{t+1}$, then

$$
\eta^j_{t+1} = \frac{\varepsilon_t}{\varepsilon_t + \omega \xi_t (1 - \varepsilon_t)}.
$$

Next, we want to show that $\eta^j_{t+1} \geq \varepsilon_{t+1}$. This condition can be rewritten as

$$
1 - \frac{\varepsilon_{t+1}}{\varepsilon_{t+1}} \geq \left(1 - \frac{\varepsilon_t}{\varepsilon_t}\right) \omega \xi_t,
$$

Using the expression for $\varepsilon_t$, we have that

$$
1 - \frac{\varepsilon_{t+1}}{\varepsilon_{t+1}} = \frac{N^U_t}{M^I - \delta T^I},
$$

and hence, condition (22) can be rewritten as

$$
\frac{N^U_t}{N^U_{t-1}} \geq \omega \xi_t,
$$

where

$$
N^U_t = (1 - \omega \xi_t \delta) \frac{\Gamma^U}{\mu}
$$

Hence, in order for (22) to be satisfied it must be that

$$
1 - \delta \omega \xi_t > (1 - \delta \omega \xi_{t-1}) \omega \xi_t,
$$

which is ensured by assumption A3.

Let us now consider managers who were working for an investor for longer than 1 period. First, notice that the investors’ beliefs about any manager who is still working but was hired at time $t - \tau$ with $\tau \in [0, t]$ must be higher than the initial belief $\varepsilon_{t-\tau}$, given that if he was not fired he never made any mistake, that is, $\eta^j_t \geq \varepsilon_{t-\tau}$. Hence, the belief about a manager who was hired at time $t - \tau$ and did not make a mistake at time $t$ is

$$
\eta^j_{t+1} = \frac{\eta^j_t}{\eta^j_t + \omega \xi_t (1 - \eta^j_t)} \geq \frac{\varepsilon_{t-\tau}}{\varepsilon_{t-\tau} + \omega \xi_{t-\tau} (1 - \varepsilon_{t-\tau})}.
$$

Hence, a sufficient condition for this guy not being fired is

$$
1 - \frac{\varepsilon_{t+1}}{\varepsilon_{t+1}} \geq \left(\frac{1 - \varepsilon_{t-\tau}}{\varepsilon_{t-\tau}}\right) \omega \xi_{t-\tau},
$$
which, by the same argument, is satisfied when assumption A3 holds. For the same reason, when \( p_t = 1/R \) no manager is fired, given that there is no information in their action, completing the proof.

**Extended model**

Here we want to show that the proposed interior equilibrium for the extended model is actually an equilibrium. The equilibrium prices are identical to the ones in the baseline model, except for the price in the regime with full revelation of default, which is now \( \tilde{p} \) instead of \( \tilde{y}/b \), where \( \bar{p} \) is such that \( V(p) = \bar{V} \). The proof that these prices are actually consistent with the equilibrium is analogous to the one for the baseline model above. The proof that the managers’ demand schedule and the investors’ firing rule are actually optimal, taking as given all the other equilibrium objects is also analogous to the one for the baseline model above, with the only caveat that now \( t + 1 = (a_{t+1}, y_t) \), \( b = b(p_t, y_t) \) and \( p \) substitutes \( \tilde{y}/b \). With the same modifications, the bond allocation is the same of the baseline model and follows from the bond market clearing as we have shown above. The new part of the proof is to show that the borrowing and default decisions of the entrepreneurs are actually optimal, taking as given the other equilibrium objects. That is, that in equilibrium (12) is satisfied, with \( \tilde{b} < b^* \) in \([0, \tilde{b}]\), and \( \chi(p_t, a_{t+1}) = 1 \{ a_{t+1} < \tilde{a}(p_t) \} \), where \( \tilde{a}(p_t) = b(p_t) / (1 - \theta) \).

Active entrepreneurs choose their default rule and how much to borrow and to consume in order to solve problem (11), taking \( p_t \) as given. Let us first consider the default decision of an old entrepreneur. For a given realization of the shock \( a_{t+1} \), she will default if and only if \( a_{t+1} - b_{t+1} < \theta a_{t+1} \). Then \( \chi(p_t, a_{t+1}) = 1 \) if \( a_{t+1} \leq \tilde{a}(p_t) \) and \( \chi(p_t, a_{t+1}) = 0 \), otherwise, with

\[
\tilde{a}(p_t) = \frac{b(p_t)}{1 - \theta}.
\]

Notice that the threshold \( \tilde{a}(p_t) \) is increasing in \( b(p_t) \), that is, as intuition suggests, the more the entrepreneurs borrow, the higher is the probability of default.

Given that \( y_t \) is uniformly distributed between 0 and 1, the signal of the uninformed \( z_t = \Gamma^I (1 - \chi_{t+1}) + y_t \) must be in \([0, \tilde{y} + \Gamma^I]\). It follows that if \( z_t \in (\tilde{y}, \tilde{y} + \Gamma^I) \), the uninformed managers will know that \( \chi_{t+1} = 0 \) and if \( z_t \in [0, \Gamma^I) \), they will know that \( \chi_{t+1} = 1 \). There are going to be two possible cases with fully revealing prices: \( z_t \in (\tilde{y}, \tilde{y} + \Gamma^I) \) and \( p_t = 1/R \), or \( z_t \in [0, \Gamma^I) \) and \( p_t = \tilde{p} \).

In the first case it must be that \( y_t \geq 1/\Gamma^I \) and \( \chi_{t+1} = 0 \). Define \( \tilde{a}^e \) the default rule expected
by the informed managers. If the price is equal to $1/R$, it must be that $a_{t+1} \geq \hat{a}^e$. In order for this to be an equilibrium it must be that when $p_t = 1/R$, entrepreneurs choose $\hat{a}^1 = \hat{a}^e$.

When $p_t = 1/R$, the entrepreneurs problem can be written as

$$b^1 = \arg \max_b u\left(\frac{b}{R} - k\right) + \frac{\beta}{1 - F(\hat{a}(b^e))} \int_{\hat{a}^e}^{\min\{\frac{b}{1-\gamma}, \hat{a}^e\}} v(\theta a) \, dF(a)$$

$$+ \frac{\beta}{1 - F(\hat{a}^e)} \int_{\min\{\frac{b}{1-\gamma}, \hat{a}^e\}}^{\hat{a}^e} v(a - b) \, dF(a),$$

given that they know that the informed managers are investing in the risky bond and hence, that they believe that $a_{t+1} \geq \hat{a}^e$. Hence to have an equilibrium, $b^1$ must be such that $b^1 = (1 - \theta) \hat{a}^e$ and such that $g(b^1) = 0$, where

$$g(b) \equiv \frac{1}{R} u'(\frac{b}{R} - k) - \frac{\beta}{1 - F(\frac{b}{1-\gamma})} \int_{\frac{b}{1-\gamma}}^{\hat{a}^e} v'(a - b) \, dF(a).$$

Moreover, to be sure that this is an equilibrium, we have to check that if prices where not revealing, entrepreneurs would default for any $a_{t+1} < \hat{a}^e$, that is, that, when $p_t \in (0, 1/R)$, they choose $b^*$ such that $b^* \leq b^1$ and $\hat{a}^e \leq \hat{a}$, where $\hat{a}^* \equiv \hat{a}(b^*)$ and $\hat{a} \equiv \hat{a}(b^1)$. We know that, for any $p_t \in (0, 1/R)$, $b^*$ solves $\tilde{g}(b^*) = 0$, where

$$\tilde{g}(b) \equiv p_t u'(p_t b - k) - \beta \int_{\frac{b}{1-\gamma}}^{\hat{a}^e} v'(a - b) \, dF(a).$$

Hence, it is enough to show that $\tilde{g}(b^1) > 0$. This comes straight from the assumption that $-cu''(c)/u'(c) > 1$.

In the second case, when $z_t \in [0, \Gamma']$, the price will be $p_t = p_z$. In this case, it must be that $y < \Gamma'$ and $x_{t+1} = 1$. Hence, if the price is equal to $p_z$, it must be that $a_{t+1} < \hat{a}^e$. In order for this to be an equilibrium it must be that when $p_t = p_z$, entrepreneurs choose $b^2 = (1 - \theta) \hat{a}^e$.

When $p_t = p_z$, the entrepreneurs problem can be written as

$$b^2 = \arg \max_b u(p z_t - k) + \frac{\beta}{F(\hat{a}(b^e))} \int_{\hat{a}^e}^{\min\{\frac{b}{1-\gamma}, \hat{a}^e\}} v(\theta a) \, dF(a)$$

$$+ \frac{\beta}{F(\hat{a}(b^e))} \int_{\min\{\frac{b}{1-\gamma}, \hat{a}^e\}}^{\hat{a}^e} v(a - b) \, dF(a),$$

given that they know that the informed managers are investing in the risk-free asset and hence, that they believe that $a_{t+1} < \hat{a}^e$. It is easy to see that when $p_t = p_z$, entrepreneurs will choose $b^2 = \bar{b}$ and, hence, will choose always to default.
In order for this to be an equilibrium, it must be that $p$ is such that

$$\bar{V} = u(p\tilde{b} - k) + \frac{\beta}{F\left(\frac{\tilde{b}}{1-\theta}\right)} \int_0^{\tilde{b}/(1-\theta)} v(\theta a) \, dF(a),$$

so that when $p_t = p$, the entrepreneurs are indifferent between undertaking the risky project, borrow $b$ and always default, or take their outside option $\bar{V}$. This, guarantees that they can choose a mixed strategy to ensure that the bond market clears. When $p_t = p$ the demand for the risky bond is given by $y_t$, so that to have market clearing it must be that the entrepreneurs choose to borrow with probability $y_t/(p\tilde{b})$, and take their outside option and borrow 0 with probability $1 - y_t/(p\tilde{b})$.

Finally, when $z_t \in [\Gamma^I, \tilde{y}]$, the price $p^*$ does not reveal any information, and the entrepreneurs have no information on the realization of $a_{t+1}$. After substituting the default decision $\chi(p_t, a_{t+1})$, problem (11) for a given $p \in (p, 1/R)$ becomes

$$\max_b u(p b - k) + \frac{\beta}{F\left(\frac{b}{1-\theta}\right)} \int_0^{b/(1-\theta)} v(\theta a_{t+1}) \, dF(a_{t+1}) + \frac{\beta}{F\left(\frac{b}{1-\theta}\right)} \int_{b/(1-\theta)}^\infty v(a_{t+1} - b) \, dF(a_{t+1}).$$

(24)

Let us define, $B(p)$ the optimal borrowing policy for a given non fully revealing price $p \in (p, 1/R)$ such that

$$pu' \left( \frac{pB(p) - k}{1-\theta} \right) - \beta \int_{B(p)}^{\infty} v'(a_{t+1} - B(p)) \, dF(a_{t+1}) = 0.$$  

(25)

Using equation (23), the default probability is $F(B(p)/(1-\theta))$. Define the function $Q(p) \equiv F(B(p)/(1-\theta))$. We will refer to $Q(p)$ as the borrowers’ optimal repayment rule. Then, when prices are not fully revealing, the equilibrium borrowing level is $b^* = B(p^*)$, the equilibrium default rule is $\hat{a}^* = b^*/(1-\theta)$, and the equilibrium default probability is $q^* = F(\hat{a}^*)$.

In order to have a well-behaved problem we need to make the following assumption.

**Assumption 1** For any $p \in (p, 1/R)$, assume that the objective function

$$\Phi(p, b) \equiv u(p b - k) + \frac{\beta}{F\left(\frac{b}{1-\theta}\right)} \int_0^{b/(1-\theta)} v(\theta a) \, dF(a) + \frac{\beta}{F\left(\frac{b}{1-\theta}\right)} \int_{b/(1-\theta)}^\infty v(a - b) \, dF(a)$$

is quasi-concave in $b$ for $b \in [0, \tilde{b}]$ and there exists an optimum $V(p) = \max_{b \in [0, \tilde{b}]} \Phi(p, b)$.

Under Assumption 1, the problem has a unique solution. Intuitively, we have to rule out the possibility that the marginal cost of default is not large enough compared to the advantage

38
of additional borrowing. In that case, the entrepreneur would always like to borrow more and default more often, so that problem (24) could not have a finite solution. This trivially implies the following proposition.

**Proposition 6** Under assumption 1, for given \( p \in (p, 1/R) \), problem (24) has a unique solution with \( pb > k \).

Assumption 1 is satisfied for many different parametric assumptions. As we show in Appendix C, it is satisfied by the example illustrated on Figure 3.

**Appendix B**

**Proof of Lemma 1**

We prove Lemma 1 for the general equilibrium when prices do not reveal any information. Let us start to prove condition (i). First, notice that

\[
B(p) \equiv \arg\max_b \Phi(p, b)
\]

Assumption 1 implies that there exists a unique \( B(p) \) and

\[
\begin{align*}
\Phi_b(p, B(p)) &= 0, \\
\Phi_{bb}(p, B(p)) &< 0.
\end{align*}
\]  

Then, by total differentiating (26), we obtain

\[
\Phi_{bp}(p, B(p)) + \Phi_{bb}(p, B(p)) B'(p) = 0,
\]

and then

\[
B'(p) = -\frac{\Phi_{bp}(p, B(p))}{\Phi_{bb}(p, B(p))}. 
\]  

Given that \( \Phi_{bb}(p, B(p)) < 0 \), to show that \( B'(p) < 0 \), it is enough to show that \( \Phi_{bp}(p, B(p)) < 0 \). We can calculate that

\[
\Phi_{bp}(p, B(p)) = u'(p_t b_{t+1} - k) + p_t b_{t+1} u''(p_t b_{t+1} - k),
\]

and hence \( \Phi_{bp}(p, B(p)) < 0 \) given that by assumption \( k > 0 \) and

\[
-\frac{cu''(c)}{u'(c)} \geq 1.
\]
From this, it is immediate that
\[ \frac{dF(\dot{\alpha})}{dp} = \frac{dF(\dot{\alpha})}{\dot{\alpha}} \left( \frac{1}{1 - \theta} \right) B'(p) \leq 0. \]
proving condition (ii).

Finally, we need to prove condition (iii). Notice that
\[ \frac{d(\dot{p}b)}{dp} = b + pB'(p), \]
where \( B'(p) \) satisfies equation (27), with \( \Phi_{bp}(p, B(p)) \) given in equation (28) and
\[ \Phi_{bb}(p, B(p)) = p^2 u''(pb - k) + \beta v' \left( \frac{\theta}{1 - \theta} b \right) + \beta \int_{\frac{b}{1 - \theta}}^{\infty} v''(a - b) dF(a) < 0, \tag{29} \]
given assumption 1. Then, after some algebra, we obtain
\[ \frac{d(\dot{p}b)}{dp} = \frac{b}{\Phi_{bb}(p, B(p))} \left\{ \beta \left[ v' \left( \frac{\theta}{1 - \theta} b \right) + \int_{\frac{b}{1 - \theta}}^{\infty} v''(a - b) dF(a) \right] - \frac{p}{b} u'(p; u + 1 - k) \right\}. \]
Hence \( \frac{d(\dot{p}b)}{dp} \geq 0 \) iff the term in parenthesis is negative. We can show that this is the case combining condition (29) and the fact that \(-cu''(c)/u'(c) \geq 1\), hence completing the proof.

**Proof of Proposition 2**

We need to show that there exists a pair \((p^*, q^*)\) that solves the fixed point defined by \(q^* = Q(p^*)\) and \(p^* = P(q^*)\). Recall that
\[ Q(p) = F \left( \frac{B(p)}{1 - \theta} \right), \]
\[ P(q) = \frac{(1 - q) (1 - \delta [\Pi(q) + q (1 - \Pi(q))])}{[1 - \delta (1 - q) (1 + \Pi(q))] R}, \]
where \( B(p) \) is implicitly define by
\[ pu' (pB(p) - k) - \beta \int_{\frac{b}{1 - \theta}}^{\infty} v'(a - B(p)) dF(a) = 0. \tag{30} \]
and \( \Pi(q) \) is implicitly defined by
\[ \Pi(q) = \psi(\Pi(q); q) \equiv \frac{[q + (1 - \tilde{q})] \gamma RM^I}{[\kappa (1 - \delta) (1 - \delta [\Pi(q) (1 - q) + q]) + \delta \gamma R] \tilde{y}}. \tag{31} \]

First, Proposition 6 shows that there exists a unique \( B(p) \) that solves equation (30) on the interval \([p, 1/R]\) and Lemma 1 shows that \( B'(p) < 0 \). Then, \( Q(p) \) is decreasing on the interval \([p, 1/R]\), given that it is immediate that \( Q'(p) \propto B'(p) \).
Next, we show that there is a unique $\Pi(q)$ that solves equation (31). First, we show that $\psi(\Pi(q); q)$ has derivative with respect to $\Pi(q)$ smaller than 1, that is,

$$\frac{\pi \delta (1 - \delta) (1 - q)}{(1 - \delta (1 - \mu)) (1 - \delta q - \delta \pi (1 - q)) < 1,$$

given that $\pi < 1$, $(1 - \delta) < (1 - \delta (1 - \mu))$, and $1 - \delta (1 - \pi) + \delta \pi q > 0$. Then, we only need to show that $\psi(0; q) > 0$ and $\psi(1; q) < 1$. With some algebra, it is easy to show that

$$\psi(0; q) = \frac{[q + (1 - \tilde{q})] \gamma R M I}{\kappa (1 - \delta) (1 - \delta q + \delta \gamma R) \tilde{y}} > 0,$$

and

$$\psi(1; q) = \frac{[q + (1 - \tilde{q})] \gamma R M I}{\kappa (1 - \delta)^2 + \delta \gamma R} \tilde{y}.$$
Second, we show that if \( q^*(\sigma') > 1/2 \), then \( q^B(\sigma') > 1/2 \). If \( q^*(\sigma') > 1/2 \), then 
\[ P^L (q^*(\sigma'); \sigma') < P^B (q^*(\sigma'); \sigma'') \]. It follows that 
\[ q^B (q^*(\sigma'); \sigma'') = q^*(\sigma'') - Q (P^B (q^*(\sigma'); \sigma'') ; \sigma'') > q^*(\sigma'') - Q (P^L (q^*(\sigma'); \sigma'') ; \sigma'') = 0, \]
and \( q^B(\sigma'') < q^*(\sigma'') \) as we wanted to show.

These first two steps immediately imply that \( q^*(\sigma') < q^B(\sigma') < 1/2 < q^*(\sigma'') < q^B(\sigma'') \), and hence \( q^*(\sigma'') - q^*(\sigma') > q^B(\sigma'') - q^B(\sigma') \). This also implies that \( p^*(\sigma') > p^B(\sigma') > 1/(2R) > p^B(\sigma'') > p^*(\sigma'') \), implying that \( p^*(\sigma'') - p^*(\sigma') > p^B(\sigma'') - p^B(\sigma') \). This completes the proof.

**Proof of Proposition 5**

**Claim 1.** Let us fix \( \alpha' \). We know that \( P^L (1; \sigma) = 0 \) for any \( \sigma \). Moreover we can calculate 
\[
\frac{dP^L (q; \sigma)}{dq} = -\frac{1}{R} \frac{\delta(1-q)}{(1-\delta(1-q))^2} < 0,
\]
and \( \lim_{R \to 0} dP^L (q; \sigma)/dq = -\infty \). Hence, if \( Q(p) > 1/2 \) and we start with \( R' \) such that \( Q (p^* (R', \alpha') | \alpha') < 1/2 \). A graphical argument available upon request implies that there exists an \( \hat{R} \), such that if \( R'' > \hat{R} \) then \( Q(p^* (R'', \alpha') | \alpha') > 1/2 \).

**Claim 2.** First, we prove that for any fixed price \( p \) there is a \( \alpha_1 \) such that for any \( \alpha < \alpha_1 \), \( \partial Q (p) / \partial \alpha < 0 \).

From \( q(b) = F (b/(1-\theta)) \), with a slight abuse of notation let us define \( b_{\alpha} (q) = (1-\theta) (F^{-1}_{\alpha-1} (q)) \) be the bond holding which implies a probability of default of \( q \) when the parameter is \( \alpha \). Then, for a fixed \( p \), we can write the first order condition of entrepreneurs as 
\[
\tilde{\Phi} (\alpha, b_{\alpha} (q)) = pu' (pb_{\alpha} (q) - k) - \beta \int_{1/(1-b)}^{\infty} v' (a_{t+1} - b_{\alpha} (q)) dF_{\alpha} (a_{t+1}).
\]

We show that there is a \( \alpha_1 \) such that 
\[
\frac{dq}{d\alpha} = -\frac{\partial \tilde{\Phi} (\alpha, b_{\alpha} (q))}{\partial \alpha} \frac{\partial b_{\alpha} (q)}{dq} + \frac{\partial \tilde{\Phi} (\alpha, b_{\alpha} (q))}{\partial \alpha} |_{\alpha<\alpha_1} < 0.
\] (32)

From the properties of a cumulative density function and our assumptions on \( \alpha \), we know that \( \partial b_{\alpha} (q) / \partial q > 0 \) and \( \partial b_{\alpha} (q) / \partial \alpha > 0 \). Furthermore, from the result of \( \Phi_{bb} (p, b) < 0 \), we know that \( \tilde{\Phi}_b (\alpha, b_{\alpha} (q)) < 0 \). Note that 
\[
\frac{\partial \tilde{\Phi} (\alpha, b_{\alpha} (q))}{\partial \alpha} = -\beta \frac{\partial \int_{1/(1-b)}^{\infty} v' (a_{t+1} - b_{\alpha} (q)) dF_{\alpha} (a_{t+1})}{\partial \alpha}.
\]
As \( \lim_{\alpha \to -\infty} F_\alpha (a_{t+1}) = 1 \), we can choose an \( \alpha_1 \) that \( \partial \left( (F_\alpha (a_{t+1}) - F_{\alpha+\varepsilon} (a_{t+1})) \right) / \partial a_{t+1} < 0 \) for any fixed \( \alpha < \alpha_1 \) and \( a_{t+1} > b_\alpha (q) / (1 - \theta) \). To show 32, it is sufficient to show that for any \( \alpha < \alpha_1 \)

\[
\frac{\partial \Phi (\alpha, b_\alpha (q))}{\partial \alpha} |_{\alpha < \alpha_1} < 0.
\]

This is a consequence of the following chain of inequalities for any \( \alpha < \alpha_1 \)

\[
\int_{\frac{b_\alpha (q)}{1 - \theta}}^{\infty} v' (a_{t+1} - b_\alpha (q)) d (F_\alpha (a_{t+1}) - F_{\alpha+\varepsilon} (a_{t+1})) =
\]

\[
= -v' \left( \frac{b_\alpha (q)}{1 - \theta} - b_\alpha (q) \right) \left( F_\alpha \left( \frac{b_\alpha (q)}{1 - \theta} \right) - F_{\alpha+\varepsilon} \left( \frac{b_\alpha (q)}{1 - \theta} \right) \right)
\]

\[
- \int_{\frac{b_\alpha (q)}{1 - \theta}}^{\infty} v'' (a_{t+1} - b_\alpha (q)) (F_\alpha (a_{t+1}) - F_{\alpha+\varepsilon} (a_{t+1})) da_{t+1}
\]

\[
< -v' \left( \frac{b_\alpha (q)}{1 - \theta} - b_\alpha (q) \right) \left( F_\alpha \left( \frac{b_\alpha (q)}{1 - \theta} \right) - F_{\alpha+\varepsilon} \left( \frac{b_\alpha (q)}{1 - \theta} \right) \right)
\]

\[
- \left( F_\alpha \left( \frac{b_\alpha (q)}{1 - \theta} \right) - F_{\alpha+\varepsilon} \left( \frac{b_\alpha (q)}{1 - \theta} \right) \right) \int_{\frac{b_\alpha (q)}{1 - \theta}}^{\infty} v'' (a_{t+1} - b_\alpha (q)) da_{t+1} =
\]

\[
= \left( F_\alpha \left( \frac{b_\alpha (q)}{1 - \theta} \right) - F_{\alpha+\varepsilon} \left( \frac{b_\alpha (q)}{1 - \theta} \right) \right) \left[ -v' \left( \frac{b_\alpha (q)}{1 - \theta} - b_\alpha (q) \right)
\]

\[
- \lim_{a_{t+1} \to \infty} \int_{\frac{b_\alpha (q)}{1 - \theta}}^{a_{t+1}} v' (a_{t+1} - b_\alpha (q)) + v' \left( \frac{b_\alpha (q)}{1 - \theta} - b_\alpha (q) \right) \right] \leq 0
\]

where the first equation is by partial integration and the inequality is the consequence of \( \partial \left( (F_\alpha (a_{t+1}) - F_{\alpha+\varepsilon} (a_{t+1})) \right) / \partial a_{t+1} < 0 \) and \( v'' (\cdot) \leq 0 \).

Moreover, if \( \lim_{\alpha \to -\infty} Q (p) \) exists, then \( \lim_{\alpha \to -\infty} Q (p) = 1 \). This is a simple consequence of the first order condition and \( \lim_{\alpha \to -\infty} F_\alpha (a_t) = 1 \).

To summarize, so far we have shown that \( \partial Q_\alpha (p) / \partial \alpha < 0 \) for any \( \alpha < \alpha_1 \) and \( \lim_{\alpha \to -\infty} Q (p) = 1 \) whenever it exists. So if \( \alpha \) is sufficiently low, then \( Q (p^* (R', \alpha) | \alpha) < \frac{1}{2} \). Then there is an \( \alpha_2 = \min \alpha \) such that \( Q (p^* (R', \alpha) | \alpha) = 1/2 \). The second claim of the proposition follows immediately if we set \( \hat{\alpha} \equiv \min (\alpha_1, \alpha_2) \).

**Appendix C**

**Example**

Assume \( u (c) = \log (c) \), \( v (c) = c \) and \( 1 - F (a) = a^\gamma a^{-\gamma} \), with \( a > 1/\beta (1 - \theta) \) and \( \gamma < 1 \). In this case we can write

\[
\Phi_b (p, b) = b^{-\gamma} \left[ \frac{pb^\gamma}{pb - k} - \beta a^\gamma (1 - \theta)^\gamma \right].
\]
For a given \( p \), there is a unique solution to \( \Phi_b(p, b) = 0 \) which solves

\[
\frac{pb^\gamma}{pb - k} = \beta a^\gamma (1 - \theta)^\gamma.
\]

Then, it is easy to verify that the left-hand side of this condition is decreasing in \( b \) whenever \( \gamma < 1 \), and it converges to \( \infty \) for \( b \to k/p \), and to 0 for \( b \to \infty \). Given that the right-hand side is a positive constant, there must be a unique optimum \( b \) that is implicitly define as follows:

\[
p = \frac{\beta a^\gamma (1 - \theta)^\gamma k}{\beta a^\gamma (1 - \theta)^\gamma b - b^\gamma}.
\]

The function \( \Phi(p, b) \) is quasi-concave because if \( b_1 < b_2 \) and \( \Phi_b(p, b_1) > 0 \), then \( \Phi_b(p, b_2) > 0 \), given that \( b^{-\gamma} > 0 \) for any \( b > k/p \). For the same reason if \( b_1 < b_2 \) and \( \Phi_b(p, b_2) < 0 \), then \( \Phi_b(p, b_1) < 0 \), completing the proof.

Moreover, we can show that there always exists a limit equilibrium for \( k \) small enough.

The fixed point problem for \((q^*, p^*)\) can be rewritten as

\[
\begin{align*}
p^* &= \frac{k \beta (1 - q^*)}{(1 - \theta) \beta a (1 - q^*)^{-\frac{1 - \gamma}{\gamma}} - 1}, \\
p^* &= \frac{(1 - q^*) (1 - \delta q^*)}{[1 - \delta (1 - q^*)] R}.
\end{align*}
\]

Let us define

\[
\begin{align*}
h_1(q) &\equiv \frac{k \beta}{(1 - \theta) \beta a (1 - q)^{-\frac{1 - \gamma}{\gamma}} - 1} \\
h_2(q) &\equiv \frac{1 - \delta q}{[1 - \delta + \delta q] R}.
\end{align*}
\]

then it is immediate that there exists a fixed point iff there exists a \( q^* \) such that \( h_1(q^*) = h_2(q^*) \). This is easy to show given that both \( h_1(q) \) and \( h_2(q) \) are decreasing in \( q \). Moreover, \( h_1(0) = k \beta / [(1 - \theta) \beta a - 1] > 0 \), \( h_2(0) = 1 / [(1 - \delta) R] > 0 \), \( h_1(1) = 0 \) and \( h_2(1) = (1 - \delta) / R > 0 \). Hence, in order for a fixed point to exists, it is enough that \( h_2(0) < h_1(0) \), that is,

\[
k < \frac{(1 - \theta) \beta a - 1}{(1 - \delta) \beta R}.
\]

Moreover, if \( k = 0 \), then there is a unique solution, given by

\[
\begin{align*}
q^* &= 1 - \left(\frac{1}{(1 - \theta) \beta a}\right)^{-\frac{\gamma}{1 - \gamma}}, \\
p^* &= \frac{(1 - q^*) (1 - \omega \delta q^*)}{[1 - \omega \delta (1 - q^*)] R}.
\end{align*}
\]

By continuity if \( k \) is small enough the equilibrium is also unique.
References


