Firm Dynamics and the Cross-Section of Equity Returns
(Preliminary and Incomplete)
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Abstract

We put forward an equilibrium model that links the cross-sectional variation in expected equity returns to firms’ life cycle dynamics. In the model, assets have different exposure to short-run and long-run consumption risks (Bansal and Yaron (2004)). An econometrician who uses the conditional CAPM regression to predict asset returns will obtain high α’s for assets that are highly exposed to low-frequency risks. Growth options have lower exposure to long-run risks than value assets because cost of exercising the growth options is highly sensitive to persistent fluctuations in aggregate consumption and, therefore, provides a hedge against risks of assets in place. Small firms exhibit high exposure to long-run risks as they are more likely to fail in bad times and, hence, in equilibrium carry a high risk premium. We calibrate the model and show that it is able to account for the observed pattern in mean returns on size and book-to-market sorted portfolios, as well as the failure of the CAPM in the data.

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Introduction

We put forward an equilibrium model that links the cross-section of expected equity returns to firms’ life cycle dynamics. The model specifies exogenously preferences of a representative agent and aggregate consumption process and focuses on the investment decisions of firms. We use the framework of Bansal and Yaron (2004) to introduce two distinct sources of risks in the economy: long-run and short-run consumption risks. In the model, assets differ in their exposures to short-run and persistent risks in consumption. Inside the model, the null hypothesis of the conditional CAPM fails – an asset has a high $\alpha$ in the one-factor conditional CAPM if it has a high exposure to long-run consumption risks.

Our model generates an endogenous mechanism, through which growth options have lower exposure to long-run risks than assets in place (i.e., value assets). Assets in the model differ by the amount of installed capital they carry: a value asset has one unit of installed capital and produces consumption goods; a growth asset does not carry any installed capital, nor does it participate in production. Growth assets are options on assets in place. Exercise of a growth option requires one unit of installed capital. The cost of installed capital, which is endogenously determined in equilibrium, has higher exposure to long-run consumption risks than value assets. Since growth options are long positions in assets in place and short positions in installed capital, the cost of installed capital acts as a hedge against long-run risks and makes growth options less risky than value assets.

We prove that the necessary and sufficient condition for the value premium to exist in the model equilibrium is that firms’ dividend payments are more exposed to long-run risks than aggregate consumption. Indeed, recent empirical evidence suggests that dividends of most assets are highly sensitive to low frequency risks in consumption (Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2006)). The intuition of our result can be understood as follows. The model assumes a fixed supply of both growth options and installed capital. In good times, value assets produce a lot of consumption goods, and become more valuable. This drives up the demand for installed capital. Because capital is inelastically supplied, its price has to increase to clear the market. In bad times, the price of value assets is low, and so are the demand for and price of installed capital. Thus, an ownership of installed capital is risky. Further, if installed capital has a higher exposure to persistent consumption risks than value assets, growth options will be less risky than assets in place. We prove that as long as dividend exposure to long-run risks exceeds that of aggregate consumption, this will always be the case. Our model, thus, provides a simple explanation of the observed value premium.
Our model also predicts that value assets with small market capitalization have higher exposure to long-run risks than those with large market capitalization. Operating a value asset requires a cost. High market capitalization assets are more productive and generate enough cash flow to cover the operating cost and, thus, are less likely to exit the industry when the economy is hit by a negative persistent shock. Small and less productive assets, however, have to respond to negative long-run risks by shutting down because of the cost of operation and, thus, are highly exposed to long-run risks in consumption. Value assets with small market capitalization, therefore, carry a high expected return as a compensation for their exposure to long-run risks.

We show that in the model, assets with high exposure to long-run risks will always have non-trivial \( \alpha \)'s in the conditional CAPM, C-CAMP and wealth-CAPM regressions. This is consistent with the well-known failure of the CAPM and consumption based CAPM in the data. Cross-sectional implications of our model are also consistent with empirical evidence in the long-run risks literature that highlights the importance of long-run consumption risks in explaining the dispersion in risk premia along size and book-to-market dimensions (Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2006), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2006), Bansal, Dittmar, and Kiku (2007), and Bansal, Kiku, Yaron (2007)). These papers, in particular, show that firms' exposures to low-frequency consumption fluctuations account for a significant portion of the size and value premium in the data. Our model rationalizes this observation endogenously as an equilibrium phenomenon. We calibrate the model and show that, quantitatively, the model-implied size and value premia match the observed pattern in mean returns on size and book-to-market sorted portfolios.

Real option based models typically imply that growth options are riskier than assets in place and, therefore, entail high risk compensations (see, for example, Berk, Green, and Naik (1999)\(^3\), Gomes, Kogan, and Zhang (2003), Carlson, Fisher, and Giammarino (2004)). This implication, however, does not confirm well with the data – empirically, low book-to-market (i.e., growth) stocks tend to pay lower returns compared to high book-to-market (or value) stocks. The argument that growth options are riskier is based on the intuition that options are long positions in the underlying asset and short positions in a strike asset. If the strike asset is risk free, then options are effectively leveraged positions in the underlying asset and must carry an amplified risk premium. For traded options, the strike asset is a cash amount and will, indeed, be risk free. For real options, strike assets are installed physical

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\(^3\)In Berk, Green, and Naik (1999), growth options could be riskier or less risky than assets in place depending on parameter values.
capital, the price of which is highly procyclical, which is supported by empirical evidence on aggregate $q$. Berk, Green, and Naik (1999) and Carlson, Fisher, and Giammarino (2004) are partial equilibrium models and the strike assets in their settings are assumed to be risk-free exogenously. In the general equilibrium model of Gomes, Kogan, and Zhang (2003), the relative price of the strike asset with respect to consumption is fixed to be one by technology and, therefore, is always risk-free. We provide a simple equilibrium mechanism that makes the installed capital risky, consistent with both the time-series evidence on marginal-$q$ and the cross-sectional evidence on the value premium.

Many other papers address the cross-sectional variation of expected returns inside partial or general equilibrium frameworks (for example, Cooper (2006), Gala (2005), Gourio (2006), Panageas and Yu (2006), Zhang (2005), among others). A key difference between our model and the above is that these models are typically built on only one source of risks. Therefore, although they are able to generate a high expected return of value firms and/or small firms, in all these models, the conditional CAPM still holds. In contrast, we allow for two independent sources of risks, long-run and short-run fluctuations in aggregate consumption growth, and are able to account the failure of the conditional CAPM in the data.

On a technical level, we obtain closed form solutions for asset prices in the economy and the cross-sectional distribution of assets’ productivity. General equilibrium models with aggregate risk and heterogenous firms are notoriously hard to solve, even numerically. Thus, for tractability, we have to sacrifice on a general equilibrium framework. Our model is not a fully-specified general equilibrium in the sense that aggregate dividend payments of firms do not add up to aggregate consumption and the determination of the discrepancy between the two is outside of the model. We focus on a sector of the economy, whose assets are publicly traded. This allows us to obtain closed form solutions and provide sharp theoretical characterization of the value premium. Similar techniques of solving for the cross-sectional distribution of firms have been used in Luttmer (2007) in a general equilibrium economy without aggregate uncertainty.

The paper is organized as follows. Section I provides a simple example to illustrate the main intuition of the model. Section II sets up the model and defines an appropriate notion of equilibrium. Section III discusses the solution to firms’ maximization problems and characterizes the cross-section distribution of assets. Section IV provides an analysis of the failure of the conditional CAPM and conditions for the value premium to exist in equilibrium. Section V calibrates the model and discusses its quantitative implications. Section VI concludes.
I A Simple Example

Here we use a simple example to demonstrate the basic intuition behind the economic mechanism in our model that generates the value premium. The key premise of our result is that the cost of installed capital is riskier than assets in place. A growth option is a long position in assets in place, and a short position in one unit of installed capital. If the cost of the installed capital is riskier, it offsets the risk in assets in place, and makes the growth option less risky. The following example illustrates the differential risk exposure of growth options and assets in place as an equilibrium phenomenon when installed capital is inelastically supplied.

Consider an economy with two dates, \( t = 1, 2 \). At \( t = 1 \), the economy is endowed with measure 1 of ideas and measure \( \frac{1}{2} \) of installed capital. The quality of ideas, denoted \( Z \), is uniformly distributed in the interval \([0, 1]\).

At date \( t = 1 \), an idea can be combined with one and only one unit of installed capital to make a production unit, which produces consumption goods. The productivity of production units is subject to a common shock \( \theta_t \), which follows a two-state Markov chain, with state space \( \{\theta_H, \theta_L\} \), where \( \theta_H > \theta_L \). The transition probabilities of the Markov chain are given by:

\[
P(\theta_2 = \theta_H | \theta_1 = \theta_H) = P(\theta_2 = \theta_L | \theta_1 = \theta_L) = p
\]

where \( p \in [0, 1] \). An idea with quality \( Z \in [0, 1] \) will produce \( Z\theta_1 \) units of consumption goods at date \( t = 1 \) once it becomes a production unit. If an idea is not matched with installed capital at date \( t = 1 \), it can be used to produce \( \delta \) units of consumption good at date \( t = 2 \), where \( \delta < \frac{1}{2} \).

For simplicity, we assume that there is no aggregate risk in this economy (or the representative agent is risk-neutral) and interest rate is 0. Although risks in \( \theta_t \) are not priced in the economy, we can still use this example to discuss the basic mechanism in our model that makes value of assets in place more sensitive to productivity shocks \( (\theta_t) \) than that of growth options. One can easily introduce a correlation of \( \theta_t \) with aggregate risk, in which case the difference in sensitivities with respect to shocks in \( \theta_t \) will translate directly into difference in expected returns.

In this economy an idea matched with a unit of installed capital is an asset in place. The value of assets in place is determined by the amount of consumption goods it produces. Therefore the value of the asset in place is given by \( Z\theta_1 \) if it involves an idea of quality \( Z \). Note that assets in place are risky, since their values depend on \( \theta_1 \). An unmatched idea is an option: it can be matched with a unit of installed capital at \( t = 1 \), or it can wait for
one period and produce consumption good at date $t = 2$. If we use $f(\theta)$ to denote the equilibrium price of installed capital, then the value of an idea with quality $Z$ is given by:

$$\max \{ Z\theta_1 - f(\theta_1), \ E[\delta\theta_2|\theta_1] \}.$$  

It is clear that "in-the-money" ideas are long positions in assets in place, and a short positions in installed capital. We are interested in the determination of $f(\theta)$, and hence the risk exposure of growth options and assets in place.

In equilibrium, only half of all ideas will be implemented at $t = 1$, since the total supply of installed capital is $\frac{1}{2}$. $f(\theta)$ will be determined by the profit maximization problem of a marginal idea that acquires a unit of installed capital. In equilibrium, high quality ideas will be implemented first. Therefore, any $Z \geq \frac{1}{2}$ will be matched with a unit of installed capital at $t = 1$, and any $Z < \frac{1}{2}$ will be used to produce consumption goods at $t = 2$. The equilibrium price of installed capital, $f(\theta)$ must be such that the marginal idea $Z = \frac{1}{2}$ is indifferent between acquiring a unit of installed capital today, and wait to produce consumption good at time 2. The total payoff of acquiring a unit of installed capital for an idea of quality $Z$ is $Z\theta_1 - f(\theta_1)$. Therefore, profit maximization of the marginal idea $Z = \frac{1}{2}$ requires:

$$\frac{1}{2}\theta_H - f(\theta_H) = (1 - p) \delta\theta_H + p\delta\theta_L$$  

$$\frac{1}{2}\theta_L - f(\theta_L) = (1 - p) \delta\theta_L + p\delta\theta_H$$

The above equations imply that the price of installed capital is given by:

$$f(\theta_H) = \frac{1}{2}\theta_H - \delta [(1 - p) \theta_H + p\theta_L]$$  

$$f(\theta_L) = \frac{1}{2}\theta_L - \delta [(1 - p) \theta_L + p\theta_H]$$

It follows that for all $Z \in [0, 1]$,

$$\frac{f(\theta_H)}{f(\theta_L)} > \frac{Z\theta_H}{Z\theta_L}$$

That is, when the economy is hit by a negative shock in $\theta$, the cost of installed capital drops by a higher percentage than the value of assets in place. Thus, the cost of installed capital is riskier than assets in place.

To understand the intuition behind equation (5), first consider the case when $p \rightarrow 0$. In
In this case, the probability of regime switching in $\theta$ is very small, and

\[ f(\theta_H) = \left(\frac{1}{2} - \delta\right) \theta_H \quad (6) \]

\[ f(\theta_L) = \left(\frac{1}{2} - \delta\right) \theta_L. \quad (7) \]

Therefore,

\[ \frac{f(\theta_H)}{f(\theta_L)} = \frac{\theta_H}{\theta_L} = \frac{\theta_H}{\theta_L}. \]

That is, installed capital is as risky as assets in place. This is not surprising. Given that the productivity of the idea in both days is proportional to $\theta$, the market clearing price $f(\theta)$ defined in equations (1) and (2) must be proportional to $\theta$. When $\theta$ changes, prices of all assets in the economy move proportionally and growth assets are as risky as value assets.

The possibility of regime switch in $\theta$ introduces an additional incentive for ideas to exercise options in the good state ($\theta_1 = \theta_H$), and a disincentive for ideas to exercise options in the bad state ($\theta_1 = \theta_L$). For the market clearing condition to hold, the price of installed capital must be higher than that in equation (6) in the good state, and lower than that in equation (7) in the bad state. Note that the left hand of equation (1) is the benefit of exercising the option at $t = 1$, and the right hand side of equation (1) is the benefit of waiting until $t = 2$ to start production. Because of the possibility of a regime switch in $\theta$, the benefit of waiting in the good state is smaller: in the case of a regime switch in $\theta$, the idea will produce less consumption good at date $t = 2$. This induces more incentives to exercise the option at $t = 1$. To deter entrance, the equilibrium price of installed capital must be higher than in equation (1). Similarly, the possibility of regime switch in $\theta$ implies that the benefit of waiting is higher in the bad state: in the case of a regime switch, the idea will produce more consumption good at $t = 2$, compared to case of no regime switch. This induces an additional incentive for ideas to wait in the bad state. Consequently, the equilibrium price of installed capital must drop to clear the market. Therefore, the market clearing price of installed capital in equation (4) is lower than that in the case of $p = 0$. To summarize, the combination of inelastic supply of installed capital and the mean reverting nature of $\theta$ creates a mechanism that makes the price of installed capital vary more than assets in place, generating the value premium.

In the rest of the paper, we imbed the above mechanism in a fully dynamic equilibrium model with long-run risks. In the model the null hypothesis of conditional CAPM fails. A similar mechanism of the above example generates a value premium that is not captured by
conditional CAPM. The model is also consistent with the recent empirical success of long-run consumption risks in explaining the cross section of equity returns. A fully dynamic model also allows us to endogenize $\delta \theta$ in the above example, which is the option value of waiting. Finally, the fully dynamic framework allows us to calibrate the model and assess the quantitative importance of the above mechanism in accounting for the value premium in the data.

II Set-up of the Model

A Preference

Consider an infinite-horizon economy with a representative consumer. The flow rate of aggregate consumption is assumed to follow the process:

$$dC_t = C_t [\theta_t dt + \sigma_C (\theta_t) dB_t]$$

where $\{B_t\}_{t \geq 0}$ is a one-dimensional standard Brownian motion. $\{\theta_t\}_{t \geq 0}$ is a two-state Markov process with state space $\Theta = \{\theta_H, \theta_L\}$, where $\theta_H > \theta_L$. The transition probability of $\theta_t$ over an infinitesimal time interval $\Delta$ is given by:

$$
\begin{bmatrix}
    e^{-\lambda_H \Delta}, & 1 - e^{-\lambda_H \Delta} \\
    1 - e^{-\lambda_L \Delta}, & e^{-\lambda_L \Delta}
\end{bmatrix}
$$

The Markov chain $\{\theta_t\}_{t \geq 0}$ is assumed to be persistent to capture the idea of long-run risks (Bansal and Yaron (2004)). We also allow the diffusion coefficient of consumption growth to depend on the state variable $\theta$. This provides a parsimonious way of modeling stochastic volatility of consumption growth. When $\theta = \theta_H$, the expected growth rate of consumption is high, the economy is in a boom. When $\theta = \theta_L$, the expected consumption growth is low, consequently, the economy is in a recession.

The representative consumer’s intertemporal preference is represented by the Kreps and Porteus (1978) utility with constant relative risk aversion parameter $\gamma > 0$ and constant intertemporal elasticity of substitution parameter $\psi > 0$. Following Duffie and Epstein (1992a) and Duffie and Epstein (1992b), we represent the preference as stochastic differential utility in continuous time. Since the aggregate consumption growth contains a persistent component $\{\theta_t\}_{t \geq 0}$, the economy is a continuous time version of the Bansal and Yaron

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(2004) economy. We use \( \{U_t\}_{t \geq 0} \) to denote the utility process of the representative agent. Given a consumption process, \( \{C_s : s \geq 0\} \), for every \( t \geq 0 \), the date-\( t \) utility of the agent, denoted \( U_t \), is defined recursively by\(^5\):

\[
U_t = E_t \left[ \int_t^\infty f(C_s, U_s) \, ds \right].
\]  
(9)

In the above equation, \( f(C, U) \) is the aggregator \(^6\) of the recursive preference and is given by:

\[
f(C, U) = \frac{\beta}{1 - 1/\psi} C^{1-1/\psi} - \frac{((1 - \gamma) U)^{1-1/\psi}}{((1 - \gamma) U)^{1-1/\psi} - 1}.
\]  
(10)

where \( \gamma, \psi > 0 \). We assume \( \gamma \neq 1 \) for simplicity. We allow \( \psi = 1 \) with the understanding that in this case,

\[
f(C, U) = \beta (1 - \gamma) U \left[ \ln C - \frac{1}{1 - \gamma} \ln [(1 - \gamma) U] \right]
\]  
(11)

The representative consumer’s preference and consumption determine the pricing kernel of the economy, denoted by \( \{\pi_t\}_{t \geq 0} \) and characterized in proposition 1 in section A.

B Life Cycle Dynamics of Firms

We focus directly on the equilibrium in the public equity sector and do not attempt to endogenize aggregate consumption. We specify the endowment and production technology of the public equity sector traded on the stock market and aim to understand the economic mechanism that generates the difference in expected asset returns.

There are two types of endowment in this economy: endowment of ideas and endowment of installed capital. Endowment of ideas arrives exogenously at rate \( m_X \) per unit of time, and endowment of installed capital arrives exogenously at rate \( m_Z \) per unit of time, where \( m_X > m_Z \).

Ideas are storable. An idea by itself does not produce any consumption good. An idea starts producing consumption good once it is matched with one unit of installed capital. In

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\(^{5}\) Representation of SDU in the infinite horizon case is discussed in Duffie and Epstein (1992b). Existence and uniqueness of SDU of the Kreps and Porteus (1978) type are discussed in Duffie and Lions (1992) and Schroder and Skiadas (1999).

\(^{6}\) In general, a recursive preference is characterized by a pair of aggregators \( (f, A) \). Duffie and Epstein (1992b) show that one can always normalize so that \( A = 0 \). The aggregator \( f \) used here is the normalized aggregator.
other words, an idea is an option on a stream of future consumption goods, which could be obtained once a unit of installed capital is acquired. Ideas capture the essence of growth assets. They do not carry any installed capital and are long-duration assets in the sense that they do not generate any cash flow immediately but are expected to generate cash-flow stream in the future. For an unmatched idea \( i \), we use the notation \( t^i \) to denote the calendar time at which it acquires a unit of installed capital.

Unmatched ideas differ by their quality. The initial quality of an idea \( i \) is drawn from a time-invariant continuous density \( u(\cdot) \) with support \([0, X]\). After birth, an unmatched idea die at Poisson rate \( \kappa_X > 0 \). Conditioning on survival, the quality of the idea, denoted \( X^i_t \) evolves randomly according to the following stochastic differential equation:

\[
dX^i_t = X^i_t [\mu_X dt + \sigma_X dB^i_t], \quad t \leq t^i, \tag{12}
\]

until the idea is matched with a unit of installed capital, or hit by the Poisson death shock. In equation (12), \( \{B^i_t\}_{t \geq 0} \) are standard Brownian motions and are independent among ideas. That is, the quality of ideas are geometric Browninan motions, and the Brownian motion risks are idosyncratic.

Installed capital is not storable unless it is matched with ideas. Each idea can be matched with at most one unit of installed capital. An idea matched with one unit of installed capital becomes a production unit and produces consumption good. Production units are value assets. They carry one unit of installed capital each and are short duration assets (when compared to unmatched ideas) – they produce consumption goods, which are paid right away as dividend to owners of production units.

Production units die at Poisson rate \( \kappa_Z > 0 \). Conditioning on survival, a production unit produces consumption goods at rate \( Z^i_tC_t \), where \( Z^i_t \) is the productivity of the production unit \( i \) at calendar time \( t \). In other words, at any time \( t \), a production unit produces a fraction of aggregate consumption, where the fraction depends on its productivity at time \( t \). The initial productivity of a production unit depends on the quality of the idea that is used to set up the production unit. For idea \( i \), which acquires one unit of installed capital and becomes a production unit at time \( t^i \), the initial productivity of the production unit at time \( t^i \) is:

\[
Z^i_{ti} = X^i_{ti}.
\]

In order to operate the production unit, a per period flow cost has to be paid in the amount of:

\[
o(\theta_t) C_t, \quad \text{for } t \geq t^i.
\]
The above specification allows the operating cost of production units to depend on the aggregate state of the economy through function \( o(\cdot) \). This allows the equilibrium failure rate of firms to differ in booms and recessions. To summarize, the rate of net cash flow generated by a production unit \( i \) is given by:

\[
(Z_i^i - o(\theta_i^i)) C_t \text{ for } t \geq t^i
\]

(13)

Claims to production units pay cash flow in the amount given in (13) as dividends.

Conditioning on survival, productivity of a production unit evolves according to the following stochastic differential equation:

\[
dZ_i^i = Z_i^i [\mu_Z (\theta_i) \, dt + \sigma_Z (\theta_i) \, dB_i^i], \quad t \geq t^i
\]

(14)

The drift and diffusion coefficient of the productivity process are identical among all firms and are assumed to depend on the long-run risk state variable \( \theta \). The Brownian motion shocks \( B_i^t \) are independent among production units. This specification captures the idea that dividend payments of firms in the public equity sector have different exposure to long-run risks than aggregate consumption. In particular, \( \mu_Z (\theta_H) > \mu_Z (\theta_L) \) implies that dividend payment has higher exposure to long-run risks than aggregate consumption, which is consistent with the cross-sectional evidence in the long-run risks literature (e.g., Bansal, Dittmar, and Lundblad (2005) and Hansen, Heaton, and Li (2006)).

A profit maximization firm who owns an unmatched idea can choose to purchase a unit of installed capital and turn the idea into a production unit at any time. The decision to match installed capital with an idea is irreversible and once matched, installed capital cannot be matched productively with a different idea later on. The price of installed capital will be determined by equilibrium conditions. A firm can choose to abandon an idea or a production unit at any time. Once abandoned, both the ideas and installed capital evaporate. The decision to abandon an idea or a production unit is irreversible.

C Notion of Competitive Equilibrium

A production unit faces an optimal stopping problem: it has an option to stop operation and exit the public equity sector. It will do so if productivity is too low to justify the operating cost. Mathematically, a production unit's optimization problem is written as:

\[
V_{VA} (i, t) = \max_t E_t \left[ \int_0^\tau e^{-\kappa (t+s)} \frac{\pi_{t+s}}{\pi_t} (Z_{t+s}^i - o(\theta_{t+s})) C_{t+s} ds \right], \quad t \geq t^i.
\]

(15)
In the above expression, \( \{\pi_t\}_{t \geq 0} \) is the state price density of the economy determined by aggregate consumption and the representative consumer’s preference. \( V_{VA}(i,t) \) denotes the value of the production unit \( i \) at calendar time \( t \geq t^i \). The maximization is taken over all stopping times \( \tau \) (that are adapted to an appropriately defined filtration). The optimal stopping time for production units can be characterized by a pair of state-contingent optimal exit thresholds \( \{Z^* (\theta_H), Z^* (\theta_L)\} \) in terms of their productivity. If the current state of the economy is \( \theta_H \), then it is optimal for the production unit \( i \) to exit if \( Z^i_t \leq Z^* (\theta_H) \). Similarly, if the current state of the economy is \( \theta_L \), then it is optimal for the production unit to exit if \( Z^i_t \leq Z^* (\theta_L) \). Mathematically, if we denote the optimal stopping time for the maximization problem in (15) as \( \tau_{VA}(i,t) \), then,

\[
\tau_{VA}(i,t) = \inf \left\{ t : \theta_t = \theta_H, \text{ and } Z^i_t < Z^* (\theta_H) \right\} \cup \left\{ t : \theta_t = \theta_L, \text{ and } Z^i_t < Z^* (\theta_L) \right\}
\]

An unmatched idea has an option to acquire a unit of installed capital and begin to produce consumption good, and an option to exit. Since no cost is needed to keep the idea alive, it is never optimal for an idea to exit voluntarily. We denote the equilibrium price of installed capital, measured in units of current period consumption numeraire as \( q_t \). The optimization problem of an idea is written as:

\[
V_{GR}(i,t) = \max \mathbb{E}_t \left[ e^{-\kappa X^i \pi_t} \frac{\pi_t}{\pi_t} \left\{ V_{VA}(i,\tau) - q_\tau \right\} \right], \quad t \leq t^i
\]

where \( V_{GR}(i,t) \) denotes the value of the stage II firm \( i \) at calendar time \( t \leq t^i \). In the class of equilibria we consider in this paper, the optimal decision rule for an idea can be characterized by an optimal exit threshold \( X^* \) in terms of its quality. It is optimal for idea \( i \) to exercise the growth option if \( X^i_t \geq X^* \). That is, if we denote the optimal stopping time for the maximization problem in (17) as \( \tau_{GR}(i,t) \), then,

\[
\tau_{GR}(i,t) = \inf \left\{ t : X^i_t > X^* \right\}
\]
threshold $X^*$. These ideas will wait and will exercise the growth option later on once their quality becomes high enough. The quality of these ideas will evolve according to the law of motion in equation (12). Some of them will eventually exercise growth option and become production units. These ideas are represented by group [4] in figure 1 and figure 2. A typical sample path of the quality of these ideas is depicted by black and blue lines. Other ideas born with quality lower than $X^*$ die prematurely at the exogenous rate $\kappa_X$. These ideas are represented by group [3] in figure 1. If the quality of an idea reaches the threshold level $X^*$, it will purchase one unit of capital and starts generating cash flow. Therefore, $X^*$ is the absorbing barrier of the cross-section distribution of the quality of unmatched ideas. There will be a continuous flow of ideas at this absorbing barrier, as depicted by group [4] in figure 1 and figure 2.

(Figure 2: Dynamics of Production Units)

Figure 2 shows the dynamics of a cohort of production units set up at time $t$. A production unit may be created by a new born idea with quality higher than $X^*$, or by an old idea that just hit the absorbing barrier $X^*$. The law of motion of productivity of a production unit is given in equation (14). Some production units die exogenously because of the Poisson shock which arrives at rate $\kappa_Z$. These are denoted by group [5] in figure 2. Other production units exit the public equity sector voluntarily if their productivity is lower than the exit threshold $Z^*(\theta)$. These ideas are represented by group [6] in figure 2.

The equilibrium price of installed capital is determined by the market clearing condition. During any time interval, the total measure of ideas that exercise growth option must be equal to the total supply of installed capital. The total measure of ideas that exercise the option will depend on the cross-sectional distribution of the quality of ideas. The cross-sectional distribution of the quality of ideas is, in general, history dependent, which makes the equilibrium hard to characterize. We focus on equilibria, in which the cross-sectional distribution of the quality of ideas has a stationary distribution. Section III of the paper shows that under very general conditions the cross-section distribution of the quality of ideas will converge to a unique stationary distribution asymptotically. We call such an equilibrium Competitive Equilibrium with Stationary Distribution of Quality of Ideas, or CE with Stationary Distribution for short. The precise definition of CE with stationary distribution is given below.

**Definition 1**: CE with Stationary Distribution

A CE with Stationary Distribution consists of:

1) Price of the firms: $\{V_{GR}(i,t), V_{VA}(i,t)\}_{i,t}$, and price of installed capital $\{q_t\}_t$.
2) Optimal stopping times for ideas and production units: $\{\tau_{GR}(i,t), \tau_{VA}(i,t)\}_{i,t}$.
3) Optimal option exercise threshold for ideas $X^*$, and optimal exit threshold for production units $\{Z^*(\theta)\}_{\theta=\theta_H,\theta_L}$.

4) Density of the stationary measure of the cross-section distribution of the quality of ideas: $\Phi(\cdot)$.

such that the above quantities satisfy:

1) Share-holder value maximization for production units.

\[ \forall i, t, \ V^{VA}(i, t) \text{ and } \tau^{VA}(i, t) \text{ are the value function and solution to the optimal stopping problem described in (15). Furthermore, } \tau^{VA}(i, t) \text{ takes form given in (16).} \]

2) Share-holder value maximization for ideas.

\[ \forall i, t, \ V^{GR}(i, t) \text{ and } \tau^{GR}(i, t) \text{ are the value function and solution to the optimal stopping problem described in (17). Furthermore, } \tau^{GR}(i, t) \text{ takes form given in (18).} \]

3) Market clearing for installed capital:

\[ \forall t, \ m_X \int_{X^*}^{Z} u(z) \, dz + m_{EXIT}^{[\Phi, X^*]} = m_Z \]  

(19)

The expression $m_{EXIT}^{[\Phi, X^*]}$ denotes the absorbing rate of measure $\Phi$ at the absorbing barrier $X^*$. The above condition says that, in equilibrium, the total rate of exercise of growth options is equal to $m_Z$.

4) Consistency of Macro- and Micro- variables. The density of the cross-sectional distribution of the quality of ideas $\Phi$ with absorbing barrier $X^*$ is consistent with the law of motion of the quality of individual ideas as described in (12).

The equilibrium requirements 1) and 2) are straightforward. The market clearing condition in (19) states that the demand and supply of installed capital must equal in equilibrium. The left hand-side of equation (19) has two terms. The first term is the demand for installed capital from the newly born ideas whose quality is high enough to exercise the growth option immediately after birth. The second term in (19) is the demand for installed capital from old ideas crossing the absorbing barrier $X^*$. The flow rate of these ideas is $m_{EXIT}^{[\Phi, X^*]}$ per unit of time. Since we have a model of heterogenous ideas, we need some consistency condition to guarantee that the variables that describe macroeconomic quantities are consistent with the individual behavior of the ideas. Technically, this implies that $\Phi$ has to satisfy a version of the Komogorov forward equation. Details of the technical conditions are discussed in appendix III.

The next section is devoted to the construction of the equilibrium defined above.
III Characterization of Equilibrium

Section III constructs the Competitive Equilibrium with Stationary Distribution defined in section C. Reader who are anxious to move on to the asset pricing implications of the model can skip the current section and go to section IV directly.

A Preliminaries

In this subsection, we outline some assumptions that are needed to ensure the existence of the Competitive Equilibrium with Stationary Distribution. We also solve for the pricing kernel of the economy implied by preferences and aggregate consumption of the representative consumer.

We first make some simplifying assumptions on the parameter values that will allow us to avoid cumbersome mathematical expressions and derive sharp and intuitive characterizations of the equilibrium. Our model can be solved for arbitrary preference parameter values of $\gamma$ and $\psi$. Assuming $\psi = 1$ greatly simplifies the math while still allows the model to capture long-run risks as long as $\gamma > 1^7$. We will maintain these assumption in section III and IV. We do allow for $\psi \neq 1$ in section V when we calibrate the model. Assumptions on preference parameters are summarized in assumption A below.

**Assumption 1:** The risk aversion and intertemporal elasticity of substitution parameters satisfy:

$$\gamma > 1, \text{ and } \psi = 1$$

In our framework, the necessary and sufficient condition for long-run risks to require a positive premium in equilibrium is that the agent has higher utility in booms (i.e., when $\theta_t = \theta_H$) than in recessions (when $\theta_t = \theta_L$). Technically, this requires that the volatility of consumption growth in booms is not too high relative to the volatility of consumption growth in recessions. This assumption is supported by the counter-cyclical variation of consumption growth volatility in the data. The technical condition on the drift and volatility of consumption growth is summarized in assumption A.

**Assumption 2:** The drift and diffusion coefficient of aggregate consumption growth satisfy:

$$\left(\theta_H - \theta_L\right) - \frac{1}{2}\gamma \left[\sigma^2_C(\theta_H) - \sigma^2_C(\theta_L)\right] > 0 \quad (20)$$

Given the above conditions, the pricing kernel, or the state price density in the language

---

7If $\gamma < \psi$, then the utility function displays preference for late resolution of uncertainty and the model will generate negative premium for long-run risks.
of Duffie (2001), is characterized in the following proposition.

**Proposition 1 (State Price Density of the Economy)**

The state price density of the economy, denoted \( \{ \pi_t \}_{t \geq 0} \), is a Levy process of the form:

\[
d\pi_t = \pi_t \left[ -r (\theta_t) \, dt - \gamma \sigma (\theta_t)^T \, dB_t - \eta_\pi (\theta_t) \, d\tilde{N}_t \right]
\]

where \( \tilde{N}_t \) is a compensated Poisson measure defined in appendix I, and

\[
\eta_\pi (\theta) = \left[ (1 - \omega^{-1}) I_{\{ \theta_H \}} (\theta), (1 - \omega) I_{\{ \theta_L \}} (\theta) \right],
\]

where \( \omega < 1 \) is a constant given in appendix I. Furthermore,

\[
r (\theta) = \beta + \theta - \gamma \sigma_C^2 (\theta)
\]

is the risk-free rate of the economy.

The quality of ideas and productivity of production units could potentially grow without bound. Together with assumption 1 and assumption 2, the following assumption guarantees that the value of ideas and production units are finite.

**Assumption 3:**

\[
\beta + \kappa_Z - \mu (\theta) > 0 \quad \text{for} \quad \theta = \theta_H, \ \theta_L \quad (24)
\]

\[
\beta + \kappa_X - \mu_X > 0 \quad (25)
\]

As long as the hazard rate of death is positive, that is \( \kappa_X, \kappa_Z > 0 \), the total measure of ideas and production units in the economy will remain finite. To guarantee that the total production in the public equity sector is always finite\(^8\), we need the following assumption to prevent the total consumption goods produced by production units from growing without bound.

**Assumption 4:**

\[
\kappa_Z > \mu_Z (\theta) \quad \text{for} \quad \theta = \theta_H, \ \theta_L
\]

Clearly, assumption A implies equation (24) in assumption A. Assumptions A-A will be maintained throughout the rest of the paper.

---

\(^8\)As long as the total production in the public equity sector is finite, we can always make sure that it is less than aggregate consumption by normalizing \( \bar{X} \). Since changing \( \bar{X} \) does not affect the asset pricing implications of the model, we do not impose any conditions on \( \bar{X} \) except in section IV when we calibrate the model.
B Firms’ Optimization Problems

We construct the equilibrium of the economy via the following procedure. For a given equilibrium price of installed capital, we solve the optimization problems of ideas for the optimal option exercise threshold \( X^* \). For a given \( X^* \), we solve for the stationary distribution of ideas. Given the stationary distribution, the market clearing condition (19) implies a unique \( X^* \). Finally, we use the optimality condition of option exercise, together with the market clearing \( X^* \) to solve for the equilibrium price of installed capital.

The optimal stopping problem of production units does not depend on the price of installed capital and can be characterized without solving for the equilibrium. The solution to the optimal stopping problem for ideas is summarized in the following proposition.

**Proposition 2 (Optimal Stopping for Production Units)**

The value function of the optimization problem for production units is of the following form:

\[
V_{VA}(i,t) = G_{VA}(Z_i^t, \theta_t) C_t,
\]

(26)

where

\[
G_{VA}(Z, \theta) = a(\theta) Z - b(\theta) o(\theta) + \sum_{i=1}^{2} e_{VA}(i, \theta) K_i Z^{\alpha_{VA}(i)},
\]

(27)

where

\[
G_{VA}(Z) = [G_{VA}(Z, \theta_H), G_{VA}(Z, \theta_L)]^T
\]

and \( \{\alpha_{VA}(i)\}_{i=1,2} \) are constants, and \( a(\theta), b(\theta), \{ e_{VA}(i, \theta) \}_{i=1,2,3,4} \) are functions of \( \theta \) given in appendix II.

Furthermore, \( Z^*(\theta_H), Z^*(\theta_L) \) along with the two constants \( K_1 \) and \( K_2 \) are jointly determined by the two value matching conditions:

\[
\begin{bmatrix}
G_{VA}(Z^*(\theta_H), \theta_H) \\
G_{VA}(Z^*(\theta_L), \theta_L)
\end{bmatrix} = 0
\]

(28)

and the smooth pasting conditions:

\[
\begin{bmatrix}
G_{VA}(Z^*(\theta_H), \theta_H) \\
G_{VA}(Z^*(\theta_L), \theta_L)
\end{bmatrix} = 0
\]

(29)

**Proof.** See Appendix.

The intuition of the above proposition is that the optimal stopping problem of production
units is characterized by a pair of optimal exit boundaries, $Z^* (\theta_H)$ and $Z^* (\theta_L)$. Production units exit the public equity sector optimally if their productivity hits $Z^* (\theta_H)$ from above in state $\theta_H$, or hits $Z^* (\theta_L)$ from above in state $\theta_L$, whichever happens earlier. The value function is homogenous in aggregate consumption $C_t$. The function $G_{VA}(Z, \theta)$ will be called the normalized value function of production units. We use the subscript $VA$ to denote this is the normalized value function for value assets. The term $a(Z)$ is the present value of cash flow $\{Z_t C_t\}_{t=0}^{\infty}$ and $b(\theta) o(\theta)$ is the present value of the operating cost of the production unit $\{o(\theta_t) C_t\}_{t=0}^{\infty}$. The term $\sum_{1=1}^{2} e_{VA}(i, \theta) K_i Z^{VA(i)}$ captures the value of the exit option.

The solution to the optimization problem of ideas will depend on the equilibrium price of installed capital. In the class of equilibria we construct in this section, the equilibrium price of installed capital will take a simple form:

$$\forall t \geq 0, \; q_t = f(\theta_t) C_t \tag{30}$$

for some function $f(\theta)$, which will be determined endogenously by equilibrium conditions. Condition (30) greatly simplifies the problem, and allows us to derive closed-form solutions to the optimization problem of ideas, which is summarized in the following proposition.

**Proposition 3 (Optimal Stopping for Ideas)**

Suppose the equilibrium price of installed capital is given in (30), then the value function of the optimization problem for ideas is of the form:

$$V_{GR}(i, t) = G_{GR}(X^i_t, \theta_t) C_t,$$

where

$$G_{GR}(X, \theta) = \sum_{i=3}^{4} e_{GR}(i, \theta) L_i X^{\alpha_{GR}(i)} \tag{31}$$

where $\{\alpha_{GR}(i)\}_{i=3,4}$ are constants, and $[e_{GR}(i, \theta)]_{i=1,2,3,4}$ are functions of $\theta$ given in appendix II.

The optimal stopping time, denoted $\tau_{GR}$ is given by:

$$\tau_{GR} = \inf \{t : \theta_t = \theta_H, \text{ and } X^i_t > X^* (\theta_H)\} \cup \{t : \theta_t = \theta_L, \text{ and } X^i_t > X^* (\theta_L)\}$$

where $X^* (\theta_H)$, $X^* (\theta_L)$ along with the two constants $[L_i]_{i=3,4}$ are jointly determined by the
value matching and smooth pasting conditions:

\[
\begin{bmatrix}
G_{GR}(X^*(\theta_H), \theta_H) \\
G_{GR}(X^*(\theta_L), \theta_L)
\end{bmatrix} =
\begin{bmatrix}
G_{VA}(X^*(\theta_H), \theta_H) - f(\theta_H) \\
G_{VA}(X^*(\theta_L), \theta_L) - f(\theta_L)
\end{bmatrix}
\] (32)

\[
\begin{bmatrix}
G_{GR}(X^*(\theta_H), \theta_H) \\
G_{GR}(X^*(\theta_L), \theta_L)
\end{bmatrix} =
\begin{bmatrix}
G_{VA}(X^*(\theta_H), \theta_H) \\
G_{VA}(X^*(\theta_L), \theta_L)
\end{bmatrix}
\] (33)

**Proof.** See appendix II. ■

The above proposition implies that if the equilibrium price of installed capital is of the form in (30), then the solution to the optimization problems of unmatched ideas can also be characterized by a pair of exit barriers, one for each state. An idea will exercise growth option once its quality is high enough, that is, once \(X^i_t\) hits the option exercise boundary \(X^*(\theta_H)\) from below in state \(\theta_H\), or \(X^i_t\) hits \(X^*(\theta_L)\) from below in state \(\theta_L\). Since ideas do not produce any consumption good, the value of ideas comes completely from their growth option feature.

The next section closes the model by imposing market clearing for installed capital and stationarity of the quality of ideas.

### C Stationary Distribution of the Quality of Ideas

In this section, we explicitly solve the stationary distribution of the quality of ideas, \(\Phi\) for a given absorbing barrier \(X^*\). This will allow us to impose the market clearing (19) and solve for the unique \(X^*\) that equates supply and demand of installed capital. The equilibrium price and quantities will be completely characterized once the market clearing \(X^*\) is determined.

To facilitate closed form solution for the stationary distribution of the quality of ideas, we assume the initial quality of ideas is of uniform distribution. That is:

**Assumption 5:**

\[u(X) = \frac{1}{X}, \quad X \in [0, \bar{X}]\]

In the equilibrium we construct, the distribution of quality of ideas is time-invariant, so is the option exercise threshold for ideas, \(X^*\). At any point in time, the total measure of ideas that enter into the economy is \(m_X\), and their initial quality is uniformly distributed on the interval \([0, \bar{X}]\). Among them, a measure \(\frac{\bar{X}-X^*}{X} m_X\) of ideas has quality higher than the option exercising barrier \(X^*\), and acquire a unit of installed capital to become a production unit immediately after birth. This is group [1] in Figure 1. A measure \(\frac{X^*}{X} m_X\) of firms has
initial quality below the option exercise boundary \( X^* \). This is group \([2]\) firms depicted in Figure 1.

Newly born ideas enter the pool of unmatched ideas at the rate of \( \frac{\lambda^*}{X} m_X \). They exit if either they are hit by the exogenous death shock at Poisson rate \( \kappa_X \), or their quality hits the option exercise threshold \( X^* \). Therefore, \( X^* \) is the absorbing barrier of the quality of ideas, and \([0, X^*]\) is the support of the stationary distribution. \( \kappa_X > 0 \) guarantees the existence of the stationary distribution of the quality of idea. Given the absorbing barrier \( X^* \), we can solve for the stationary distribution of the quality of ideas in closed form, which is done in appendix III. Once the distribution of the quality of ideas is known, we can find the exit rate of ideas at the given absorbing barrier \( X^* \). Together with the market clearing condition for installed capital, this determines the equilibrium option exercise threshold \( X^* \). The exit rate at a given absorbing barrier \( X^* \) is characterized by the following proposition:

**Proposition 4 (Rate of Option Exercise for Unmatched Ideas)**

Suppose the option exercise barrier of ideas is a constant \( X^* \), then the stationary distribution of the quality of ideas exits. If \( X^* \leq \overline{X} \), then the exit rate of ideas at the absorbing barrier of the stationary distribution, \( X^* \) is:

\[
m_{EXIT} [\Phi, X^*] = \frac{1}{1 - \eta_2} \frac{X^*}{\overline{X}} m_X, \tag{34}
\]

where \( \eta_2 \) is a function of the parameters \((\kappa_X, \mu_X, \sigma_X)\) given in appendix III.

If \( X^* > \overline{X} \), then the exit rate at the absorbing barrier, \( X^* \), is given by:

\[
m_{EXIT} [\Phi, X^*] = \frac{1}{1 - \eta_2} \left( \frac{X^*}{\overline{X}} \right)^{\eta_2} m_X \tag{35}
\]

**Proof.** See appendix III.

Using equations (34) and (35), the market clearing condition (19) can be written in closed form. We can then solve for the market clearing option exercise threshold \( X^* \) as a function of the primitive parameters of the model. This is stated below as a Corollary of the above proposition.

**Corollary 1 (Market Clearing Option Exercise Threshold)**

The unique option exercise threshold \( X^* \) determined by the market clearing condition (19) is given by:

\[
X^* = \delta \overline{X}
\]
If \( \frac{m_Z}{m_X} \in \left[ \frac{1}{1-\eta_2}, 1 \right) \), then \( \delta \) is given by:

\[
\delta = \frac{1 - \frac{m_Z}{m_X}}{\frac{1}{1-\eta_2}} \in (0, 1]
\]

If \( \frac{m_Z}{m_X} \in \left( 0, \frac{1}{1-\eta_2} \right) \), then \( \delta > 1 \) is determined by the unique solution to the following equation on \((1, \infty)\):

\[
\frac{1}{1-\eta_2} \delta^{\eta_2} = \delta - 1 + \frac{m_Z}{m_X}
\]

Since \( m_Z < m_X \), and installed capital is not storable unless matched immediately with ideas, in any Competitive Equilibrium with Stationary Distribution, there will always be a surplus of ideas, but no idle installed capital. The above Corollary implies that if the supply of installed capital is relatively abundant, that is if \( \frac{m_Z}{m_X} \geq \frac{1}{1-\eta_2} \), then the optimal threshold level is below \( \bar{X} \). In this case, there is enough supply of installed capital to absorb some new ideas of high quality, so that \( X^* < \bar{X} \). In this scenario, some new-born ideas always become production units immediately after birth. If the supply of installed capital is small relative to the supply of ideas, that is, when \( \frac{m_Z}{m_X} < \frac{1}{1-\eta_2} \), then all newly born ideas have to wait before they could become production units.

The equilibrium price and quantity can be obtained through the following procedure. We first solve the optimization problem of production units and obtain the value function \( G_{VA}(Z, \theta) \) using proposition 2. We use the above corollary to obtain the equilibrium level of the optimal option exercise threshold \( X^* \). Given \( X^* \), we use proposition 3 to solve for the equilibrium price \( q_t \) and the optimal stopping problem for ideas. In fact, once we impose \( X^*(\theta_H) = X^*(\theta_L) = X^* \) the two value matching conditions (32) and the smooth pasting conditions (33) can be used to jointly to determine \( f(\theta_H), f(\theta_L) \), and the constants \( L_3, L_4 \).

**IV Asset Pricing Implications**

There are two sources of aggregate risks in this economy, short-run consumption risks, represented by the Brownian motion \( B_{Ct} \), and long-run consumption risks, represented by \( \theta_t \). In general, the expected return of an asset will depend on both its \( \beta \) on long-run risks, and its \( \beta \) on short-run risks. A one-factor conditional CAPM is bound to fail in this economy. Section A discusses the determination of risk premium in the economy. We characterize asset \( \alpha \)'s in conditional CAPM regressions in section B and provide conditions...
under which assets with high exposure to long-run risks obtain high $\alpha$'s. In section C, we prove the key theorem of the paper: value premium exists in equilibrium if and only if firm's dividend payments have higher exposure to long-run risks than aggregate consumption.

A Risk Premium of Firms’ Assets

In the economy constructed above, the value of a generic asset $i$ is of the form:

$$V_t^i = G(i, \theta_t) C_t$$

with the understanding that for ideas, $G(i, \theta_t) = G_{GR}(X^i_t, \theta_t)$, and for production units, $G(i, \theta_t) = G_{VA}(Z^i_t, \theta_t)$. Let $D_{i,t}$ denote the rate of the dividend payment of asset $i$ at time $t$. We can define the cumulative return process of asset $i$, $\{R_{i,t}\}_{t \geq 0}$ via the following equation:

$$dR_t^i = \frac{1}{V_t^i} [dV_t^i + D_t^i dt]$$

(37)

The expected return of holding the asset during a small interval $[t, t + \Delta]$ is therefore given by:

$$E_t \left[ \frac{R_{t+\Delta}^i}{R_t^i} - 1 \right]$$

The risk premium of asset $i$, denoted $RP(i, t)$ is defined by:

$$RP(i, t) = \lim_{\Delta \to 0} \frac{1}{\Delta} E_t \left[ \frac{R_{t+\Delta}^i}{R_t^i} - 1 \right] - r(\theta_t)$$

where $r(\theta_t)$ is the instantaneous risk-free rate of this economy as given in equation (23). The risk premium of asset $i$ with value of the form in (36) is given by the following proposition.

**Proposition 5 (Risk Premium)**

This risk premium of firm $i$ whose value given by (36) is:

$$RP(i, t) = \gamma \sigma_C^2 (\theta_L) + \lambda_L (1 - \omega) \left[ \frac{G(i, \theta_H)}{G(i, \theta_L)} - 1 \right]$$

if $\theta_t = \theta_L$  

(38)

$$RP(i, t) = \gamma \sigma_C^2 (\theta_H) + \lambda_H (1 - \omega^{-1}) \left[ \frac{G(i, \theta_L)}{G(i, \theta_H)} - 1 \right]$$

if $\theta_t = \theta_H$  

(39)

**Proof.** Appendix IV. □
The above proposition has a simple intuitive interpretation. The risk premium of a firm’s asset has two components: compensation for short-run risks and compensation for long-run risks. The first term in (38) and (39) can be written as:

$$\gamma \sigma_C^2(\theta_t) = \gamma \lim_{\Delta \to 0} Cov_t \left( \frac{V_{t+\Delta}^i - V_t^i}{V_t^i}, \frac{C_{t+\Delta} - C_t}{C_t} \right).$$

Clearly, this term is the compensation for the covariance of return of the asset with contemporaneous innovations in consumption growth, that is, the compensation for the asset’s exposure to short-run consumption risks. The second component of the risk premium is the compensation for the covariance of the asset return with innovations in expected consumption growth: $\theta_t$. Note that $\lambda_L(1 - \omega) > 0$. Therefore, the higher the size of the jump in the firm’s asset value associated with a regime switch in $\theta_t$, the higher is the second component of risk premium. In fact, the second component is proportional to:

$$\lim_{\Delta \to 0} Cov_t \left( \frac{V_{t+\Delta}^i - V_t^i}{V_t^i}, \frac{\theta_{t+\Delta} - \theta_t}{\theta_t} \right).$$

In the model economy all firms have the same exposure to short-run consumption risks, but differ in their exposure to long-run consumption risks. The exposure of an asset to long-run risks depends on the asset’s characteristics, which is captured by the $G(i, \theta_t)$ function. Ideas are growth assets and have different $G(i, \theta_t)$ functions from production units, which are already assets in place. Production units with different productivity levels also differ in their exposure to long-run risks, which is captured by the $G(i, \theta_t)$ function. In the rest of section IV, we will show that this feature of the model makes the conditional CAPM fail, and generates the value and the size premia as an equilibrium phenomenon.

**B Failure of Conditional CAPM**

As shown in section A, the risk premium of an asset is determined by both the asset’s exposure to short-run and long-run consumption risks. The null hypothesis of the conditional CAPM fails because it is a one-factor model. The key result in this section is proposition 6, in which we provide conditions under which assets with high exposure to long-run risks will obtain a high $\alpha$ in the conditional CAPM regressions. These conditions are in fact empirically plausible and are satisfied in most calibrated long-run risks models.

For any asset $i$, let $R_t^i$ denote the cumulative return process of the asset as defined in (37). We consider the following conditional CAPM regressions in the model economy:
1. Conditional CAPM with a reference asset:

\[ R_{t,t+\Delta}^i - r_{t,t+\Delta} = \alpha_{t,\Delta}^i + \left( R_{t,t+\Delta}^{ref} - r_{t,t+\Delta} \right) \beta_{t,\Delta}^i + \varepsilon_{t,t+\Delta}. \]  

(40)

In the above specification,

\[ R_{t,t+\Delta}^i \equiv \frac{R_{t+\Delta}^i - R_t^i}{R_t^i}, \]

where \( R_t^i \) is the cumulative return process of asset \( i \) defined in equation (37). Therefore, \( R_{t,t+\Delta}^i \) is the return of asset \( i \) during the time interval \([t, t+\Delta]\). \( r_{t,t+\Delta} \) is the return of the locally risk-free asset during the same time interval.

\[ R_{t,t+\Delta}^{ref} = \frac{R_{t+\Delta}^{ref} - R_t^{ref}}{R_t^{ref}} \]

is the return to a reference asset with cumulative return process \( \left\{ R_t^{ref} \right\}_{t \geq 0} \). The reference asset could be the aggregate wealth of the economy, in which case the regression in (40) will be called Wealth CAPM. The reference asset could be a constructed market index, in which case the regression will be called Market CAPM.

2. Conditional Consumption CAMP (CCAPM):

\[ R_{t,t+\Delta}^i - r_{t,t+\Delta} = \alpha_{t,\Delta}^i + \left( \frac{C_{t+\Delta} - C_t}{C_t} \right) \beta_{t,\Delta}^i + \varepsilon_{t,t+\Delta} \]  

(41)

Under the null hypothesis of the above CAPM regressions, the \( \alpha' \)'s will not depend on asset’s individual characteristics, that is, \( \alpha' \)'s do not depend on \( i \). This will be true if the right hand variable are perfectly correlated with the true state price density of the economy. However, as shown in equation (21), innovations in the state price density are driven by two risk factors: long-run risks and short-run risks. The CAPM regressions are one factor models and in general will not be able to capture differential exposures of assets to the two sources of consumption risks.

Consider an econometrician who has infinitely many observations over arbitrary small time interval, and is able to recover the true value of \( \alpha_{t,\Delta}^i, \beta_{t,\Delta}^i \). We know that

\[ \beta_{t,\Delta}^i = \frac{Cov_t \left( R_{t,t+\Delta}^i, R_{t,t+\Delta}^{ref} \right)}{Var_t \left( R_{t,t+\Delta}^{ref} \right)} \]  

(42)
in the CAPM with a reference asset, and

\[ \beta_{i,t}^{\Delta} = \frac{Cov_t \left( R^i_{t,t+\Delta}, \frac{C_{t+\Delta} - C_t}{C_t} \right)}{Var_t \left( \frac{C_{t+\Delta} - C_t}{C_t} \right)} \]

in the CCAPM. Therefore, the theoretical values of \( \alpha \)'s are

\[ \alpha_{i,t}^{\Delta} = \left( E_t \left[ R^i_{t,t+\Delta} \right] - r_{t,t+\Delta} \right) - \frac{Cov_t \left( R^i_{t,t+\Delta}, R^{ref}_{t,t+\Delta} \right)}{Var_t \left( R^{ref}_{t,t+\Delta} \right)} \left( E_t \left[ R^{ref}_{t,t+\Delta} \right] - r_{t,t+\Delta} \right) \]  \( (43) \)

in the CAPM with a reference asset, and

\[ \alpha_{i,t}^{\Delta} = \left( E_t \left[ R^i_{t,t+\Delta} \right] - r_{t,t+\Delta} \right) - \frac{Cov_t \left( R^i_{t,t+\Delta}, \frac{C_{t+\Delta} - C_t}{C_t} \right)}{Var_t \left( \frac{C_{t+\Delta} - C_t}{C_t} \right)} \left( \frac{C_{t+\Delta} - C_t}{C_t} \right) \]  \( (44) \)

in the CCAPM. To characterize the \( \alpha \) is the CAPM regression, assume that the value of the reference asset takes the following form:

\[ V^{ref}_t = s \left( \theta_t \right) C_t \]  \( (45) \)

This specification includes Wealth CAPM as a special case. In fact, the total value of aggregate wealth in this economy is equal to \( \frac{1}{\beta} C_t \). It will also include the Market CAPM as a special case, if the market index is calculated using a fixed set of equity. Since our model is a continuous time model, it is more convenient to deal with the continuous time limit:

\[ \alpha_{i}^{\Delta} = \lim_{\Delta \to 0} \frac{1}{\Delta} \alpha_{i,t}^{\Delta}. \]

**Proposition 6** In the CCAPM, and Wealth CAPM,

\[ \alpha_{i}^{\Delta} = \left( \gamma - \chi_t \right) \sigma^2_C \left( \theta_H \right) + \lambda_H \left( \omega^{-1} - 1 \right) \left[ 1 - \frac{G(i, \theta_L)}{G(i, \theta_H)} \right] \quad \text{if} \quad \theta_t = \theta_H \]  \( (46) \)

and

\[ \alpha_{i}^{\Delta} = \left( \gamma - \chi_t \right) \sigma^2_C \left( \theta_L \right) + \lambda_L \left( 1 - \omega \right) \left[ \frac{G(i, \theta_H)}{G(i, \theta_L)} - 1 \right] \quad \text{if} \quad \theta_t = \theta_L \]  \( (47) \)

The total value of the equity traded in the public equity sector in the model, however, depends on the distribution of production units, which is a function of the history of \( \theta_t \). Therefore, the total market value in the model is not of the form in (45).
In general, In the CAPM with a reference asset, whose value is of the form in (45),

$$\alpha_t^i = (\gamma - \chi_t) \sigma_C^2 (\theta_H) + \lambda_H \left[ (1 - \omega^{-1}) - \chi_t \left( \frac{s(\theta_L)}{s(\theta_H)} - 1 \right) \right] \left[ \frac{G(i, \theta_L)}{G(i, \theta_H)} - 1 \right] \quad \text{if } \theta_t = \theta_H$$

and

$$\alpha_t^i = (\gamma - \chi_t) \sigma_C^2 (\theta_L) + \lambda_L \left[ (1 - \omega) - \chi_t \left( \frac{s(\theta_H)}{s(\theta_L)} - 1 \right) \right] \left[ \frac{G(i, \theta_H)}{G(i, \theta_L)} - 1 \right] \quad \text{if } \theta_t = \theta_L$$

where

$$\chi_t = \lim_{\Delta \to 0} \frac{E_t \left[ R_{t,t+\Delta}^M \right] - r_{t,t+\Delta}}{\text{Var}_t \left( R_{t,t+\Delta}^M \right)}$$

**Proof.** Use equation (43) and (44). ■

Because IES is assumed to be 1, consumption-to-wealth ratio is constant. Therefore, CCAPM and Wealth CAPM are equivalent in the model. The above proposition implies that in the CCAPM and the Wealth CAPM, assets with high exposure to long-run risks, that is a high \( G(i, \theta_H) \) ratio, will always have high \( \alpha^i \)'s. In general, assets with high exposure to long-run risks will obtain a high \( \alpha^i \) if

$$\left( \omega^{-1} - 1 \right) > \chi_t \left( 1 - \frac{s(\theta_L)}{s(\theta_H)} \right)$$

and

$$\left( 1 - \omega \right) > \chi_t \left( \frac{s(\theta_H)}{s(\theta_L)} - 1 \right)$$

Note that \( \omega^{-1} - 1 \) is the size of the jump of the state price density when \( \theta_t \) jumps from \( \theta_H \) to \( \theta_L \), and \( 1 - \omega \) is the size of the jump of the state price density when \( \theta_t \) changes from \( \theta_L \) to \( \theta_H \). Therefore \( \omega^{-1} - 1 \) and \( 1 - \omega \) measure the sensitivity of the state price density with respect to shifts in \( \theta_t \). Similarly, \( 1 - \frac{s(\theta_L)}{s(\theta_H)} \) and \( \frac{s(\theta_H)}{s(\theta_L)} - 1 \) measure the sensitivity of the return to the reference asset with respect to the jumps in \( \theta_t \). Intuitively, condition (51) and (52) imply that the state price density is \( \chi_t \) times more sensitive to long-run risks than the return to the reference asset.

Note that

$$\chi_t = \lim_{\Delta \to 0} \frac{E_t \left[ R_{t,t+\Delta}^M \right] - r_{t,t+\Delta}}{\text{Var}_t \left( R_{t,t+\Delta}^M \right)}$$

is the Sharpe ratio of the market divided by the volatility of the market return. Assume a
market risk-premium of 6 – 8%, and a volatility of the market return of 15 – 20%, then the plausible value of $\chi_t$ is between 1.5 and 3.5. The sensitivity of the pricing kernel exceeds the sensitivity of the market return with respect to long-run risks by more than this amount in most of the calibrated long-run risks models. For example, in Bansal and Yaron (2004), the pricing kernel is 4.5 times more sensitive to the long-run risks than the market return. Thus in the model the conditional CAPM fails and assets with high exposures to long-run risks have high $\alpha$’s in conditional CAPM regressions.

C Value and Size Premium

In this section, we provide conditions under which ideas have less exposure to long-run risks than production units. Since we interpret ideas as growth assets, and production units as value assets, this implies the existence of the value premium. The key result in this section is proposition 7, which proves that value premium exists if and only if dividends are more sensitive to long-run risks than aggregate consumption. We also show that the model implies that value assets with small capitalization have higher exposure to long-run risks than large value assets.

To compare the riskiness of value and growth assets, first note that by proposition 5, if a production unit with productivity $Z_i$ is riskier with respect to long-run risks than an idea with quality $X_j$, then

$$\frac{G_{VA}(Z_i, \theta_H)}{G_{VA}(Z_i, \theta_L)} > \frac{G_{GR}(X_j, \theta_H)}{G_{GR}(X_j, \theta_L)}. \tag{53}$$

For simplicity, consider a special case in which $X_j = Z_i = X^*$. That is, we compare the riskiness of an idea just before the option exercise and immediately after the option exercise. In this case, the value of the idea after the option exercise is the value of the production unit it makes. The value of this production unit is equal to the value of the same idea before the option exercise plus the value of the installed capital, that is:

$$G_{VA}(X^*, \theta) = G_{GR}(X^*, \theta) + f(\theta), \quad \text{for} \quad \theta = \theta_H, \theta_L \tag{54}$$

Using equation (54), we can write equation (53) as:

$$\frac{G_{VA}(X^*, \theta_H)}{G_{VA}(X^*, \theta_L)} > \frac{G_{VA}(X^*, \theta_H) - f(\theta_H)}{G_{VA}(X^*, \theta_L) - f(\theta_L)}. \tag{55}$$

26
It is clear that equation (53) is equivalent to:

\[ \frac{f(\theta_H)}{f(\theta_L)} > \frac{G_{VA}(X^*, \theta_H)}{G_{VA}(X^*, \theta_L)}. \]  

(56)

That is, the growth option is less risky than the asset in place precisely when the cost of installed capital is more riskier than the asset in place.

The above argument is illustrated in figure 3. Figure 3 plots the normalized value functions for ideas and production units and illustrates the mechanism of the model. The dashed lines are the normalized value functions for production units in the high state (red line) and in the low state (blue line). The solid lines are the normalized value functions of ideas. Red lines are higher than blue lines, indicating that both ideas and production units are risky. At the option exercise threshold \( X^* \), the distance between the red and blue dashed line is higher than that between the red and blue solid line. This means that when there is a regime switch in the long-run risk state variable \( \theta \), the size of the jump in the value of production units is higher than the jump in the value of ideas. The reason for this, is that the price of installed capital, \( f(\theta_t) C_t \) also jumps when there is a regime switch in \( \theta \). The distance between red dot and red circle is \( f(\theta_H) \), and the distance between the blue dot and the blue circle is \( f(\theta_L) \). \( f(\theta_H) > f(\theta_L) \) reflects the riskiness of installed capital. In fact, it is riskier than production units as shown in equation (56). Consequently, the risk in installed capital more than offsets the risk in assets in place, making growth options less risky.

Implication (56) arises due to the fixed supply of installed capital in our model and the mean reverting nature of long-run risks. The economic intuition behind this is demonstrated in the simple example in section I of the paper. Essentially, if the probability of regime switching in \( \theta \) is close to 0, installed capital must be as risky as assets in place to clear the market. When the economy switches from \( \theta_L \) to \( \theta_H \), the cost of installed capital must increase by the same proportion as the value of assets in place to equate supply and demand of installed capital. The possibility of regime switch introduces more incentives to exercise the option when \( \theta_t = \theta_H \) and more disincentives to exercise the option when \( \theta_t = \theta_L \). Intuitively, in the good state, holders of ideas want to exercise the option early, because they are worried that the long-run risk may hit the economy in the next period, in which case waiting for one more period will lower the value of the option. To deter entance and restore equilibrium, the price of installed capital must be higher in the good state relative to case of no regime switch in \( \theta \). Similarly, the possibility of regime switch introduces more

\(^{10}\text{Figure 3 and figure 4 are based on the parameter values in the calibration exercise in section V.}\)
incentive for holders of ideas to delay the exercise of the option: they want to wait for a regime switch to happen, in which case the idea will have a higher value.

To facilitate analytically simple results and make the intuition as transparent as possible, we make a simplifying assumption that \( o(\theta_H) = o(\theta_L) = 0 \), that is, the operating cost for value assets is assumed to be 0. In this case, growth assets are less risky with respect to long-run risks if and only if dividends paid by production units have more exposure to long-run risks than the aggregate consumption. The formal statement is given by the following proposition.

**Proposition 7** Suppose \( o(\theta_H) = o(\theta_L) = 0 \), then

\[
\frac{G_{VA}(Z, \theta_H)}{G_{VA}(Z, \theta_L)} > \frac{G_{GR}(X^*, \theta_H)}{G_{GR}(X^*, \theta_L)}
\]

for all \( Z > 0 \) if and only if \( \mu_Z(\theta_H) > \mu_Z(\theta_L) \).

**Proof.** Appendix ■

Figure 4\(^1\) plots the risk premia of ideas and production units against the quality of ideas (productivity of production units). The dashed lines are the risk premia of production units, and the solid lines are the risk premia of ideas. The red lines are risk premia of assets in the state \( \theta_t = \theta_H \), and the blue lines are risk premia of assets when \( \theta_t = \theta_L \). It is clear that production units (value assets) are riskier than ideas (growth assets) uniformly. Also, as the productivity of a production unit gets smaller, the risk premium of the production unit increases to infinity. This will generate a size premium in our calibration exercise, as market capitalization is increasing in productivity for value assets. The size effect is due to the presence of the operating cost. Production units have an option to exit the industry. They will do so when the productivity is too low and the option value of waiting is not high enough to cover the operating cost. Low productivity production units in our model are more likely to fail when \( \theta_t = \theta_L \). This implies that low productivity production units will have higher exposure to long-run risks than high productivity ones, generating the size premium.

\(^1\)Note that figures 3 and figure 4 assumes non-zero operating cost parameters.
V Calibration

A Parameter Configuration

We evaluate asset pricing implications of the model by calibrating it to the US economy. As we show, the model is able to quantitatively account for the value and size premia, as well as replicate the failure of the CAPM observed in the data. We simulate the model on the monthly frequency but aim to match time-series dynamics of annual consumption and aggregate dividends. We focus on annual moments in order to avoid seasonal and measurement biases in the data. To be specific, we simulate monthly series over 100 years, aggregate the simulated variables to the annual frequency and report various moments of the resulting annual data. To remove the effect of initial conditions, we effectively run the simulation for 150 years and discard the first fifty years. We find that further increasing the size of the initial simulation sample, does not alter the results.

We choose preference parameters following the long-run risks literature (see Bansal-Yaron (2004)):

$$\begin{align*}
\beta & = 0.01 \\
\gamma & = 10 \\
\psi & = 1.5
\end{align*}$$

This choice of preferences (together with technology parameters, discussed below) allows the model to match dynamics and the level of the risk-free rate, as well as the magnitude of the risk premium in the economy. In particular, the model delivers a stable and reasonably low risk-free rate of about 1.8%, on average (see Table 3).

Consumption growth parameters are chosen as to match time-series properties of observed consumption data. We target moments of real annual per-capita consumption of non-durables and services from the NIPA tables available from the Bureau of Economic Analysis. Our consumption series span the period from 1929 to 2006. We choose the following values of aggregate consumption process:

$$\begin{align*}
\theta_H & = 0.032 \\
\theta_L & = 0.00 \\
\lambda_H & = 0.28 \\
\lambda_L & = 0.38 \\
\sigma_H & = 0.012 \\
\sigma_L & = 0.020
\end{align*}$$

The implied dynamics of annualized simulated consumption are presented in Table 1. For comparison, we also report the corresponding moments of the observed consumption data. Our calibration implies volatility of consumption growth of about 1.8%, comparable to about

\footnote{All the underlying parameters of consumption growth dynamics, as well as cash-flow parameters, are stated in annual terms.}
2.2% in the data. We are also able to match the first-order autocorrelation of consumption growth rates of about 0.44. Note that our calibration implies a longer duration of expansions than that of recessions, which is consistent with business cycle fluctuations defined by the NBER indicator.

We choose firm’s cash flow parameters to match the dynamics of dividend growth rates of the aggregate market portfolio. In particular,

$$
\begin{align*}
\mu_z(\theta_H) & \quad \mu_z(\theta_L) & \quad \sigma_z(\theta_H) & \quad \sigma_z(\theta_L) & \quad \rho(\theta_H) & \quad \rho(\theta_L) \\
0.11 & \quad -0.10 & \quad 0.20 & \quad 0.30 & \quad 0.12 & \quad 0.15
\end{align*}
$$

Table 2 illustrates the fit of the calibrated model to the observed dividend and return dynamics of the aggregate stock market. The aggregate market portfolio data, as well as cross-sectional firm data come from the Center for Research in Securities Prices (CRSP) and the Compustat database. We work with the 1964-2006 sample given the availability of the latter data set. The key moments of dividend growth rates and returns of the market portfolio are presented in the left panel of Table 2. Newey-West(1987) standard errors are reported in parentheses. Overall, the model-implied dynamics of the aggregate market portfolio matches well with the data. Note, however, that the model understates somewhat the volatility of aggregate dividends. In particular, in our sample, the standard deviation of dividend growth rates is about 4.8%, while the model generates about 3.4% only. Consequently, the model-implied correlation between aggregate consumption and market dividends is higher than that in the data (compare 0.57 in the model to 0.31 in the data). Although this implication of the model is not desirable, it can always be improved by allowing for a non-trivial exposure of productivity shocks to short-run consumption risks. Recall that, for simplicity, we set the correlation between productivity innovation and contemporaneous consumption news to zero. Relaxing this assumption will introduce another common component in firm’s cash flows and, consequently, amplify the volatility of aggregate dividends. As expected, the channel of long-run risks induces positive serial correlation in dividend growth dynamics – the first-order autocorrelation of growth rates in the model is about 0.25, which is within the two-standard error from the data estimate.

The bottom panel of Table 2 shows that the model has no difficulties in accounting for the historically high equity premium, as in Bansal-Yaron (2004). In particular, the model-implied average return on the market portfolio is above 8%. As in the data, the correlation between equity returns and consumption growth is low. The model also correctly predicts higher volatilities of asset prices relative to dividends – the standard deviation of market returns is about 13.6%. This statistic, though, is lower than the data estimate, which once
again is driven by the low volatility of dividend growth rates in the model (something that can be easily fixed).

In addition, we need to choose entry measure parameters, $\overline{X}$, $m_X$ and $m_Z$. As far as the model’s implications are concerned, only $\frac{m_Z}{m_X}$ matters. We, therefore, set $\frac{m_Z}{m_X}$ at 0.5, which implies that 50% of new ideas eventually obtain capital and become production units and the other half dies prematurely. We find that asset pricing implications of the model are not particularly sensitive to these parameters. We further choose to set $\overline{X}$ at 1, which is purely a normalization that has no effect on any results.

### B Cross-Sectional Distribution of Firm Returns

Relying on the assumed dynamics of productivity and derived analytical solutions, we simulate a large pool of ideas and production units. To take the model to the data, we then staple the simulated assets into firms. In particular, each firm is created by first randomly sampling from the pool of production units. Guided by the cross-sectional dispersion of sales and market capitalization in the data, we assume that the initial distribution of production units across firms is Pareto. Initially, each firm is assumed to have at least one production unit. We then randomly grant simulated idea to created firms. We track each firm over time, while replacing extinct units with brand-new ideas. Our cross-section consists of 4,000 firms. We sort the simulated sample into five size and book-to-market portfolios following the standard sorting procedure as in Fama-French(1992,1993) and others.

Table 3 presents average returns on the simulated portfolios along with their empirical counterparts. Consistent with the data, the model features both the value and the size premia. The mean return on small market capitalization firms is much higher than the average compensation for holding large firms. Quantitatively, the model-implied size premium is about 2.8%. Similarly, value firms in the model carry a higher risk premium than growth firms. The difference in mean returns on high and low book-to-market portfolios is about 5%. For comparison, the value premium in the data amounts to slightly more than 6%. Thus, the model can account for a significant portion of the cross-sectional variation in mean return on size and book-to-market sorted portfolios.

As in the data, the standard CAPM fails to explain the cross-sectional dispersion in risk premia. The intuition behind this result has been discussed above. We quantify deviations of the model from the market-CAPM predictions in Table 4. For each portfolio in the data and our simulation, we report the CAPM alpha and the corresponding t-statistic. As the table shows, inside the model, the CAPM is strongly rejected – all the alpha’s are significant and most of them are economically sizable. For example, the CAPM severely underprices
growth stocks by more than 3% per annum and overvalues value stocks. Overall, the pattern in simulated alpha’s matches the pattern of the CAPM mis-pricing in the data. Similarly, inside our model, the CAPM brakes down along the size dimension. Note that although the failure of the CAPM to price size sorted portfolios in the data is not statistically significant, many alpha’s are quite sizable.

To summarize, we show that the model calibrated to match the observed time-series distribution of aggregate consumption and market dividends, is able to generate a cross-section of firm returns, which is consistent with some stylized features of the observed asset prices. In particular, the model is able to simultaneously reconcile the high value and size premia and the empirical failure of the CAPM regressions.

VI Conclusion

We put forward a general equilibrium model that links the cross-sectional variation of expected returns to firm’s life cycle dynamics. In the model, all assets have the same exposure to short-run consumption risks, but differ in their exposure to long-run consumption risks (Bansal and Yaron (2004)). An econometrician who uses conditional CAPM regression to predict asset returns will obtain high $\alpha$ for assets with high exposure to long-run risks. In the model, growth options have low exposure to long-run risks than assets in place because the cost of exercising growth options is highly sensitive to low-frequency fluctuations in aggregate consumption and, therefore, provides a hedge against risks in assets in place. The size premium in the model is driven by high operating leverage of small firms.

VII Appendix

A Appendix I: Pricing Kernel of the Economy

Proof of proposition ??

It is notationally convenient to represent the Markov chain as a compound Poisson process. In particular, let $\{N_{t}\}_{t\geq 0}$ be a Poisson process with intensity $\lambda_H$, and $\{N_{L,t}\}_{t\geq 0}$ be a Poisson process with intensity $\lambda_L$. let $I_{\{x\}}$ be the indicator function, that is,

$$I_{\{x\}}(y) = \begin{cases} 
1 & \text{if } y = x \\
0 & \text{if } y \neq x 
\end{cases}$$
Then $\{\theta_t\}$ can be represented as the following compound Poisson process:

$$d\theta_t = \Delta \eta (\theta^-)_t^T dN_t$$  \hspace{1cm} (58)

where

$$\Delta = \theta_H - \theta_L$$

and $\eta (\theta)$, and $N (t)$ are vector notations:

$$\eta (\theta) = [- I_{\{\theta_H\}} (\theta), I_{\{\theta_L\}} (\theta)]^T$$  \hspace{1cm} (59)

$$N (t) = [N_{Ht}, N_{Lt}]^T$$  \hspace{1cm} (60)

Here we use the convention that $\{\theta_t\}$ is right-continuous with left limits, and use the notation

$$\theta^-_t = \lim_{s \to t, s < t} \theta_s$$

We first guess the utility function of the representative agent is of the following form:

$$v_t = V (\theta_t, C_t) = \frac{1}{1 - \gamma} H (\theta_t) C_t^{1-\gamma}$$  \hspace{1cm} (61)

where

$$H (\theta_H) = \exp \left\{ \frac{1}{\beta} \left[ (1 - \gamma) \theta_H - \frac{1}{2} \gamma (1 - \gamma) \sigma^2_C (\theta_H) + \lambda_H (\omega^{-1} - 1) \right] \right\}$$  \hspace{1cm} (62)

$$H (\theta_L) = \exp \left\{ \frac{1}{\beta} \left[ (1 - \gamma) \theta_L - \frac{1}{2} \gamma (1 - \gamma) \sigma^2_C (\theta_L) + \lambda_L (\omega - 1) \right] \right\}$$  \hspace{1cm} (63)

where $\omega$ is the unique solution on $(0, \infty)$ to the following equation:

$$\beta \ln \omega = (1 - \gamma) \left[ (\theta_H - \theta_L) - \frac{1}{2} \gamma (\sigma^2_C (\theta_H) - \sigma^2_C (\theta_L)) \right] + \lambda_H (\omega^{-1} - 1) + \lambda_L (1 - \omega)$$  \hspace{1cm} (64)

One can show that equation (64) has a unique solution on $(0, \infty)$, and the solution satisfies $\omega \in (0, 1)$. Define

$$LHS (\omega) = \beta \ln \omega$$

and

$$RHS (\omega) = (1 - \gamma) \left[ (\theta_H - \theta_L) - \frac{1}{2} \gamma (\sigma^2_C (\theta_H) - \sigma^2_C (\theta_L)) \right] + \lambda_H (\omega^{-1} - 1) + \lambda_L (1 - \omega)$$
Then \( LHS (\omega) \) is strictly increasing on \((0, \infty)\), and \( LHS (\omega) \to -\infty \) as \( \omega \to 0 \); \( LHS (1) = 0 \). Also, \( RHS (\omega) \) is strictly decreasing on \((0, \infty)\). \( RHS (\omega) \to +\infty \) as \( \omega \to 0 \), and

\[
RHS (1) = (1 - \gamma) \left[ (\theta_H - \theta_L) - \frac{1}{2} \gamma \left( \sigma_C^2 (\theta_H) - \sigma_C^2 (\theta_L) \right) \right] < 0
\]

under assumption A and A. It then follows from (62), (63) and (64)

\[
\frac{H (\theta_H)}{H (\theta_L)} = \omega < 1
\]

The state price process of this economy is given by (Duffie and Epstein (1992b)):

\[
\frac{d\pi_t}{\pi_t} = \frac{df_C (C_t, V_t)}{f_C (C_t, V_t)} + f_V (C_t, V_t) \ dt
\]

Using generalized Ito’s formula (Oksendal and Sulem (2004)), one can show \( \{\pi_t\} \) is a Levy process of the form:

\[
d\pi_t = \pi_t \left[ -\bar{r} (\theta_t) \ dt - \gamma \sigma_C (\theta) dB_t - \eta_{\bar{\pi}} (\theta_t^-)^T dN_t \right]
\]

where

\[
\bar{r} (\theta) = \beta + \theta - \gamma \sigma_C^2 (\theta) - \left[ \lambda_H I_{\{H\}} (\theta) (1 - \omega^{-1}) + \lambda_L I_{\{L\}} (\theta) (1 - \omega) \right]
\]

and

\[
\eta_{\bar{\pi}} (\theta) = \left[ (1 - \omega^{-1}) I_{\{H\}} (\theta), (1 - \omega) I_{\{L\}} (\theta) \right]
\]

Furthermore, the risk-free rate of this economy is given by:

\[
r (\theta) = \beta + \theta - \gamma \sigma_C^2 (\theta)
\]

B Appendix II: Firms’ Optimal Stopping Problems

Lemma 1 General Solutions to the Optimal Stopping Problem
Consider the following general formulation of the optimal stopping problem:

\[ V(t;\xi_t, C_t) = E \left[ \int_t^\tau e^{-\kappa(s-t)} \frac{\pi_s}{\pi_t} D(\xi_s, \theta_s) C_s ds + e^{-\kappa(\tau-t)} \frac{\pi_t}{\pi_t} B(\xi_\tau, \theta_\tau) C_\tau \bigg| t \right] \]  

(67)

\[ \begin{align*}
    d\xi &= \xi [\mu(\theta_t) \, dt + \sigma(\theta_t) \, dB_t] \\
    dC_t &= C_t [\theta_t \, dt + \sigma_C(\theta_t) \, dBC_t] \\
    d\pi_t &= \pi_t \left[-\ddot{r}(\theta_t) \, dt - \gamma \sigma_C(\theta_t) \, dB_{Ct} - \eta_\pi(\theta_t)^T \, dN_t \right] \\
    d\theta_t &= (\theta_H - \theta_L) \times \eta(\theta_t)^T \, dN_t,
\end{align*} \]

where \( \tau \) is a \( \{\mathcal{F}_t\}_{t \geq 0} \) adapted stopping time. Then

\[ V(t;\xi, C) = G(\xi, \theta) C \]

where \( G(\xi, \theta) \) is the solution to the following coupled ODE:

\[ D(\xi, \theta_H) - G(\xi, \theta_H) \beta_H + G'(\xi, \theta_H) \xi \mu(\theta_H) + \frac{1}{2} G''(\xi, \theta_H) \xi^2 \sigma^2(\theta_H) + \lambda_H \left[ \omega^{-1} G(\xi, \theta_H) - G(\xi, \theta_H) \right] = 0 \]  

(68)

and

\[ D(\xi, \theta_L) - G(\xi, \theta_L) \beta_L + G'(\xi, \theta_L) \xi \mu(\theta_L) + \frac{1}{2} G''(\xi, \theta_L) \xi^2 \sigma^2(\theta_L) + \lambda_L \left[ \omega G(\xi, \theta_H) - G(\xi, \theta_L) \right] = 0 \]  

(69)

where we denote

\[ \begin{align*}
    \beta_H &= \kappa + \ddot{r}(\theta_H) - \theta_H + \gamma \sigma^2_C(\theta_H) = \beta + \kappa - \lambda_H \left( 1 - \omega^{-1} \right) \\
    \beta_L &= \kappa + \ddot{r}(\theta_L) - \theta_L + \gamma \sigma^2_C(\theta_L) = \beta + \kappa - \lambda_L \left( 1 - \omega \right)
\end{align*} \]  

(70, 71)

**Proof.** By equation 8, we have:

\[ e^{-\kappa t} \pi_t V(\xi_t, \theta_t, C_t) + \int_0^t e^{-\kappa s} \pi_s D(\xi_s, \theta_s) C_s ds \]

\[ = \max_{\tau} E_t \left[ \int_0^\tau e^{-\kappa(s-t)} \frac{\pi_s}{\pi_t} D(\xi_s, \theta_s) C_s ds + e^{-\kappa(\tau-t)} \frac{\pi_t}{\pi_t} B(\xi_\tau, \theta_\tau) C_\tau \bigg| \theta_t, \xi_t, C_t \right] \]
is a martingale. Therefore,

$$e^{-\kappa t} D (\xi_t, \theta_t) C_t + \mathcal{L} \left\{ e^{-\kappa t} \pi_t V (\xi_t, \theta_t, C_t) \right\} = 0$$

(72)

where $\mathcal{L}$ is the generator associated with the vector Markov process $\{\xi_t, C_t, \pi_t, \theta_t\}$. Using homogeneity, $V (\theta_t, \xi_t, C_t) = G (\xi_t, \theta_t) C_t$, we have

$$\mathcal{L} \left\{ e^{-\kappa t} \pi_t V (\xi_t, \theta_t, C_t) \right\} = -\kappa e^{-\kappa t} \pi_t C_t G (\xi_t, \theta_t) + e^{-\kappa t} \mathcal{L} \left\{ \pi_t C_t G (\xi_t, \theta_t) \right\}$$

(73)

Using generalized Ito’s formula (?)), we have:

$$\frac{1}{\pi_t C_t} \mathcal{L} \left\{ \pi_t C_t G (\xi_t, \theta_t) \right\} = G (\xi_t, \theta_t) \left[ -\bar{r} (\theta_t) + \theta_t - \gamma \sigma^2 (\theta_t) \right]$$

$$+ G' (\xi_t, \theta_t) \xi_t \mu (\theta_t) + \frac{1}{2} \lambda (\xi_t, \theta_t) \xi_t^2 \sigma^2 (\theta_t)$$

$$+ I_{\theta_H} (\theta_t) \lambda_H \left[ \omega^{-1} G (\xi_t, \theta_H) - G (\xi_t, \theta_H) \right]$$

$$+ I_{\theta_L} (\theta_t) \lambda_L \left[ \omega G (\xi_t, \theta_H) - G (\xi_t, \theta_L) \right]$$

Therefore equation (72) is written as:

$$D (\xi_t, \theta_t) + G (\xi_t, \theta_t) \left[ -\kappa - \bar{r} (\theta_t) + \theta_t - \gamma \sigma^2 (\theta_t) \right]$$

$$+ G' (\xi_t, \theta_t) \xi_t \mu (\theta_t) + \frac{1}{2} \lambda (\xi_t, \theta_t) \xi_t^2 \sigma^2 (\theta_t)$$

$$+ I_{\theta_H} (\theta_t) \lambda_H \left[ \omega^{-1} G (\xi_t, \theta_L) - G (\xi_t, \theta_H) \right]$$

$$+ I_{\theta_L} (\theta_t) \lambda_L \left[ \omega G (\xi_t, \theta_H) - G (\xi_t, \theta_L) \right] = 0$$

This is the same as (68) and (69). 

Lemma 2 Consider the homogenous part of the coupled ODE (68) and (69):

$$- G (\xi, \theta_H) \beta_H + G' (\xi, \theta_H) \xi \mu (\theta_H) + \frac{1}{2} G'' (\xi, \theta_H) \xi^2 \sigma^2 (\theta_H)$$

$$+ \lambda_H \left[ \omega^{-1} G (\xi, \theta_L) - G (\xi, \theta_H) \right] = 0$$

(74)

$$- G (\xi, \theta_L) \beta_L + G' (\xi, \theta_L) \xi \mu (\theta_L) + \frac{1}{2} G'' (\xi, \theta_L) \xi^2 \sigma^2 (\theta_L)$$

$$+ \lambda_L \left[ \omega G (\xi, \theta_H) - G (\xi, \theta_L) \right] = 0$$

(75)
is of the following form:

\[
G (\xi, \theta_H) = \sum_{i=1}^{4} K_i e_{i,H} \xi^{\alpha_i}, \quad G (\xi, \theta_L) = \sum_{i=1}^{4} K_i e_{i,L} \xi^{\alpha_i}
\]  

(76)

where \( \{K_i\}_{i=1,2,3,4} \) are constants. \( \{\alpha_i\}_{i=1,2,3,4} \) are the eigenvalues of the following quadratic eigenvalue problem:

\[
\frac{1}{2} M e^{2} + N e + L e = 0
\]

(77)

where

\[
M = \begin{bmatrix}
\sigma^2 (\theta_H) & 0 \\
0 & \sigma^2 (\theta_L)
\end{bmatrix}, \quad N = \begin{bmatrix}
\mu (\theta_H) - \frac{1}{2} \sigma^2 (\theta_H) & 0 \\
0 & \mu (\theta_L) - \frac{1}{2} \sigma^2 (\theta_L)
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
-(\lambda_H + \beta_H) & \lambda_H \omega^{-1} \\
\lambda_L \omega & -(\lambda_L + \beta_L)
\end{bmatrix}
\]

and

\[
e_i = [e_{i,H}, e_{i,L}]^T, \quad \text{for} \quad i = 1, 2, 3, 4
\]

are the eigenvectors associated with \( \{\alpha_i\}_{i=1,2,3,4} \).

Without loss of generality, we will always normalize the eigenvectors \([e (i,H), e (i,L)]^T\) so that

\[
e (i,H) = 1, \quad i = 1, 2, 3, 4
\]

Proof. We seek a solution to the homogenous part of (68) and (69) of the following form:

\[
G (\xi, \theta_H) = e_H \xi^\alpha, \quad G (\xi, \theta_L) = e_L \xi^\alpha
\]

Then the homogenous part of equation (68) and (69) are written as:

\[
-\beta_H e_H \xi^\alpha + \mu (\theta_H) e_H \alpha e_H \xi^\alpha + \frac{1}{2} \sigma^2 (\theta_H) e_H (\alpha - 1) \xi^\alpha + \lambda_H [\omega^{-1} e_L \xi^\alpha - e_H \xi^\alpha] = 0
\]

\[
-\beta_L e_L \xi^\alpha + \mu_L e_L \alpha e_L \xi^\alpha + \frac{1}{2} \sigma^2 (\theta_L) e_L (\alpha - 1) \xi^\alpha + \lambda_L [\omega e_H \xi^\alpha - e_L \xi^\alpha] = 0
\]

Divide by \( \xi^\alpha \) on both sides and re-arrange, we have:

\[
\frac{1}{2} \sigma^2 (\theta_H) e_H \alpha^2 + \left[ \mu (\theta_H) - \frac{1}{2} \sigma^2 (\theta_H) \right] e_H \alpha - (\beta_H + \lambda_H) e_H + \lambda_H \omega^{-1} e_L = 0 \quad (78)
\]

\[
\frac{1}{2} \sigma^2 (\theta_L) e_L \alpha^2 + \left[ \mu (\theta_L) - \frac{1}{2} \sigma^2 (\theta_L) \right] e_L \alpha - (\beta_L + \lambda_L) e_L + \lambda_L \omega e_H = 0 \quad (79)
\]

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Clearly, equation (78) and (79) can be written in matrix notation as (77).

**Corollary 2** Consider a stage-II firm. Suppose \( a(\theta_H) = a(\theta_L) = 0 \), then the value of the stage-II firm is of the form in (26), where

\[
G_{V_A}(Z, \theta_H) = a(\theta_H) Z, \quad G_{V_A}(Z, \theta_L) = a(\theta_L) Z
\]

and

\[
a(\theta_H) = \frac{\beta + \kappa Z + \lambda_L \omega + \lambda_H \omega^{-1} - \mu_Z(\theta_L)}{[\beta + \kappa Z + \lambda_L \omega - \mu_Z(\theta_L)] - \lambda_H \lambda_L} \quad (80)
\]

\[
a(\theta_L) = \frac{\beta + \kappa Z + \lambda_L \omega + \lambda_H \omega^{-1} - \mu_Z(\theta_H)}{[\beta + \kappa Z + \lambda_L \omega - \mu_Z(\theta_H)] - \lambda_H \lambda_L} \quad (81)
\]

**Proof.** Let

\[
\xi = Z_{i_i}, \quad (82)
\]

\[
\kappa = \kappa_Z, \quad \mu(\theta) = \mu_Z(\theta), \quad \sigma(\theta) = \sigma_Z(\theta) \quad \text{for} \quad \theta = \theta_H, \theta_L, \quad (83)
\]

\[
D(Z_{i_i}, \theta) = Z_{i_i}, \quad B(Z_{i_i}, \theta) = 0, \quad (84)
\]

\[
\tau = \infty, \quad (85)
\]

and apply lemma 1 and lemma 2.

**Corollary 3** Consider a stage-II firm with dividend given in equation (13). Suppose it does not have the option to exit, then the value of the stage-II firm is of the form in (26), where

\[
G_{V_A}(Z, \theta_H) = a(\theta_H) Z - b(\theta_H), \quad G_{V_A}(Z, \theta_L) = a(\theta_L) Z - b(\theta_L)
\]

and

\[
b(\theta_H) = \frac{o(\theta_H)[\beta + \kappa + \lambda_L \omega] + o(\theta_L) \lambda_H \omega^{-1}}{[\beta + \kappa + \lambda_L \omega - \mu_Z(\theta_L)] - \lambda_H \lambda_L} \quad (86)
\]

\[
b(\theta_L) = \frac{o(\theta_L)[\beta + \kappa + \lambda_L \omega] + o(\theta_H) \lambda_H \omega}{[\beta + \kappa + \lambda_L \omega - \mu_Z(\theta_L)] - \lambda_H \lambda_L} \quad (87)
\]

**Proof.** Apply lemma 1 and lemma 2 to the special case in which (82), (83), (85) and

\[
D(Z_{i_i}, \theta) = Z_{i_i} - o(\theta), \quad B(Z_{i_i}, \theta) = 0. \quad (88)
\]
Proof of Proposition 2

Note the optimization problem for production units stated in equation (15) is a special case of (67) with (82), (83), (85), (88), and

\[ \tau = \tau_{VA}(i,t) \]  

where \( \tau_{VA}(i,t) \) is the solution to the optimal stopping problem (15). By lemma 1 and lemma 2, the general solution to the optimal stopping problem in (15) must be of the form as given in (26) and (27), where \( a(\theta_H), a(\theta_L) \) are given in (80) and (81), \( b(\theta_H), b(\theta_L) \) are given in (86) and (87), and \( \{\alpha_{VA}(i), e_{VA}(i,\theta_H), e_{VA}(i,\theta_L)\} \) are eigenvalues and eigenvectors of the quadratic eigenvalue problem in (77) with parameter restrictions (83). Assumption 3 implies we can arrange the eigenvalues so that \( \alpha_{VA}(1) < \alpha_{VA}(2) < 0 < \alpha_{VA}(3) < \alpha_{VA}(4) \) without loss of generality (For proof, see Guo (2001)). Furthermore, boundary conditions at \( Z_i^t \rightarrow \infty \) implies \( K_3 = K_4 = 0 \) in (76). Optimality requires the value matching and smooth pasting conditions (28) and (29) to be satisfied, which determine the coefficients \( K_1, K_2 \) and the optimal option exercise barrier \( Z^*(\theta_H), Z^*(\theta_L) \).

Proof of Proposition 3

Note the optimization problem for ideas stated in equation (17) is a special case of (67) with

\[ \xi = X_i^i, \]
\[ \kappa = \kappa_X, \quad \mu(\theta_H) = \mu(\theta_L) = \mu_X, \quad \sigma(\theta_H) = \sigma(\theta_L) = \sigma_X, \]  

and

\[ D(X_i^i, \theta) = 0, \quad B(X_i^i, \theta) = G_{VA}(X_i^i, \theta) - f(\theta) \quad \text{for} \quad \theta = \theta_H, \theta_L. \]  

Lemma ?? implies \( G_{GR}(X, \theta) \) is of the form:

\[ G_{GR}(X, \theta_H) = \sum_{i=1}^{4} L_i e_{GR}(i, H) X^{\alpha_{GR}(i)}, \quad G_{GR}(X, \theta_L) = \sum_{i=1}^{4} L_i e_{GR}(i, L) X^{\alpha_{GR}(i)} \]

where \( [\alpha_{GR}(i)]_{i=1,2,3,4} \) and \( [e_{GR}(i, H), e_{GR}(i, L)]_{i=1,2,3,4} \) are the eigenvalues and their corresponding eigenvectors of the quadratic eigenvalue problem in (77) with parameter restrictions (90). Again, \( [\alpha_{GR}(i)]_{i=1,2,3,4} \) can be ordered so that \( \alpha_{GR}(1) < \alpha_{GR}(2) < 0 < 1 < \alpha_{GR}(3) < \alpha_{GR}(4) \) (Proof provided in lemma XXX in appendix XXX.). Since
\( \alpha_{GR}(1), \alpha_{GR}(2) < 0 \), the boundary condition

\[
\lim_{z \to 0} G_{GR}(X, \theta) \geq 0 \quad \text{for} \quad \theta = \theta_H, \theta_L
\]

implies \( L_1 = L_2 = 0 \). \( L_3, L_4 \), together with the option exercise threshold \( X^*(\theta_H) \), \( X^*(\theta_L) \) are determined by the value matching and smooth pasting conditions, (32) and (33).

\section*{C Appendix III: The Cross-Section Distribution of Firms’ Characteristics}

We first observe the characteristic quadratic associated with the homogenous part of the forward equation in (??),

\[
\kappa_X + \left( \mu_X - \frac{1}{2}\sigma_X^2 \right) \eta - \frac{1}{2}\sigma_X^2 \eta^2 = 0,
\]

has two roots:

\[
\eta_1 = \left( \frac{\mu_X}{\sigma_X^2} - \frac{1}{2} \right) + \sqrt{\left( \frac{\mu_X}{\sigma_X^2} - \frac{1}{2} \right)^2 + \frac{2\kappa}{\sigma_X^2}} > 0 \tag{93}
\]

\[
\eta_2 = \left( \frac{\mu_X}{\sigma_X^2} - \frac{1}{2} \right) - \sqrt{\left( \frac{\mu_X}{\sigma_X^2} - \frac{1}{2} \right)^2 + \frac{2\kappa}{\sigma_X^2}} < 0. \tag{94}
\]

It is more convenient to consider the distribution of

\[
x_i^j = \ln X_i^j. \tag{95}
\]

Let \( a \) denote the age of the idea, and let \( \nu(a, \cdot) \) denote the density of the logarithm of the quality of ideas of age \( a \). \( \nu(0, \cdot) \) is therefore the density of the initial distribution of \( \ln X \).

For \( a > 0 \), \( \nu(a, \cdot) \) has to satisfy the following forward equation:

\[
D_a \nu(a, x) = -\kappa_X \nu(a, x) - \left( \mu_X - \frac{1}{2}\sigma_X^2 \right) D_y \nu(a, x) + \frac{1}{2}\sigma_X^2 D_{yy} \nu(a, x) \tag{96}
\]

along with the boundary condition at the absorbing barrier:

\[
\forall a > 0, \quad \nu(a, \ln X^*) = 0 \tag{97}
\]
Lemma 3  The solution to (96) with the boundary condition (97) is:

\[
\nu(a, y) = \int_{-\infty}^{+\infty} \left[ e^{-\kappa X a} \int_{-\infty}^{+\infty} h \left( t \sigma X^2, y - x - \left( \mu X - \frac{1}{2} \sigma X^2 \right) a \right) \right] \nu(0, x) \, dx
\]

and

\[
h(t, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}
\]

The solution can be found in Luttmer (2007). The density of the logarithm of quality of ideas owned by all existing firms can be found by integrating \( \nu(a, y) \) over all \( a \):

\[
\phi(y) = \int_{0}^{\infty} \nu(a, y) \, da
\]

(98)

Recall that the initial quality of ideas are drawn from the uniform distribution on \([0, X] \). This implies that the density of \( \ln X \) is given by:

\[
\frac{1}{X} m X e^x, \quad x \in (-\infty, \ln X].
\]

(99)

There are two possibilities: \( X^* \leq X \) and \( X^* \geq X \). If \( X^* < X \), that is the absorbing barrier \( X^* \) is below \( X \). This implies not all entrant ideas enter into stage I directly. The density of the ideas that enter into stage I is:

\[
\nu(0, x) = \frac{1}{X} m X e^x, \quad x \in (-\infty, \ln X^*].
\]

If \( X^* \geq X \), then all ideas enter into stage I, and the density of the initial entrance to stage I is:

\[
\nu(0, x) = \frac{1}{X} m X e^x, \quad x \in (-\infty, \ln X]
\]

The density of the stationary distribution of the quality of ideas are given by the following lemma.

Lemma 4  \( K_{GR} > 0 \) implies the integral in (98) exists and density of the stationary distribution of \( \ln X \) is given by:

Case 1: \( X^* < X \):

\[
\phi(y) = \frac{m X}{X} \frac{1}{\kappa X + \mu X - \sigma X^2} \left[ e^y - e^{(1-\eta)\ln X^* + \eta y} \right], \quad y \in (-\infty, \ln X^*]
\]
Case 2: $X^* \geq \overline{X}$: For $y \in (\ln \overline{X}, \ln X^*)$

$$\phi(y) = \frac{m_X}{\overline{X}^{\eta_2}} \frac{1}{(1 - \eta_2) \sqrt{(\mu_X - \frac{1}{2} \sigma_X^2)^2 + 2\kappa GR \sigma_X^2}} \left[ e^{\eta_2 y} - e^{(\eta_2 - \eta_1) \ln X^* + \eta_1 y} \right],$$

and for $y \in (-\infty, \ln \overline{X}]$,

$$\phi(y) = \frac{m_X}{\overline{X}} \left\{ \frac{1}{\sqrt{(\mu_X - \frac{1}{2} \sigma_X^2)^2 + 2\kappa GR \sigma_X^2}} \left[ \frac{1}{1 - \eta_1} - \frac{1}{1 - \eta_2} \left( \frac{X^*}{X} \right)^{\eta_2 - \eta_1} \right] e^{\eta_1 y} \right\}$$

where $\eta_1$ and $\eta_2$ are defined in (93) and (94).

Using lemma 4, the density of the stationary distribution of $X$ can be written as:

Case 1: $X^* < \overline{X}$:

$$\Phi(X) = \frac{m_X}{(\kappa + \mu_X - \sigma_X^2) \overline{X}} \left[ 1 - \left( \frac{X}{\overline{X}} \right)^{\eta_1 - 1} \right], \quad X \in [0, X^*]$$

Case 2: $X^* > \overline{X}$. For $X \in (\overline{X}, X^*)$:

$$\Phi(X) = \frac{m_X}{\overline{X}} \frac{1}{(1 - \eta_2) \sqrt{(\mu_X - \frac{1}{2} \sigma_X^2)^2 + 2\kappa GR \sigma_X^2}} \left[ \left( \frac{X}{\overline{X}} \right)^{\eta_2 - 1} - \left( \frac{X^*}{\overline{X}} \right)^{(\eta_2 - \eta_1)} \left( \frac{X}{\overline{X}} \right)^{\eta_1 - 1} \right],$$

and for $X \in (0, \overline{X})$:

$$\Phi(X) = \frac{m_X}{\overline{X}} \left\{ \frac{1}{\sqrt{(\mu_X - \frac{1}{2} \sigma_X^2)^2 + 2\kappa GR \sigma_X^2}} \left[ \frac{1}{1 - \eta_1} - \frac{1}{1 - \eta_2} \left( \frac{X^*}{X} \right)^{\eta_2 - \eta_1} \right] \left( \frac{X}{\overline{X}} \right)^{\eta_1 - 1} \right\}.$$

Furthermore, $\Phi$ can be proved by noting that the absorbing rate at the absorbing barrier must be given by:

$$\frac{1}{2} \sigma_X^2 |\phi'(\ln X^*)|$$

D Appendix IV: Asset Pricing Implications

Proof of Proposition 5

Since $\{\pi_t\}_{t \geq 0}$ is the state price density, for any asset $i$, $\pi_t R_t^i$ is a martingale, where $R_t^i$ is the cumulative return process of the asset defined in equation (100). Using generalized Ito's
formula (\(?\)),
\[
\lim_{\Delta \to 0} \frac{1}{\Delta} E_t \left[ \frac{R_{i,t+\Delta}}{R_{i,t}} - 1 \right] = C \Omega_t \left[ \frac{d\pi_t}{\pi_t}, \frac{dR^i_t}{R^i_t} \right]
\]
One can therefore prove proposition 5 accordingly by using the functional form of \(\pi_t\) in (21) and \(V^i_t\) in (36).

**Proof of Proposition 6**

Take limit as \(\Delta \to 0\) on both sides of equation (43), we have:
\[
\alpha^i_t = RP (i,t) - \chi_t \Omega_t \left( \frac{dR^i_t}{R^i_t}, \frac{dR^{ref}_t}{R^{ref}_t} \right) \tag{103}
\]
where \(\chi_t\) is as defined in (50) and \(RP (ref,t)\) is the risk premium of the reference asset at time \(t\). Using proposition 5,
\[
RP (i,t) = \gamma \sigma_C^2 (\theta_t) + \sum_{j=H,L} \lambda_j \eta_{\pi,j} \eta_{i,j}
\]
where
\[
\begin{bmatrix}
\eta_{\pi,H} \\
\eta_{\pi,L}
\end{bmatrix} = \begin{bmatrix}
1 - \omega^{-1} \\
1 - \omega
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
\eta_{i,H} \\
\eta_{i,L}
\end{bmatrix} = \begin{bmatrix}
G(i,\theta_L) - 1 \\
G(i,\theta_H) - 1
\end{bmatrix}
\]

Also, since the value of the reference asset is of the form in (45), the cumulative return process of the reference asset can be written as:
\[
\frac{dR^i_t}{R^i_t} = \left[ EP (ref,t) + r(\theta_t) \right] dt + \sigma_C (\theta_t) dB_t + \eta_{ref} (\theta_t) dN_t
\]
where
\[
\eta_{ref} (\theta) = \begin{bmatrix}
\eta_{ref,H} I_{\{\theta_H\}} (\theta) \\
\eta_{ref,L} I_{\{\theta_L\}} (\theta)
\end{bmatrix} = \begin{bmatrix}
\frac{s(\theta_L)}{s(\theta_H)} - 1 \\
\frac{s(\theta_L)}{s(\theta_H)} - 1
\end{bmatrix} I_{\{\theta_H\}} (\theta)
\]
Therefore,
\[
\Omega_t \left( \frac{dR^i_t}{R^i_t}, \frac{dR^{ref}_t}{R^{ref}_t} \right) = \sigma_C^2 (\theta_t) + \sum_{j=H,L} \lambda_j \eta_{ref,j} \eta_{i,j}
\]

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Therefore (103) can be written as:

$$\alpha_t^i = \gamma \sigma^2_C(\theta_t) + \sum_{j=H,L} \lambda_j \eta_{ij} - \chi_t \left[ \sigma^2_C(\theta_t) + \sum_{j=H,L} \lambda_j \eta_{ref ij} \right]$$

$$= (\gamma - \chi_t) \sigma^2_C(\theta_t) + \sum_{j=H,L} \lambda_j \eta_{ij} \left[ \eta_{ij} - \chi_t \eta_{ref ij} \right]$$

This proves equations (48) and (49).

To prove (46) and (47), note that in this economy, consumption-to-wealth ratio is constant and equal to $\beta$. Therefore the Consumption CAPM and CAPM with aggregate wealth as the reference asset should produce the same $\alpha$. With aggregate wealth as the reference asset, we have

$$s(\theta_H) = s(\theta_L) = \frac{1}{\beta}$$

Equations (46) and (47) can therefore be proved by imposing condition (104) in equations (48) and (49).

E Appendix V: Value Premium

In this section, we prove of proposition 7 through a series of lemmas. First note that the optimization problem for stage I firms in (17) is a special case of (67) with the parameter restriction (90) and (91). We first state and prove a lemma that gives a complete characterization of the eigenvalue and eigenvectors of the quadratic eigenvalue problem for stage I firms.

**Lemma 5** The eigenvalues of the quadratic eigenvalue problem in (67) with the parameter restriction (90) and (91) satisfies $a_{GR}(1) < a_{GR}(2) < 0 < 1 < a_{GR}(3) < a_{GR}(4)$. The corresponding normalized eigenvectors are given by:

$$\begin{bmatrix}
e_{GR}(1,H) \\
e_{GR}(1,L)
\end{bmatrix} = \begin{bmatrix}
e_{GR}(4,H) \\
e_{GR}(4,L)
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\lambda_{GR}} \omega^2 \\
-\frac{1}{\lambda_{GR}} \omega^2
\end{bmatrix}$$

and

$$\begin{bmatrix}
e_{GR}(2,H) \\
e_{GR}(2,L)
\end{bmatrix} = \begin{bmatrix}
e_{GR}(3,H) \\
e_{GR}(3,L)
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix}$$

**Proof.** Note that the optimization problem for stage I firms in (17) is a special case of (67) with the parameter restriction (90) and (91). Therefore the quadratic eigenvalue problem...
in (77) can be written as:

\[
\frac{1}{2} \sigma_{GR} e_H \alpha^2 + \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] e_H \alpha - (\beta_H + \lambda_H) e_H + \lambda_H \omega^{-1} e_L = 0 \quad (105)
\]

\[
\frac{1}{2} \sigma_{GR} e_L \alpha^2 + \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] e_L \alpha - (\beta_L + \lambda_L) e_L + \lambda_L \omega e_H = 0 \quad (106)
\]

Divide both sides of equation (105) by \(e_H\), and both sides of equation (106) by \(e_L\), we have:

\[
\frac{1}{2} \sigma_{GR} \alpha^2 + \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] \alpha - (\beta_H + \lambda_H) + \lambda_H \omega^{-1} \frac{e_L}{e_H} = 0 \quad (107)
\]

\[
\frac{1}{2} \sigma_{GR} \alpha^2 + \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] \alpha - (\beta_L + \lambda_L) + \lambda_L \omega \frac{e_H}{e_L} = 0 \quad (108)
\]

We can rearrange (107) and (108) and get:

\[
e_H = -\frac{\frac{1}{2} \sigma_{GR} \alpha^2 - \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] \alpha + (\beta_L + \lambda_L)}{\lambda_L \omega} \cdot \frac{\lambda_H \omega^{-1}}{-\frac{1}{2} \sigma_{GR} \alpha^2 - \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] \alpha + (\beta_H + \lambda_H)}
\]

It is therefore easy to see that the eigenvectors and eigenvalues can be constructed in the following way:

1. \(\alpha\) is an eigenvalue if \(\exists x\) such that

\[
\frac{\beta_L + \lambda_L - x}{\lambda_L \omega} = \frac{\lambda_H \omega^{-1}}{\beta_H + \lambda_H - x} \quad (109)
\]

and

\[
\frac{1}{2} \sigma_{GR} \alpha^2 + \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] \alpha = x \quad (110)
\]

2. The corresponding normalized eigenvector can be constructed by:

\[
\frac{e_H}{e_L} = \frac{\beta_L + \lambda_L - x}{\lambda_L \omega}, \quad e_H = 1 \quad (111)
\]

Use equation (70) and (71), equation (109) can be written as:

\[
\frac{\beta + \kappa + \lambda_L \omega - x}{\lambda_L \omega} = \frac{\lambda_H \omega^{-1}}{\beta + \kappa + \lambda_H \omega^{-1} - x} \quad (112)
\]
and equation (111) can be written as:

\[
\frac{e_H}{e_L} = \frac{\beta + \kappa + \lambda L \omega - x}{\lambda L \omega}, \quad e_H = 1
\]

Equation (112) has two solutions,

\[
x_1 = \beta + \kappa, \quad x_2 = \beta + \kappa + \lambda H \omega^{-1} + \lambda L \omega
\]

Therefore there are two independent eigenvectors:

\[
\frac{e_{A,H}}{e_{A,L}} = 1
\]

and

\[
\frac{e_{B,H}}{e_{B,L}} = -\frac{\lambda H}{\lambda L} \omega^{-2}
\]

There are two eigenvalues corresponding to the eigenvector \([e_{A,H}, e_{A,L}]\), denoted \(\alpha_{A,1}, \alpha_{A,2}\), which are solutions to the following quadratic:

\[
\frac{1}{2} \sigma_{GR} \alpha^2 + \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] \alpha = \beta + \kappa
\]

Without loss of generality, we require \(\alpha_{A,1} < \alpha_{A,2}\). There are two eigenvalues corresponding to the eigenvector \([e_{B,H}, e_{B,L}]\), denoted \(\alpha_{B,1}, \alpha_{B,2}\), which are solutions to the following quadratic:

\[
\frac{1}{2} \sigma_{GR} \alpha^2 + \left[ \mu_{GR} - \frac{1}{2} \sigma_{GR} \right] \alpha = \beta + \kappa + \lambda H \omega^{-1} + \lambda L \omega
\]

Without loss of generality, we require \(\alpha_{B,1} < \alpha_{B,2}\). It is straightforward to show \(\alpha_{B,1} < \alpha_{A,1} < 0 < \alpha_{A,2} < \alpha_{B,2}\). Furthermore, assumption 3 implies \(\alpha_{A,2} < \alpha_{B,2}\). This proves the lemma.

The following lemma links \(a(\theta)\) to the sign of the coefficient of the value function for ideas, \(L_4\), in equation (31).

**Lemma 6** \(L_4 > 0\) if and only if

\[
a(\theta_H) > a(\theta_L)
\]

where \(a(\theta_H)\) and \(a(\theta_L)\) are given by equation (80) and (81), and \(L_4\) is the coefficient in equation (31).
Proof. Given the functional form of $G_{GR}(Z,\theta)$ in (31), the value matching and smooth pasting conditions, (32) and (33), can therefore be written as:

\[
e_{GR}(3,H)(X^{*})^{\alpha_{GR}(3)}L_{3} + e_{GR}(4,H)(X^{*})^{\alpha_{GR}(4)}L_{4} = a(\theta_{H})X^{*} - f(\theta_{H}) \quad (116)
\]

\[
e_{GR}(3,L)(X^{*})^{\alpha_{GR}(3)}L_{3} + e_{GR}(4,L)(X^{*})^{\alpha_{GR}(4)}L_{4} = a(\theta_{L})X^{*} - f(\theta_{L}) \quad (117)
\]

and

\[
a_{GR}(3)e_{GR}(3,H)(X^{*})^{\alpha_{GR}(3)}L_{3} + a_{GR}(4)e_{GR}(4,H)(X^{*})^{\alpha_{GR}(4)}L_{4} = a(\theta_{H})X \quad (118)
\]

\[
a_{GR}(3)e_{GR}(3,L)(X^{*})^{\alpha_{GR}(3)}L_{3} + a_{GR}(4)e_{GR}(4,L)(X^{*})^{\alpha_{GR}(4)}L_{4} = a(\theta_{L})X \quad (119)
\]

Given $X^{*}$, equation (118) and (119) determine $L_{3}$ and $L_{4}$, and equation (116) and (117) determine $f(\theta_{H})$ and $f(\theta_{L})$. Using equation (118) and (119), we have:

\[
L_{4} = \frac{a(\theta_{L})e_{GR}(3,H) - a(\theta_{H})e_{GR}(3,L)}{[e_{GR}(4,L)e_{GR}(3,H) - e_{GR}(4,H)e_{GR}(3,L)] a_{GR}(4)(X^{*})^{\alpha_{GR}(4)-1}}
\]

\[
= \frac{a(\theta_{H}) - a(\theta_{L})}{\left[\frac{\lambda_{L}}{\lambda_{H}}\omega^{2} + 1\right] a_{GR}(4)(X^{*})^{\alpha_{GR}(4)-1}}
\]

where the second line above uses lemma 5. This proves the lemma since $a_{GR}(4) > 1$ by lemma 5.

The following lemma gives a necessary and sufficient condition for the inequality (57) in terms of the eigenvectors of (77).

Lemma 7 (57) is true if and only if (115) is true.

Proof. $o(\theta_{H}) = o(\theta_{L}) = 0$ implies

\[
G_{VA}(Z,\theta_{H}) = a(\theta_{H})Z; \quad G_{VA}(Z,\theta_{L}) = a(\theta_{L})Z \quad (120)
\]

Equation (120), along with the value matching condition (32) implies that condition (57) is equivalent to

\[
\frac{a(\theta_{H})X^{*} - f(\theta_{H})}{a(\theta_{L})X^{*} - f(\theta_{L})} < \frac{a(\theta_{H})}{a(\theta_{L})}. \quad (121)
\]

Therefore we only need to prove the equivalence between (115) and (121).
Combine equation (116) and (118), we have:

\[ \alpha_{GR}(3) [a(\theta_H) X^* - f(\theta_H)] = a(\theta_H) X^* - L_4 [\alpha_{GR}(4) - \alpha_{GR}(3)] e_{GR}(4, H) (X^*)^{\alpha_{GR}(4)} \]  

(122)

Similarly, we combine equation (117) and (119) to get:

\[ \alpha_{GR}(3) [a(\theta_L) X^* - f(\theta_L)] = a(\theta_L) X^* - L_4 [\alpha_{GR}(4) - \alpha_{GR}(3)] e_{GR}(4, L) (X^*)^{\alpha_{GR}(4)} \]  

(123)

Note both sides of equation (122) and (123) must be positive by the value matching conditions. Combine equation (122) and (123), we have:

\[
\frac{a(\theta_H) X^* - f(\theta_H)}{a(\theta_L) X^* - f(\theta_L)} = \frac{a(\theta_H) X^* - L_4 [\alpha_{GR}(4) - \alpha_{GR}(3)] e_{GR}(4, H) (X^*)^{\alpha_{GR}(4)}}{a(\theta_L) X^* - L_4 [\alpha_{GR}(4) - \alpha_{GR}(3)] (X^*)^{\alpha_{GR}(4)}}
\]

(124)

where the second equality makes use of lemma 5. Equation (124) implies

\[
\frac{a(\theta_H) X^* - f(\theta_H)}{a(\theta_L) X^* - f(\theta_L)} < \frac{a(\theta_H)}{a(\theta_L)}
\]

if and only if \(L_4 > 0\). By lemma 6, this is equivalent to \(a(\theta_H) > a(\theta_L)\), as needed.

To complete the proof of proposition 7, note that lemma 7 implies that (57) is true if and only if

\[
\frac{a(\theta_H)}{a(\theta_L)} > 1.
\]

(125)

Using the definition of \(a_H\) and \(a_L\) are given by equation (80) and (81), it is easy to see that (125) is equivalent to \(\mu_{VA}(\theta_H) > \mu_{VA}(\theta_L)\).

**References**


Bansal, R. and A. Yaron (2004, August). Risk for the long run: A potential resolution of


Table 1: Consumption Growth Dynamics

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<thead>
<tr>
<th>Statistic</th>
<th>Data</th>
<th>Model</th>
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<tbody>
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<td>$E[\Delta c]$</td>
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<tr>
<td>$\sigma(\Delta c)$</td>
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<td>$AC(1)$</td>
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Table 2: Dynamics of Aggregate Market Portfolio

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<th>Data</th>
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<td>(1.01)</td>
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<tr>
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<td>(0.64)</td>
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<td>(0.11)</td>
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<tr>
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<tr>
<td>Return</td>
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<tr>
<td>$E[R]$</td>
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<td>(2.22)</td>
</tr>
<tr>
<td>$\sigma(R)$</td>
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<td>(1.79)</td>
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<tr>
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<tr>
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<td>(0.13)</td>
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Table 3: Average Returns

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<tr>
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Figure 1: Dynamics of Ideas
Figure 2: Dynamics of Production Units
Figure 3: Normalized Value Functions of Ideas and Production Units
Figure 4: Equity Premia of Ideas and Production Units