

Long-Run Effects of Fiscal Policy under Altruism and Endogenous Fertility

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January 1986
revised February 1986

An earlier draft of this paper was circulated as "The Modified Golden Rule under Endogenous Fertility and Concave Altruism" and was presented at a Research Meeting of the Financial Markets and Monetary Economics program of the NBER. I thank N. Gregory Mankiw for helpful discussion and the National Science Foundation for financial support.

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Abstract

The steady state of an economy with bequests motivated by altruism is characterized by the Modified Golden Rule, $R\beta = G$, where R is the gross rate of return on capital, G is the gross rate population growth, and β is an intertemporal discount factor reflecting the degree of altruism. In conventional growth models, G and β are fixed exogenously and therefore R is invariant to various types of changes which do not directly affect the rate of return to capital. Under concave altruism, β is specified to be a decreasing function of utility; in this case, if R and G are fixed exogenously, then steady state utility is invariant to various changes. Alternatively, β may be a function only of G , and G may be determined endogenously; in this case, if R is fixed exogenously, then steady state fertility is invariant to various types of changes. In particular, Becker and Barro (1985) have derived the remarkable result that, under linear altruism (β independent of the level of utility) and exogenous R , steady state fertility is invariant to changes in the cost of raising children.

This paper develops a model in which R , β and G are all determined endogenously. The model contains many existing models as special cases and illustrates the degree to which many well-known results depend on assuming that one or more of R , β and G are fixed. This model is used to analyze the steady state effects of various fiscal policies and productivity shocks.

The long-run effects of fiscal policy depend on the length of the planning horizon of individual consumers as well as on certain features of the specification of their intertemporal preferences. If consumers have infinite horizons and time-separable utility as in the Cass (1965) model, then the steady state capital stock is determined by the Modified Golden Rule and is invariant to changes in government spending financed by lump-sum taxes. In addition, the steady state capital stock is invariant to the level of government debt outstanding.

Alternatively, if consumers have finite horizons as in Diamond's (1965) overlapping generations model, then the steady state capital stock can be affected by changes in steady state government spending or by changes in the steady state level of government debt outstanding, even if these changes are financed by lump-sum taxes. However, Barro (1974) showed that if consumers have finite lives but have operative altruistic bequest motives of a certain form (linear altruism), then they effectively have infinite planning horizons with time-separable preferences; therefore, the steady state invariance of the capital stock derived in the Cass model continues to hold.

An alternative extension of the Cass model retains the infinite horizon of individual consumers but relaxes the assumption that preferences are additively separable over time. (See Obstfeld (1981), Svensson and Razin (1983), Epstein and Hines (1983), Epstein (1985), and Lucas and Stokey (1984).) Without the assumption of time-separability, the rate of time preference is endogenous. In particular, under the common assumption of "increasing marginal impatience" it can be shown that an increase in government spending financed by lump-sum taxes leads to an increase in the steady state capital labor ratio. Recently, I have (Abel (1985)) analyzed a more general specification of altruism (concave altruism) and have shown that in certain cases it is formally identical to the recursive preferences of infinitely lived consumers.

Most of the literature with altruistic consumers has assumed that the rate of population growth is exogenous. Recently, Barro and Becker (1984) and Becker and Barro (1985) have introduced the fertility decision into the optimization problem of a representative altruistic

consumer. They have derived the remarkable result that if the interest rate is exogenously fixed (as would be the case for a small open economy), then the steady state rate of fertility is independent of the cost of raising children. In particular, a tax on fertility would not affect steady state fertility.

The various strands of the literature described above can be unified by the observation that if consumers (or a social planner) effectively have infinite planning horizons, then the steady state of the economy can be described by the Modified Golden Rule. The discrete-time version of the Modified Golden Rule, which was presented by Samuelson (1968), states that $R\beta = G$ where R is the gross rate of return on capital, G is the gross rate of population growth, and β is an intertemporal preference parameter. The precise interpretation of β depends on the particular formulation of the model: In the case of an infinitely-lived consumer, β reflects the consumer's rate of time preference; in the case of a social planner in an economy of overlapping generations of finitely-lived consumers, β reflects the social planner's preference for one generation vis-a-vis the subsequent generation; in the case of a dynastic family with finitely-lived altruistic consumers, β reflects the degree of altruism of a consumer toward his children.

The simple relation in the Modified Golden Rule has been used to derive various steady state invariance results. Under the standard assumptions of exogenous fertility and time-separable preferences with a constant rate of time preference, both β and G are constant. In this case, the *steady state rate of return to capital is invariant to various sorts of policy changes*. In particular, if R is a decreasing function of capital labor ratio, then the steady state capital labor ratio and the steady state rate of return on capital are invariant to changes in lump-sum taxes. In the context of a monetary growth model, the invariance of the steady state marginal product of capital leads to long-run superneutrality of money as in Sidrauski (1967).

Relaxing the assumption that β is an exogenous preference parameter, there is an alternative steady state invariance result when G is given exogenously. In the context of infinitely-lived consumers, Obstfeld (1981), Svensson and Razin (1983), Epstein and Hines (1983), Epstein (1985), and Lucas and Stokey (1984) have specified the rate of time preference to depend on

the level of utility. The result which emerges from this literature is that steady state utility is invariant to lump-sum fiscal policy if the rate of return on capital, R , is fixed exogenously. In the context of a dynastic family of finitely-lived consumers, there is a similar invariance result if consumers have concave altruism. Under concave altruism, β is a decreasing function of V , the utility of a representative heir. In this case, the Modified Golden Rule can be written as $R\beta(V) = G$, which implies that steady state utility is invariant to lump-sum fiscal policy if R and G are fixed exogenously.

In the recent work of Barro and Becker (1984) and Becker and Barro (1985) the number of children per parent, G , enters the individual optimization problem through two channels: (1) it is costly to raise children; and (2) the number of children affects utility. Formally, Barro and Becker specify the altruism parameter β to be an increasing concave function of G ; in addition β is specified to be independent of V so that there is linear altruism, as discussed in section I below. In this case, the Modified Golden Rule can be written as $R\beta(G) = G$ so that if R is fixed exogenously, then the steady state population growth rate is independent of lump-sum fiscal policy; indeed it is even independent of changes in the cost of raising children.

The goal of the present paper is twofold. First, I will develop and analyze a model with endogenous fertility, an endogenous rate of return to capital and concave altruism which will unify the various strands of the infinite horizon and altruism literatures described above. Special cases of this model will display various invariance results, but, in general, steady state capital intensity and steady state fertility will respond to various sorts of exogenous changes. The second goal of the paper is to analyze the long-run effects of fiscal policies in the unified model presented in this paper. The model will be used to analyze both lump-sum and distortionary tax policies. It will be used to analyze policies in which the tax revenue is used by the government to purchase output as well as policies in which the tax revenue is rebated to consumers via lump-sum subsidies.

The decision problem of the individual consumer is analyzed in section I, and competitive factor prices are presented in section II. In section III I discuss the steady state of the economy

and introduce two loci of pairs of steady state fertility and steady state capital intensity which must characterize the steady state: the Modified Golden Rule (MGR) locus and the Optimal Fertility (OF) locus. Sections IV and V use these loci to analyze the long-run effects of changes in the cost of child-rearing and productivity changes, respectively. This framework is then used to examine the long-run effects of fiscal policies in Section VI. Concluding remarks are presented in Section VII.

1. The Consumer's Optimization Problem

Consider an economy in which each consumer lives for one period as a child and for one period as an adult. Only adults make economic decisions. A generation t consumer enters adulthood at the beginning of period t and receives an inheritance l_t at this time. During period t , the consumer inelastically supplies one unit of labor for which he receives a wage w_t . Also during period t he consumes c_t and spends $\phi(G_{t+1}; \alpha)$ in order to raise G_{t+1} children, where $\phi' > 0$ and $\phi'' < 0$. His wealth at the end of period t is $l_t + w_t - c_t - \phi(G_{t+1}; \alpha)$ and this wealth is held in the form of capital to be used in production in period $t+1$. This capital is divided equally among his G_{t+1} children, so that the capital labor ratio in period $t+1$, k_{t+1} , is

$$k_{t+1} = [l_t + w_t - c_t - \phi(G_{t+1}; \alpha)] / G_{t+1} \quad (1)$$

Let R_{t+1} be the gross rate of return on capital held at the beginning of period $t+1$. The inheritance received by each generation $t+1$ consumer consists of the capital stock per capita, k_{t+1} , with its accrued interest,

$$l_{t+1} = R_{t+1} k_{t+1} = (R_{t+1} / G_{t+1}) [l_t + w_t - c_t - \phi(G_{t+1}; \alpha)] \quad (2)$$

Let V_t denote the utility of a generation t consumer and observe that for a given path of interest rates and wage rates, V_t can be expressed as a function of the inheritance received by the generation t consumer, $V_t = V_t(l_t)$. The consumer is assumed to be altruistic toward his children, and the utility function is specified to be

$$V_t(l_t) = \max \{ u(c_t) + h(V_{t+1}(l_{t+1}), G_{t+1}) \} \quad (3)$$

where the maximization in (3) is with respect to c_t and G_{t+1} and is subject to the constraint in (2). The utility of consumption, $u(c)$, is assumed to be strictly increasing and strictly concave ($u' > 0$ and $u'' < 0$). The altruism function is assumed to be strictly increasing in the utility of the representative child and in the number of children ($h_V > 0$ and $h_G > 0$). It is also assumed that $h_V < 1$ so that at the margin a consumer values his own utility more than his representative heir's utility. The altruism function is also assumed to be concave which implies that $h_{VV} \leq 0$ and $h_{GG} \leq 0$. The cross partial derivative h_{VG} is assumed to be positive which implies that an additional child is more desirable if the child has a higher level of utility. Finally, in order to satisfy the second-order conditions we place three additional restrictions on the utility function

$$\theta \equiv 1 - Gh_{VG}/h_V > 0 \quad (4a)$$

$$\Gamma \equiv -\{\theta + (1-h_V)[Gh_G/h_V]u''/(u'^2)\} > 0 \quad (4b)$$

$$\Psi \equiv -Gh_{GG}/h_G - \theta \geq 0 \quad (4c)$$

Condition (4a) states that the elasticity of h_V with respect to G is less than one. Condition (4b) is equivalent to the second-order condition given in the Appendix of Becker and Barro (1985). In particular, Becker and Barro specify $h(V,G)$ as $\alpha G^{1-\epsilon} V$ with $\alpha > 0$ and $0 < \epsilon < 1$. In this case, $\theta = \epsilon$ which satisfies (4a), and in the steady state $\Gamma = -\epsilon - (1-\epsilon)uu''/u'^2$; thus, the condition $\Gamma > 0$ is identical to the second-order condition given in the Appendix of Becker and Barro. In the Becker-Barro formulation Gh_{GG}/h_G is equal to $-\epsilon$ so that $\Psi = 0$, which satisfies (4c). Conditions (4a,b,c) insure that the matrix of second derivatives in Appendix A is negative definite.

The distinction between linear altruism ($h_{VV} = 0$) and concave altruism ($h_{VV} < 0$) discussed in Abel (1985) has important implications for the comparative steady state analysis in sections III through VI. A linear altruism function (more precisely, an altruism function which is linear in V) can be specified as $h(V,G) = h^*(G)V$ so that the utility function of an individual consumer is

$$V_t(l_t) = \max \{u(c_t) + h^*(G_{t+1})V_{t+1}(l_{t+1})\} \quad (5)$$

Note that if the number of children is an exogenous constant, then $h^*(G_{t+1})$ is simply a constant and the linear altruism function in (5) is identical to the standard formulation of altruism analyzed by Barro (1974). This formulation is (in the absence of binding constraints on bequests) equivalent to the standard time-separable utility functions of infinitely-lived consumers in optimal growth models. Barro and Becker (1984) and Becker and Barro (1985) analyzed the behavior of consumers with the altruism function in (5) but, since they allow G_{t+1} to be a decision variable, $h^*(G_{t+1})$ is not a parametric constant in their formulation. In addition to analyzing behavior under linear altruism as in (5), we also analyze behavior under concave altruism ($h_{yy} < 0$) which is the analogue of endogenous time preference in the infinite-horizon optimal growth literature.

The first-order conditions for the maximization problem in (3) are obtained by substituting (2) into (3) and differentiating with respect to c_t and G_{t+1} , respectively, to obtain

$$u'(c_t) = h_V(V_{t+1}, G_{t+1}) V'(I_{t+1})(R_{t+1}/G_{t+1}) \quad (6a)$$

$$h_V(V_{t+1}, G_{t+1}) V'(I_{t+1})(R_{t+1}/G_{t+1}) [k_{t+1} + \phi'(G_{t+1}; \alpha)] = h_G(V_{t+1}, G_{t+1}) \quad (6b)$$

In addition, to prevent the consumer from running a Ponzi game we impose the condition

$\lim_{T \rightarrow \infty} [(G_{t+1}/R_{t+1}) \cdots (G_T/R_T)] I_T \geq 0$. Using (6a) to simplify (6b) yields

$$u'(c_t) [k_{t+1} + \phi'(G_{t+1}; \alpha)] = h_G(V_{t+1}, G_{t+1}) \quad (7)$$

It follows immediately from the envelope theorem that $V'(I_{t+1}) = u'(c_{t+1})$ so that (6a) can be rewritten as

$$u'(c_t) = h_V(V_{t+1}, G_{t+1}) u'(c_{t+1})(R_{t+1}/G_{t+1}) \quad (8)$$

II. Competitive Factor Prices

In this section we determine factor prices as a function of the capital labor ratio. The production function is assumed to be a linearly homogeneous function of capital and labor and is written in intensive form as $y = f(k, \delta)$ where $f_k > 0$, $f_{kk} < 0$, $f_\delta > 0$, $f_{k\delta} \geq 0$ and y is output per capita, k is capital per capita, and δ is a technological shift parameter. The gross rate of return

to capital, R , is equal to one plus the marginal product of capital

$$R(k, \delta) = 1 + f_k(k, \delta) \quad (9a)$$

$$R_k(k, \delta) = f_{kk}(k, \delta) < 0 \quad (9b)$$

$$R_\delta(k, \delta) = f_{k\delta}(k, \delta) \geq 0 \quad (9c)$$

The competitive wage rate, w , is equal to the marginal product of labor

$$w(k, \delta) = f(k, \delta) - kf_k(k, \delta) \quad (10a)$$

$$w_k(k, \delta) = -kf_{kk}(k, \delta) > 0 \quad (10b)$$

$$w_\delta(k, \delta) = f_\delta(k, \delta) - kf_{k\delta}(k, \delta) \quad (10c)$$

Equations (9b) and (10b) imply the following useful relation

$$kR_k + w_k = 0 \quad (11)$$

Equations (9c) and (10c) imply

$$kR_\delta + w_\delta = f_\delta \quad (12)$$

The relations in (11) and (12) will be useful when we linearize the model around the steady state and examine the effect of a technological shock.

III. The Steady State

In this section we characterize the steady state of the economy and then linearize around the steady state in order to lay the groundwork for comparative steady state analysis in subsequent sections. In the steady state consumption per capita (c), capital per capita (k), children per parent (G), and utility (V), are all constant. Setting $c_t = c_{t+1}$ in (8) yields the Modified Golden Rule

$$R(k, \delta)h_V(V, G) = G \quad (13)$$

Note that h_V is equivalent to β in the formulation of the Modified Golden Rule presented in the introduction. Since we have assumed that $0 < h_V < 1$, the Modified Golden Rule in (13) implies that $R > G$, i.e., that the steady state is dynamically efficient. Equation (7) immediately delivers the steady state relation

$$u'(c) [k + \phi'(G; \alpha)] = h_G(V, G) \quad (14)$$

It follows from the specification of the utility function in (3) that in the steady state

$$V = u(c) + h(V, G) \quad (15)$$

Evaluating (2) in the steady state yields the steady state market clearing condition

$$[R(k, \delta) - G]k + w(k, \delta) - c - \phi(G; \alpha) = 0 \quad (16)$$

Equations (13)-(16) are four equations in the four variables c , V , k , and G and in the two shift parameters α and δ . To analyze comparative steady states we linearize this system of equations around the steady state to obtain

$$\begin{array}{ccc|c|c|c} Rk h_{VV} & 0 & -\theta & R_k h_V & dV & -R_G h_V d\delta \\ \hline -h_{GV} & (k + \phi') u'' & u' \phi'' - h_{GG} & u' & dc & -u' \phi'_\alpha d\alpha \\ \hline h_V - 1 & u' & h_G & 0 & dG & 0 \\ \hline 0 & -1 & -(k + \phi') & R - G & dk & \phi_\alpha d\alpha - f_\delta d\delta \end{array} \quad (17)$$

Let N be the 4×4 matrix on the left hand side of (17) and let Δ denote the determinant of N . In Appendix B we show that Δ is positive. Also in Appendix B we present the third and fourth rows of N^{-1} which allow us to calculate the effects on G and k of changes in the parameters α and δ .

Before presenting the formal results based on the linearized system in (17), we characterize the steady state by two equations in the steady state capital stock, k , and the steady state number of children per parent, G . It follows immediately from the steady state market clearing condition (16) that steady state consumption c can be expressed as a function of steady state fertility, the steady state capital labor ratio and the parameters α and δ

$$c = c(k, G, \alpha, \delta) \quad (18)$$

where $c_k = R - G > 0$, $c_G = -(k + \phi') < 0$, $c_\alpha = -\phi_\alpha$ and $c_\delta = f_\delta$. The partial derivatives of $c(\cdot)$ have straightforward interpretations. An increase in the steady state capital labor ratio increases steady state consumption per capita by the excess of the interest rate over the population growth rate, as is well-known from the Golden Rule literature; hence, $c_k = R - G$. An

increase in steady state fertility decreases steady state consumption per capita for two reasons: savings must increase in order to provide the extra children with the per capita capital stock, k , and the cost of child-rearing increases by ϕ' ; hence $c_G = -(k + \phi')$. For given steady state fertility and a given steady state capital labor ratio, an increase in the child-rearing cost parameter α reduces steady state resources available for consumption by ϕ_α , and an increase in the technological parameter δ raises steady state resources available for consumption by f_δ ; hence $c_\alpha = -\phi_\alpha$ and $c_\delta = f_\delta$.

Steady state utility per capita, V , can also be written as a function of k and G and the parameters α and δ

$$V = V(k, G, \alpha, \delta) \quad (19)$$

where $V_k = Ru' > 0$, $V_G = 0$, $V_\alpha = -u'R\phi_\alpha/[R-G]$ and $V_\delta = u'Rf_\delta/[R-G]$. The partial derivatives of $V(\)$ can be calculated from the third and fourth rows of (17). To interpret these partial derivatives, note that a permanent increase in steady state consumption increases steady state utility by $u'/[1-h_V] = Ru'/[R-G]$. Therefore, multiplying the partial derivatives c_k , c_α , and c_δ by $Ru'/[R-G]$ yields the corresponding partial derivatives of V . However, to calculate V_G we must recall that G affects utility directly as well as through its effect on consumption. Indeed, since G is chosen to maximize utility, it follows that $V_G = 0$.

Equations (18) and (19) can be used to write the steady state first-order conditions (13) and (14) as functions of k and G . We denote these steady state relations as the MGR locus (Modified Golden Rule) and the OF locus (Optimal Fertility)

$$R(k, \delta) h_V(V(k, G, \alpha, \delta), G) = G \quad (\text{MGR locus}) \quad (20a)$$

$$[k + \phi'(G; \alpha)] u'(c(k, G, \alpha, \delta)) = h_G(V(k, G, \alpha, \delta), G) \quad (\text{OF locus}) \quad (20b)$$

The MGR and OF loci are shown in Figure 1. To determine the slope of the MGR locus, suppose that we start from a point on the MGR locus and then increase G . The increase in G causes Rh_V to increase as well. In the neighborhood of the steady state $Rh_{VG} = Gh_{VG}/h_V$ so that assumption (4a) implies that $Rh_{VG} < 1$. Therefore, an increase in G causes Rh_V to increase by less than G increases. If there is concave altruism, or if R is a decreasing function of k , then a decrease in k

will increase Rh_Y and restore the Modified Golden Rule condition. There are two channels by which a decrease in k causes Rh_Y to increase: (1) if $R_k < 0$ then a decrease in k will increase R ; and (2) a decrease in k reduces V and, if the altruism function is strictly concave in V , increases h_Y . Therefore, the MGR locus is downward sloping if there is concave altruism or if R is a decreasing function of k . However, with linear altruism and a fixed interest rate as in Becker and Barro (1985), both of these channels are absent and Rh_Y is independent of k . Therefore, there is a unique value of G which satisfies the MGR condition and hence the MGR locus is horizontal in this case (locus MGR' in Figure 1).

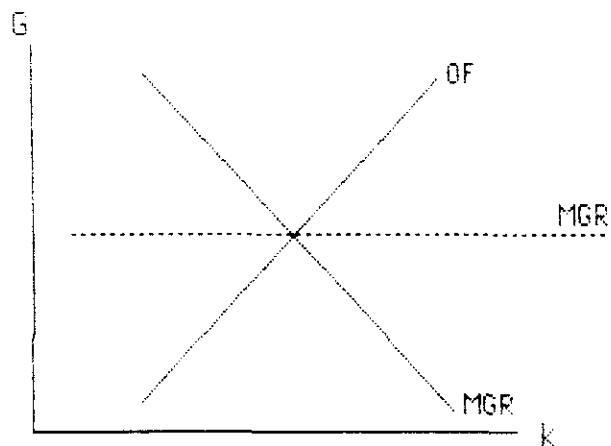


Figure 1

Now consider the OF locus. The left hand side of the OF condition, $(k + \phi')u'$, can be interpreted as the marginal cost of an additional child and the right hand side of this condition, h_G , can be interpreted as the marginal utility of an additional child. The marginal cost of an additional child has two components: (1) the direct child-rearing cost ϕ' ; and (2) the cost of providing the additional child with the per capita capital stock provided to the other children. The marginal cost of an additional child is increasing in G , and the marginal utility of an additional child is non-increasing in G . Therefore, starting from a point on the OF locus, an increase in G causes the marginal cost $(k + \phi')u'$ to exceed the marginal utility h_G . We will show below that an increase in k causes the marginal utility of an additional child to increase by more than the marginal cost increases. Therefore, the equality of marginal cost and marginal utility can be

restored by an increase in k . Hence the OF locus is upward sloping.

To establish that an increase in k causes the marginal utility h_G to increase by more than the marginal cost $[k+\phi']u'$ increases, recall from (18) and (19) that a unit increase in k causes c to increase by $R-G$ and causes V to increase by Ru' . Therefore in response to a unit increase in k , the marginal utility, h_G , increases by $u'Rh_{GV}$ which, in the neighborhood of the steady state, is equal to $u'Gh_{VG}/h_V$. In response to a unit increase in k , there are two opposing effects operating on the marginal cost $[k+\phi']u'$: the increase in c reduces u' and tends to reduce the marginal cost; the increase in k increases the marginal cost. The net effect on the marginal cost of a unit increase in k is $u' + [k+\phi'](R-G)u''$ which, using the MGR and OF conditions, is equal to $u' + Gh_G(h_V^{-1}-1)u''/u'$. Now observe that assumption (4b) is equivalent to the statement that the increase in the marginal utility, $u'Gh_{VG}/h_V$, exceeds the increase in the marginal cost, $u' + Gh_G(h_V^{-1}-1)u''/u'$. Therefore, an increase in k causes that marginal utility to increase by more than the marginal cost increases. Hence the OF schedule is upward sloping as shown in Figure 1.

Figure 1 clearly illustrates an important difference between the Becker-Barro (1985) model, which has a fixed rate of return on capital and linear altruism, and the more general model presented in this paper. The MGR locus is, in general, downward sloping except in the Becker-Barro case with a fixed interest rate and linear altruism in which case it is horizontal. It is the horizontal MGR locus which drives the Becker-Barro result that steady state fertility is independent of the cost of raising children.

In the next two sections we analyze the steady state effects of changes in the parameters α and δ . The exposition will be based on shifts in the MGR and OF loci. More formal results will also be presented but the derivations of these results are relegated to Appendix B. Unless stated otherwise, we will assume that $R_k < 0$ and /or $h_{VV} < 0$ so that the MGR locus slopes downward.

IV. The Effects of an Increase in the Cost of Raising Children

To analyze the effects of an increase in the cost of raising children, we assume that $\phi_{\alpha} > 0$ and that $\phi'_{\alpha} \geq 0$, and then we analyze the effects of an increase in α . Thus, the change we examine is one which, for all positive values of G , increases the total cost of raising children and does not decrease the marginal cost of raising children. Recall from (18) that an increase in α reduces steady state consumption per capita. The reduction in steady state consumption per capita raises the marginal utility of consumption u' and hence increases the marginal cost of an additional child. In addition, the marginal child-rearing cost ϕ' may increase which further increases the marginal cost of an additional child. In order to restore the equality of marginal cost and marginal utility at the initial level of fertility, k must increase. Therefore, the OF locus shifts to the right.

Under linear altruism, the MGR condition continues to hold at the initial steady state (point A in Figure 2). Therefore, the MGR locus does not shift and the new steady state is at point B with a higher steady state capital labor ratio and lower fertility. Alternatively, with concave altruism, the MGR locus shifts to the right. Recall from (19) that an increase in α reduces V which, under concave altruism, increases h_Y and makes Rh_Y greater than G . To restore the equality of Rh_Y and G , k must increase at the initial level of fertility. Therefore, the MGR schedule shifts to the right.

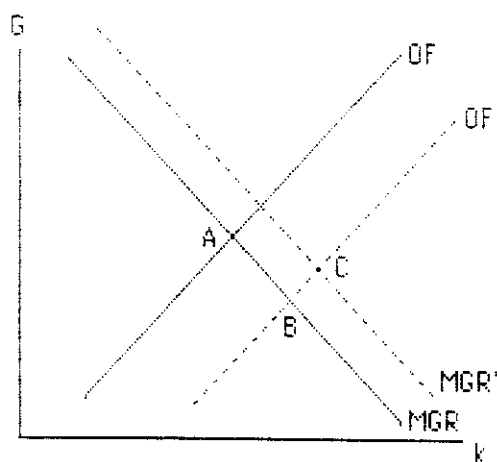


Figure 2

Effects of an Increase in Child-Rearing Costs
(α increases)

Linear Altruism: B; Concave Altruism: C

We have shown that under concave altruism an increase in the parameter α causes both the MGR and the OF loci to shift to the right. Therefore, the steady state capital labor ratio, k , unambiguously increases. However, the analysis so far does not indicate the direction of the effect, if any, on steady state fertility. We will demonstrate below that at the initial level of fertility, the OF locus shifts further to the right than does the MGR locus so that steady state fertility must fall.

To determine how far to the right the MGR and OF loci shift at the initial level of fertility, suppose $dG = 0$ and $dk = \phi_{\alpha} d\alpha / (R-G)$ and examine whether this new point is to the right or the left of each of the new loci. Note that if $dk = \phi_{\alpha} d\alpha / (R-G)$, then $dc = dV = 0$. Therefore, if $R_k \equiv 0$, the new point lies on the MGR locus. More generally, if $R_k < 0$, the increase in k reduces R so that R_h is less than G at the new point. In order to satisfy the MGR condition, k must be smaller than at the new point; that is, at the initial level of fertility, k rises from the initial steady state by less than $\phi_{\alpha} d\alpha / (R-G)$. As for condition OF, $dk = \phi_{\alpha} d\alpha / (R-G)$ implies that at the new point the marginal cost $[k + \phi']u'$ is greater than the marginal utility h_G . In order to restore the equality of marginal utility and marginal cost, k must rise even more. Therefore, in order to satisfy the OF condition at the initial level of fertility, k must rise by more than $\phi_{\alpha} d\alpha / (R-G)$. Therefore, at the initial level of steady state fertility the OF locus shifts further

to the right than does the MGR locus; hence steady state fertility falls as illustrated in Figure 2.

The formal comparative steady state analysis appears in Appendix B. In the text we will present the results. Let m_{ij} be the (i,j) minor of matrix N . Then it can be shown that

$$dG/d\alpha = \{u'\phi_\alpha m_{23} - \phi_\alpha m_{43}\}/\Delta \leq 0 \quad (21)$$

where $m_{23} = (R-G)\{Ru'h_{YY} + (R_k/R)h_Y\} \leq 0$ and $m_{43} = -\{Ru'h_{YY} + (R_k/R)(1+\Gamma)h_Y\}u' \geq 0$. Note that in the Becker-Barro (1985) model with linear altruism ($h_{YY}=0$) and a fixed interest rate ($R_k=0$) both m_{23} and m_{43} are equal to zero and hence $dG/d\alpha$ is equal to zero. More generally, if $h_{YY} < 0$ and/or if $R_k < 0$, then $dG/d\alpha < 0$. Barro and Becker (1984) observed that if $R_k < 0$, then an increase in the cost of child-rearing would reduce steady state fertility. Their analysis was restricted to linear altruism ($h_{YY} \equiv 0$) and a proportional child-rearing cost function ($\phi(0) = 0$; $\phi'' \equiv 0$). The results presented above allow for a more general child-rearing cost function and, more importantly, demonstrate the role of concave altruism in determining the steady state effects on fertility of a change in the cost of child-rearing.

It can also be shown that

$$dk/d\alpha = \{-\phi_\alpha u' m_{24} + \phi_\alpha m_{44}\}/\Delta > 0 \quad (22)$$

where $m_{24} < 0$ and $m_{44} > 0$. Even with linear altruism and a fixed interest rate, an increase in α leads to an increase in the steady state capital stock.

V. The Effects of a Change in Productivity

In this section we consider the effects of an increase in the productivity parameter δ , which at all levels of the capital labor ratio, increases output per capita and does not decrease the marginal product of capital. Formally, we examine the effects of an increase in δ and assume that $f_\delta > 0$ and $R_\delta = f_{k\delta} \geq 0$. At the initial steady state, an increase in δ leads to an increase in steady state consumption and steady state utility. The increase in steady state consumption leads to a decrease in u' , which reduces the marginal cost of an additional child; the increase in steady

state utility leads to an increase in the marginal utility of an additional child, h_G . Therefore, at the initial steady state, the marginal utility of an additional child h_G exceeds the marginal cost $[k + \phi']u'$. The equality of marginal utility and marginal cost can be restored by an increase in G or by a decrease in k . Therefore, the OF locus shifts up and to the left as shown in Figures 3 and 4.

In the Becker-Barro (1985) case with linear altruism and an exogenously fixed interest rate, the MGR schedule is horizontal. If the exogenous interest rate is not affected by the change in δ , then the MGR schedule remains at its initial position. In this case the effect of the productivity increase is to reduce steady state capital per capita and to leave fertility unchanged.

If the MGR schedule is downward sloping (either because of concave altruism or $R_k < 0$), then, in general, the increase in δ can shift the MGR locus either to the right or to the left. This ambiguity arises because the effects operating through concave altruism and the shift in the marginal product of capital schedule tend to move Rh_Y in opposite directions: the increase in δ increases R at the initial steady state if $R_\delta > 0$; the increase in δ increases V and hence reduces h_Y under concave altruism. For clarity of exposition, we analyze two cases which isolate these opposing effects on the MGR schedule. First, we examine the effects under concave altruism for the case in which the productivity shift does not affect the marginal product of capital schedule. Then we examine the effects of a shift in the marginal product of capital under linear altruism.

Case 1: Concave altruism and $R_\delta \equiv 0$.

At the initial steady state, an increase in δ increases steady state utility, thereby reducing h_Y and making Rh_Y less than G . In order to increase Rh_Y at the initial level of fertility, k must fall. Therefore the MGR locus shifts to the left. Since both loci shift to the left, it is clear from Figure 3 that the steady state capital stock must fall. The effect on steady state fertility depends on which schedule shifts further to the left at the initial level of fertility. We will show below that the OF locus shifts further to the left than does the MGR locus so the steady state fertility must rise.

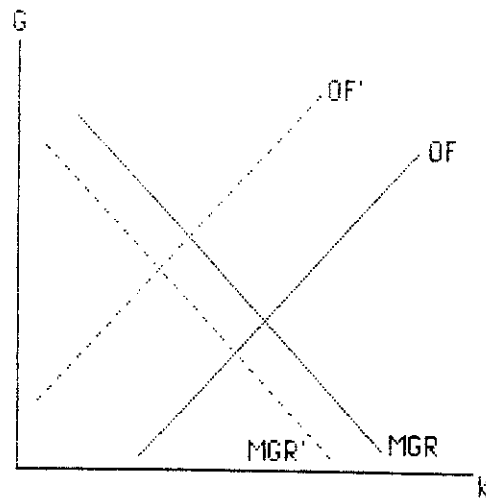


Figure 3

Effects of an Additive Productivity Increase ($R_\delta \equiv 0$) under Concave Altruism
(δ increases)

Suppose that after the increase in δ , G remains unchanged but k decreases by $f_\delta \delta / (R - G)$. This decrease in k would allow consumers to attain the initial steady state level of consumption and utility, thereby leaving h_V unchanged. If $R_k \equiv 0$, then R would be unchanged and $R h_V$ would be equal to G at the new point; therefore, in response to the increase in δ , the MGR locus shifts leftward by $f_\delta \delta / (R - G)$. More generally, if $R_k < 0$, then the decrease in k would increase R and would make $R h_V > G$ at the new point. In order to restore the MGR condition at the initial level of fertility, k would have to increase from the new point, i.e., it would have to decrease from the initial steady state by an amount smaller than $f_\delta \delta / (R - G)$; therefore, in response to the increase in δ the MGR locus shifts leftward by less than $f_\delta \delta / (R - G)$. As for the OF locus, if k were to decrease by $f_\delta \delta / (R - G)$, then both u' and h_G would remain unchanged at the initial level of fertility because consumption and utility would be unchanged. However the decrease in k , and hence in $(k + \phi')u'$, decreases the marginal cost of an additional child. In order to restore the equality of marginal cost and marginal utility at the initial level of fertility, k would have to decrease by even more than $f_\delta \delta / (R - G)$. Therefore, the OF locus shifts further to the left than does the MGR locus and steady state fertility rises as illustrated in Figure 3.

Case 2: Linear Altruism and $R_\delta > 0$

In this case, an increase in the technological parameter δ increases the marginal product of

capital at each capital labor ratio, so that $Rh_y > G$ at the initial steady state. An increase in k at the initial level of fertility is required in order to restore the equality of Rh_y and G . Therefore, the MGR locus shifts to the right. Since the OF locus shifts to the left, it is clear from Figure 4 that steady state fertility must rise. The effect on the steady state capital labor ratio is ambiguous.

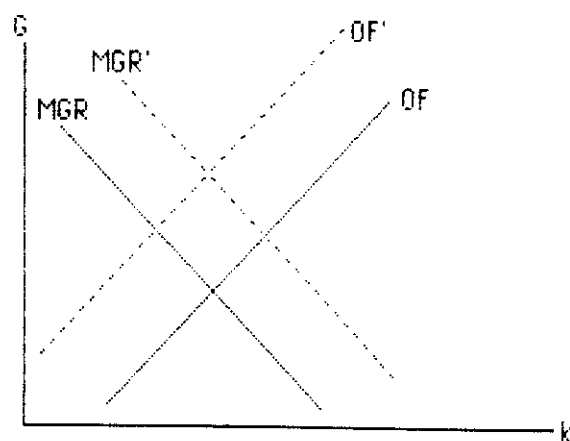


Figure 4
Effects of an Increase in Marginal Productivity of Capital under Linear Altruism
(δ increases)

Despite the ambiguity of the effect on the steady state capital labor ratio, it can be shown that the steady state marginal product of capital rises unambiguously. Recall that under linear altruism, $h(V, G) = h^*(G)V$, and the MGR condition can be written as $Rh^*(G) = G$. Since the elasticity of $h^*(G)$ with respect to G is less than 1 (recall the assumption in (4a)), it is clear that $R = G/h^*(G)$ is an increasing function of G . Since we have already demonstrated that an increase in productivity leads to an increase in steady state fertility, it follows that the steady state value of R must increase also. This result is consistent with the result of Barro and Becker (1984) that (under linear altruism) an adverse Harrod-neutral technological shock leads to a decrease in steady state fertility and a decrease in the steady state interest rate. However, the results presented above are more general in that they are not limited to Harrod-neutral changes.

We now return to the general case with concave altruism and a declining marginal product of capital which can be shifted by changes in the technological parameter δ . It is shown in Appendix

It is that

$$d\delta/d\delta = \{-R_\delta h_Y m_{13} + f_\delta m_{43}\}/\Delta \geq 0 \quad (23)$$

where $m_{13} = -(1-h_Y)\Gamma u' < 0$ and $m_{43} = -\{R u' h_{YY} + (R_K/R)(1+\Gamma)h_Y\}u' \geq 0$. It follows from (23) that under linear altruism and a fixed interest rate, a change in the technological parameter δ which does not shift the marginal product of capital has no effect on steady state fertility. More generally, however, an increase in δ leads to an increase in steady state fertility if one or more of the following is satisfied: (a) the marginal product of capital is shifted upward ($R_\delta > 0$); (b) there is concave altruism ($h_{YY} < 0$); or (c) the marginal product of capital is a decreasing function of the capital labor ratio ($R_K < 0$).

The effect on the steady state capital labor ratio is, in general ambiguous

$$dk/d\delta = \{R_\delta h_Y m_{14} - f_\delta m_{44}\}/\Delta \quad (24)$$

where $m_{14} > 0$ and $m_{44} > 0$, as shown in Appendix B. If $R_\delta \equiv 0$, then $dk/d\delta$ is unambiguously negative. However, if $R_\delta > 0$, then the sign of $dk/d\delta$ in (24) is indeterminate.

VI. Balanced Budget Fiscal Policies

In this section we consider the long-run effects of various balanced budget fiscal policies. We will distinguish between two classes of fiscal policy which we call "tax and spending" policies and "tax and transfer" policies. In the tax and spending policies, the tax revenue raised by a particular tax is used by the government to purchase output which is then disposed of in a manner which yields no utility to private consumers. Alternatively, in the tax and transfer policies, the tax revenue raised by a particular tax is rebated to consumers in the same period in the form of lump-sum subsidies. The analysis will exploit the equivalence, discussed by Abel and Blanchard (1983), between fiscal policies and appropriately specified technological shocks.

A. Tax and Spending Policies

1. Head Tax

We first consider the effect of a head tax levied on all consumers. A head tax used to finance

government spending is equivalent to downward shift in the production function which reduces output per capita by the same amount at each capital labor ratio. A head tax is also equivalent to an upward shift in the child-rearing cost function which increases the cost of raising children by the same amount at each level of fertility. Formally, if $R_g \equiv 0$, then an increase of $d\delta$ in the technological parameter is equivalent to a decrease in the head tax of $f_g d\delta$. Alternatively, if $\phi_\alpha' \equiv 0$, then an increase of $d\alpha$ in the child-rearing cost parameter is equivalent to an increase in the head tax of $\phi_\alpha d\alpha$. Therefore, setting ϕ_α' equal to zero in (21) and (22), and letting τ denote the head tax, yields

$$dG/d\tau = -m_{43}/\Delta \leq 0 \quad (25)$$

$$dk/d\tau = m_{44}/\Delta > 0 \quad (26)$$

Alternatively, equations (25) and (26) could be obtained by setting R_g equal to zero in equations (23) and (24). In the Becker-Barro formulation with $h_{yy} \equiv R_k \equiv 0$, the minor m_{43} is equal to zero and hence steady state fertility is invariant to changes in the head tax. More generally, if $h_{yy} < 0$ or $R_k < 0$ (so that the MGR schedule is downward sloping), then an increase in the head tax reduces steady state fertility.

Equation (26) indicates that an increase in the head tax increases steady state capital per capita even under linear altruism and a fixed interest rate. The increase in the steady state capital labor ratio under linear altruism stands in sharp contrast to the well-known result that the steady state capital labor ratio is invariant to lump-sum taxes under time-separable preferences and exogenous fertility. (See, for example, Abel and Blanchard (1983).)

To understand why the steady state capital labor ratio increases under linear altruism and endogenous fertility and yet remains unchanged under the conventional formulation of optimal growth models with linear altruism and exogenous fertility, examine Figure 5. For a given k and G , the increase in the head tax reduces steady state consumption, c , and steady state utility, V . The reduction in c increases the marginal utility of consumption u' , thereby increasing the marginal cost of an additional child; the reduction in V reduces the marginal utility of an additional child h_G . Therefore, the marginal cost $[k + \phi']u'$ exceeds the marginal utility h_G . In

order to restore the equality of marginal cost and marginal utility at the initial level of fertility, k must increase. Therefore, the OF schedule shifts to the right.

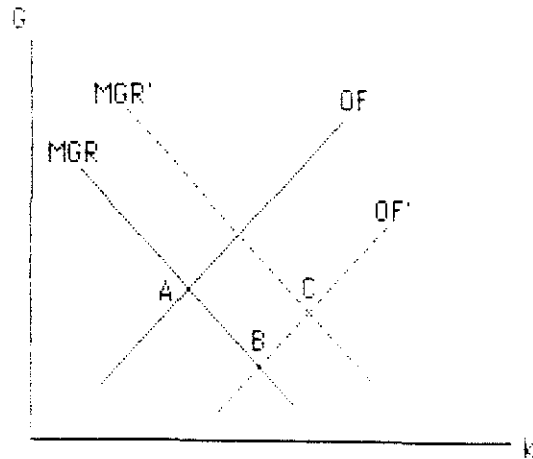


Figure 5
Effects of an Increase in Lump-Sum Taxes
Linear Altruism: B ; Concave Altruism: C

Now consider the effects of lump-sum taxes on the MGR schedule. Under linear altruism, the MGR condition is independent of steady state utility V and hence the MGR schedule does not shift when the lump-sum tax is increased. In this case, the steady state equilibrium moves from point A to point B in Figure 5; the steady state capital labor ratio increases and steady state fertility falls. By contrast, in conventional optimal growth models with linear altruism and exogenous fertility, there is no change in G . Since the MGR condition is also satisfied in conventional optimal growth models, the facts that the MGR schedule does not shift and that G is unchanged together imply that the steady state capital labor ratio is unchanged. Alternatively, under concave altruism, the MGR schedule shifts upward and to the right because the decrease in steady state utility leads to an increase in h_V at the initial equilibrium (point A), which requires an increase in G or an increase in k to restore the MGR condition. Thus, with concave altruism, the new steady state is at point C with a higher capital labor ratio and (in view of (25)) a lower level of fertility.

2. Capital Income Tax

Now consider the effects of an increase in the tax on capital income which the government

uses to purchase goods which yield no utility to individual consumers. For a given capital labor ratio, an increase in the capital income tax reduces the rate of return to capital R and reduces total output per capita available to the private sector. For an appropriate specification of the production function, an increase in the capital income tax is equivalent to a decrease in δ when $R_{\delta} > 0$ and $f_{\delta} > 0$. Therefore, it follows from (23) and (24) that an increase in the capital income tax reduces steady state fertility, but, in general, has an ambiguous effect on the steady state capital labor ratio.

3. Fertility Tax

Now consider the effects of an increase in the tax on fertility which the government uses to purchase goods which yield no utility to individual consumers. An increase in the tax on fertility increases both the total cost of child-rearing ϕ and the marginal cost of child-rearing ϕ' . For an appropriate specification of the child-rearing cost function, an increase in the fertility tax is equivalent to an increase in α where ϕ_{α} and ϕ'_{α} are both positive. It follows from equation (21) that an increase in the fertility tax has no effect on steady state fertility in the Becker-Barro (1985) case with linear altruism and a fixed rate of return to capital. More generally, with concave altruism or declining marginal product of capital, an increase in the tax on fertility reduces steady state fertility. As for the effect on the steady state capital labor ratio, it follows from (22) that an increase in the fertility tax increases the capital labor ratio in the steady state, even in the Becker-Barro case.

B. Tax and Transfer Policies

1. Capital Income Tax

Now we consider the effects of tax policies in which the tax revenue is rebated to consumers via lump-sum subsidies. The rebating of tax revenues implies that for any given steady state capital stock and steady state fertility, the associated steady state level of consumption and utility are unchanged. First we consider a tax on capital income. The capital income tax reduces the rate of return to capital at any given capital labor ratio and hence shifts the MGR schedule downward. Because the capital income tax is accompanied by lump-sum tax rebates which

maintain steady state consumption and utility at each (k, G) , the OF schedule does not shift. Therefore, the effect of the capital income tax is to decrease both steady state fertility and the steady state capital labor ratio.

2. Fertility Tax

A fertility tax accompanied by lump-sum rebates has no effect on the MGR schedule. However, an increase in the fertility tax raises the marginal cost of child-rearing, ϕ' , and hence increases the marginal cost of an additional child. In order to restore the equality of marginal cost and marginal utility at the initial level of fertility, k must increase. Thus, the OF schedule shifts to the right. The long run effect of an increase in the fertility tax is to increase the capital labor ratio and to decrease fertility.

VII. Conclusion

The Modified Golden Rule, $R\beta = G$, characterizes the steady state of an economy with intergenerationally altruistic consumers. In general, R , β and G can all be endogenous, and in this paper we examine four endogenous effects: (1) the rate of return R may be a decreasing function of the capital labor ratio; (2) the discount factor β may be a concave function of the number of children; (3) the discount factor β may be a decreasing function of the utility of the representative child; and (4) the number of children G may be chosen to maximize utility taking account of the altruism of parents and the costs of raising children. All previous work has assumed that at least one of these four effects is not operative. By restricting various sets of these effects to be inoperative, various steady state invariance results have been derived in the literature. In this paper, we have allowed all four of these effects to be operative and have examined the steady state effects of various changes in technology and in fiscal policy.

Rather than systematically summarize the results derived in the paper, I will simply highlight one result which differs dramatically from well-known results in the literature. It is well-known in optimal growth models with time-separable utility that the steady state capital

labor ratio is invariant to changes in lump-sum taxes which are used to finance useless government spending. This result also holds in a model with finitely-lived consumers with linear altruism and exogenous fertility. However, we have shown that this invariance result does not hold under linear altruism and endogenous fertility. More precisely, we have shown that an increase in government spending financed by lump-sum taxes increases the steady state capital labor ratio and reduces steady state fertility.

Appendix A

In this appendix, we present second-order conditions which must be satisfied in order for the first-order conditions to locate a local maximum. In order to avoid having to specify these second-order conditions in terms of the value function $V(\cdot)$ which is the solution of a functional equation, we consider the following perturbation around the optimal path. Let c_t^* , c_{t+1}^* , c_{t+2}^* , \dots ; G_t^* , G_{t+1}^* , G_{t+2}^* , \dots ; and l_t^* , l_{t+1}^* , l_{t+2}^* denote the sequences of per capita consumption, number of children per parent, and inheritance received per capita, respectively, along the optimal path. Now consider the following perturbation: increase c_t by x and increase the number of children per generation t consumer by y . Then adjust c_{t+1} and G_{t+2} so that the total number of generation $t+2$ consumers and the inheritance received by each of these consumers is unchanged. More precisely, let $c_t = c_t^* + x$, $G_{t+1} = G_{t+1}^* + y$ and $G_{t+2} = (G_{t+2}^*)(G_{t+1}^*)/G_{t+1}$. It follows from equation (2) that in order to provide an inheritance of

$$c_{t+1} = [l_t^* + w_t - c_t^* - x - \phi(G_{t+1}^* + y; \alpha)] [R_{t+1}/(G_{t+1}^* + y)] + w_{t+1} - \phi([G_{t+1}^* G_{t+2}^*/(G_{t+1}^* + y)]; \alpha) - [G_{t+1}^* G_{t+2}^*/(G_{t+1}^* + y)] l_{t+2}^*/R_{t+2} \quad (A1)$$

Equation (A1) gives the value of c_{t+1} which maintains the inheritance received by each generation $t+2$ consumer. Clearly this value of c_{t+1} is a function of x and y . Differentiating (A1) with respect to x and y , and evaluating the derivatives in the steady state ($G_{t+1} = G_{t+2}$, $R_{t+1} = R_{t+2}$) at the optimal values of x and y ($x = y = 0$) yields

$$\partial c_{t+1} / \partial x = -R/G \quad (A2)$$

$$\partial c_{t+1} / \partial y = (1 - R/G)(k + \phi') \quad (A3)$$

$$\partial^2 c_{t+1} / \partial x^2 = 0 \quad (A4)$$

$$\partial^2 c_{t+1} / \partial x \partial y = R/G^2 \quad (A5)$$

$$\partial^2 c_{t+1} / \partial y^2 = -(2/G)[k + \phi'] [1 - R/G] - [1 + R/G] \phi'' \quad (A6)$$

Now define $Z(x, y)$ to be the value of utility corresponding to the perturbation around the optimal path parametrized by x and y

$$Z(x, y) = u(c_t^* + x) + h(u(c_{t+1})) + h(V_{t+2}^*, G_{t+1}^* G_{t+2}^*/(G_{t+1}^* + y), G_{t+1}^* + y) \quad (A7)$$

The first partial derivatives of (A7) with respect to x and y are easily calculated using (A2) and (A3). Setting these derivatives equal to zero in the steady state with $x = y = 0$ yields

$$Rh_V(V,G) = G \quad (A8)$$

$$(k+\phi')u'(c) = h_G(V,G) \quad (A9)$$

Observe that (A8) and (A9) are identical to (13) and (14) in the text.

Let H denote the matrix of second partial derivatives of $Z(x,y)$ with respect to x and y . Using (A2) - (A6), H can be calculated as

$$H = A + B \quad (A10)$$

$$\text{where } A = \begin{vmatrix} (Ru'/G)^2 h_{VV} & [h_G/h_V](Ru'/G)h_{VV} \\ [h_G/h_V](Ru'/G)h_{VV} & [h_G/h_V]^2 h_{VV} - [1+R/G]h_V u' \phi'' \end{vmatrix} \quad (A11)$$

$$B = \begin{vmatrix} [1+R/G]u'' & -\Gamma u'/G \\ -\Gamma u'/G & -\{[1-G/R]\Gamma + [1+G/R]\Psi\}h_G/G \end{vmatrix} \quad (A12)$$

Note that the trace of $A \leq 0$ with strict inequality if h_{VV} or ϕ'' is positive. The determinant of A is $-[1+R/G]u'^3 h_V \phi'' (R/G)^2 h_{VV} \leq 0$ with strict inequality if $\phi'' > 0$ and $h_{VV} < 0$. Therefore, the symmetric matrix A is negative semi-definite.

Recall that we have assumed that $\Gamma > 0$ and $\Psi \geq 0$ (equations (4b) and (4c) in the text).

Under these assumptions, it is clear that the trace of B is negative. The determinant of B can be calculated to be $-[1+R/G]^2 (G/R)u'' \Psi h_G/G - (u'/G)^2 \Gamma \{Rh_{VG} - 1 + (G/R)(R-G)(k+\phi')u''/u'\}$.

Under the assumption that $Rh_{VG} - 1$ is negative (which follows from equation (4a) and the Modified Golden Rule $R = G/h_V$), the determinant of B is positive. Therefore, under conditions (4a)-(4c), B is negative definite. Since A is negative semi-definite and B is negative definite, H is negative definite. Therefore, the first-order conditions for the perturbations x and y locate a local maximum at $x = y = 0$.

The perturbation examined above is, of course, only one of many possible perturbations. This particular perturbation was chosen because in the special case examined by Becker and Barro (1985) with $h(V,G) = \alpha G^{1-\epsilon_V}$ the conditions for the matrix H to be negative definite are

the same as the second-order condition presented by Becker and Barro. In general, the negative definiteness of H is a necessary, but not a sufficient, condition for the consumer's maximization problem.

Appendix B

In this Appendix we calculate the third and fourth rows of the inverse of the matrix N defined in equation (17). We then use these rows of N^{-1} to calculate the effects on steady state fertility and capital per capita of permanent changes in the parameters α and δ . Recall from (17) that

$$N = \begin{pmatrix} Rh_{yy} & 0 & -\theta & R_k h_y \\ -h_{Gy} & (k+\phi')u'' & u'\phi'' - h_{GG} & u' \\ h_y^{-1} & u' & h_G & 0 \\ 0 & -1 & -(k+\phi') & R-G \end{pmatrix} \quad (B1)$$

Before calculating the determinant of N , it is useful to calculate the determinants of two 2×2 matrices embedded in N . It follows from (14) that

$$\det \begin{pmatrix} u' & h_G \\ -1 & -(k+\phi') \end{pmatrix} = 0 \quad (B2)$$

Also note that

$$\Omega \equiv - \det \begin{pmatrix} (k+\phi')u'' & u'\phi'' - h_{GG} \\ u' & h_G \end{pmatrix} > 0 \quad (B3)$$

Let m_{ij} be the (i,j) minor of the matrix N . Using (B2) and (B3), we can calculate the relevant minors of N shown below

$$\begin{aligned} m_{11} &= -[R-G]\Omega < 0 \\ m_{13} &= -[1-G/R]\Gamma u' < 0 \\ m_{14} &= [1-G/R]\Omega/u' > 0 \\ m_{23} &= [R-G]\{Rh_{yy}u' + R_k h_y/R\} \leq 0 \\ m_{24} &= -[1-G/R]\theta < 0 \\ m_{43} &= -R_k h_y u'[1+\Gamma]/R - Rh_{yy}u'^2 \geq 0 \\ m_{44} &= -Rh_{yy}\Omega + \theta[1+\Gamma]u'/R > 0 \end{aligned}$$

The determinant of N, denoted Δ , is calculated by expanding along the first row of N

$$\Delta = R h_{VV} m_{11} - \theta m_{13} - R_K h_V m_{14} > 0 \quad (B4)$$

Note that in the Becker-Barro (1985) formulation $h_{VV} \equiv 0 \equiv R_K$ and, in this case, (B4) indicates that $\Delta = \theta [1 - G/R] \Gamma u' > 0$. Since Becker and Barro assume at the outset that $\theta > 0$, the determinant Δ will be positive in the Becker-Barro formulation if and only if Γ is positive which is identical to the condition they derive in the appendix to their paper.

Using the fact that the inverse of N is equal to the adjoint of N divided by the determinant of N we have

$$N^{-1} = \Delta^{-1} \begin{vmatrix} m_{11} & -m_{21} & m_{31} & -m_{41} \\ -m_{12} & m_{22} & -m_{32} & m_{42} \\ m_{13} & -m_{23} & m_{33} & -m_{43} \\ -m_{14} & m_{24} & -m_{34} & m_{44} \end{vmatrix} \quad (B5)$$

Now observe from (17) that

$$\begin{vmatrix} dV \\ dc \\ dG \\ dk \end{vmatrix} = N^{-1} \begin{vmatrix} -R_{\delta} h_V d\delta \\ -u' \phi_{\alpha}' d\alpha \\ 0 \\ \phi_{\alpha} d\alpha - f_{\delta} d\delta \end{vmatrix} \quad (B6)$$

It follows from (B5) and (B6) that

$$dG = \{-R_{\delta} h_V m_{13} d\delta + u' \phi_{\alpha}' m_{23} d\alpha - m_{43} [\phi_{\alpha} d\alpha - f_{\delta} d\delta]\} / \Delta \quad (B7)$$

$$dk = \{R_{\delta} h_V m_{14} d\delta - u' \phi_{\alpha}' m_{24} d\alpha + m_{44} [\phi_{\alpha} d\alpha - f_{\delta} d\delta]\} / \Delta \quad (B8)$$

so that

$$dG/d\delta = \{-R_{\delta} h_V m_{13} + f_{\delta} m_{43}\} / \Delta \geq 0 \quad (B9)$$

$$dG/d\alpha = \{u' \phi_{\alpha}' m_{23} - \phi_{\alpha} m_{43}\} / \Delta \leq 0 \quad (B10)$$

$$dk/d\delta = \{R_{\delta} h_{\nu} m_{14} - f_{\delta} m_{44}\} / \Delta \quad (\text{B11})$$

$$dk/d\alpha = \{-u' \phi_{\alpha}' m_{24} + \phi_{\alpha} m_{44}\} / \Delta > 0 \quad (\text{B12})$$

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