A Production-Based Model for the Term Structure

Urban J. Jermann

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Production-based asset pricing in the literature

- General equilibrium with endogenous capital
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- "Pure" production-based:
Production-based asset pricing in the literature

- General equilibrium with endogenous capital
- "Pure" production-based:
  - Firm’s return function of investment, productivity ... (Cochrane 1991)
Production-based asset pricing in the literature

- General equilibrium with endogenous capital
- "Pure" production-based:
  - Firm’s return function of investment, productivity ... (Cochrane 1991)
What is done

- Present a production-based model for pricing nominal bonds
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- Present a production-based model for pricing nominal bonds
- Examine implied term structure quantitatively and analytically
Findings

- Match average and standard deviation of longer term yields
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- Time-varying premiums, partially match Fama-Bliss
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- Match average and standard deviation of longer term yields
- Time-varying premiums, partially match Fama-Bliss
- Depreciation rates are important for term premium
Real Model, 1

- Uncertainty: $s \in (s_1, s_2)$, current realization $s_t$, history $s^t$
Real Model, 1

- Uncertainty: \( s \in (s_1, s_2) \), current realization \( s_t \), history \( s^t \)
- Firms solve

\[
\max_{\{I, K'\}} \sum_{t=0}^{\infty} \sum_{s^t} P(s^t) \left[ F:\left(\{K_j(s^{t-1})\}_{j\in\{1,2\}}, s^t\right) \right. \\
\left. \quad - \sum_{j=1}^{2} H_j(K_j(s^{t-1}), I_j(s^t)) \right]
\]

s.t. \( K_j(s^t) = K_j(s^{t-1})(1 - \delta_j) + I_j(s^t) \), \( \forall s^t, j, \)
Real Model, 1

- Uncertainty: \( s \in (s_1, s_2) \), current realization \( s_t \), history \( s^t \)
- Firms solve

\[
\max_{\{l, K'\}} \sum_{t=0}^{\infty} \sum_{s^t} P(s^t) \begin{bmatrix}
F \left( \{ K_j(s^{t-1}) \}_{j \in \{1,2\}, s^t} \right) \\
- \sum_{j=1}^{2} H_j \left( K_j(s^{t-1}), l_j(s^t) \right)
\end{bmatrix}
\]

s.t. \( K_j(s^t) = K_j(s^{t-1})(1 - \delta_j) + l_j(s^t), \ \forall s^t, j, \)
- \( F(...)= \sum_{j=1}^{2} A_j(s^t) K_j(s^{t-1}) \)
Real Model, 1

- Uncertainty: $s \in (s_1, s_2)$, current realization $s_t$, history $s^t$
- Firms solve

$$\max_{\{l, K'\}} \sum_{t=0}^{\infty} \sum_{s^t} P(s^t) \left[ F \left( \{ K_j(s^{t-1}) \}_{j \in (1,2)}, s^t \right) \right. \left. - \sum_{j=1}^{2} H_j \left( K_j(s^{t-1}), l_j(s^t) \right) \right]$$

s.t. $K_j(s^t) = K_j(s^{t-1})(1 - \delta_j) + l_j(s^t), \ \forall s^t, j,$

- $F(\ldots) = \sum_{j=1}^{2} A_j(s^t) K_j(s^{t-1})$
- $H_j(\ldots) = \left\{ \frac{b_j}{\nu_j} \left( l_j(s^t) / K_j(s^{t-1}) \right)^{\nu_j} + c_j \right\} K_j(s^{t-1})$
Real Model, 2

First-order conditions

\[ 1 = \sum_{s_{t+1}} P(s_{t+1}|s^t) \, R^l_j (s^t, s_{t+1}) \quad \text{for } j = 1, 2 \]

with

\[ R^l_j (s^t, s_{t+1}) \equiv \left( \frac{F_{K_j}(s^t, s_{t+1}) - H_{j,1}(s^t, s_{t+1}) + (1 - \delta_j) q_j(s^t, s_{t+1})}{q_j(s^t)} \right) \]

and

\[ q_j (s^t) = H_{j,2} (...) = b_j \left( \frac{l_j(s^t)}{K_j(s^{t-1})} \right)^{v_j-1} \]
Real Model, 3

- Recovering state prices

\[
\begin{bmatrix}
R_1^l (s^t, s_1) & R_1^l (s^t, s_2) \\
R_2^l (s^t, s_1) & R_2^l (s^t, s_2)
\end{bmatrix}
\begin{bmatrix}
P (s_1 | s^t) \\
P (s_2 | s^t)
\end{bmatrix} = 1
\]
Real Model, 3

- Recovering state prices

\[
\begin{bmatrix}
R^I_1(s^t, s_1) & R^I_1(s^t, s_2) \\
R^I_2(s^t, s_1) & R^I_2(s^t, s_2)
\end{bmatrix}
\begin{bmatrix}
P(s_1 | s^t) \\
P(s_2 | s^t)
\end{bmatrix} = 1
\]

- so that state prices depend on

\[
\left(\frac{l_1(s^t)}{K_1(s^{t-1})}, \frac{l_2(s^t)}{K_2(s^{t-1})}, \lambda^I_1(s^{t+1}), \lambda^I_2(s^{t+1}), A_j(s^{t+1})\right)
\]
Nominal bonds

- Assume $\lambda^P (z_t)$, with $z_t \in (\delta_1, \delta_2)$
Nominal bonds

- Assume \( \lambda^P (z_t) \), with \( z_t \in (\hat{z}_1, \hat{z}_2) \)
- Assume investment and technology not contingent on inflation. For instance,

\[
P (s_1 | s^t) = P (s_1 | s^t, z_t) = P (s_1, \hat{z}_1 | s^t, z_t) + P (s_1, \hat{z}_2 | s^t, z_t)
\]
Nominal bonds

- Assume $\lambda^P(z_t)$, with $z_t \in (\bar{z}_1, \bar{z}_2)$
- Assume investment and technology not contingent on inflation. For instance,

$$P(s_1|s^t) = P(s_1|s^t, z_t) = P(s_1, \bar{z}_1|s^t, z_t) + P(s_1, \bar{z}_2|s^t, z_t)$$

- Inflation not directly priced. For instance,

$$P(s_1, \bar{z}_1|s^t, z_t) = \left( \frac{\Pr(s_1, \bar{z}_1|s^t, z_t)}{\Pr(s_1, \bar{z}_1|s^t, z_t) + \Pr(s_1, \bar{z}_2|s^t, z_t)} \right) P(s_1|s^t), \text{ and}$$

$$P(s_1, \bar{z}_2|s^t, z_t) = \left( 1 - \frac{\Pr(s_1, \bar{z}_1|s^t, z_t)}{\Pr(s_1, \bar{z}_1|s^t, z_t) + \Pr(s_1, \bar{z}_2|s^t, z_t)} \right) P(s_1|s^t)$$
Nominal bonds

- Assume $\lambda^P(z_t)$, with $z_t \in (\delta_1, \delta_2)$
- Assume investment and technology not contingent on inflation. For instance,

$$P(s_1 | s^t) = P(s_1 | s^t, z_t) = P(s_1, \delta_1 | s^t, z_t) + P(s_1, \delta_2 | s^t, z_t)$$

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$$P(s_1, \delta_1 | s^t, z_t) = \left( \frac{\Pr(s_1, \delta_1 | s^t, z_t)}{\Pr(s_1, \delta_1 | s^t, z_t) + \Pr(s_1, \delta_2 | s^t, z_t)} \right) P(s_1 | s^t), \text{ and}$$

$$P(s_1, \delta_2 | s^t, z_t) = \left( 1 - \frac{\Pr(s_1, \delta_1 | s^t, z_t)}{\Pr(s_1, \delta_1 | s^t, z_t) + \Pr(s_1, \delta_2 | s^t, z_t)} \right) P(s_1 | s^t)$$

- If inflation and investment independent

$$V_t^{s(1)}(s^t, z_t) = \{ P(s_1' | s^t) + P(s_2' | s^t) \} E\left( \frac{1}{\lambda^P} | s^t, z_t \right)$$
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment rates</td>
<td>$\lambda^I(s_1), \lambda^I(s_2)$</td>
<td>0.9497, 1.1109</td>
</tr>
<tr>
<td>Serial correlation</td>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td>Relative freq. of low</td>
<td></td>
<td>0.8</td>
</tr>
<tr>
<td>Inflation rates</td>
<td>$\lambda^P(\delta_1), \lambda^P(\delta_2)$</td>
<td>1.0169, 1.0763</td>
</tr>
<tr>
<td>Serial correlation</td>
<td></td>
<td>0.8</td>
</tr>
<tr>
<td>Relative freq. of low</td>
<td></td>
<td>1.9</td>
</tr>
<tr>
<td>Depreciation rates</td>
<td>$\delta_E, \delta_S$</td>
<td>0.112, 0.031</td>
</tr>
<tr>
<td>Relative value of cap.</td>
<td>$K_E/K_S$</td>
<td>0.6</td>
</tr>
<tr>
<td>Adjustment cost par.</td>
<td>$b_E, b_S, c_E, c_S$ so that $\bar{q}$</td>
<td>1.5</td>
</tr>
<tr>
<td>Adjustment cost curv.</td>
<td>$\nu_E, \nu_S$</td>
<td>2.2385, 4.1080</td>
</tr>
<tr>
<td>Marginal prod. of cap.</td>
<td>$A_E, A_S$ so that $\bar{R}_E, \bar{R}_S$</td>
<td>1.04515, 1.05773</td>
</tr>
</tbody>
</table>
Table 2: Equity returns and short term yields

<table>
<thead>
<tr>
<th></th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E \left( r_M - y^{(1)} \right) %$</td>
<td>4.64</td>
<td>4.64</td>
</tr>
<tr>
<td>$\sigma \left( r_M, r \right) %$</td>
<td>17.13</td>
<td>17.13</td>
</tr>
<tr>
<td>$E \left( y^{(1)} \right) %$</td>
<td>5.29</td>
<td>5.29</td>
</tr>
<tr>
<td>$\sigma \left( y^{(1)} \right) %$</td>
<td>2.98</td>
<td>2.98</td>
</tr>
</tbody>
</table>

Yields, $y$, are from Fama and Bliss, defined as $- \ln \left( \frac{\text{price}}{\text{maturity}} \right)$, stock returns are the logs of value-weighted returns from CRSP, $r_M, r$ is the stock return deflated by the CPI-U. All data is 1952-2010.
<table>
<thead>
<tr>
<th>Nominal yields</th>
<th>Maturity (years)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean - Model %</td>
<td></td>
<td>5.29</td>
<td>5.44</td>
<td>5.58</td>
<td>5.72</td>
<td>5.86</td>
</tr>
<tr>
<td>Mean - Data %</td>
<td></td>
<td>5.29</td>
<td>5.49</td>
<td>5.67</td>
<td>5.81</td>
<td>5.90</td>
</tr>
<tr>
<td>Std - Model %</td>
<td></td>
<td>2.98</td>
<td>2.73</td>
<td>2.51</td>
<td>2.33</td>
<td>2.17</td>
</tr>
<tr>
<td>Std - Data %</td>
<td></td>
<td>2.98</td>
<td>2.93</td>
<td>2.85</td>
<td>2.80</td>
<td>2.75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Real yields</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean - Model %</td>
<td></td>
<td>1.68</td>
<td>1.84</td>
<td>2.00</td>
<td>2.15</td>
<td>2.31</td>
</tr>
<tr>
<td>Std - Model %</td>
<td></td>
<td>2.06</td>
<td>1.92</td>
<td>1.81</td>
<td>1.71</td>
<td>1.62</td>
</tr>
</tbody>
</table>
Table 4: Fama-Bliss excess return regressions

\[ rx_{t+1}^{(n)} = \alpha + \beta \left( f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+1}^{(n)} \]

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model - ( \beta )</td>
<td>.3050</td>
<td>.3906</td>
<td>.5144</td>
<td>.6135</td>
</tr>
<tr>
<td>Data - ( \beta )</td>
<td>.7606</td>
<td>1.0007</td>
<td>1.2723</td>
<td>.9952</td>
</tr>
</tbody>
</table>

Yields are from Fama and Bliss 1952-2010, \( rx_{t+1}^{(n)} \) is the excess return of a \( n \)-period discount bond, \( f_t^{(n)} \) is the forward rate, \( (p_t^{(n-1)} - p_t^{(n)}) \), \( p_t^{(n)} \) the log of the price discount bond, and \( y_t^{(1)} \) is the 1 period yield.
Table 5: Fama-Bliss excess return regressions

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Model - $\beta$ no inflation risk</th>
<th>Model - $\beta$ real forward premium</th>
<th>Model - $\beta$ benchmark</th>
<th>Data - $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.4656</td>
<td>.4667</td>
<td>.3050</td>
<td>.7606</td>
</tr>
<tr>
<td>3</td>
<td>.6101</td>
<td>.6039</td>
<td>.3906</td>
<td>1.0007</td>
</tr>
<tr>
<td>4</td>
<td>.7881</td>
<td>.7866</td>
<td>.5144</td>
<td>1.2723</td>
</tr>
<tr>
<td>5</td>
<td>.9465</td>
<td>.9473</td>
<td>.6135</td>
<td>.9952</td>
</tr>
</tbody>
</table>

No inflation risk

\[ r_{x_t}^{(n)} = \alpha + \beta \left( f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_t^{(n)} \]
Continuous-time

Assume univariate $dz$ with discount factor process

$$\frac{d\Lambda}{\Lambda} = -r(.)\,dt - \sigma(.)\,dz$$

with given returns for the two types of capital

$$\frac{dR_j}{R_j} = \mu_j(.)\,dt + \sigma_j(.)\,dz, \text{ for } j = 1, 2$$
Continuous-time

- Assume univariate $dz$ with discount factor process
  \[
  \frac{d\Lambda}{\Lambda} = -r(\cdot)\, dt - \sigma(\cdot)\, dz
  \]
  with given returns for the two types of capital
  \[
  \frac{dR_j}{R_j} = \mu_j(\cdot)\, dt + \sigma_j(\cdot)\, dz, \text{ for } j = 1, 2
  \]
- The absence of arbitrage implies that
  \[
  0 = -r + \mu_j - \sigma_j\sigma, \text{ for } j = 1, 2
  \]
  so that
  \[
  r = \frac{\sigma_2}{\sigma_2 - \sigma_1} \mu_1 - \frac{\sigma_1}{\sigma_2 - \sigma_1} \mu_2
  \]
  \[
  \sigma = \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}
  \]
Capital return

The return to a given capital stock equals

\[
\left\{ \begin{aligned}
&\frac{A_j - c_j}{b_j \left( \frac{l_{j,t}}{K_{j,t}} \right)^{v_j - 1}} - (v_j - 1) \left( 1 - \frac{1}{v_j} \right) \frac{l_{j,t}}{K_{j,t}} - \delta_j \\
&+ (v_j - 1) \left[ \left( \lambda^{l,j} - 1 \right) + \delta_j + \frac{1}{2} (v_j - 2) \sigma^2_{l,j} \right] \\
&\text{dt} \\
&+ (v_j - 1) \sigma_{l,j} \text{dz} \\
\end{aligned}\right\}
\]

\[
\mu_j(.)
\]

\[
\sigma_j(.)
\]
Sharpe ratio

At steady state, \( I/K = \lambda^l - 1 + \delta \), and with \( \sigma_{I,j} = \sigma_I \), the Sharpe ratio is given by

\[
\sigma|_{ss} = \frac{\mu_j - r}{\sigma_j} = \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} = \frac{\bar{R}_2 - \bar{R}_1}{(v_2 - v_1) \sigma_I} + \frac{v_1 + v_2 - 3}{2} \sigma_I
\]

with

\[
\bar{R} = \frac{A - c}{b \left( \lambda^l - (1 - \delta) \right)^{v-1}} + \left( 1 - \frac{1}{v} \right) \lambda^l + \frac{1}{v} (1 - \delta)
\]
Dynamics of the short rate

- The short rate equals

\[ r = \frac{\sigma_2}{\sigma_2 - \sigma_1} \mu_1 - \frac{\sigma_1}{\sigma_2 - \sigma_1} \mu_2 \]
Dynamics of the short rate

- The short rate equals

\[ r = \frac{\sigma_2}{\sigma_2 - \sigma_1} \mu_1 - \frac{\sigma_1}{\sigma_2 - \sigma_1} \mu_2 \]

- Specializing to the case \( \sigma_{ij} = \sigma_I \)

\[ r = \frac{\nu_2 - 1}{\nu_2 - \nu_1} \mu_1 - \frac{\nu_1 - 1}{\nu_2 - \nu_1} \mu_2 \]
Dynamics of the short rate

- The short rate equals

\[ r = \frac{\sigma_2}{\sigma_2 - \sigma_1} \mu_1 - \frac{\sigma_1}{\sigma_2 - \sigma_1} \mu_2 \]

- Specializing to the case \( \sigma_{I,j} = \sigma_I \)

\[ r = \frac{v_2 - 1}{v_2 - v_1} \mu_1 - \frac{v_1 - 1}{v_2 - v_1} \mu_2 \]

- \( dr = \mu_r (.) \ dt + \sigma_r (.) \ dz \): at steady state, for \( \sigma_{I,j} = \sigma_I \), and \( \lambda_{I,j} \) and \( \sigma_I \) constant,

\[ \sigma_r|_{ss} = \frac{(v_2 - 1)(v_1 - 1)}{v_2 - v_1} \left[ \bar{R}_2 - \bar{R}_1 + \delta_2 - \delta_1 \right] \sigma_I \]
<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Cont.-time</th>
<th>Discrete-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>$-\sigma_r \sigma$</td>
<td>$E_t \left( r_{t+1}^{(2)} - y_t^{(1)} \right)$</td>
</tr>
<tr>
<td>$\delta_1 = \delta_2, \bar{R}_1 = \bar{R}_2,$</td>
<td>.0024</td>
<td>.0022</td>
</tr>
<tr>
<td>$\delta_1 = \delta_2$</td>
<td>0</td>
<td>0.00001</td>
</tr>
<tr>
<td>$\bar{R}_1 = \bar{R}_2, \delta_1 = .112 &gt; \delta_2 = .0313$</td>
<td>$-0.00044$</td>
<td>$-.00036$</td>
</tr>
<tr>
<td>$\bar{R}_1 = \bar{R}_2, \delta_1 = .0313 &lt; \delta_2 = .112$</td>
<td>$.0017$</td>
<td>$.0015$</td>
</tr>
</tbody>
</table>

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Term Structure
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Conclusion

- Two-sector q-theoretical model can do a good job replicating averages and volatilities of longer term US yields
- Time-varying term premiums are evidenced through Fama-Bliss regressions
Conclusion

- Two-sector q-theoretical model can do a good job replicating averages and volatilities of longer term US yields
- Time-varying term premiums are evidenced through Fama-Bliss regressions
- Even with homoscedastic investment and inflation, the market price of risk and the volatility of the short rate are naturally time-varying, driven by time-varying investment to capital ratios