

EFFICIENCY, EQUILIBRIUM, AND ASSET PRICING WITH RISK OF DEFAULT

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We introduce a new equilibrium concept and study its efficiency and asset pricing implications for the environment analyzed by Kehoe and Levine (1993) and Kocherlakota (1996). Our equilibrium concept has complete markets and endogenous solvency constraints. These solvency constraints prevent default at the cost of reducing risk sharing. We show versions of the welfare theorems. We characterize the preferences and endowments that lead to equilibria with incomplete risk sharing. We compare the resulting pricing kernel with the one for economies without participation constraints: interest rates are lower and risk premia depend on the covariance of the idiosyncratic and aggregate shocks. Additionally, we show that asset prices depend only on the valuation of agents with substantial idiosyncratic risk.

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1. INTRODUCTION

IN MODELS WITH INCOMPLETE financial markets agents cannot eliminate all idiosyncratic risk. For this reason, pricing kernels are generally more volatile than they would have been if markets were complete. Such models have two drawbacks, however, when compared with complete markets models. First, they require more or less arbitrary assumptions about which set of securities is available, and the conclusions may in turn crucially depend on these assumptions. The second problem is tractability. Because finding equilibria of these models involves solving a complicated fixed point problem, it has been difficult to analyze the case of many assets or many agents. In this paper, we study a class of models whose equilibria, in general, entail limited risk sharing and avoid these drawbacks.

Our approach for limiting risk sharing builds on work by Kehoe and Levine (1993) and Kocherlakota (1996).² These authors present and study efficient allocations in endowment economies where *participation constraints* ensure that agents would at no time be better off reverting permanently to autarchy. We show that these participation constraints can be modelled as portfolio constraints. We imagine a world, where, if agents default on some debt, they can be punished by seizing all the assets that they may own, but they cannot be punished by garnishing their labor income. In such an environment, risk sharing may be effectively

reduced because agents with low income realizations can only borrow up to the amount they are willing to pay back in the future.

A main contribution of this paper is the introduction of a new equilibrium concept that emphasizes portfolio constraints: a competitive equilibrium with *solvency constraints*. Specifically, we focus on constraints that are tight enough to prevent default but allow as much risk sharing as possible. These *endogenously* determined solvency constraints are agent and state specific and ensure that the participation constraints are fulfilled. This means that wealth that agents carry to any particular date and event will never be small enough to make them choose to default and to revert to autarchy. At the same time, these constraints allow as much risk sharing as possible.

We show versions of the first and second welfare theorems for our equilibrium concept. Our decentralization differs from the one in Kehoe and Levine (1993) where the participation constraints are modelled as direct restrictions on the consumption possibility sets. Instead our decentralization relates more closely to the existing asset pricing literature that focuses on portfolio restrictions such as the work by He and Modest (1995) and Luttmer (1996) among others. Indeed, our decentralization provides a justification for such solvency constraints. Additionally, with our definition of equilibrium we obtain a simpler and intuitive representation of the prices of securities.

We use the welfare theorems to find conditions for the time discount factors, risk aversion, variance and persistence of endowments, under which autarchy is the only feasible allocation. These properties are important for quantitative predictions. Indeed, quantitative studies such as Mehra and Prescott (1985) and Cochrane and Hansen (1992) have shown that the same parameters are important for asset pricing results in representative agent models.

We also present some properties of the pricing kernel. We show that one-period contingent claims (Arrow securities) are priced by the agent with the highest marginal rate of substitution, which is the agent that is not constrained with respect to his holding of this asset. Thus the price of an Arrow security is equal to the highest marginal valuation across agents. The price of an arbitrary asset is calculated by adding up the prices of the corresponding contingent claims. Interestingly, this price need not coincide with the highest marginal valuation of the security across agents. As we mentioned before, this framework has two advantages for the purpose of asset pricing over the standard incomplete markets specifications. First, allocations do not depend on a particular arbitrary set of assets that is considered to be available. And second, since markets are complete, any security can be priced.

Finally, we compare the asset prices in an equilibrium with solvency constraints with those obtained in an identical economy, but without the solvency

constraints. First, we show that interest rates are necessarily lower in the economy with solvency constraints. This conclusion holds regardless of whether there is a precautionary motive ($u''' > 0$). Second, we examine the following measure of the market price of risk: the excess return on a one-period security whose payments are a function of aggregate endowment. We show that unless relative endowment shocks are correlated with the aggregate shocks, this premium is the same in the two economies. Third, we show that asset prices are determined by those agents with substantial individual risk. In particular, the presence of an agent whose endowment is very similar to the aggregate endowment is irrelevant for asset pricing.

The paper is organized as follows. Section 2 presents the environment. Section 3 analyzes some characteristics of efficient allocations and introduces the equilibrium with solvency constraints. Section 4 relates the equilibrium concept to the one by Kehoe and Levine, shows versions of the welfare theorems, and describes the preferences and endowments that leads to equilibrium with no risk sharing. Section 5 analyzes the resulting pricing kernel. Section 6 concludes. The appendix contains the proofs.

2. THE ENVIRONMENT

We consider a pure exchange economy with I agents and one good. Agents' endowments follow a finite state Markov process, agents' preferences are identical and given by time-separable expected discounted utility. We add to this simple environment participation constraints of the following form: the continuation utility implied by any allocation should be at least as high as the one implied by autarchy at any time and for any history.

Formally, we let $\{1, 2, \dots, I\} = \mathbf{I}$, be the set of agents, with typical element i . We use $\{z_t\}$ to denote a Markov process with generic elements z , a member of the finite set Z , and transition probabilities given by matrix Π . We use $z^t = (z_0, z_1, z_2, \dots, z_t)$ to denote a length t history of z . The matrix Π generates conditional probabilities for histories that we denote as $\pi(z^t | z_0 = z)$. We use the symbol \succeq for the partial order $z^{t'} \succeq z^t$ for $t' \geq t$ to indicate that $z^{t'}$ is a possible continuation of z^t , that is, that there exists a history z^s such that $z^{t'} = (z^t, z^s)$ for $s = t' - t$. We use the notation $\{c_i\}$ and $\{e_i\}$ for the stochastic process of consumption and endowment of each agent, hence $\{c_i\} = \{c_{i,t}(z^t) : \forall t \geq 0, z^t \in Z^t\}$. We define aggregate endowment as $e_t(z^t) \equiv \sum_{i \in \mathbf{I}} e_{i,t}(z^t)$. Individual endowments are given by a function ϵ_i , that depends only on z_t , so that $e_{i,t}(z^t) = \epsilon_i(z_t)$. We assume that $\epsilon_i(z) > 0$ for all i and z .

The utility for an agent corresponding to the consumption process $\{c\}$ starting at time t at history z^t is

$$U(c)(z^t) \equiv \sum_{s=t}^{\infty} \sum_{z^s \in Z^s} \delta_{t,s}(z^{s-1}) u(c_s(z^s)) \pi(z^s|z^t). \quad (2.1)$$

We assume that the period utility function $u : R_+ \rightarrow R$ is strictly increasing, strictly concave and C^1 . The multi-period time discount factors are defined recursively using the one period, state contingent discount factor $\beta : Z \rightarrow (0, 1)$. Specifically, $\delta_{t,t}(z^{t-1}) \equiv 1$ for all z^{t-1} ,

$$\delta_{t,s}(z^{s-1}) = \delta_{t,s-1}(z^{s-2}) \beta(z_{s-1}) \quad \text{all } z^s \succeq z^{t+1}.$$

Hence $U(\cdot)$ satisfies

$$U(c)(z^t) = u(c(z_t)) + \beta(z_t) \sum_{z_{t+1} \in Z} U(c)(z^t, z_{t+1}) \pi(z_{t+1}|z_t).$$

Note that the standard case with constant discount factor $\beta(z) = \bar{\beta}$ corresponds to $\delta_{t,s}(z^{s-1}) = \bar{\beta}^{s-t}$. As we will show below, permitting the time discount factor to be state contingent, allows us to incorporate stochastic growth.

An allocation $\{c_i\}_{i \in \mathbf{I}}$ is resource feasible if

$$\sum_{i=1}^I c_{i,t}(z^t) = e_t(z^t) \quad \forall t \geq 0, z^t \in Z^t, \quad (2.2)$$

and it satisfies the Participation Constraints if :

$$U(c_i)(z^t) \geq U(e_i)(z^t) \quad \forall t \geq 0, z^t \in Z^t, i \in \mathbf{I}. \quad (2.3)$$

Notice that $U(e_i)(z^t)$ depends only on z_t because $e_{i,t}(z^t)$ is a function of z_t .

Except for the state-contingent time discount factor our environment is a special case of the one by Kehoe and Levine (1993). In particular, we consider the case with one good and where the participation constraints have autarchy as the outside option.

2.1. Aggregate Growth

Our specification of the time discount factor allows for growth in the aggregate endowment as in Mehra-Prescott (1985) and in much of the asset pricing literature. In particular, an economy with stochastic growth, constant time discount factor and constant relative risk aversion can be expressed as an economy with constant aggregate endowment and state contingent discount factor, fitting the

assumptions of the previous section. Let

$$e_{t+1}(z^t, z_{t+1}) = e_t(z^t) \cdot \lambda(z_{t+1}) \quad \text{and} \quad e_{i,t}(z^t) = e_t(z^t) \cdot \epsilon_i(z_t) \quad \text{for } i \in \mathbf{I}.$$

Assume a constant time discount factor β , and let the period utility function be of the form $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some positive γ (for simplicity assume that $\gamma \neq 1$).

Define the normalized consumptions, probabilities and discount factors as

$$\begin{aligned} \hat{c}_{i,t}(z^t) &\equiv c_{i,t}(z^t) / e_t(z^t), \quad \hat{\pi}(z'|z) \equiv \frac{\pi(z'|z) \cdot \lambda(z')^{1-\gamma}}{\sum_{z'} \pi(z'|z) \cdot \lambda(z')^{1-\gamma}}, \quad \text{and} \\ \hat{\beta}(z) &\equiv \beta \cdot \sum_{z'} \pi(z'|z) \cdot \lambda(z')^{1-\gamma}. \end{aligned}$$

It is straightforward to check that resource feasibility and participation constraints are satisfied for an allocation $\{c_i\}_{i \in \mathbf{I}}$ in an economy with aggregate growth $\lambda(\cdot)$ and constant discount factor β *if and only if* they are satisfied for the corresponding $\{\hat{c}_i\}_{i \in \mathbf{I}}$ allocation in the economy with constant aggregate endowment, discount factor $\hat{\beta}(\cdot)$ and probabilities $\hat{\pi}$.³ Moreover, the preference orderings are identical in the two corresponding economies.

3. EFFICIENT ALLOCATIONS AND EQUILIBRIUM WITH SOLVENCY CONSTRAINTS

An allocation $\{c_i\}$ is feasible if it is resource feasible and satisfies the participation constraints. Efficient allocations are feasible allocations that are not Pareto dominated by other feasible allocation.

Kocherlakota (1996) analyzes efficient allocations for a similar environment. Relative to his environment, we consider the case with state specific time discount factors , $I \geq 2$ agents and serially correlated shocks. As in Kocherlakota (1996), efficient allocations may be characterized by risk sharing regimes of three types, depending on the parametric environment: Pareto efficiency in the usual unconstrained sense (full risk sharing), autarchy (no risk sharing) or limited risk sharing. In Proposition 4.9 we extend his characterization by showing under which conditions on the time discount factors, risk aversion, variance and persistence of endowments, autarchy is the only feasible allocation.

We first show a property of the marginal rates of substitution for efficient allocations that will be crucial for asset pricing. We show that an unconstrained agent has the highest marginal rate of substitution—and hence, all unconstrained agents equalize their marginal rates of substitution.

Proposition 3.1. *Let $\{c_i\}_{i \in \mathbf{I}}$ be constrained efficient. Then*

$$U(c_j)(z^t, z_{t+1}) > U(e_j)(z^t, z_{t+1}), \text{ implies} \quad (3.1)$$

$$\frac{u'(c_{j,t+1}(z^t, z_{t+1}))}{u'(c_{j,t}(z^t))} = \max_{i \in \mathbf{I}} \frac{u'(c_{i,t+1}(z^t, z_{t+1}))}{u'(c_{i,t}(z^t))}, \text{ all } z^t \in Z^t, t \geq 0.$$

Next we define a competitive equilibrium with complete markets in Arrow securities and with solvency constraints. The solvency constraints prohibit agents from holding large amounts of contingent debt, hence preventing default. In general, these solvency constraints will be state-contingent, since the value of default (reverting to autarchy) varies with the state.⁴

Let $q_t(z^t, z')$ denote the period t , state z^t , price of one unit of the consumption good delivered at $t + 1$, contingent on the realization of $z_{t+1} = z'$, in terms of period t consumption goods. The holdings of agent i at t of this security are denoted by $a_{i,t+1}(z^t, z')$, and the lower limit on the holdings of agent i is denoted by $B_{i,t+1}(z^t, z')$. Following our notational convention, we use $\{q\}$, $\{a_i\}$ and $\{B_i\}$ for the corresponding stochastic processes.

Definition 3.2. *An equilibrium with Solvency Constraints $\{B_i\}$ for initial conditions $\{a_{i,0}\}$ has quantities $\{c_i, a_i\}$ and prices $\{q\}$ such that:*

(1) for each i , $\{c_i, a_i\}$ solves

$$J_{i,t}(a, z^t) = \max_{c, \{a_{z'}\}_{z' \in Z}} \left\{ u(c) + \beta(z) \sum_{z'} J_{i,t+1}(a_{z'}, (z^t, z')) \pi(z'|z_t) \right\} \quad (3.2)$$

$$e_{i,t}(z^t) + a = \sum_{z' \in Z} a_{z'} q_t(z^t, z') + c \quad (3.3)$$

$$a_{z'} \geq B_{i,t+1}(z^t, z') \quad \text{all } z' \in Z, \text{ and} \quad (3.4)$$

(2) markets clear,

$$\sum_{i \in \mathbf{I}} c_{i,t}(z^t) = e_t(z^t), \quad \text{all } t, \text{ all } z^t$$

$$\sum_{i \in \mathbf{I}} a_{i,t+1}(z^t, z') = 0, \quad \text{all } t, \text{ all } z^t, \text{ all } z'.$$

Notice that in problem (3.2) agents are not given the option to default. Since the problem in (3.2) - (3.4) has a concave objective function and a convex feasible set, the following Euler and Transversality conditions are sufficient for a maximum:

$$-u'(c_{i,t}(z^t)) q_t(z^t, z') + \beta(z) \pi(z'|z) u'(c_{i,t+1}(z^t, z')) \leq 0 \quad (3.5)$$

with equality if $a_{i,t+1} > B_{i,t+1}(z^t, z')$ and,

$$\lim_{t \rightarrow \infty} \sum_{z^t \in Z^t} \delta_{0,t}(z^{t-1}) u'(c_{i,t}(z^t)) \cdot [a_{i,t}(z^t) - B_{i,t}(z^t)] \cdot \pi(z^t | z_0) = 0. \quad (3.6)$$

Now we move to the analysis of the decision of default, which is the main contribution of our equilibrium concept. We consider equilibria in which the solvency constraints are determined by the risk of default in the following way:

Definition 3.3. *An equilibrium has solvency constraints that are not too tight if*

$$J_{i,t+1}(B_{i,t+1}(z^{t+1}), z^{t+1}) = U(e_i)(z^{t+1}), \text{ all } t \geq 0, z^{t+1} \in Z^{t+1}. \quad (3.7)$$

Solvency constraints that satisfy this condition prevent default by prohibiting agents from accumulating more contingent debt that they are willing to pay back, but at the same time allow as much insurance as possible. By contrast, if the solvency constraints were to bind and the continuation utility were strictly higher than the value of autarchy, then the constraint could be relaxed a bit, without inducing the agent to default.⁵ Thus these conditions imply that

$$\begin{aligned} U(c_i)(z^t) &\geq U(e_i)(z^t) \quad \text{and} \\ U(c_i)(z^t) &= U(e_i)(z^t) \Leftrightarrow a_{i,t}(z^t) = B_{i,t}(z^t). \end{aligned} \quad (3.8)$$

To enforce any solvency constraint the agent's entire portfolio must be known. To determine whether solvency constraints are not too tight, *i.e.* whether (3.7) holds, information about preferences and endowment process of agent i must be used. Thus, implementing the equilibrium allocation using these constraints requires information about the agent's type and portfolio.

Finally, we introduce the concept of *high implied interest rates*. This condition ensures finiteness of the value of endowment implied by a given allocation. Given an allocation $\{c_i^*\}$, for $i = 1, 2, \dots, I$, we define

$$q_t^*(z^t, z') \equiv \max_{i \in \mathbf{I}} \left\{ \beta(z) \frac{u'(c_{i,t+1}^*(z^t, z'))}{u'(c_{i,t}^*(z^t))} \pi(z'|z) \right\}, \quad (3.9)$$

and

$$Q_0^*(z^t|z_0) = q_0^*(z_0, z_1) \cdot q_1^*(z_0, z_1, z_2) \cdots q_{t-1}^*(z^{t-1}, z_t). \quad (3.10)$$

Note, $Q_0^*(z^t|z_0)$ is the price of one unit of consumption delivered in state z^t in units of consumption of $t = 0$.

Definition 3.4. *The implied interest rates for the allocation $\{c_i^*\}$ are high if*

$$\sum_{t \geq 0} \sum_{z^t \in Z^t} Q_0^*(z^t|z_0) \left(\sum_{i \in \mathbf{I}} c_{i,t}^*(z^t) \right) < +\infty. \quad (3.11)$$

4. WELFARE THEOREMS AND THE EXTENT OF RISK SHARING

In this section we show the conditions under which efficient allocations can be decentralized as a competitive equilibrium with solvency constraints, and the conditions under which the first welfare theorem holds. We then use these results to characterize the extent of risk sharing.

4.1. Second Welfare Theorem

In this section we characterize the allocations that can be supported as competitive equilibria with solvency constraints. In particular, we decentralize constrained efficient allocations.

Proposition 4.1. *Suppose the allocation $\{c_i^*\}$ satisfies: (a) the resource constraint (2.2), (b) the participation constraints (2.3), (c) the high MRS condition (3.1) for all $t \geq 0$, $z^t \in Z^t$, and (d) the high implied interest rates condition in (3.9)-(3.11). Then there exist solvency constraints $\{B_i\}$, initial wealth $a_{i,0}$ and asset holdings $\{a_i^*\}$ such that the quantities $\{a_i^*, c_i^*\}$ are a competitive equilibrium. Moreover, the solvency constraints $\{B_i\}$ can be chosen so that they are not too tight, i.e., (3.7) holds for all agents.*

As a corollary of the previous proposition we get the second welfare theorem, since efficient allocations satisfy properties (a), (b) and (c) of proposition (4.1).

Corollary 4.2. *Any constrained efficient allocation that has high implied interest rates can be decentralized as a competitive equilibrium with solvency constraints that are not too tight.*

The connection between equilibrium allocations and efficient allocations is clear. In the equilibrium, solvency constraints that are not too tight take the place of the participation constraints of the efficient allocations. In equilibrium allocations, an agent whose solvency constraint is not binding has the highest marginal rate of substitution. In efficient allocations, an agent whose participation constraint does not bind has the highest marginal rate of substitution. Later we will show that for any constrained optimal allocation where there is some risk sharing the implied interest rates are high.

We finally establish that autarchy can always be decentralized as an equilibrium with solvency constraints that are not too tight.

Proposition 4.3. *The quantities and prices*

$$\begin{aligned} c_{a,i,t}(z^t) &= e_{i,t}(z^t), \quad a_{a,i,t+1}(z^t, z') = B_{a,i,t}(z^t) = 0, \\ q_{a,t}(z^t, z') &\equiv \max_{i \in \mathbf{I}} \beta(z) \frac{u'(e_{i,t+1}(z^t, z'))}{u'(e_{i,t}(z^t))} \pi(z'|z_t) \end{aligned} \quad (4.1)$$

for all $t \geq 0, z^t \in Z^t, z' \in Z$ and the initial conditions $a_{i,0} = 0$, for $i \in \mathbf{I}$, is an equilibrium with solvency constraint that are not too tight.

Notice that even though autarchy allocations can always be decentralized, in general, these are not constrained efficient allocations. Also notice that the implied interest rate in autarchy can be low, so that (3.11) may be violated.

4.2. First Welfare Theorem

In this section we show the connection between the Kehoe-Levine (henceforth K-L) decentralization and an equilibrium with solvency constraints that are not too tight. First, we show that the implied Arrow prices in the K-L decentralization are equal to the highest marginal rate of substitution across agents, as in our equilibrium with solvency constraints that are not too tight. Second, we show that under weak conditions, the allocation of an equilibrium with solvency constraints corresponds to the allocations of a K-L equilibrium. An immediate consequence of these results is that we can use the first welfare theorem shown by K-L. And thus, we obtain a (qualified) version of the first welfare theorem for our equilibrium concept. That is, allocations corresponding to an equilibrium with solvency constraints not too tight (3.7) and that have high implied interest rates (3.11) are constrained efficient.

The problem for agent i in the K-L decentralization is

$$\max_{\{c_i\}} U(c_i)(z_0) \quad (4.2)$$

$$s.t. \quad : \quad p_0(c_i - e_i) \leq a_{i,0} \quad (4.3)$$

$$U(c_i)(z^t) \geq U(e_i)(z^t) \text{ for all } t \geq 0 \text{ and } z^t \in Z^t \quad (4.4)$$

where p_0 is a non-negative linear function. Thus, with the inequalities in (4.4) defining the consumption possibility set, an equilibrium is a standard Arrow-Debreu equilibrium, henceforth A-D .

Given p_0 , the dot product representation of the A-D prices is defined as follows.

For any t and $z^t \in Z^t$ define

$$Q_0(z^t|z_0) \equiv p_0(\tilde{c})$$

where $\tilde{c}_s(z^s) = 0$ if $s \neq t$ and $z^s \neq z^t$ and otherwise $\tilde{c}_t(z^t) = 1$.

Definition 4.4. *The A-D prices are said to have a dot product representation if*

$$p_0(c) = \sum_{t \geq 0} \sum_{z^t \in Z^t} c_t(z^t) Q_0(z^t|z_0).$$

With this notation we can write the budget constraint of agent i as

$$\sum_{t \geq 0} \sum_{z^t \in Z^t} (c_{i,t}(z^t) - e_{i,t}(z^t)) Q_0(z^t | z_0) \leq a_{i,0}.$$

The corresponding Arrow prices are defined as

$$q_{0,t}(z^t, z') = \frac{Q_0(z^t, z' | z_0)}{Q_0(z^t | z_0)}.$$

Our equilibrium concept differs from the one by K-L along two dimensions. First, they include the participation constraints in the consumption possibility sets, as opposed to our limits on borrowing (solvency constraints). Our approach makes the form of the pricing kernel immediate and allows us to relate directly to the literature on empirical asset pricing with solvency constraints. Second, in the K-L decentralization agents are not given the option to default and walk away from their debts. Our equilibria with solvency constraints that are not too tight allow agents the option to default, although in equilibrium they prefer not to do so. We find this interpretation of the solvency constraints as enforcing individual rationality an attractive feature of our equilibrium concept.

The next proposition characterizes the Arrow prices in a K-L equilibrium.

Proposition 4.5. *Let $\{c_i\}, i = 1, \dots, I$, and $\{Q_0\}$ be the allocations and A-D*

prices corresponding to a K-L equilibrium. Let q_0 be the corresponding Arrow prices. Then

$$q_{0,t}(z^t, z') = \max_{i \in \mathbf{I}} \left\{ \beta(z) \frac{u'(c_{i,t+1}(z^t, z'))}{u'(c_{i,t}(z^t))} \pi(z'|z_t) \right\}, \quad (4.5)$$

and if

$$U(c_i)(z^t, z') > U(e_i)(z^t, z')$$

then

$$q_{0,t}(z^t, z') = \beta(z) \frac{u'(c_{i,t+1}(z^t, z'))}{u'(c_{i,t}(z^t))} \pi(z'|z_t).$$

Now we show that for any equilibrium with solvency constraints, if the implied interest rate are high and the constraints are not too tight, then the implied A-D prices and consumption allocations constitute a K-L equilibrium.

Proposition 4.6. *Let $\{q, c_i, a_i\}$ be an equilibrium given the solvency constraints $\{B_i\}$ and the initial wealth a_{i0} . Suppose (a) the implied interest rates are high, i.e. (3.11) holds, (b) the solvency constraints are not too tight, i.e. (3.7) holds, and (c) that for each $i \in \mathbf{I}$ there is a constant ξ_i such that for all t, z^t ,*

$$\left| u(c_{i,t}(z^t)) \right| \leq \xi_i \cdot u'(c_{i,t}(z^t)) \cdot c_{i,t}(z^t). \quad (4.6)$$

Then the consumption allocations $\{c_i\}$ and the A-D prices $\{Q_t\}$ derived from $\{q_t\}$ are a K-L equilibrium.

Remark 1. The condition (4.6) is a joint requirement on the consumption allocation and the utility functions. It is satisfied automatically in several relevant cases. For instance, it holds if (i) $u(\cdot)$ has relative risk aversion different from one at zero consumption, *i.e.*:

$$\lim_{c \rightarrow 0} -\frac{c u''(c)}{u'(c)} \neq 1$$

which can be verified by repeated application of L'Hôpital's rule, or (ii) $u'(0) < +\infty$, or (iii) if consumption for an agent is uniformly bounded away from zero.

We obtain the first welfare theorem as a corollary of the previous proposition.

Corollary 4.7. *Since a K-L equilibrium is a standard Arrow-Debreu equilibrium, its consumption allocation is efficient. Then by Proposition 4.6, if (4.6) is satisfied, an equilibrium with solvency constraints that are not too tight, *i.e.* (3.7) holds, and with implied high interest rates, *i.e.* (3.11) holds, is efficient.*

4.3. Solvency Constraints and the Extent of Risk Sharing

In this section we use the equivalence between a K-L equilibrium, equilibrium with solvency constraints that are not too tight and optimal allocations to address

three issues. The first is to characterize the circumstances under which autarchy is the only feasible allocation. The second is to show that if an optimal allocation is different from autarchy, then it has high implied interest rates. The third is to show that solvency constraints are negative.

Proposition 4.8. *If the implied interest rates for the autarchy allocations $\{c_i = e_i\}$, are high, i.e., satisfy (3.11), then autarchy is a constrained efficient allocation, and hence is the only feasible allocation.*

In the next proposition we give sufficient conditions so that autarchy is the only feasible allocation. For this proposition let

$$\hat{\Pi}(\alpha) \equiv \alpha I + (1 - \alpha) \Pi$$

where α indexes the persistence of $\hat{\Pi}(\alpha)$.

Proposition 4.9. *Autarchy is the only feasible allocation in any of the following cases: (i) the time discount factor is sufficiently small (i.e. $\max_z \beta(z) \downarrow 0$), (ii) risk aversion is sufficiently small uniformly, (iii) the variance of the idiosyncratic shock is sufficiently close to zero ($\max_{i,z,\tilde{z}} |\epsilon_i(\tilde{z}) - \epsilon_i(z)| \rightarrow 0$), and (iv) the transition probability matrix is sufficiently close to identity, (i.e. $\alpha \uparrow 1$).*

Proposition 4.9 suggests the type of parameter values needed so that idiosyncratic shocks generate high and volatile pricing kernels: high discount rates, low risk aversion, low variance and high persistence of the shocks.⁶

Next we show that for any constrained efficient allocation where some risk sharing is possible, the implied interest rates are high. This result, together with our statement of the second welfare theorem implies that any efficient allocation with some risk sharing can be decentralized as an equilibrium with solvency constraints that are not too tight.

Proposition 4.10. *Let $\{c_i\}$ be a constrained efficient allocation. Assume that some risk sharing is possible, i.e. for all agents i there is a z^t such that:*

$$\pi(z^t|z_0) \left[U(c_i)(z^t) - U(e_i)(z^t) \right] > 0. \quad (4.7)$$

Then (3.11) holds (i.e. the implied interest rates are high).

In the definition of an equilibrium with solvency constraints there is no requirement that the constraints are negative. Notice that a positive $B_{i,t+1}$ means that agent i has to *save* a minimum amount. We end this section by showing that in equilibrium with high implied interest rates, the solvency constraints are negative, i.e. they satisfy $B_{i,t+1} < 0$. Thus, with high interest rates, solvency constraints are effectively constraints on contingent borrowing.

Proposition 4.11. *If $\{q^*, c_i^*, a_i^*, J_i^*, B_i^*, \}, a_{i,0}^*$ is an equilibrium with solvency constraints that are not too tight, where the implied interest rates are high, then there is an equilibrium with solvency constraints that are not too tight with $\{q^*, c_i^*, a_i^*, \bar{J}_i, \bar{B}_i, a_{i,0}^*\}, a_{i,0}^*$ where $\bar{B}_{i,t+1}(z^t, z') \leq 0$, with strict inequality if $U(c_i^*)(z^t) = U(e_i)(z^t)$ and $\exists z^s \succeq z^t$ $c_{i,s}^*(z^s) \neq e_{i,s}(z^s)$.*

5. ASSET PRICES IN ECONOMIES WITH SOLVENCY CONSTRAINTS

In this section we analyze some properties of the pricing kernels in economies with solvency constraints. We start by describing the pricing of securities that are more complex than Arrow securities. We then compare pricing implications in economies with and without participation constraints by considering marginal valuations, interest rates and risk premia. Finally we present a simple irrelevance result.

5.1. Pricing Complex Securities

Instead of allowing agents to trade only one-period contingent claims, suppose that they can trade other securities, including multiperiod securities such as stocks and bonds. A straightforward extension of our framework allows this possibility. Under certain conditions stated below, the price of any security can be obtained

by pricing the corresponding portfolio of Arrow securities.

We assume that at time t the set of available securities from dates and states prior to z^t is given by $K_t(z^t)$. These securities have prices $q_{t,k}$ and may pay dividends $d_{t,k}$ at multiple dates. Analogously $\{K_{t+1}(z^t, z') : z' \in Z\}$ is the set of securities that can be traded at z^t . In this case, agents face the following sequence of budget constraints for all $t = 0, 1, \dots$ and for all $z^t \in Z^t$:

$$\begin{aligned} & \sum_{k' \in K_{t+1}(z^t, z')} a_{i,t+1,k'}(z^t) q_{t,k'}(z^t) + c_{i,t}(z^t) \\ \leq & \sum_{k \in K_t(z^t)} a_{i,t,k}(z^{t-1}) \left[q_{t,k}(z^t) + d_{t,k}(z^t) \right] + e_{i,t}(z^t). \end{aligned}$$

To have the same budget set as in the case with Arrow securities we need two conditions: first, that this set of securities be rich enough so that markets are dynamically complete, and second, that the portfolio choice be restricted so that the value of an agent's portfolio be high enough in the different states. Specifically, for each period, a set of solvency constraints that limits the minimum value of next period's portfolio for each state is defined as :

$$\sum_{k' \in K_{t+1}(z^t, z')} \left[q_{t+1,k'}(z^t, z') + d_{t+1,k'}(z^t, z') \right] a_{i,t+1,k'}(z^t, z') \geq B_{i,t+1}(z^t, z')$$

for all $t = 0, 1, \dots$ and for all $z^t \in Z^t$. Clearly, every security is just a portfolio of

contingent claims and can be valued as such.

5.2. Prices and Marginal Valuations

Given the frictions introduced by solvency constraints, we find that the prices of securities with non-negative payoffs are generally higher than any of the agents' valuation. Superficially, this result may seem to imply an arbitrage opportunity, in the sense that the prices are too high for everyone, but recall that agents can short sell only limited amounts of securities.⁷

Assume that k' is a security available at time t , *i.e.* $k' \in K_{t+1}(z^t, z')$ and that

$$d_{t+s, k'}(z^{t+s}) \geq 0 \text{ for all } s > 0.$$

Let us denote the price of this security for the equilibrium with solvency constraints by $q_{t, k'}$ and the marginal valuation of agent i by $MV_{i, t, k'}$ where the marginal valuation is defined as

$$MV_{i, t, k'}(z^t) \equiv \sum_{s>0} \sum_{z^s \in Z^s} \delta_{t, t+s} (z^{t+s-1}) d_{t+s, k'}(z^t, z^s) \frac{u'(c_{i, t+s}(z^t, z^s))}{u'(c_{i, t}(z^t))} \pi(z^s | z^t).$$

This quantity measures the marginal change in utility, in terms of time t consumption, produced by an increase in the i^{th} agent's consumption that is proportional to the dividends of security k' at each future date. When agents are never constrained,

or equivalently for the case of perfect insurance, we have that for all i ,

$$q_{t,k'}(z^t) = MV_{i,t,k'}(z^t).$$

In our environment we have the following result:

Lemma 5.1. *In an equilibrium with solvency constraints, when there is only limited risk sharing, then*

$$q_{t,k'}(z^t) \geq \max_{i \in \mathbf{I}} MV_{i,t,k'}(z^t)$$

for all agents i , securities k' , $t \geq 0$, and $z^t \in Z^t$. With strict inequality for an agent i if he is constrained at least once between t and $t + s$.

Notice that since we assume that the dividends are non-negative, increasing the holding of security k' can never lead to default in the future if the original plan did not already contemplate default. One may therefore be tempted to conclude that in equilibrium $q_{t,k'}(z^t) = \max_{i \in \mathbf{I}} MV_{i,t,k'}(z^t)$. The misleading part of this conjecture is that this equality holds only for Arrow securities and not for general securities. The latter may pay in different states, but the pricing kernel is defined by the *max* across agents of the marginal rate of substitution, state by state. That is, the agent whose marginal rate of substitution is equal to the price of the

Arrow security in a given state may not price the Arrow security for a different state.

5.3. Interest Rates and Security Prices

The next result shows that in several interesting cases, interest rates are lower in economies with solvency constraints than in corresponding economies without such constraints. In contrast to many models with incomplete markets, this effect does not rely on the precautionary savings motive (convexity of the marginal utility).

Proposition 5.2. *Arrow prices in an economy with solvency constraints are higher than in an otherwise identical economy without solvency constraints in any one of the following cases: (i) $\sum_i \epsilon_i(z) = 1$, all z , i.e., constant aggregate endowment, (ii) $u(\cdot)$ has constant relative risk aversion, or (iii) $u(\cdot)$ is quadratic.*

Notice that since the proposition holds for constant relative risk aversion, it holds for the case with aggregate growth described in section 2.1.

For the cases described in Proposition (5.2), the price of a one-period bond is higher in an economy with solvency constraints than in an otherwise identical economy without these constraints.⁸ It is strictly so, if at least one agent is constrained in each period. By the same logic as in the proposition, in these

cases, any security with non-negative payouts will have a weakly higher price in the solvency constraints economy compared to the corresponding representative agent one.

5.4. The Premium for Aggregate Risk with Idiosyncratic Shocks

A major issue for any heterogenous-agent model of asset pricing is the mechanism through which idiosyncratic shocks can generate a risk premium for claims contingent on the aggregate shock, that is, the market risk premium. In this section we consider the endowment specification with aggregate growth, constant relative risk aversion and constant discount factor described in section 2.1. We derive sufficient conditions under which the economy with idiosyncratic shocks generates a risk premium on a one-period risky strip identical to the representative agent economy.

We specialize the endowment process so that $\lambda(\cdot)$ and, $\epsilon(\cdot) = \{\epsilon_i(\cdot)\}_{i \in \mathbf{I}}$ are statistically independent and $\lambda(\cdot)$ is i.i.d. Assume that z can be written as $z = (x, y) \in Z = X \times Y$ and that λ and ϵ are functions of y and x , specifically: $\lambda : Y \rightarrow R$ and $\epsilon : X \rightarrow \Delta^I$, the I -th dimensional simplex.

Definition 5.3. *We say that the aggregate shock is i.i.d. and independent of the idiosyncratic shock if there exists a probability distribution ϕ and a stochastic*

matrix ψ such that

$$\pi(z'|z) \equiv \pi((x', y') | (x, y)) = \phi(y') \cdot \psi(x'|x) \text{ for all } z, z'.$$

We consider the risk premium for one-period risky strips. A strip is a security that pays a random dividend d_{t+1} only at one period. We consider a one-period strip whose payout is function only of the aggregate output, y_{t+1} , and hence whose price equals $\sum_{z_{t+1}} q_t(z^{t+1})d_{t+1}(y_{t+1})$. Equation (5.1) defines the premium as the expected strip return divided by the risk free rate,

$$\left[\frac{\sum_{z_{t+1}} d_{t+1}(y_{t+1})\pi(z_{t+1}|z_t)}{\sum_{z_{t+1}} q_t(z^{t+1})d_{t+1}(y_{t+1})} \right] / \left[\frac{1}{\sum_{z_{t+1}} q_t(z^{t+1})} \right], \quad (5.1)$$

where $\left[1 / \sum_{z_{t+1}} q_t(z^{t+1}) \right]$ is the risk free rate. This is sometimes called the “multiplicative excess return of the one-period risky strip.” We choose this premium as opposed to the equity premium for its tractability. Indeed, the equity premium is a weighted sum of the entire infinite sequence of strips, one for each period.

Proposition 5.4. *If endowments and preferences are specified as in section 2.1 and if the aggregate shock is i.i.d. and independent of the idiosyncratic shock, then the multiplicative premium on a one-period risky strip with payout contingent on the aggregate output is the same in an economy with and without participation*

constraints.

This proposition shows that dependence of the cross sectional distribution of earnings on the aggregate is required to obtain interesting results for excess returns. This result complements similar results by Mankiw (1986) and Constantinides and Duffie (1996) that were obtained in environments with exogenous asset market incompleteness and non-binding borrowing constraints.

5.5. Irrelevance of the Average Agent

The next result is important for thinking about quantitative implementation of this equilibrium concept. Consider first an equilibrium with solvency constraints that are not too tight and add one agent to that economy. If the marginal rates of substitution under autarchy of this agent are smaller than the equilibrium prices, then there is an equilibrium for the expanded economy with the same prices. Additionally, the extra agent will be constrained all the time.

Proposition 5.5. *Let the processes $\{c_i^*, a_i^*, q^*, B_i^*\}$, and the initial conditions $a_{i,0}^*$ for $i = 1, 2, \dots, I$ be an equilibrium with solvency constraints that are not too tight. Add to the previous economy one agent with endowment process $\{e_{I+1}\}$. Assume that for all z^t and z_{t+1}*

$$\beta(z) \frac{u'(e_{I+1,t+1}(z^t, z_{t+1}))}{u'(e_{I+1,t}(z^t))} \pi(z_{t+1}|z_t) \leq q_t^*(z^t, z_{t+1}). \quad (5.2)$$

Then $\{c_i^*, a_i^*, q^*, B_i^*\}$, for $i = 1, 2, \dots, I + 1$ and the initial condition $a_{i,0}^*$ for $i = 1, 2, \dots, I + 1$ is an equilibrium with solvency constraints that are not too tight where $\{c_{I+1}^*\} = \{e_{I+1}\}$, $\{a_{I+1}^*\} = \{0\}$ and $\{B_{I+1}^*\} = \{0\}$ and initial condition $a_{I+1,0}^* = 0$.

The example in the next remark says that adding an extra agent whose endowment is perfectly correlated with aggregate endowment will have no effect on equilibrium prices.

Remark 2. Let $\mu > 0$ be a constant and $e_{I+1,t}(z^t) = \mu \cdot e_t(z^t)$ for all t and z^t where e_t is the aggregate endowment for the economy with I agents. Assume that $u(\cdot)$ has constant relative risk aversion. Then, by using proposition (5.2) one obtains immediately that assumption (5.2) holds.

Agent $I + 1$ has no reason to participate in future insurance arrangements with the I agents, and thus he will default if he acquires any debt. By continuity, agents with individual endowment processes that are very similar to the aggregate endowment are not important for the determination of asset prices. Instead, only the agents with substantial idiosyncratic risk are key to determining asset prices.

6. CONCLUSIONS

We have studied a market equilibrium with explicit endogenous portfolio constraints for an environment with risk of default. We have derived the first and second welfare theorems, related our equilibrium definition to the one by Kehoe and Levine, and characterized the corresponding asset pricing kernel. Risk sharing is reduced, and hence the pricing kernel is more variable, the more impatient and risk tolerant agents are, and the more persistent and less variable their individual income processes are. We have compared the pricing kernel with the one for economies without solvency constraints. First, interest rates are lower, regardless of the precautionary savings motive. Second, the risk premium for a claim that depends only on next period's aggregate endowment is identical if individual and aggregate shocks are independent. Third, asset prices are determined by agents with substantial individual risks. In particular, agents whose income process is similar to the aggregate are irrelevant for asset prices.

We view this paper as a first step in exploring asset pricing relationships in an economy where the possibility of default limits risk sharing. In a companion paper (Alvarez and Jermann (1998)) we analyze in depth some special cases of the environment presented here. In particular we characterize efficient allocations, develop computational algorithms, and document quantitative implications for

asset prices.

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APPENDIX

Proof of Proposition 3.1: The proof is based on a simple variational argument, so we only sketch it here. Let $i \neq j$ be the agent with the highest marginal rate of substitution. One can increase current consumption and decrease future consumption of agent j in state z_{t+1} so as to keep $U(c_j)(z^t)$ constant. This is feasible since there is slack in (3.1). By decreasing current consumption and increasing future consumption of agent i in state z_{t+1} while keeping material balance, $U(c_i)(z^t)$ increases since i 's marginal rate of substitution is the higher of the two, which is in contradiction to constrained efficiency. ■

Proof of Proposition 4.1: The proof is by construction. Given the allocation, use (3.9) and (3.10) to define prices. Since by assumption the implied interest rates are high, we use the budget constraint (3.3) to construct asset holdings at each time and event so that $\{c_i^*\}$ is budget feasible. When an agent's marginal rate of substitution is not the highest one, let the solvency constraint equal the holding of the corresponding Arrow security. When an agent's marginal rate of substitution is the highest, we will define, as an intermediate step, solvency constraints $\{\tilde{B}_i\}$ such that $\tilde{B}_{i,t+1}(z^t, z') = -\sum_{z^s \succeq z^{t+1}} Q_s(z^s | z^{t+1}) e_{i,s}(z^s)$, we will redefine the solvency constraints later. The sufficient Euler and Transversality conditions for individual maximization can now be checked. The Euler conditions

(3.5) follows from the definition of the Arrow prices (3.9) and the assumption that the unconstrained agent has the highest marginal rate of substitution. Now we can check the transversality condition:

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \sum_{z^T \in Z^T} \delta_{0,T} (z^{T-1}) u' (c_{i,T}^* (z^T)) \left[a_{i,T}^* (z^T) - \tilde{B}_{i,T} (z^T) \right] \pi (z^T | z_0) \\
& \leq \lim_{T \rightarrow \infty} \sum_{z^T \in Z^T} \delta_{0,T} (z^{T-1}) u' (c_{i,T}^* (z^T)) \left[\sum_{z^s \succeq z^T} Q_s^* (z^s | z^T) c_{i,s}^* (z^s) \right] \pi (z^T | z_0) \\
& \leq \lim_{T \rightarrow \infty} \sum_{z^T \in Z^T} \delta_{0,T} (z^{T-1}) u' (c_{i,T}^* (z^T)) \left[\sum_{z^s \succeq z^T} Q_s^* (z^s | z^T) e_s (z^s) \right] \pi (z^T | z_0) \\
& \leq u' (c_{i,0} (z_0)) \lim_{T \rightarrow \infty} \sum_{z^T \in Z^T} Q^* (z^T | z_0) \sum_{z^s \succeq z^T} Q_s^* (z^s | z^T) e_s (z^s) = 0
\end{aligned}$$

where the first inequality follows since by construction $a_{i,T}^* (z^T) - \tilde{B}_{i,T} (z^T) = 0$ if the participation constraint binds or equal to $\sum_{z^s \succeq z^T} Q_s^* (z^s | z^T) c_{i,s}^* (z^s) \geq 0$ otherwise, the second because $c_{i,s}^* (z^s) \leq e_s (z^s)$ since the allocation is feasible, the third follows from the definition of Q_t^* 's in terms of product of the q_t^* and the relationship of the q_t^* 's with the marginal rate of substitution, and the final equality from the assumption that the implied interest rates are high.

Using the prices and solvency constraints we construct the function $J_{i,t}$, which is attained by $\{c_i^*\}$. Finally we use $J_{i,t}$ to redefine the values of the solvency constraints for the cases where an agent has the highest MRS, that is, when the constraints do not bind. In particular, for every period and state, we de-

find a new solvency constraint $B_i^{*,1}$ that solves $J_{i,t+1}(B_{i,t+1}^{*,1}(z^t, z'), (z^t, z')) = U(e_i)(z^t, z')$, which has a solution since the $J_{i,t}(\cdot, z^t)$ are strictly decreasing. We iterate on the sequence of solvency constraints until convergence, at $\{B_i^*\}$.

Clearly $\{c_i^*\}$ and $\{a_i^*\}$ still solve the problem (3.2) for the same prices and solvency constraints $\{B_i^*\}$, since the feasible set is smaller, but the original plan is still feasible. ■

Proof of Proposition 4.3: It is immediate to verify that $\{c_{a,i}, a_{a,i}, q_a, B_{a,i}\}$ satisfy the sufficient conditions (3.5) and (3.6), resource feasibility and market clearing. Moreover $\{B_{a,i}\}$ are not too tight, since trivially $J_{i,t}(a_{a,i,t}(z^t), z^t) = J_{i,t}(B_{a,i,t}(z^t), z^t) = U(e_i)(z^t)$ for all t, z^t . ■

Proof of Proposition 4.5: It follows from the following variational argument around the consumption allocation in a K-L equilibrium. For an arbitrary date (“the current date”) allow consumption to decrease at the current date and to increase in the next at a particular state, so as to keep the same expenditure at the K-L prices. Choose the values of current and future consumption to maximize utility. Notice that the participation constraints at the future date will be satisfied since consumption can only increase in the future. Also, the participation constraints at the current date will be satisfied (at the maximum), since current continuation utility cannot decrease, given that no deviation is feasible for the modified problem. This problem has a strictly concave differentiable objec-

tive function and linear constraints, so the Kuhn-Tucker conditions are necessary.

The result follows immediately from the Kuhn-Tucker conditions. ■

Proof of Proposition 4.6: In an equilibrium, for given solvency constraints, the consumption allocation is feasible. If the solvency constraints are not too tight, the consumption allocations satisfy the participation constraints for each agent at all times and states. It only remains to be shown that given the A-D prices implied by $\{q\}$ the consumption allocation maximizes utility subject to budget and participation constraints. It will suffice to find non-negative shadow values (multipliers) associated with the budget constraints and participation constraints and verify that they are a saddle. The following algorithm defines these multipliers as a function of the consumption allocation, Arrow prices and participation constraints. The multiplier on the A-D budget constraint (4.3) is set to $\zeta_i = \frac{u'(c_{i,0}(z_0))}{Q(z_0|z_0)} = u'(c_{i,0}(z_0))$. The multiplier for the participation constraints (4.4) at time $t = 0$ is $\eta_{i,0}(z_0) = 0$. The multipliers for the participation constraints (4.4) for all $t > 0$ and $z^t \in Z^t$ are defined recursively by

$$\delta_{0,t}(z^{t-1}) u'(c_{i,t}(z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}(z^r) \right] \pi(z^t|z_0) = \zeta_i Q(z^t|z_0). \quad (6.1)$$

Finally, one verifies that these multipliers together with the consumption allocation are indeed a saddle. This is accomplished by verifying the first order

conditions for a saddle. The fact that Arrow prices are defined as in (4.5) is used to show that the multipliers are non-negative.

$$\begin{aligned} \text{We define the Lagrangian as follows: } L(\{\bar{c}_i\}, \bar{\zeta}_i, \{\bar{\eta}_i\}) &= \\ &= U(\bar{c}_i)(z_0) + \bar{\zeta}_i \left[a_{i0} + \sum_{z^t \succeq z_0} Q(z^t|z_0) (e_i(z^t) - \bar{c}_{i,t}(z^t)) \right] \\ &\quad + \sum_{z^t \succeq z_0} \delta_{0,t} (z^{t-1}) \bar{\eta}_{i,t}(z^t) \left[U(\bar{c}_i)(z^t) - U(e_i)(z^t) \right] \pi(z^t|z_0) \end{aligned}$$

for any non-negative consumption plan $\{\bar{c}_i\}$ and non-negative multipliers $\bar{\zeta}_i, \{\bar{\eta}_i\}$.

First, by construction of the multiplier, it is straightforward to see that $\bar{\zeta}_i, \{\bar{\eta}_i\}$ minimizes $L(\{c_i\}, \cdot)$. Second, we turn to establish that $\{c_i\}$ maximizes $L(\cdot, \zeta_i, \{\eta_i\})$. In order to do so we interchange the order of summations, so as a technical requirement, we first check that each of the terms of the sum

$$\sum_{z^t \succeq z_0} \delta_{0,t} (z^{t-1}) \eta_{i,t}(z^t) U(e_i)(z^t) \pi(z^t|z_0) \quad \text{and} \quad \sum_{z^t \succeq z_0} \delta_{0,t} (z^{t-1}) \eta_{i,t}(z^t) U(c_i)(z^t) \pi(z^t|z_0)$$

converges to the same finite limit.

Using the definitions of $U(c_i)(z^t)$, using (6.1) for the multipliers $\{\eta_i\}$, collecting terms on $u(c_{i,t}(z^t))$ and using the assumption (4.6), one obtains

$$\left| U(c_i)(z_0) + \sum_{t \geq 0} \sum_{z^t \in Z^t} \delta_{0,t} (z^{t-1}) \eta_{i,t}(z^t) U(c_i)(z^t) \pi(z^t|z_0) \right| \quad (6.2)$$

$$\begin{aligned}
&= \left| \sum_{t \geq 0} \sum_{z^t \in Z^t} \delta_{0,t} (z^{t-1}) u (c_{i,t} (z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r} (z^r) \right] \pi (z^t | z_0) \right| \\
&\leq \xi_i \cdot u' (c_{i,0} (z_0)) \cdot \sum_{t \geq 0} \sum_{z^t \in Z^t} Q (z^t | z_0) c_{i,t} (z^t)
\end{aligned}$$

which is finite by the assumption that the implied interest rates are high.

By the previous arguments about the convergence of each component of the sums it is equivalent to verify that

$$\begin{aligned}
&U (c_i) (z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q (z^t | z_0) c_{i,t} (z^t) \right] + \sum_{z^t \succeq z_0} \delta_{0,t} (z^{t-1}) \eta_{i,t} (z^t) U (c_i) (z^t) \pi (z^t | z_0) \\
&\geq U (\bar{c}_i) (z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q (z^t | z_0) \bar{c}_{i,t} (z^t) \right] + \sum_{z^t \succeq z_0} \delta_{0,t} (z^{t-1}) \eta_{i,t} (z^t) U (\bar{c}_i) (z^t) \pi (z^t | z_0).
\end{aligned}$$

By collecting terms on $u (\bar{c}_{i,t} (z^t))$ we can write

$$\begin{aligned}
&\sum_{z^t \succeq z_0} \beta_{0,t} (z^{t-1}) u (c_{i,t} (z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r} (z^r) \right] \pi (z^t | z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q (z^t | z_0) c_{i,t} (z^t) \right] \\
&\geq \sum_{z^t \succeq z_0} \beta_{0,t} (z^{t-1}) u (\bar{c}_{i,t} (z^t)) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r} (z^r) \right] \pi (z^t | z_0) - \zeta_i \left[\sum_{z^t \succeq z_0} Q (z^t | z_0) \bar{c}_{i,t} (z^t) \right].
\end{aligned}$$

By concavity and differentiability of u we have

$$u (\bar{c}_{i,t} (z^t)) \leq u (c_{i,t} (z^t)) + u' (c_{i,t} (z^t)) [\bar{c}_{i,t} (z^t) - c_{i,t} (z^t)] \quad (6.3)$$

and by definition of the $\{\eta_i\}$

$$\delta_{0,t} \left(z^{t-1} \right) u' \left(c_{i,t} \left(z^t \right) \right) \left[\sum_{z^r \preceq z^t} \eta_{i,r} \left(z^r \right) \right] \pi \left(z^t | z_0 \right) - \zeta_i Q \left(z^t | z_0 \right) = 0. \quad (6.4)$$

Thus using the inequality (6.3), the equality (6.4) and rearranging, we obtain the desired inequality:

$$\begin{aligned} & \sum_{z^t \succeq z_0} \delta_{0,t} \left(z^{t-1} \right) u \left(\bar{c}_{i,t} \left(z^t \right) \right) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r} \left(z^r \right) \right] \pi \left(z^t | z_0 \right) - \zeta_i \left[\sum_{z^t \succeq z_0} Q \left(z^t | z_0 \right) \bar{c}_{i,t} \left(z^t \right) \right] \\ \leq & \sum_{z^t \succeq z_0} \delta_{0,t} \left(z^{t-1} \right) u \left(c_{i,t} \left(z^t \right) \right) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r} \left(z^r \right) \right] \pi \left(z^t | z_0 \right) \\ & + \sum_{z^t \succeq z_0} \delta_{0,t} \left(z^{t-1} \right) u' \left(c_{i,t} \left(z^t \right) \right) \left[\bar{c}_{i,t} \left(z^t \right) - c_{i,t} \left(z^t \right) \right] \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r} \left(z^r \right) \right] \pi \left(z^t | z_0 \right) \\ & - \zeta_i \left[\sum_{z^t \succeq z_0} Q \left(z^t | z_0 \right) \bar{c}_{i,t} \left(z \right) \right] \\ = & \sum_{z^t \succeq z_0} \delta_{0,t} \left(z^{t-1} \right) u \left(c_{i,t} \left(z^t \right) \right) \left[1 + \sum_{z^r \preceq z^t} \eta_{i,r} \left(z^r \right) \right] \pi \left(z^t | z_0 \right) - \zeta_i \left[\sum_{z^t \succeq z_0} Q \left(z^t | z_0 \right) c_{i,t} \left(z \right) \right]. \end{aligned}$$

■

Proof of Proposition 4.8: Recall that Proposition 4.3 shows that autarchy is an equilibrium with solvency constraints that are not too tight for $B_{i,t} = 0$ and $a_{i,0} = 0$. Then the result follows using the first welfare theorem (4.7) since by assumption the implied interest rates are high. ■

Proof of Proposition 4.9: We need to show that under the given condition

the autarchy-value of endowment converges to a finite limit. In autarchy, Arrow prices depend only on the current and past z , so they are Markov. Thus the value of the aggregate endowment satisfies a simple recursion. This recursion satisfies monotonicity and discounts at a rate that depends on assumptions (i) to (iv). We show below that this rate is strictly smaller than the one for the limit values in (i) to (iv).

Let's denote the value of aggregate endowment evaluated at autarchy prices $q_a(z, z')$ conditional on current shock z as $A(z)$. This value satisfies the following recursion: $q_a(z, z') \equiv \beta(z) \max_{i \in \mathbf{I}} \left\{ \frac{u'(\epsilon_i(z'))}{u'(\epsilon_i(z))} \right\} \pi(z'|z)$, and $A(z) = \sum_{z' \in Z} q_a(z, z') [e(z') + A(z')]$. Equivalently, $A(z) = \beta^*(z) \sum_{z' \in Z} [e(z') + A(z')] \pi^*(z'|z)$, with $\pi^*(z'|z) \equiv \max_{i \in \mathbf{I}} \left\{ \frac{u'(\epsilon_i(z'))}{u'(\epsilon_i(z))} \right\} \pi(z'|z) / \sum_{z' \in Z} \max_{i \in \mathbf{I}} \left\{ \frac{u'(\epsilon_i(z'))}{u'(\epsilon_i(z))} \right\} \pi(z'|z)$, and $\beta^*(z) = \beta(z) \cdot \sum_{z' \in Z} \max_{i \in \mathbf{I}} \left\{ \frac{u'(\epsilon_i(z'))}{u'(\epsilon_i(z))} \right\} \pi(z'|z)$. Notice that the operator defined by that recursion is monotone and satisfies discounting at a rate less than one if $\beta^*(z) < 1$ for all z , in which case the value of aggregate endowment is finite.

We show that for arbitrary z , as we take each of the four limits, $\beta^*(z)$ tends to a number smaller than one. First, as $\max_z \beta(z) \rightarrow 0$ then $\max_z \beta^*(z)$ can be made arbitrarily small. For cases (ii) and (iii) $\max_{i \in I, z, \tilde{z} \in Z} \frac{u'(\epsilon_i(\tilde{z}))}{u'(\epsilon_i(z))} \rightarrow 1$ hence $\beta^*(z) \rightarrow \beta(z)$. In case (iv) $\lim_{\alpha \rightarrow 1} \pi^*(z'|z) = 1$ if $z' = z$ and 0 otherwise, and since $\frac{u'(\epsilon_i(z))}{u'(\epsilon_i(z))} = 1$, then $\beta^*(z) \rightarrow \hat{\beta}(z) < 1$. ■

Proof of Proposition 4.10: A K-L economy is a standard Arrow-Debreu

convex economy, hence by the 2nd welfare theorem any efficient allocation can be supported as a quasi-equilibrium by a linear function p_0 . Assumption (4.7) implies that for all z^t there is an i such that $U(c_i)(z^t) > U(e_i)(z^t)$. Adapting the argument in Proposition 4.5 and using the implication of (4.7), the Arrow prices implicit in the supporting prices p_0 satisfy (4.5), and are strictly positive. Denote the corresponding A-D prices by Q_0 . By separability of preferences, $\{c_i\}_{z^s \succeq z^t}$ minimizes $p_t(c)$ subject to $U(c)(z^t) \geq U(c_i)(z^t)$ and to c satisfying the participation constraints, where p_t denotes the projection of p_0 to $z^s \succeq z^t$. By (4.7) agent i has a cheaper point in his consumption possibility set at z^t , hence $\{c_i\}_{z^s \succeq z^t}$ maximizes $U(c)(z^t)$ subject to $p_t(c) \leq p_t(c_i)$ and the participation constraints. Thus $p_t(c_i) < +\infty$. Since \mathbf{I} is finite, $\{c_i\}_{z^s \succeq z^t}$ differs from $\{c_i\}$ on a finite number of components, hence $p_0(c_i) < +\infty$ for all i . By monotonicity of preferences, $\sum_{z^t \succeq z} Q_0(z^t|z_0) c_{i,t}(z^t) \leq p_0(c_i)$. Adding up across agents and using resource feasibility we obtain the desired result. ■

Proof of Proposition 4.11: Consider an equilibrium $\{q^*, c_i^*, a_i^*, J_i^*, B_i^*\}$, $a_{i,0}^*$ with solvency constraints that are not too tight and with high interest rates. By Proposition (4.6) there is a K-L equilibrium with the same Arrow prices, with prices p_0 given by the implied A-D prices. Define $\bar{J}_{i,0}(a, z_0)$ as the maximized value of Equation (4.2) subject to (4.3) and (4.4). Notice the following two properties of the function $\bar{J}_{i,0}$. First, $\bar{J}_{i,0}(0, z_0) \geq U(e_i)(z_0)$ because $c_i = e_i$ is budget

feasible for $a_{i,0} = 0$. Second, if $\bar{J}_{i,0}(a, z_0) = U(e_i)(z_0)$, and if the optimal consumption $c_i \neq e_i$ then $a_{i,0} < 0$, because the objective function is strictly concave and the feasible set is convex. Since the continuation of a K-L equilibrium at node z^t is itself a K-L equilibrium then we can define price systems p_t and functions $\bar{J}_{i,t}(\cdot, z^t)$ for the economy starting at node z^t . These functions satisfy the same two properties that in the case of $t = 0$. Define $\bar{B}_{i,t+1}(z^{t+1})$ as the solution of $\bar{J}_{i,t+1}(\cdot, z^{t+1}) = U(e_i)(z^{t+1})$ for each z^{t+1} . It is immediate to verify that $\{q_t^*, c_{i,t}^*, a_{i,t+1}^*, \bar{J}_{i,t}, \bar{B}_{i,t+1}, a_{i,0}^*\}$ constitute an equilibrium with solvency constraints that are not too tight. Finally, from the two properties of the functions \bar{J} we obtain the desired result for $\bar{B}_{i,t+1}$. ■

Proof of Lemma 5.1: It follows directly from the fact that Arrow prices are equal to the highest marginal rate of substitution across agents. ■

Proof of Proposition 5.2: Case (i): In the case with no solvency constraints all the marginal rates of substitution are equated, and because of constant aggregate endowment, the ratio of the marginal utilities equals one. If in the economy with solvency constraints Arrow prices are smaller, then $u'(c_{i,t+1}(z^{t+1})) < u'(c_{i,t}(z^t))$, for all i , thus since u' is decreasing we arrive to $\sum_i c_{i,t+1}(z^{t+1}) > \sum_i c_{i,t}(z^t)$, a contradiction. Case (ii) and (iii) are shown through straightforward adaptation of these arguments. ■

Proof of Proposition 5.4: We first prove an intermediate result for any

constrained efficient allocation. Let $z^t \equiv (x^t, y^t)$. For all i , and for all z^t , $\hat{c}_{i,t}(z^t) \equiv \hat{c}_{i,t}(x^t, y^t)$ is the same for all y^t . Notice that in this case, by direct computation, $\hat{\beta}$ is constant, and thus $U(\hat{e}_i)(x^t, y^t)$ does not depend on y^t . Hence, neither the resource constraints, the participation constraints nor the discount factor depend on y^t . Thus, given the strict concavity of the utility function, it would not be efficient to make $\hat{c}_{i,t}(x^t, \cdot)$ depend on y^t .

Now we use the intermediate result to show the proposition. By the first welfare theorem and the representation of Arrow prices it suffices to analyze the marginal rate of substitution: $\beta [\lambda(y_{t+1})]^{-\gamma} \phi(y_{t+1}) \cdot \left(\frac{\hat{c}_{i,t+1}(x^{t+1}, y^{t+1})}{\hat{c}_{i,t}(x^t, y^t)} \right)^{-\gamma} \psi(x_{t+1}|x_t)$, which can be written as the product of two functions. The first one depends on y_{t+1} , which is equal to the marginal rate of substitution for the economy without participation constraints, $\beta [\lambda(y_{t+1})]^{-\gamma} \phi(y_{t+1})$. The second one, $\left(\frac{\hat{c}_{i,t+1}(x^{t+1}, y^{t+1})}{\hat{c}_{i,t}(x^t, y^t)} \right)^{-\gamma} \psi(x_{t+1}|x_t)$, by the intermediate result, does not depend on y^{t+1} , and hence cancels out in the formula for the strip premium. ■

Proof of Proposition 5.5: The allocation is clearly resource feasible and satisfies the participation constraints. It suffices to show that $\{c_i^*, a_i^*\}$ solve (3.2) given $\{q^*\}$ and $\{B_i\}$ and $a_{i,0}^*$ for $i = I + 1$. It is immediate to verify that given (5.2) the Euler equations (3.5) are satisfied for all t and that since $a_{i,t+1}(z^{t+1}) = B_{i,t+1}(z^{t+1})$ for all z^{t+1} the Transversality condition (3.6) is satisfied. ■

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Footnotes

2. Earlier work in the sovereign debt literature by Eaton and Gersovitz (1981) first formalized the main idea of this approach.
3. A sufficient condition for $U(c)$ to be finite for any c that satisfy the participation constraints is that $\hat{\beta}(z) < 1$ for all z .
4. There is a related literature on how to model portfolio constraints in infinite horizon incomplete markets models such as Levine and Zame (1996), Hernandez and Santos (1994), Santos and Woodford (1997) and Magill and Quinzii (1994). None of these paper focuses on the link between portfolio constraints and the risk of default.
5. In some related work Zhang (1997) computes numerical examples of an equilibrium subject to a borrowing constraint chosen so that agents are better off than in autarchy. Our model differs from Zhang's in that he exogenously limits agents to trade only one-period riskless bonds and that his borrowing constraint is neither time nor state dependent.
6. These are the properties of the kernels found in empirical studies, such as Hansen and Cochrane (1992).
7. We thank George Constantinides for hihglighting this point.

8. See Kehoe and Levine (1993) for an example with a constant aggregate endowment and periodic individual endowments where interest rates are strictly lower than in the corresponding economy without participation constraints.