Rare Booms and Disasters in a Multisector Endowment Economy

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Why do value stocks have higher average returns than growth stocks, despite having lower risk? Why do these stocks exhibit positive abnormal performance, while growth stocks exhibit negative abnormal performance? This paper offers a rare-event-based explanation that can also account for the high equity premium and volatility of the aggregate market. The model explains other puzzling aspects of the data, such as joint patterns in time-series predictablity of aggregate market and value and growth returns, long periods in which growth outperforms value, and the association between positive skewness and low realized returns. (*JEL* G12)

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Among the myriad of facts that characterize the cross-section of stock returns, the value premium stands out both for its empirical robustness and for the problem it poses for theory. The value premium is the finding that stocks with high book-to-market ratios (value) have higher expected returns than stocks with low book-to-market ratios (growth). By itself, this finding would not constitute a puzzle, for it could be that value firms are more risky. Such firms would then have high expected returns in equilibrium, which would simultaneously explain both their high realized returns and their low valuations. The problem with this otherwise appealing explanation is that value stocks are not riskier according to conventional measures. Over the postwar period, which is long enough to measure second moments, value stocks have lower covariance with the market, and lower standard deviations. And while one could argue that neither definition of risk is appropriate in a complex world, the challenge still remains to find a measure of risk that does not, in equilibrium, essentially amount to covariance or standard deviation. Over a decade of theoretical

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research on the value premium demonstrates that this is a significant challenge indeed.

This paper proposes an explanation of the value premium that is not risk based but rather is based on rare events. We introduce a representative-agent asset pricing model in which the endowment is subject to positive and negative events that are much larger than what would be expected under a normal distribution. One of our theoretical contributions is to show an asymmetry in how disasters and booms affect average returns. The possibility of a disaster raises risk premiums. While realized returns are lower in samples with disasters than in those without, these two types of samples are similar in that, in both of them, an econometrician would calculate a positive disaster premium. The possibility of a boom also raises risk premiums, because it also is a source of risk. Samples with and without booms look different, however. The econometrician would calculate a positive boom premium in the first type of sample but a negative boom premium in the second. We use this simple theoretical observation to account for the value premium. In our model, the growth sector consists of stocks that capture the benefits of a large consumption boom. We show that a a value premium will be observed if booms were expected but did not occur.

What is the source of the asymmetry between disasters and booms? Why, in other words, is the measured premium for bearing boom risk positive in population but negative in samples without rare events? Consider first the case of a risk-neutral investor, and assume (reasonably) that asset prices rise in booms and fall in disasters. To hold an asset exposed to disasters, the risk-neutral investor must be compensated by higher realized returns in the event a disaster does not occur. Likewise, when holding an asset exposed to booms, he is willing to tolerate lower returns in the event the boom does not occur. If returns were also higher when booms did not occur, no-arbitrage would be violated.

Now consider the more realistic case of a risk-averse investor. This investor requires a premium to hold assets exposed to disasters. We would thus expect such assets to have higher returns, even in samples that contain the "correct" number of disasters. In samples that, ex post, have no disasters, we would expect these assets to have yet higher returns because of the no-arbitrage effect discussed in the previous paragraph. As a result, the econometrician would measure a positive disaster premium in both types of samples: the true premium is positive, and the *observed* premium in no-disaster samples is also positive.

As in the case of disasters, the risk-averse investor requires a positive risk premium to hold assets exposed to booms. However, by no-arbitrage, these assets must have lower returns in samples in which booms do not occur. How can these two statements be reconciled? It must be that *the higher returns due to the risk premium come about when the boom itself is realized*. Samples with and without booms look qualitatively different: assets exposed to booms have higher true expected returns, but, on average, lower realized returns when booms do not occur.

This reasoning explains why one should expect to observe a negative premium for boom risk in samples without rare events. However, it says nothing about the magnitude of the effect. To obtain quantitatively relevant results, a second mechanism is important. In our model, we make the standard assumption of constant relative risk aversion (CRRA), which leads to stationary rates of return. This standard assumption also implies that boom risk has a lower price than disaster risk. Because any risk premium for booms works against our main mechanism, this second source of asymmetry between disasters and booms combines with the first to produce an economically significant value premium.

Specifically, in our calibrated model, the average excess market return is 5%. However, the average return on growth stocks is only 3%, and the average return on value stocks is 6%. Moreover, our model also explains why value stocks have strong abnormal performance, and growth stocks poor abnormal performance relative to the CAPM. The relative abnormal performance of value stocks implied by our model is 5%, as it is in the data. Indeed, our model naturally explains aspects of the data on value and growth that have posed a challenge to previous general equilibrium models. Namely,

1. growth stocks have higher variance and covariance with the market despite having lower observed returns.
2. growth stocks have yet higher covariances with the market, and greater returns than value stocks, during periods of high market valuations (for example, the late 1990s).
3. the value-minus-growth return, unlike the market excess return, cannot be predicted by the price-dividend ratio. It can however be predicted by the value spread.

Moreover, while a full explanation of skewness puzzles is outside the scope of this study, our model does imply that high valuation stocks have high skewness and low expected returns, as in the data. Our model also implies that assets with high upside betas and low downside betas also have low excess returns, as in the data.

In explaining these facts, we tie our hands by assuming that value and growth cash flows have the same exposure to disaster risk. While differential exposure is plausible and in the spirit of the model, we assume it away to focus on our main mechanism. Moreover, we assume that booms affect consumption as well as dividends; this implies boom risk is priced, and this works against us in finding a value premium.

Finally, because of the presence of disasters, the model explains a high equity premium and equity volatility, along with low observed volatility of consumption growth. The model achieves this with a risk aversion coefficient of three. Low risk aversion helps in explaining the value premium puzzle in our setting; if risk aversion were too high, growth would carry a higher premium in population, and we would not be able to match low observed returns over the sample. The model generates realistic stock return volatility through the mechanism of time-varying disaster and boom risk, combined with recursive preferences. Without this mechanism, equity claims would have counterfactually low volatility during normal times.

In our focus on the underlying dynamics separating value and growth, our model follows a substantial literature that explicitly models the cash-flow dynamics of firms, or sectors, and how these relate to risk premiums in the cross-section (Ai and Kiku 2013; Ai, Croce, and Li 2013; Berk, Green, and Naik 1999; Carlson, Fisher, and Giammarino 2004; Gârleanu, Kogan, and Panageas 2012; Gomes, Kogan, and Zhang 2003; Kogan, Papanikolaou, and Stoffman 2013; Novy-Marx 2010; Zhang 2005). These papers show how endogenous investment dynamics can lead to a value premium. Ultimately, however, the value premium arises in these models because of greater risk. Thus these models do not explain the observed pattern in variances and covariances. A second branch of the literature relates cash-flow dynamics of portfolios, as opposed to underlying firms, to risk premiums (Bansal, Dittmar, and Lundblad 2005; Da 2009; Hansen, Heaton, and Li 2008; Kiku 2006). This literature finds that dividends on the value portfolio are more correlated with a long-run component of consumption than dividends and returns on the growth portfolio. In the context of a model in which risk to this long-run component is priced (Bansal and Yaron 2004) this covariance leads to a higher premium for value. However, for this long-run component to be an important source of risk in equilibrium, it also must be present in the market portfolio, and it must be an important source of variation in these returns themselves. Again, this would seem to imply, counterfactually, that the covariance with the market return and volatility of returns would be greater for value than for growth. Moreover, if the long-run component of consumption growth is an important source of risk in the market portfolio, consumption growth should be forecastable by stock prices; however, it is not (Beeler and Campbell 2012).

To capture the disconnect between risk and return in the cross-section, shocks associated with growth stocks should have a low price of risk. As shown by Lettau and Wachter (2007), Santos and Veronesi (2010), and Binsbergen, Brandt, and Koijen (2012), achieving this pricing poses a challenge for general equilibrium models.[1] Kogan and Papanikolaou (2013) endogenously generate a cross-section of firms through differences in investment opportunities, but, like Lettau and Wachter, they assume an exogenous stochastic discount factor. Papanikolaou (2011) presents an equilibrium model in which investment shocks have a negative price of risk. This is achieved by assuming that the representative agent has a preference for late resolution of uncertainty. While this assumption allows the model to explain the cross-section, it implies an

[1] Campbell and Vuolteenaho (2004) and Lettau and Wachter (2007) consider the role of duration in generating a value premium when discount rate shocks carry a zero or negative price. McQuade (2013) shows that stochastic volatility in production can generate a value premium, depending on how the risk of volatility is priced. These models are in partial equilibrium.

equity premium that is counterfactually low. In our model, growth stocks are exposed to a source of risk that, endogenously, has a low price. Nonetheless, our model has a reasonable equity premium.

Our model features rare disasters, as do the models of Rietz (1988), Longstaff and Piazzesi (2004), Veronesi (2004), and Barro (2006). Time variation in disaster risk is the primary driver of stock market volatility, as in the models of Gabaix (2012), Gourio (2012), and Wachter (2013). Jorion and Goetzmann (1999) argue that focusing on the United States, which has not had disasters, leads to an estimate of the equity premium that is too high. These papers do not study rare booms. Our rare booms are similar to the technological innovations modeled by Pástor and Veronesi (2009), and Jovanovic and Rousseau (2003).[2] Bekaert and Engstrom (2013) also assume a two-sided risk structure, but propose a model of the representative agent motivated by habit formation. These papers do not address the cross-section of stock returns which is the primary focus of our paper.

Thus far, the literature has shown that one-sided rare events, namely, disasters explain the equity premium. We show, however, that the presence of booms has a large affect on the cross-section if some assets are exposed to them and some are not. That is, by introducing booms as well as disasters, one can explain not only the equity premium puzzle but also the value premium puzzle.

1. Model

1.1 Endowment and preferences

We assume an endowment economy with an infinitely lived representative agent. Aggregate consumption (the endowment) follows a diffusion process with time-varying drift:

$$\frac{dC_t}{C_t} = \mu_{Ct}\,dt + \sigma\,dB_{Ct}, \tag{1}$$

where B_{Ct} is a standard Brownian motion. The drift of the consumption process is given by

$$\mu_{Ct} = \bar{\mu}_C + \mu_{1t} + \mu_{2t}, \tag{2}$$

where

$$d\mu_{jt} = -\kappa_{\mu_j}\mu_{jt}\,dt + Z_{jt}\,dN_{jt}, \tag{3}$$

for $j = 1,2$. Rare events are captured by the Poisson variables N_{jt}. Absent rare events, the drift rate of consumption is $\bar{\mu}_C$ and the volatility is σ_c. This model implies that consumption adjusts smoothly as in the data (see Nakamura et al. 2013), but that it also can undergo periods of extreme growth in either direction.

[2] Pástor and Veronesi (2009) show how the transition from idiosyncratic to systematic risk can explain the time series patterns of returns around technological revolutions. In the present paper, the risk of the technology is systematic from the start. Jovanovic and Rousseau (2003) show that technological revolutions can have long-lived effects on market valuations, consistent with our model.

We assume $Z_{1t} < 0$ and $Z_{2t} > 0$. Namely, N_{1t} represents disasters and N_{2t} represents booms. Let ν_1 denote the (time-invariant) disaster distribution and ν_2 the boom distribution. We write E_{ν_j} to denote expectations taken over the distribution ν_j.

We let λ_{jt} denote the intensity of N_{jt}. We will refer to λ_{jt} as the probability of rare event j in what follows; given our calibration, the intensity is a good approximation of the annual probability. We assume λ_{jt} follows the process

$$d\lambda_{jt} = \kappa_{\lambda_j}(\bar{\lambda}_j - \lambda_{jt})dt + \sigma_{\lambda_j}\sqrt{\lambda_{jt}}\,dB_{\lambda_j t}, \quad j = 1,2, \tag{4}$$

where $B_{\lambda_1 t}$, $B_{\lambda_2 t}$, and B_{Ct} are independent Brownian motions. For convenience, we define the vector notation $\lambda_t = [\lambda_{1t},\ \lambda_{2t}]^\top$, $\mu_t = [\mu_{1t}, \mu_{2t}]^\top$, $B_{\lambda t} = [B_{\lambda_1 t}, B_{\lambda_2 t}]^\top$, and $B_t = [B_{Ct}, B_{\lambda t}^\top]^\top$.

We assume the continuous-time analog of the utility function, defined by Epstein and Zin (1989) and Weil (1990), that generalizes power utility to allow for preferences over the timing of the resolution of uncertainty. The continuous-time version is formulated by Duffie and Epstein (1992). We assume that the elasticity of intertemporal substitution (EIS) is equal to one. That is, the utility function V_t for the representative agent is defined using the following recursion:

$$V_t = E_t \int_t^\infty f(C_s, V_s)ds, \tag{5}$$

where

$$f(C_t, V_t) = \beta(1-\gamma)V_t\left(\log C_t - \frac{1}{1-\gamma}\log((1-\gamma)V_t)\right). \tag{6}$$

The parameter γ represents relative risk aversion and β the rate of time preference. The assumption of EIS equal to one implies a closed-form solution up to ordinary differential equations.[3]

1.2 The state-price density

We start by establishing how the various sources of risk are priced in the economy. We use the notation $\mathcal{J}_j(\cdot)$ to denote how a process changes in response to a rare event of type j. For example, for the state-price density π_t, $\mathcal{J}_j(\pi_t) = \pi_t - \pi_{t^-}$ if a type-j jump occurs at time t. In our complete-markets endowment economy, the state-price density represents the marginal utility of the representative agent.

Theorem 1. The state-price density π_t follows the process

$$\frac{d\pi_t}{\pi_{t^-}} = \mu_{\pi t}dt + \sigma_{\pi t}dB_t + \sum_{j=1,2}\frac{\mathcal{J}_j(\pi_t)}{\pi_{t^-}}dN_{jt}, \tag{7}$$

[3] Using log-linearization, Eraker and Shaliastovich (2008) and Benzoni, Collin-Dufresne, and Goldstein (2011) find approximate solutions to related continuous-time jump-diffusion models when the EIS is not equal to one.

where

$$\sigma_{\pi t} = \left[-\gamma\sigma, \;\; b_{\lambda_1}\sigma_{\lambda_1}\sqrt{\lambda_{1t}}, \;\; b_{\lambda_2}\sigma_{\lambda_2}\sqrt{\lambda_{2t}} \right], \tag{8}$$

and

$$\frac{\mathcal{J}_j(\pi_t)}{\pi_t} = e^{b_{\mu_j} Z_{jt}} - 1, \tag{9}$$

$$b_{\mu_j} = \frac{1-\gamma}{\kappa_{\mu_j}+\beta}, \tag{10}$$

$$b_{\lambda_j} = \frac{1}{\sigma^2_{\lambda_j}} \left(\beta + \kappa_{\lambda_j} - \sqrt{\left(\beta+\kappa_{\lambda_j}\right)^2 - 2E_{\nu_j}\left[e^{b_{\mu_j} Z_{jt}} - 1 \right] \sigma^2_{\lambda_j}} \right), \tag{11}$$

for $j = 1,2$. Moreover, for $\gamma > 1$, $b_{\lambda_1} > 0$, $b_{\lambda_2} < 0$, and $b_{\mu_j} < 0$ for $j = 1,2$.

Proof. See Appendix B.2. ■

Equations (9) and (10) imply that marginal utility jumps up in a disaster and down in a boom, with the upward jump larger than the downward jump for the same size shock Z_{jt}. Equation (8) shows that changes in the rare event probabilities also affect marginal utility: marginal utility rises when the probability of a disaster rises, and falls when the probability of a boom rises. All else equal, marginal utility rises more in the case of a disaster than it falls in the case of a boom.

Because the EIS is equal to one, and because only expected consumption (not realized consumption) is subject to jumps, the risk-free rate in this economy is standard.

Corollary 2. Let r_t denote the instantaneous risk-free rate in this economy. Then

$$r_t = \beta + \mu_{Ct} - \gamma\sigma^2. \tag{12}$$

Proof. See Appendix B.2 ■

Looking ahead, Table 1 summarizes the effects of state variables on prices and returns. We derive these effects in the sections that follow.

1.3 The aggregate market

Here, we derive results for the price-dividend ratio and the equity premium on the aggregate market. Unless otherwise stated, proofs can be found in Appendix B.4.

Table 1
The effects of shocks on prices and returns

	High λ_1	High λ_2	Low μ_1	High μ_2
Risk-free rate	0	0	−	+
Market price-dividend ratio	−	+	−	+
Value price-dividend ratio	−	−	−	−
Population equity premium	+	+	0	0
Population risk premium for value	+	−	0	0
Average excess market return in samples without rare events	+	−	0	0
Average excess value return in samples without rare events	+	+	0	0

Signs of the effect of shocks to each state variable on the risk-free rate, price dividend ratios, and risk premiums. "High λ_1" refers to an increase in the disaster probability. "High λ_2" refers to an increase in the boom probability. "Low μ_1" refers to a decrease in the component of expected consumption growth due to disasters. "High μ_2" refers to an increase in the component of expected consumption growth due to booms. The true and observed risk premiums are all relative to the risk-free rate. The observed premiums refer to the expected excess return that would be observed in a sample without rare events.

Let D_t denote the dividend on the aggregate market. Assume that dividends follow the process

$$\frac{dD_t}{D_t} = \mu_{Dt}\,dt + \phi\sigma\,dB_{Ct}, \tag{13}$$

where

$$\mu_{Dt} = \bar{\mu}_D + \phi\mu_{1t} + \phi\mu_{2t}.$$

This structure allows dividends to respond by a greater amount than consumption to booms and disasters. For simplicity, we assume that the same parameter ϕ governs the dividend response to normal shocks, booms and disasters. This ϕ is analogous to leverage as in Abel (1999), and we will refer to it as leverage in what follows.

1.3.1 Valuation. Our first result gives the formula for the price of the aggregate market. By no-arbitrage,

$$F(D_t, \mu_t, \lambda_t) = E_t \int_t^\infty \frac{\pi_s}{\pi_t} D_s\,ds,$$

where π_s is the state-price density. Valuing the market amounts to calculating the expectation on the right-hand side.

Theorem 3. Let $F(D_t, \mu_t, \lambda_t)$ denote the value of the market portfolio. Then

$$F(D_t, \mu_t, \lambda_t) = \int_0^\infty H(D_t, \mu_t, \lambda_t, \tau)\,d\tau, \tag{14}$$

where

$$H(D_t, \mu_t, \lambda_t, \tau) = D_t \exp\left\{a_\phi(\tau) + b_{\phi\mu}(\tau)^\top \mu_t + b_{\phi\lambda}(\tau)^\top \lambda_t\right\}, \tag{15}$$

$$b_{\phi\mu_j}(\tau) = \frac{\phi - 1}{\kappa_{\mu_j}}\left(1 - e^{-\kappa_{\mu_j}\tau}\right), \qquad j = 1, 2 \tag{16}$$

and the remaining terms satisfy

$$\frac{db_{\phi\lambda_j}}{d\tau} = \frac{1}{2}\sigma_{\lambda_j}^2 b_{\phi\lambda_j}(\tau)^2 + \left(b_{\lambda_j}\sigma_{\lambda_j}^2 - \kappa_{\lambda_j}\right) b_{\phi\lambda_j}(\tau) + E_{\nu_j}\left[e^{b_{\mu_j} Z_{jt}}\left(e^{b_{\phi\mu_j}(\tau) Z_{jt}} - 1\right)\right], \tag{17}$$

$$\frac{da_\phi}{d\tau} = \bar{\mu}_D - \bar{\mu}_C - \beta + \gamma\sigma^2(1-\phi) + b_{\phi\lambda}(\tau)^\top \left(\kappa_\lambda * \bar{\lambda}\right), \tag{18}$$

with boundary conditions $b_{\phi\lambda_j}(0) = a_\phi(0) = 0$. Furthermore, the price-dividend ratio on the market portfolio is given by

$$G(\mu_t, \lambda_t) = \int_0^\infty \exp\left\{a_\phi(\tau) + b_{\phi\mu}(\tau)^\top \mu_t + b_{\phi_\lambda}(\tau)^\top \lambda_t\right\} d\tau. \tag{19}$$

Here, and in what follows, $b_{\phi\mu}(\tau) = [b_{\phi\mu_1}(\tau), b_{\phi\mu_2}(\tau)]^\top$, $b_{\phi\lambda}(\tau) = [b_{\phi\lambda_1}(\tau), b_{\phi\lambda_2}(\tau)]^\top$, and $*$ denotes element-by-element multiplication.

Equation 14 expresses the value of the aggregate market as an integral of prices of zero-coupon equity claims. H gives the values of these claims as functions of the disaster and boom terms μ_{1t}, μ_{2t}, the probabilities of a disaster and boom $\lambda_{1t}, \lambda_{2t}$ and the time τ until the dividend is paid.

These individual dividend prices (and, by extension, the price of the market as a whole) have interpretations based on the primitive parameters. As (16) shows, prices are increasing in μ_{1t} and μ_{2t}. There is a tradeoff between the effect of expected consumption growth on future cash flows and on the risk-free rate. Because leverage ϕ is greater than the EIS (namely, 1), the cash-flow effect dominates and the valuation of the market falls during disasters and rises during booms. Moreover, the more persistent is the process (the lower is κ_{μ_j}), the greater is the effect of a change in μ_{jt} on prices.[4]

The probability of rare events also affect prices, but the intuition is more subtle. The functions $b_{\phi\lambda_1}(\tau)$ and $b_{\phi\lambda_2}(\tau)$ would be identically zero without the last term in the ODE (17). It is this term that determines the sign of $b_{\phi\lambda_j}(\tau)$, and thus how prices respond to changes in probabilities. We can decompose this last term as follows:

$$E_{\nu_j}\left[e^{b_{\mu_j} Z_{jt}}\left(e^{b_{\phi\mu_j}(\tau) Z_{jt}} - 1\right)\right] = \\ -\underbrace{E_{\nu_j}\left[\left(e^{b_{\mu_j} Z_{jt}} - 1\right)\left(1 - e^{b_{\phi\mu_j}(\tau) Z_{jt}}\right)\right]}_{\text{Risk premium effect}} + \underbrace{E_{\nu_j}\left[e^{b_{\phi\mu_j}(\tau) Z_{jt}} - 1\right]}_{\text{Cash-flow and risk-free rate effect}}. \tag{20}$$

The first term in (20) is one component of the equity premium, namely, the static rare-event premium (we discuss this terminology in the next section).[5]

[4] The derivative of (16) with respect to κ_{μ_j} is proportional to $(\kappa_{\mu_j}\tau + 1)e^{-\kappa_{\mu_j}\tau} - 1$ which is negative, because $e^{\kappa_{\mu_j}\tau} > \kappa_{\mu_j}\tau + 1$.

[5] More precisely, this is the static rare-event premium for zero-coupon equity with maturity τ.

Because an increase in the discount rate lowers the price-dividend ratio, this risk premium appears with a negative sign. The second term is the expected price response if the rare event occurs, representing the combined effect of changes in expected future cash flows and risk-free rates. Thus, the response of equity values to a change in the rare-event probability is determined by a risk premium effect, and a (joint) cash-flow and risk-free rate effect.

These effects have different implications depending on whether the rare event is a disaster or boom. First, consider disasters ($j=1$). When the risk of a disaster increases, the equity premium increases (the first term in (20) is negative). Expectations of future cash flows and risk-free rates decrease, with the cash-flow effect dominating (the second term in (20) is also negative). Thus, an increase in the disaster probability lowers valuations. Now consider booms ($j=2$). When the probability of a boom increases, the equity premium increases (the first term in (20) is again negative). Expectations of future cash flows and risk-free rates increase with the cash-flow effect dominating (the second term in (20) is positive). An increase in the probability of a boom increases the price because the cash-flow/risk-free rate effect outweighs the risk premium effect. The next corollary summarizes these results.

Corollary 4. The price-dividend ratio $G(\mu_t, \lambda_t)$ is increasing in μ_{jt} (for $j=1,2$), decreasing in λ_{1t}, and increasing in λ_{2t}.

Proof. The result for μ_{jt} follows immediately from the form of the function. The result for λ_{jt} follows from Corollary B.10. ■

The left panel of Figure 1 shows $b_{\phi\mu_j}(\tau)$ and $b_{\phi\lambda_j}(\tau)$ for the calibration discussed below. We see that $b_{\phi\mu_j}(\tau)$ is positive and increasing for $j=1,2$, and quickly converges. This reflects the fact that disasters and booms are short lived in our calibration. We also see that $b_{\phi\lambda_1}(\tau)$ is negative, and $b_{\phi\lambda_2}(\tau)$ is positive. Both take a longer time to converge because rare-event probabilities are more persistent than the rare events themselves. Also interesting is the fact that $b_{\phi\lambda_2}(\tau)$ is so much smaller than $b_{\phi\lambda_1}(\tau)$.[6] This occurs because the cash-flow/risk-free rate effect and the risk premium effect operate in the same direction for disasters but in opposite directions for booms.

1.3.2 The equity premium. We now turn to the equity premium. For our quantitative results, we will average excess returns in a simulation, where returns are calculated over a finite time interval that matches the data. However, we can gain intuition by examining instantaneous returns.

By Ito's lemma, we can write the price process for the aggregate market as

$$\frac{dF_t}{F_{t^-}} = \mu_{Ft}\,dt + \sigma_{Ft}\,dB_t + \sum_{j=1,2} \frac{\mathcal{J}_j(F_t)}{F_{t^-}}\,dN_{jt},$$

6 Booms are smaller than disasters in our calibration; however, this result holds even when they are the same size.

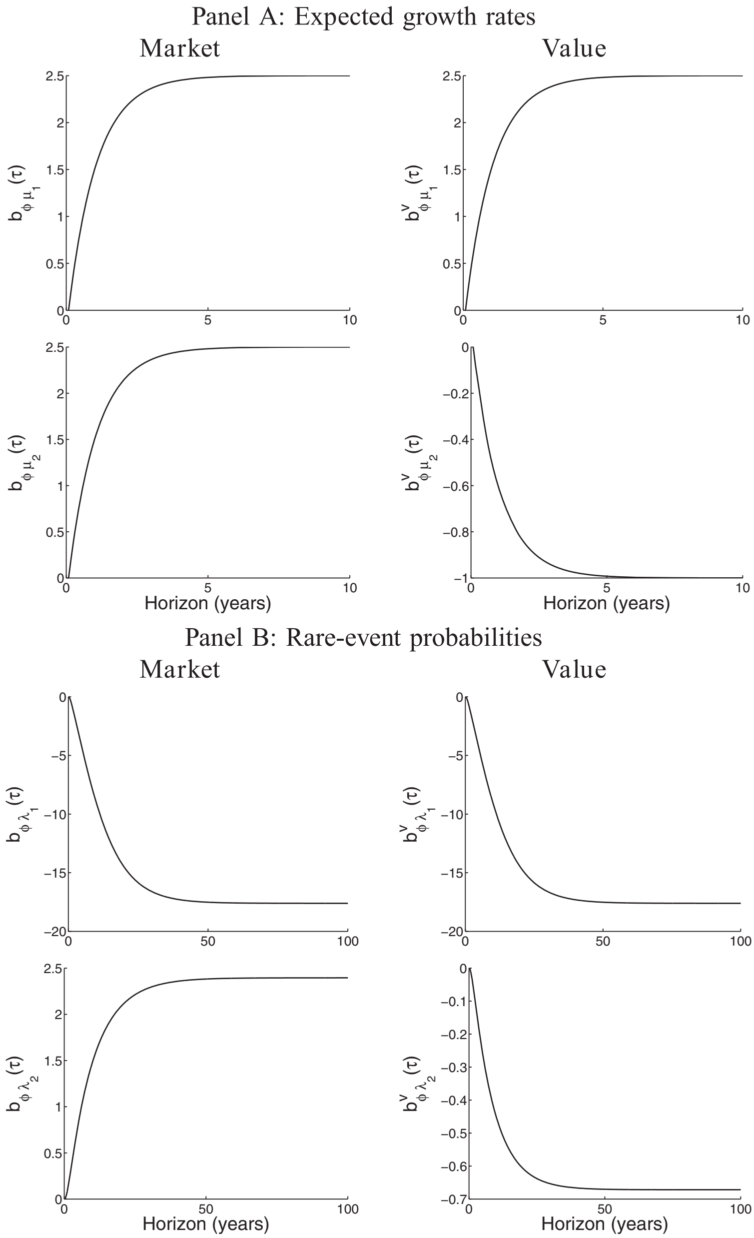

Figure 1
Solution for the price-dividend ratio
Panel A shows the coefficients multiplying μ_1 and μ_2 (the disaster and boom components in expected consumption growth, respectively) in the price-dividend ratio. Panel B shows the coefficients multiplying λ_{1t} (the probability of a disaster) and λ_{2t} (the probability of a boom). The left panel shows results for the market; the right shows results for the value premium. The scales on the left and the right may differ.

for some drift term μ_{Ft}, a (row) vector of diffusion terms σ_{Ft}, and terms $\frac{\mathcal{J}_j(F_t)}{F_{t^-}}$ that denote percent change in price due to the rare event. Note that $\frac{\mathcal{J}_j(F_t)}{F_{t^-}}=\frac{\mathcal{J}_j(G_t)}{G_{t^-}}$, because dividends themselves do not jump in this model; only the price-dividend ratio does.

The instantaneous expected return is defined as the expected percent price appreciation, plus the dividend yield. In the notation above,

$$r_t^m = \underbrace{\mu_{Ft} + \frac{1}{F_t}\sum_{j=1,2}\lambda_{jt}E_{\nu_j}[\mathcal{J}_t(F_t)] +}_{\text{expected price appreciation}} \underbrace{\frac{D_t}{F_t}}_{\text{dividend yield}}. \tag{21}$$

Using this characterization of returns, we can calculate the equity premium.

Theorem 5. The the instantaneous equity premium relative to the risk-free rate is

$$r_t^m - r_t = \phi\gamma\sigma^2 \underbrace{-\sum_{j=1,2}\lambda_{jt}E_{\nu_j}\left[\left(e^{b_{\mu_j}Z_{jt}}-1\right)\frac{\mathcal{J}_j(G_t)}{G_t}\right]}_{\text{static rare-event premium}} \underbrace{-\sum_{j=1,2}\lambda_{jt}\frac{1}{G_t}\frac{\partial G}{\partial \lambda_j}b_{\lambda_j}\sigma_{\lambda_j}^2}_{\lambda\text{-premium}}. \tag{22}$$

Theorem 5 divides the equity premium into three components: The first is the standard term arising from the consumption CAPM (Breeden 1979). The second component is the sum of the premiums directly attributable to disasters ($j=1$) and to booms ($j=2$). These are covariances between state prices and market returns during rare events, multiplied by probabilities that the rare events occur.[7] We call this second term the static rare-event premium because it is there regardless of whether the probabilities of rare events are constant or time varying.[8]

The third component in (22) represents the compensation the investor requires for bearing the risk of changes in the rare-event probabilities. Accordingly, we call this the λ-premium. This term can also be divided into the compensation for time-varying disaster probability (the λ_1-premium) and compensation for time-varying boom probability (the λ_2-premium). The following corollary shows that all terms in (22) are positive. A closely related result is that the increases in both the disaster and the boom probabilities increase the equity premium, as indicated in Table 1 and discussed in what follows.[9]

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[7] These terms take the form of uncentered second moments, but they are indeed covariances; this is because the jump occurs instantaneously and so the expected change in the variable is negligible.

[8] Note, however, that if we assumed time-additive utility, this static premium would also disappear because it arises from shocks to the consumption distribution, not to consumption itself.

[9] While Table 1 shows that there is no effect of μ_{1t} and μ_{2t} on risk premiums, there is in fact a second-order effect that arises from changes in duration of the claims. This size of this effect is negligible in our calibration.

Corollary 6. 1. The static disaster and boom premiums are positive.

2. The premiums for time-varying disaster and boom probabilities (the λ_j-premiums) are also positive.

Proof. For the first result, recall that $b_{\mu_j} < 0$ for $j = 1, 2$ (Theorem 1). First, consider disasters ($j = 1$). Note $Z_{1t} < 0$, so $e^{b_{\mu_1} Z_{1t}} - 1 > 0$. Furthermore, because G_t is increasing in μ_{1t} (Corollary 4), $\mathcal{J}_1(G_t) < 0$. It follows that the static disaster premium is positive. Now consider booms ($j = 2$). Because $Z_{2t} > 0$, $e^{b_{\mu_2} Z_{2t}} - 1 < 0$. Because G_t is increasing in μ_{2t}, $\mathcal{J}_2(G_t) > 0$. Therefore, the static boom premium is also positive.

For the second result, first consider disasters ($j = 1$). Recall that $b_{\lambda_1} > 0$ (Theorem 1). Further, $\partial G / \partial \lambda_1 < 0$ (Corollary 4). For booms ($j = 2$), each of these quantities takes the opposite sign. The result follows. ■

The static premiums are positive because marginal utilities and valuations move in opposite directions during rare events: during disasters, marginal utility is high, but valuations are low, and during booms, the opposite is true. Thus, disasters and booms have a direct positive impact on the equity premium.

Exposure to disasters and booms also increases the equity premium indirectly through the dynamic effect of time-varying probabilities. An increase in disaster risk raises marginal utility and lowers valuations, likewise an increase in boom risk lowers marginal utility and raises market valuations. Thus, exposure to time-varying probabilities of rare events further increases the equity premium.

Figure 2 (top left panel) shows these terms as a function of the disaster probability for the calibration discussed later in the paper. The dotted line that is essentially at zero shows the CCAPM. The dash-dotted line shows the static disaster premium; lying above it is the full static premium, including the premium due to booms. Finally, the solid line is the full equity premium, which includes the λ-premium. Whereas the λ-premium due to disasters is substantial, the λ-premium due to booms is extremely small. We discuss this result further in the next section.[10]

1.3.3 Observed returns in samples without rare events. We now consider the average return the econometrician would observe in an sample without rare events. To distinguish these average returns from true population returns, we use the subscript nj (no jump).[11] This average return is simply given by the

10 This figure also shows that the static boom premium is small. This is not a general result; it arises in our calibration because booms are smaller than disasters. While booms have a smaller effect on marginal utility, and thus on state prices, they have a larger effect on asset prices because of Jensen's inequality. If the leverage parameter ϕ and risk aversion are equal, and booms and disasters are symmetric, then the static premiums would be of the same size. On the other hand, the λ-premium due to booms is smaller than for disasters, even under these conditions.

11 This "ideal" average return is what one would obtain by averaging over an infinite number of samples that do not contain rare events.

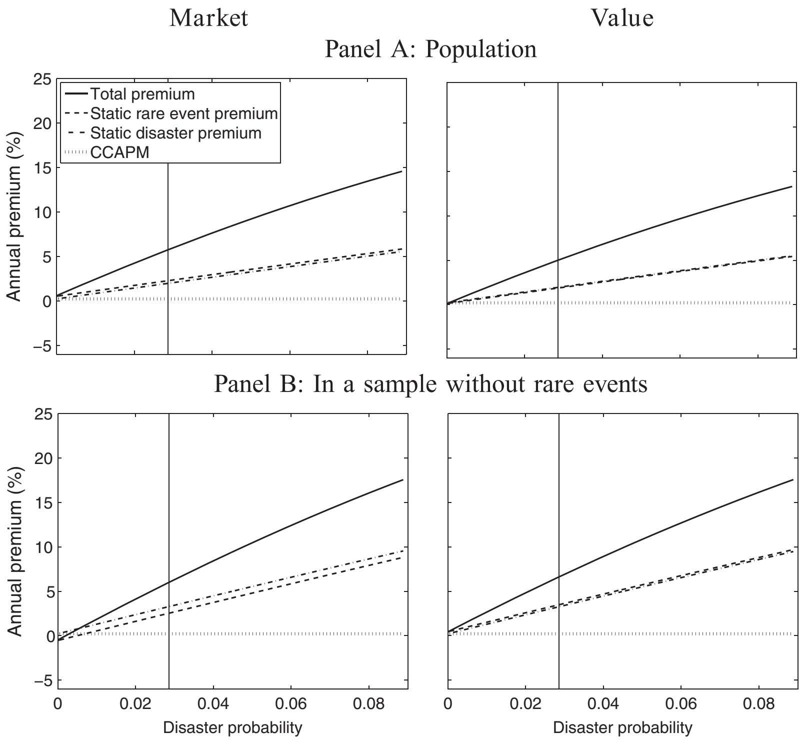

Figure 2
Risk premiums as a function of disaster probability
Panel A shows population risk premiums as functions of the disaster probability with the boom probability fixed at its mean. The vertical line represents the mean of the disaster probability. Panel B shows what the average realized excess return would be in a sample without rare events.

drift rate in the price, plus the dividend yield:

$$r^m_{\text{nj},t} = \mu_{Ft} + \frac{D_t}{F_t}. \tag{23}$$

The expression for these average realized returns follows from Theorem 5.

Corollary 7. The average excess market return in a sample without rare events is given by

$$r^m_{\text{nj},t} - r_t = \phi\gamma\sigma^2 \underbrace{-\sum_{j=1,2}\lambda_{jt} E_{\nu_j}\left[e^{b_{\mu_j} Z_{jt}} \frac{\mathcal{J}_j(G_t)}{G_t}\right]}_{\text{observed static rare-event premium}} \underbrace{-\sum_{j=1,2}\lambda_{jt}\frac{1}{G_t}\frac{\partial G}{\partial \lambda_j} b_{\lambda_j}\sigma^2_{\lambda j}}_{\lambda\text{-premium}}. \tag{24}$$

As in the true risk premium, there are components of the observed premium attributable to disasters ($j=1$) and to booms ($j=2$). The premium for time-varying λ risk takes the same form for both the observed and true premium cases, because rare-event risk varies whether or not a rare event occurs. It is the static premium, or the premium due to the rare event itself, that differs. It follows immediately from (24) (and indeed, it can be inferred from the definition (23)) that the observed static premium for disasters is higher than the true static premium, whereas the observed static premium for booms is lower. In fact, for booms, the observed static premium is negative.

Corollary 8. The observed static disaster premium in a sample without jumps is positive. The observed static boom premium in a sample without jumps is negative.

Proof. The result follows from (24), from $e^{b_{\mu_j} Z_{jt}} > 0$, and from $\mathcal{J}_1(G_t) < 0$ and $\mathcal{J}_2(G_t) > 0$ (because prices are increasing in μ_{jt}). ■

We now return to a question raised in the Introduction: why does the average excess return associated with booms switch signs depending on whether booms are present in the sample? Consider first the samples without booms. The intuition in the introduction was based on no-arbitrage. This intuition is reflected in the very simple proof of Corollary 8. First, the relevant component of state prices $e^{b_{\mu_j} Z_{jt}}$ is positive, regardless of parameter values (this reflects the absence of arbitrage in the model). Second, during booms, asset prices rise. The observed (static) boom premium is equal to the negative of the percent change in asset prices multiplied by the relevant component of the state price. In other words, the observed premium due to booms must be negative to compensate for the positive returns when booms are realized; otherwise no-arbitrage would be violated. This effect is mitigated by risk aversion. The greater is γ, the closer to zero is the observed premium.[12] Of course, as shown in Corollary 6, the true premium due to booms arises from the covariance of state prices with asset prices and must be positive. This risk premium is realized by the investor when the boom actually occurs.

We can see the difference between booms and disasters by contrasting the left panels in Figure 2 with those of Figure 3. Figure 2 shows risk premiums as a function of disaster probability; Figure 3 shows risk premiums as a function of boom probability, and hence better highlights the role of booms. In Figure 2 there is very little difference between true risk premiums and observed risk premiums in samples without rare events. In Figure 3, true and observed risk premiums are qualitatively different. The true boom premium is positive and increasing in the boom probability, whereas the observed boom premium is negative and decreasing in the probability.

12 More precisely, what matters is $\gamma - 1$, or more generally, the difference between γ and the inverse of the EIS. The reason is that the rare events change the consumption distribution rather than consumption itself. The relevant notion of risk neutrality is thus time-additive utility, in which the agent is indifferent over the timing of the resolution of uncertainty.

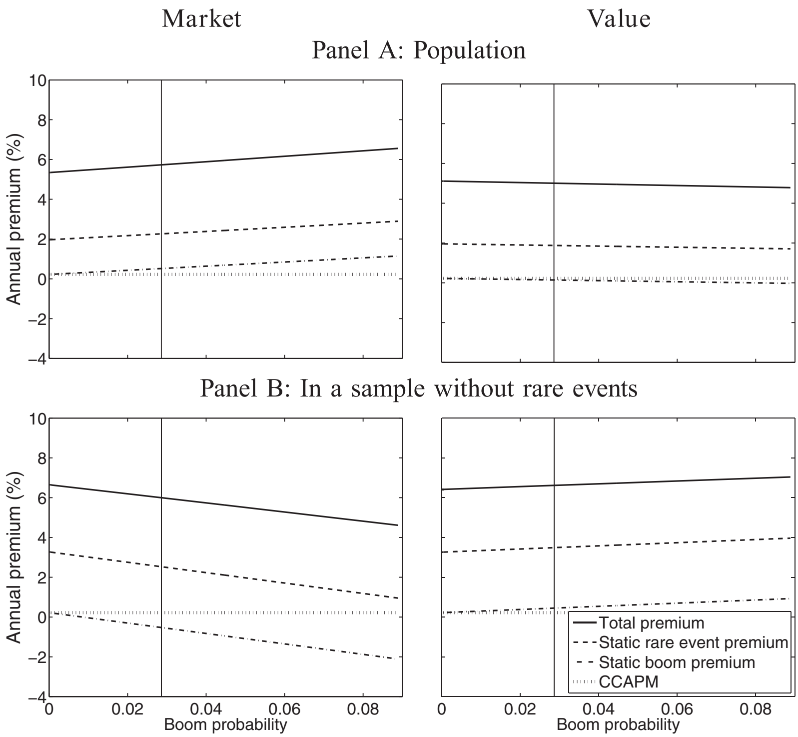

Figure 3
Risk premiums as a function of boom probability
Panel A shows population risk premiums as functions of the boom probability with the disaster probability fixed at its mean. The vertical line represents the mean of the boom probability. Panel B shows what the average realized excess return would be in a sample without rare events.

Before leaving the section on risk premiums, we note the importance of asymmetry in the price of boom versus disaster risk. The discussion above pertains to just the static part of the premium, not the λ-premium. If the λ-premium for booms were large enough it could reduce, or even override our results on the observed static premium. However, the λ-premium for booms, unlike that for disasters, is negligible. There are two reasons for this. One is that the price of risk for λ_{2t} is small in magnitude compared to the price of risk for λ_{1t}, that is, $b_{\lambda_1} > -b_{\lambda_2}$. The other is that changes in the probability of booms have a smaller effect on prices, as explained in the discussion following Theorem 3.

To summarize, this section shows that, although the true premiums for both disaster and boom risk are positive, the observed premium for disaster risk is positive whereas the observed premium for boom risk is negative. These results are directly relevant for the cross-section because, as we will see, value and growth claims differ based on their exposure to these risks.

1.4 Growth and value sectors

We now turn to the pricing of assets that differ in their exposure to the sources of uncertainty in the economy. See Appendix B.5 for proofs not given below.

1.4.1 The value sector. It makes intuitive sense that firms and industries will differ in their ability to directly profit from technological progress. For simplicity, we define a sector that does not directly benefit from a boom, but is otherwise identical to the market. A second sector, one that captures the benefits of the boom, is simply defined as what remains in the market portfolio after we subtract the first sector.

Consider an asset with cash flows following the process

$$\frac{dD_t^v}{D_t^v} = \mu_{Dt}^v dt + \phi\sigma dB_{Ct}, \tag{25}$$

where $\mu_{Dt}^v = \bar{\mu}_D + \phi\mu_{1t}$. We use the superscript v to denote "value." As we will show, this asset will have a lower ratio of price to fundamentals than the market as a whole. This is the defining characteristic of value in the data.[13]

Corollary 9. The price-dividend ratio for value is below that of the market.

Dividend growth for the market is weakly greater than dividend growth for value in every state of the world, and strictly greater in some states of the world. It follows that the price-dividend ratio, which is the present discounted value of future dividends scaled by current dividends, is lower for value than for the market.

Pricing for the value claim is directly analogous to that of the market (Theorem 3). The price of the claim to the dividend stream (25) is given by

$$F^v\left(D_t^v, \mu_t, \lambda_t\right) = \int_0^\infty H^v\left(D_t^v, \mu_t, \lambda_t, \tau\right) d\tau, \tag{26}$$

where

$$H^v\left(D_t^v, \mu_t, \lambda_t, \tau\right) = D_t^v \exp\left\{a_\phi^v(\tau) + b_{\phi\mu}^v(\tau)^\top \mu_t + b_{\phi\lambda}^v(\tau)^\top \lambda_t\right\}. \tag{27}$$

The price-dividend ratio for the value claim is therefore

$$G^v(\mu_t, \lambda_t) = \int_0^\infty \exp\left\{a_\phi^v(\tau) + b_{\phi\mu}^v(\tau)^\top \mu_t + b_{\phi_\lambda}^v(\tau)^\top \lambda_t\right\} d\tau, \tag{28}$$

with $a_\phi^v(\tau)$, $b_{\phi\mu}^v(\tau)$, and $b_{\phi\lambda}^v(\tau)$ given in Appendix B.5. We highlight an important difference between these terms and their counterparts for the market

13 Our assumptions imply that observed dividend growth is only higher for the growth sector if a rare boom actually occurs. Thus, our model is consistent with the results of Chen (2012), who finds relatively small differences in the measured dividend growth rate on growth stocks as compared to value stocks.

portfolio. The sensitivity of the price to booms is given by

$$b^v_{\phi\mu_2}(\tau) = -\frac{1}{\kappa_{\mu_2}}\left(1-e^{-\kappa_{\mu_2}\tau}\right). \tag{29}$$

The analogous term for the market is $b_{\phi\mu_2}(\tau) = \frac{\phi-1}{\kappa_{\mu_2}}\left(1-e^{-\kappa_{\mu_2}\tau}\right)$. From (29), we see that the price of the value claim fluctuates with booms, even though the cash-flow process does not itself depend on booms. The reason is that, when a boom occurs, the risk-free rate rises because the representative agent has a greater desire to borrow. This causes asset prices to fall. This effect is present for the aggregate market, but it is dominated by the expected cash flow effect. For the value claim, this is the only effect booms have on prices.

Naturally, the difference in $b^v_{\phi\mu_2}(\tau)$ carries over to $b^v_{\phi\lambda_2}(\tau)$, which reflects how the price responds to changes in the probability of a boom. An increase in the probability of a boom decreases the price of the value claim because the risk premium effect and combined cash-flow and risk-free rate effect work in the same direction.

Corollary 10. The price-dividend ratio for the value claim $G^v(\mu_t,\lambda_t)$ is increasing in μ_{1t}, decreasing in μ_{2t}, and decreasing in the probability of a rare event λ_{jt}, for $j=1,2$.

Proof. See Corollaries B.12 and B.13. ■

Figure 1 compares the coefficients on value with those for the market. We see that the response of the value claim to disasters and to changes in the disaster probability are almost indistinguishable. The response for booms is quite different. The function $b^v_{\phi\mu_2}(\tau)$ is negative and decreasing in τ, rather than positive and increasing as it is for the market. It is also about half the magnitude of the market coefficient, because the risk-free rate effect alone is small compared with the (combined) cash-flow and risk-free effect for the market. We see this also when considering the response of the price of the value claim to changes in the boom probability. Again, $b^v_{\phi\lambda_2}(\tau)$ is negative and decreasing, and small in magnitude when compared with the corresponding function for the market.

These results lead directly to formulas for the risk premium on the value claim.

Corollary 11. 1. The value sector premium relative to the risk-free rate is

$$r^v_t - r_t = \phi\gamma\sigma^2 - \sum_{j=1,2}\lambda_{jt}E_{\nu_j}\left[\left(e^{b_{\mu_j}Z_{jt}}-1\right)\frac{\mathcal{J}_j(G^v_t)}{G^v_t}\right]$$
$$-\sum_{j=1,2}\lambda_{jt}\frac{1}{G^v_t}\frac{\partial G^v}{\partial\lambda_j}b_{\lambda_j}\sigma^2_{\lambda j}. \tag{30}$$

2. The observed premium on the value sector in a sample without rare events is

$$r^v_{\mathrm{nj},t} - r_t = \phi\gamma\sigma^2 - \sum_{j=1,2} \lambda_{jt} E_{\nu_j}\left[e^{b_{\mu_j} Z_{jt}} \frac{\mathcal{J}_j(G^v_t)}{G^v_t} \right] - \sum_{j=1,2} \lambda_{jt} \frac{1}{G^v_t} \frac{\partial G^v}{\partial \lambda_j} b_{\lambda_j} \sigma^2_{\lambda j}. \tag{31}$$

Proof. The result follows from Lemma B.6 and (26). See the proof of Theorem 5 for more detail. ■

Corollary 12. 1. The static boom premium for the value sector is negative (it is positive for the market).
2. The λ_2-premium is negative (it is positive for the market).
3. The observed static boom premium for the value sector is positive (it is negative for the market).

Other components of the risk premium and observed risk premium on value take the same sign as the market.

Proof. The result follows from Corollary 11 and the reasoning in the proof of Corollary 6. ■

We show the components of the value sector premium next to the market as a function of disaster probability (Figure 2) and as a function of boom probability (Figure 3). The difference is most apparent when we consider risk premiums as a function of the boom probability. For the market portfolio, the static observed boom premium is negative in samples without rare events. For the value sector, the static observed boom premium is slightly positive. This reflects the intuition in the Introduction: when investors are expecting booms and they do not occur, the observed returns on assets exposed to booms will be lower than on assets not exposed to booms (or assets that fall in price when booms occur).

1.4.2 The growth sector. Given this definition of the value sector, the growth sector is defined as the residual. Define D^g_t to be the dividend on the growth claim and F^g_t the price. By definition, $D^g_t = D_t - D^v_t$, and by no-arbitrage,

$$F^g_t = F_t - F^v_t. \tag{32}$$

Figure 4 shows dividends (panel A) and prices (panel B) for value and for the market in a typical simulation that contains a boom. The dividend on value is

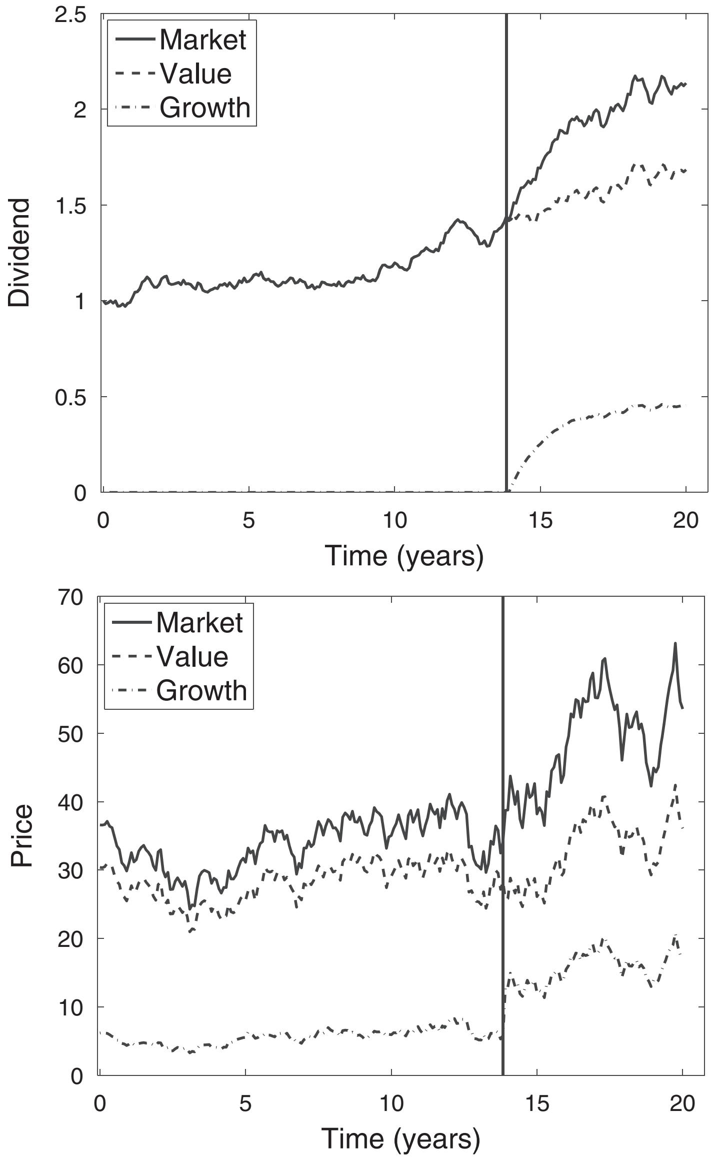

Figure 4
Sample simulation path of asset dividends and prices
This figure shows results from a time series simulated from the model that includes a boom. The top figure shows dividends (initialized at one), and the bottom panel shows prices. The solid line marks the onset of the boom.

normalized to that of the market at the start of the simulation. When a boom occurs, a wedge opens up between the market dividend and the value dividend. This wedge is the dividend on the growth claim.

Figure 4, panel B, shows the price of a claim to the value sector, the market, and the growth sector. Though the growth sector pays no dividends prior to the boom, it has a positive price because investors anticipate the possibility of future dividends. When a boom occurs, the price of the growth claim immediately rises, the value of the aggregate market also rises, but by less, and the price of the value claim falls slightly. After the boom, the price of the growth claim and the overall market remain high relative to value, reflecting permanently higher dividends.[14]

We can use the basic accounting identity (32) to derive properties of the growth claim.

Corollary 13. The dividend-price ratio for the growth sector is below the dividend-price ratio for the value sector.

Proof. It follows from the definition of the growth dividend and from the accounting identity (32) that

$$\begin{aligned}\frac{D_t^g}{F_t^g} &= \frac{D_t - D_t^v}{F_t - F_t^v} \\ &= \frac{D_t - D_t^v}{D_t G_t - D_t^v G_t^v} \\ &= \frac{1}{G_t^v}\frac{D_t - D_t^v}{D_t \frac{G_t}{G_t^v} - D_t^v}.\end{aligned} \tag{33}$$

Note that $1/G_t^v$ is the dividend-price ratio on the value claim. Because value has a lower price-dividend ratio than the market as a whole (Corollary 9), $\frac{G_t}{G_t^v} > 1$. Furthermore, $D_t > D_t^v$. Therefore, $(D_t - D_t^v)/(D_t \frac{G_t}{G_t^v} - D_t^v) < 1$. The result follows. ■

Corollary 14. The price of the growth sector is increasing in μ_1, decreasing in λ_1, and increasing in μ_2 and λ_2.

It is not surprising that the price of the growth claim is increasing in the probability and expected size of a boom. Less obvious is the fact that growth is also exposed to the risk of a disaster. Prior to a boom, growth has no cash flows to fall in the case of disaster. However, after a boom takes place, the cash flows that accrue to growth fall by the same percentage amount in the event of disaster as the rest of the dividends in the economy. Anticipating this, investors price the effect of a disaster into growth stocks before the boom occurs.

14 The figure also shows prices of all claims rising after the boom; this is because aggregate dividends are growing.

Finally, a concern one might have with this model is whether the size of the value sector relative to the market is nonstationary. It might seem that the value sector would grow ever smaller as a proportion of the market. This would be a problem, as the size of the value sector in the data does not appear to be trending downward.

It turns out that there is a simple way to avoid the problem of nonstationarity. At each time t, we assume that a new value sector is created so that the dividend on the value claim is equal to the dividend of the market.[15] The price of this value sector at each time t is still the no-arbitrage value of the dividend stream (25), and so is equal to (26) with $D_t^v = D_t$ in the first argument.[16] Because the price-dividend ratios on value and on the market are stationary, this normalization yields a stationary value sector. The calculation of returns is invariant to this normalization, as shown in Appendix C.

The key property of the growth claim that emerges from this section is that it is a levered bet on large booms. Relative to the market as a whole, the growth claim is small. However, it bears the entire risk of a boom.

2. Quantitative Results

2.1 Data

This section describes our data sources. We will compare our rare events in the model to tail events in the data, using international consumption data described in detail in Barro and Ursúa (2008). These data contain annual observations on real, per capita consumption for 43 countries; start dates vary from early in the 19th century to the middle of the 20th century.

Our aggregate market data come from CRSP. We define the market return to be the gross return on the value-weighted CRSP index. Dividend growth is computed from the dividends on this index. The price-dividend ratio is price divided by the previous 12 months of dividends to remove the effect of seasonality in dividend payments (in computing this dividend stream, we assume that dividends on the market are not reinvested). We compute market returns and dividend growth in real terms by adjusting for inflation using changes in the consumer price index (also available from CRSP). For the government bill rate, we use real returns on the three-month Treasury bill. We also use real, per capita, expenditures on nondurables and services for the United States, available from the Bureau of Economic Analysis. These data are

[15] To be precise, the dividends on the value sector evolve according to

$$\frac{dD_s^v}{D_s^v} = \mu_{Ds}^v dt + \phi\sigma dB_{Cs}, \quad s \geq t \tag{34}$$

with boundary condition $D_t^v = D_t$.

[16] Note that Figure 4 shows the time path of prices without redefining value's dividends; it is therefore what cash flows and price appreciation look like from the point of view of the owner of each of the claims.

annual, begin in 1947, and end in 2010. Focusing on postwar data allows for a clean comparison between U.S. data and hypothetical samples in which no rare events take place.

Data on value and growth portfolios are from Kenneth French's Web site. CRSP stocks are sorted annually into deciles based on their book-to-market ratios. Our growth claim is an extreme example of a growth stock; it is purely a claim to positive extreme events and nothing else. In the data, it is more likely that growth stocks are a combination of this claim and the value claim. To avoid modeling complicated share dynamics, we identify the growth claim with the decile that has the lowest book-to-market ratio, while the value claim consists of a portfolio (with weights defined by market equity) of the remaining nine deciles. A standard definition of the value spread is the log book-to-market ratio of the value portfolio minus the log book-to-market ratio of the growth portfolio (Cohen, Polk, and Vuolteenaho 2003). In our endowment economy, book value can be thought of as the dividend. However, the dividend on the growth claim is identically equal to zero (though of course this claim has future dividends), and for this reason, there is no direct analog of the value spread. We therefore compute the value spread in the model as the log dividend-price ratio on the value portfolio minus the log dividend-price ratio on the aggregate market. For comparability, we use the log book-to-market ratio on value minus the log book-to-market ratio on the market in the data. The predictability results that we report are nearly the same when we use the more standard definition.

2.2 Calibration

The parameter set consists of the normal-times parameters $\bar{\mu}_C$, σ and $\bar{\mu}_D$, leverage ϕ, the preference parameters β and γ, the parameters determining the duration of disasters and booms (κ_{μ_1} and κ_{μ_2} respectively), the parameters determining the disaster and boom processes ($\bar{\lambda}_j$, κ_{λ_j}, and σ_{λ_j} for $j=1,2$) and finally the distributions of the disasters and booms themselves. Some of these parameters define latent processes, for which direct measurement is difficult. The fact that these processes relate to rare events makes the problem even harder.

For this reason, we proceed by dividing the parameters into groups and impose reasonable restrictions on the parameter space. First, the mean and standard deviation of consumption growth during normal times are clearly determined by $\bar{\mu}_C$ and σ. We can immediately eliminate two free parameters by setting these equal to their values in the postwar data (see Tables 2 and 3).

Second, to discipline our calibration, we assume that consumption growth after a disaster reverts to normal at the same rate as consumption growth following a boom, namely, $\kappa_{\mu_1}=\kappa_{\mu_2}$. Further, we assume that the rare-event processes are symmetric. That is, we assume that the average probability of a boom equals that of a disaster ($\bar{\lambda}_1=\bar{\lambda}_2$), and that the processes have the same mean reversion and volatility parameters ($\kappa_{\lambda_1}=\kappa_{\lambda_2}$ and $\sigma_{\lambda_1}=\sigma_{\lambda_2}$).

Table 2
Parameter values

Panel A: Basic parameters	
Average growth in consumption (normal times) $\bar{\mu}_C$ (%)	1.96
Average growth in dividend (normal times) $\bar{\mu}_D$ (%)	3.03
Volatility of consumption growth (normal times) σ (%)	1.45
Leverage ϕ	3.5
Rate of time preference β	0.003
Relative risk aversion γ	3.0
Panel B: Disaster parameters	
Average probability of disaster $\bar{\lambda}_1$ (%)	2.86
Mean reversion in disaster probability κ_{λ_1}	0.11
Volatility parameter for disasters σ_{λ_1}	0.081
Mean reversion in expected consumption growth κ_{μ_1}	1.00
Minimum consumption disaster (%)	10
Power law parameter for consumption disaster	6.27
Panel C: Boom parameters	
Average probability of boom $\bar{\lambda}_2$ (%)	2.86
Mean reversion in boom probability κ_{λ_2}	0.11
Volatility parameter for booms σ_{λ_2}	0.081
Mean reversion in expected consumption growth κ_{μ_2}	1.00
Minimum consumption boom (%)	5
Power law parameter for consumption booms	15.00

Parameter values for the main calibration are expressed in annual terms.

Third, we calibrate the average disaster probability and the disaster distribution to international consumption data. Barro and Ursúa (2008) estimate that the probability of a rare disaster in OECD countries is 2.86%.[17] We use this number as our average disaster probability, $\bar{\lambda}_1$. Following Barro and Jin (2011), we assume a power law distribution for rare events (see Gabaix 2009 for a discussion of the properties of power law distributions). Using maximum likelihood, Barro and Jin estimate a tail parameter of 6.27. They also argue that the distribution of disasters is better characterized by a double power law, with a lower exponent for larger disasters. Incorporating this more complicated specification would lead to a fatter tail and a higher equity premium and volatility. Thus, our parameter choice is conservative.[18] Following Barro and Ursúa (2008), we assume a 10% minimum disaster size.

The power law distribution for booms is quite difficult to observe directly. We could use international data on consumption growth, and in fact such data provide plenty of evidence of extreme positive growth rates. However, one could reasonably ask whether these data are directly applicable to a developed

[17] We calibrate the size of the disasters to the full set of countries and the average probability to the OECD subsample. In both cases, we choose the more conservative measure, because the OECD subsample has rarer, but more severe, disasters.

[18] A potential concern is that the consumption data on disasters and booms are international, while our stock market data are from the United States. However, many of the facts that we seek to explain have been reported as robust features of the international data (e.g. Campbell 2003; Fama and French 1992). We view the international data as a means to discipline the choice of distribution of the rare events, as the data from the United States are extremely limited in this regard.

Table 3
Log consumption and dividend growth moments

Panel A: Consumption growth

		No-jump simulations			All simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
Mean	1.95	1.65	1.95	2.26	−0.31	1.65	2.70	1.50
Standard Deviation	1.45	1.22	1.44	1.66	1.47	3.16	7.52	4.24
Skewness	−0.37	−0.50	0.00	0.48	−4.56	−1.63	2.21	−4.80
Kurtosis	3.22	2.20	2.80	3.87	2.85	10.33	28.09	55.34

Panel B: Dividend growth

		No-jump simulations			All simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
Mean	1.67	1.84	2.91	3.98	−5.01	1.86	5.52	1.31
Standard Deviation	6.46	4.28	5.04	5.82	5.15	11.05	26.33	14.84
Skewness	0.10	−0.50	0.00	0.48	−4.56	−1.63	2.21	−4.80
Kurtosis	4.66	2.20	2.80	3.87	2.85	10.33	28.09	55.34

Data moments are calculated using annual data from 1947 to 2010. Population moments are calculated from simulating data from the model at a monthly frequency for 600,000 years and then aggregating monthly growth rates to an annual frequency. We also simulate 100,000 60-year samples and report the 5th, 50th and 95th percentile for each statistic, both from the full set of simulations and for the subset of samples for which no rare events occurred.

country like the United States. We thus turn to asset markets and, in particular, to the size of the growth sector. The size of the growth sector in the model is sensitive to the thickness of the tail of the power law distribution, a thicker tail implying a larger growth sector. We can therefore infer tail thickness by matching the size of the growth sector in the model to the size of the growth sector in the data.[19]

As discussed in Section 2.1, we identify the growth sector in the model with the lowest book-to-market decile in the data. We use an annual growth rate of 5% as our minimum jump size. This would be an unusually high observation for an annual growth rate, so it is a reasonable choice for the starting point of the upper tail of the consumption growth distribution. Given this minimum jump size, we require the model to match the relative book-to-market ratio of value (deciles 2–9) as compared with the market as a whole.[20] Given our other parameter choices, this implies a power law parameter of 15, corresponding to a thinner tail than for disasters. As we later discuss, the value premium is quite insensitive to the choice of this parameter.

Despite the fact that we use asset market data to infer our distribution for booms, we still want to compare these booms to those we see in international data. The disaster distribution and the boom distribution in the model and in the data are reported in Figures 5 and 6. To focus on the tails of the distribution, we

[19] See David and Veronesi (2013) for the use of asset prices to estimate high growth states that may not have been realized in sample.

[20] In the model, the "book" values of the market and of value are the same. Requiring the model to match relative market valuations produces very similar answers.

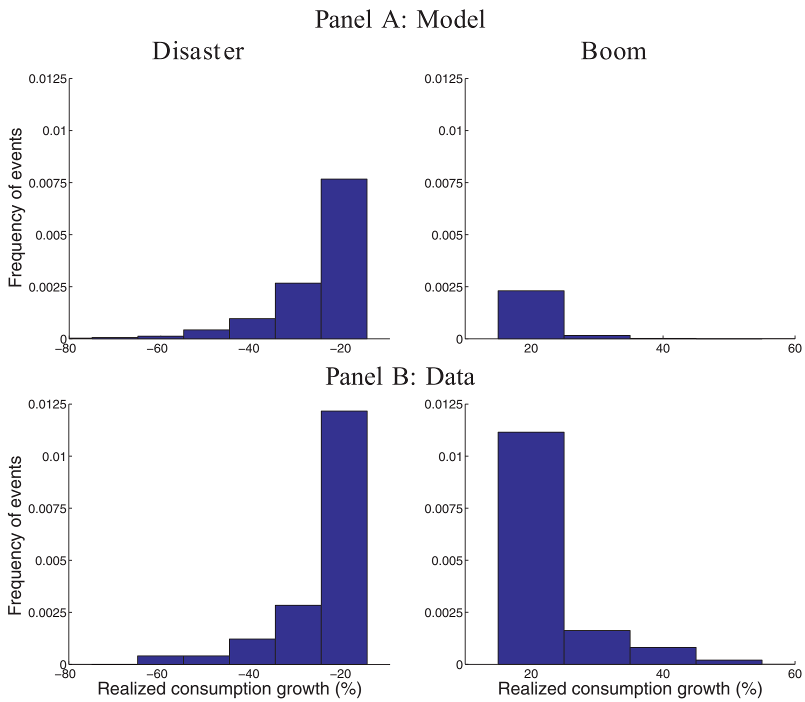

Figure 5
Tails of the one-year consumption growth rate distribution
This figure shows histograms of one-year consumption growth rates. The right panel considers growth rates above 15%. The left panel considers growth rates below −15%. The frequency is calculated by the number of observations within a range, divided by the total number of observations in the sample. Panel A shows results from simulated data from the model. Panel B shows results from the data. Data are from Barro and Ursúa (2008). For the consumption booms, we exclude observations between 1944 and 1953.

consider consumption changes of greater than 15% for one-year consumption growth rates and consumption changes of greater than 45% for consumption growth rates that are cumulative across five years. These figures show that, except for small disasters at the five-year horizon, our assumptions imply less extreme distributions than the data.[21] In particular, our model implies fewer, and smaller, booms than observed in the international data.

The remaining parameters are the dividend process parameters μ_D, and ϕ, preference parameters β and γ, and rare-event parameters $\kappa_{\lambda_1} = \kappa_{\lambda_2}$ and $\sigma_{\lambda_1} = \sigma_{\lambda_2}$. We choose these parameters to minimize the distance between the mean value of various statistics in a sample without rare events and the corresponding statistic in the postwar data. We also impose some reasonable economic limits on the parameter choices from this search.

[21] It is the case that our power law distributions are unbounded, thus placing some probability on events that are greater in magnitude than what has occurred in the data. Truncating the distributions has little effect on the results.

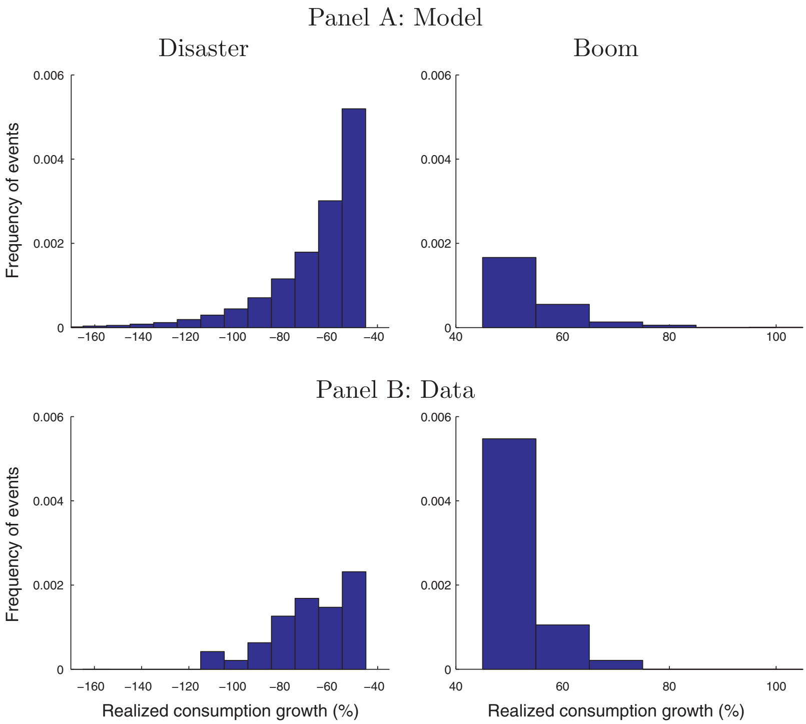

Figure 6
Tails of the five-year consumption growth rate distribution
This figure shows histograms of five-year consumption growth rates. The right panel considers growth rates above 45%. The left panel considers growth rates below -45%. Panel A shows results from simulated data from the model. Panel B shows results from the data. Data are from Barro and Ursúa (2008). For the consumption booms, we exclude five-year periods beginning between 1940 and 1948.

The first requirement is that the solution to the agent's problem exists. It follows from Theorem 1, that parameters must satisfy

$$\frac{1}{2\sigma_{\lambda_1}^2}\left(\kappa_{\lambda_1}+\beta\right)^2 \geq E_{\nu_1}\left[e^{b_{\mu_1} Z_1}-1\right] \tag{35}$$

(see also Appendix A). Equation 35 is a joint restriction on the size of a disaster, on the agent's risk aversion, on the discount rate of the agent, and on the persistence and volatility of the disaster probability process.[22] Our second requirement is that the discount rate β be greater than zero. Because of positive consumption growth and a elasticity of intertemporal substitution equal to 1,

[22] Why, intuitively, is there such a constraint? Note that utility is a solution to a recursive equation; the above discussion reflects the fact that there is no guarantee in general that a solution to this recursion exists. In this particular case, it appears that the problematic region of the parameter space is one in which there is a lot of uncertainty that is resolved very slowly. A sufficiently slow resolution of uncertainty could lead to infinitely negative utility for our recursive utility agent.

matching the low risk-free rate of 1.25 will be a challenge. We discuss this aspect of the model's fit in more detail in a later section. We choose a small positive number for the lower bound of β, and our minimization procedure selects this as optimal on account of the risk-free rate.

Our third requirement is that leverage (ϕ) be reasonable. High leverage helps the model match the equity premium and volatility, but allowing these data points to determine ϕ might lead to a value that is unreasonably high. Cash-flow data, on the other hand, does not clearly pin down a value of ϕ.[23] We choose $\phi = 3.5$, in line with values considered in the literature (for example, Bansal and Yaron 2004 assume a value of 3.0, and Backus, Chernov, and Martin 2011 assume a value of 5.1). Given that high values of ϕ are helpful for the moments of equity returns, lower values of ϕ will result in an inferior fit.[24]

The restrictions above imply that we have three free parameters remaining. We search over values of $\mu_D, \gamma, \kappa_{\lambda_1}$ as other parameters are determined by these. We seek to match average dividend growth, the equity premium, the volatility of the market return, the average price-dividend ratio, and the persistence of the value spread.[25] We measure the model's fit by simulating 1,500 60-year samples and taking only those without rare events. We minimize the sum of squared differences between the mean across samples and the data moment, normalizing by the variance across samples. The criterion function is minimized for average dividend growth $\mu_D = 3\%$, risk aversion $\gamma = 3$, and mean reversion $\kappa_{\lambda_1} = 0.11$. Value spread moments are reported in Table 4, and aggregate market moments are reported in Table 5. With these parameters, the probability of not observing a boom in a 60-year period is about 20%. Thus, there is no need to assume that the postwar period is exceptional in that a boom has not been observed.[26]

23 The ratio of dividend to consumption volatility during normal times implies a value of 4.7 (this assumes, however, that dividends are perfectly correlated with consumption in the data). Using the decline in earnings relative to the decline in consumption during the Great Depression leads to an even higher number, as earnings fell by nearly 100% (Longstaff and Piazzesi 2004); however, this decline might have reasonably been expected by market participants to be temporary, while our model, for simplicity, assumes such declines are permanent.

24 We have simulated from a calibration in which the normal-times standard deviation of dividend growth is twice that of consumption (rather than 3.5 times, as in our benchmark calibration), but where everything else is the same. The results are very similar to what is reported here, not surprisingly, because it is the risk of rare events, rather than the normal-times consumption risk, that drives our results. Lowering ϕ itself does lead to somewhat lower observed equity and value premiums, but the difference is not large. A ϕ of three implies an equity premium of 5.1% (as compared with 5.4 in our main calibration) and an observed value premium of 2.6% (as compared with 2.7 in our main calibration).

25 Attempting to match the very high persistence of the price-dividend ratio leads to unstable results.

26 Asness, Moskowitz, and Pedersen (2013) report the existence of a value premium in international equities (Fama and French 1992 also report an international value premium, but over a shorter sample). The data on individual stocks in Asness, Moskowitz, and Pedersen (2013) come from the United States, the United Kingdom, and Japan. Given that a large boom would have worldwide implications, impacting at least the major developed markets, adding data from the United Kingdom and from Japan does not necessarily help us in observing the correct number or size of the booms. The other data they consider are international equity indices. Our model is a natural fit for explaining these data as well, since the stock markets of some countries might be expected to outperform in the event of a large global boom; these would be "growth" according to their measure and would have lower

Table 4
Value spread moments

	Data	No-jump simulations 0.05	No-jump simulations 0.50	No-jump simulations 0.95	All simulations 0.05	All simulations 0.50	All simulations 0.95	Population
exp(E[log(value spread)])	1.23	1.18	1.21	1.27	1.18	1.23	1.34	1.24
σ(log(value spread))	0.08	0.02	0.04	0.07	0.03	0.07	0.16	0.10
Value spread autocorrelation	0.79	0.57	0.78	0.90	0.39	0.69	0.89	0.71

Data moments are calculated using annual data from 1947 to 2010. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 100,000 60-year samples and report the 5th, 50th and 95th percentile for each statistic, both from the full set of simulations and for the subset of samples for which no rare events occurred. The value spread is defined as the log of the book-to-market ratio for the value sector minus the book-to-market ratio for the aggregate market in the data, and as log price-dividend ratio for the aggregate market minus the log price-dividend ratio for the value sector in the model. In the data, the growth portfolio is the lowest book-to-market decile. The remaining nine deciles comprise the value portfolio.

2.3 Simulation results

To evaluate the quantitative success of the model, we simulate monthly data for 600,000 years, and also simulate 100,000 60-year samples. For each sample, we initialize the λ_{jt} processes using a draw from the stationary distribution.[27] Given a simulated series of the state variables, we obtain price-dividend ratios and one-period dividend growth rates on the market and on value. Using these, we simulate returns as described in Appendix C. In the tables, we report population values for each statistic, percentile values from the small-sample simulations, and percentile value for the subset of small-sample simulations that do not contain rare events. It is this subset of simulations that is the most interesting comparison for postwar data.

2.3.1 The aggregate market. Table 3 reports moments of log growth rates of consumption and dividends. There is little skewness or kurtosis in postwar annual consumption data. Postwar dividend growth exhibits somewhat more skewness and kurtosis. The simulated paths of consumption and dividends for the no-jump samples are, by definition, normal, and the results reflect this. However, the full set of simulations does show significant nonnormality; the median kurtosis is seven for consumption and dividend growth. Kurtosis exhibits a substantial small-sample bias. The last column of the table reports the population value of this measure, which is 55.

Table 5 reports simulation results for the aggregate market. The model is capable of explaining most of the equity premium: the median value among the simulations with no disaster risk is 5.4%; in the data it is 7.2%. Moreover, the data value is below the 95th percentile of the values drawn from the model, indicating that the data value does not reject the model at the 10% level.

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observed returns. This would also explain the links between the value effects from the international equity indices and the individual equities.

27 The stationary distribution for λ_{jt} is gamma with shape parameter $2\kappa_j\bar{\lambda}_j/\sigma^2_{\lambda_j}$ and scale parameter $\sigma^2_{\lambda_j}/(2\kappa_j)$ (Cox, Ingersoll, and Ross 1985).

Table 5
Aggregate market moments

	Data	No-jump simulations 0.05	No-jump simulations 0.50	No-jump simulations 0.95	All simulations 0.05	All simulations 0.50	All simulations 0.95	Population
$E[R^b]$	1.25	1.65	1.95	2.10	−0.12	1.62	2.57	1.49
$\sigma(R^b)$	2.75	0.13	0.25	0.50	0.38	2.43	5.76	3.24
$E[R^m - R^b]$	7.25	3.66	5.44	7.94	2.88	5.97	10.28	6.27
$\sigma(R^m)$	17.8	10.4	14.5	20.1	13.1	20.5	31.9	22.3
Sharpe ratio	0.41	0.27	0.38	0.51	0.14	0.30	0.46	0.28
$\exp(E[p-d])$	32.5	25.0	30.7	34.4	20.0	28.6	34.3	27.8
$\sigma(p-d)$	0.43	0.10	0.19	0.34	0.14	0.27	0.50	0.35
$\text{AR1}(p-d)$	0.92	0.57	0.78	0.91	0.53	0.78	0.91	0.85

Data moments are calculated using annual data from 1947 to 2010. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 100,000 60-year samples and report the 5th, 50th and 95th percentile for each statistic, both from the full set of simulations and for the subset of samples for which no rare events occurred. R^b denotes the government bond return, R^m denotes the return on the aggregate market, and $p-d$ denotes the log price-dividend ratio.

The model can also explain high return volatility, and low volatility of the government bond yield. Note that we define disasters as large deviations in expected consumption growth. Observed consumption growth is smooth, and it takes several years for disasters to unfold. Thus, the critique of Constantinides (2008), Julliard and Ghosh (2012) and Mehra and Prescott (1988) concerning the instantaneous nature of disasters in many models of rare events does not apply here.

Before moving on to the cross-section, we note two limitations to the model's fit to the data. First, the average government bond yield in the model is higher than in the data (1.95% vs. 1.25%). This fit could be improved by allowing some of the disaster-related decline in consumption to take place immediately. This modification would be straightforward to implement but would substantially complicate the notation and exposition without changing any of the underlying economics. Moreover, Treasury-bill returns may in part reflect liquidity at the very short end of the yield curve (Longstaff 2000); the model does a better job of explaining the return on the one-year bond.[28] Second, while the model can account for a substantial fraction of the volatility of the price-dividend ratio (the volatility puzzle, reviewed in Campbell 2003), it cannot explain all of it, at least if we take the view that the postwar series is a sample without rare events. This is a drawback that the model shares with other models attempting to explain aggregate prices using time-varying moments (see the discussion in Bansal, Kiku, and Yaron 2012 and Beeler and Campbell 2012) but parsimonious preferences. It arises from strong general equilibrium effects: time-varying moments imply cash flow, risk-free rate, and risk premium effects, and one of these generally acts as an offset to the other two, limiting the effect time-varying moments have on prices. Some behavior of asset prices (i.e.,the "bubble" in

[28] The model predicts a near-zero volatility for returns on this bill in samples without disasters. This is not a limitation, since volatility in returns in the data is due to inflation, which is not present in the model.

Table 6
Cross-sectional moments

	Data	No-jump simulations 0.05	No-jump simulations 0.50	No-jump simulations 0.95	All simulations 0.05	All simulations 0.50	All simulations 0.95	Population
$E[R^v - R^b]$	7.95	4.34	6.06	8.52	2.59	5.26	8.28	5.34
$E[R^g - R^b]$	6.62	1.04	3.37	6.41	1.10	7.90	24.79	9.97
$E[R^v - R^g]$	1.34	1.26	2.74	4.03	−19.55	−2.42	3.46	−4.63
$\sigma(R^v)$	17.0	9.7	13.5	18.7	11.3	17.4	25.2	18.1
$\sigma(R^g)$	21.0	16.6	22.8	32.6	21.8	41.8	117.3	64.0
$\sigma(R^v - R^g)$	11.7	10.7	14.6	19.9	13.1	35.1	116.1	60.7
Sharpe ratio, value	0.48	0.34	0.45	0.60	0.13	0.31	0.51	0.29
Sharpe ratio, growth	0.32	0.05	0.15	0.24	0.04	0.18	0.29	0.16
Sharpe ratio, value-growth	0.11	0.07	0.19	0.33	−0.20	−0.07	0.25	−0.08
Alpha, value	1.26	0.79	1.08	1.50	−0.22	0.86	2.82	1.23
Alpha, growth	−1.26	−5.96	−4.32	−3.10	−9.62	−3.17	1.36	−4.28
Alpha, value-growth	2.53	3.97	5.41	7.33	−1.53	4.07	12.29	5.51
Beta, value	0.92	0.86	0.92	0.96	0.26	0.83	0.96	0.66
Beta, growth	1.09	1.21	1.41	1.63	1.23	1.69	3.64	2.27
Beta, value-growth	−0.16	−0.75	−0.49	−0.25	−3.32	−0.87	−0.28	−1.62

Data moments are calculated using annual data from 1947 to 2010. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 100,000 60-year samples and report the 5th, 50th and 95th percentile for each statistic, both from the full set of simulations and for the subset of samples for which no rare events occurred. R^v denotes the gross return on the value sector, R^g denotes the gross return on the growth sector, alpha denotes the loading of the constant term of the CAPM regression, and beta denotes the loading on the market equity excess return of the CAPM regression. In the data, the growth portfolio is the lowest book-to-market decile. The remaining nine deciles comprise the value portfolio.

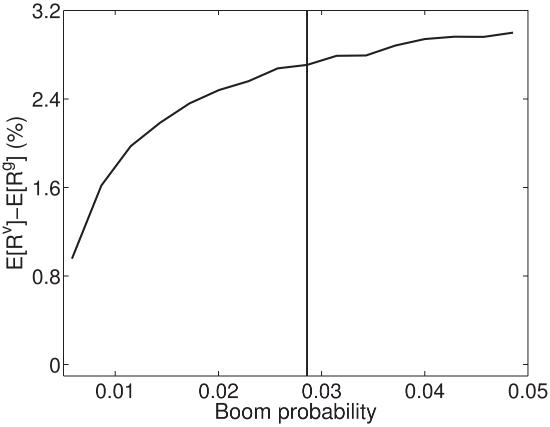
the late 1990s) may be beyond the reach of this type of model. This is a fruitful area for further research.

2.3.2 Unconditional moments of value and growth portfolios. Tables 6 reports cross-sectional moments in the model. As a tight data comparison, we take the growth portfolio as the bottom decile formed by sorting on book-to-market and the value portfolio as the remaining nine deciles. This comparison has the advantage that, in both the model and in the data, the two portfolios considered sum to the market. However, we also report excess returns for more traditional measures of value and growth in Table 8.

Table 6 shows that our model can account for an observed value premium of 2.74%, a substantial fraction of the data value of 4.28%. This value corresponds to the median in simulations without rare events. The population value premium is negative, as shown in Section 1. Yet even looking across the full set of simulations implies that it is not unlikely to observe a value premium in any particular sample.

Table 6 also shows that value stocks have lower standard deviations than growth stocks and higher Sharpe ratios. Both of these results hold across the full set of simulations, as well as in the samples without rare events. Both of these affects are strongly present in the data. The reason the model can capture these effects is that the observed high average return on value stocks does not represent a return for bearing risk. As explained in Section 1, because investors

Panel A: Value premium as a function of average boom probability ($\bar{\lambda}_2$)



Panel B: Value premium as a function of the power law tail parameter (α)

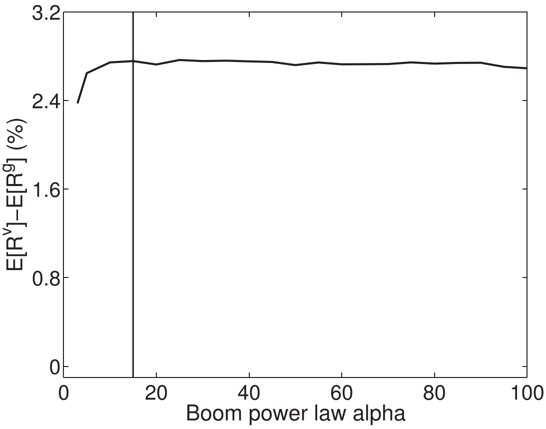


Figure 7
Sensitivity of the observed value premium to parameter choices
Panel A reports the observed value premium (defined as the average difference between the return on the value sector and the return on the growth sector across simulations with no rare events) as a function of the average probability of a boom. All other parameters are unchanged. Panel B reports the average value premium across simulations with no rare events as a function of the power law parameter α. Lower values of α correspond to thicker tails.

are willing to accept a lower return on growth in most periods, in return for an occasional very high payout.

Perhaps surprisingly, the model's predictions for the observed value premium are largely insensitive to the size of booms. In Figure 7, we show the observed value premium for different specifications of the boom distribution. In panel A, we vary the probability of the boom, and in panel B we vary the size of the tail parameter for the power law distribution. The observed value premium is indeed increasing in the probability of a boom: if the probability of a boom were zero, then so would be the observed value premium.[29] Conversely, a high probability of a boom leads to a high observed value premium. In contrast, panel B shows that the observed value premium is quite flat as a function of thickness of the tail. Lower values of the tail parameter imply thicker tails. At extremely low values expected dividend growth is high enough so that prices fail to converge (see Assumption 2 in Appendix A). Within the range of 3 to nearly 100, there is little noticeable change in the observed value premium.

Why is it that the observed value premium is so insensitive to the shape of the boom distribution? The reason lies in the two opposing forces described in Section 1. On the one hand, the greater the probability of large booms, the riskier growth stocks become, and the more negative is the true value premium in population. On the other hand, the greater the probability of large booms, the lower is the return on growth stocks a risk neutral investor is willing to accept in samples without booms. These two effects roughly cancel.

Table 6 also shows that our model can explain the relative alphas and betas for value and growth stocks. Growth stocks have a high covariance with the market, because they are a levered bet on the occurrence of booms. Shocks to the probability of a boom move the market price and the growth price in the same direction. The same is true when a boom actually occurs. Given that growth stocks have higher betas and lower average returns than value stocks, it is of course not surprising that they have negative alphas. In fact, they have negative alphas in population, as well as in samples without rare events, because a large part of their risk comes from changes to the probability of a boom, and the premium associated with this risk is low. Thus, unlike previous models of the value premium, our model is able to explain the patterns in betas on growth and value in the data.

2.3.3 Return predictability. In a recent survey, Cochrane (2011) notes that time-varying risk premiums are a common feature across asset classes. However, variables that predict excess returns in one asset class often fail in another, suggesting that more than one economic mechanism lies behind this common predictability.[30] For example, the price-dividend ratio is a significant predictor of aggregate market returns, but fails to predict the value-minus-growth return. On the other hand, the value spread predicts the value-minus-growth return, but it is less successful than the price-dividend ratio at predicting the aggregate market return.

Panel A of Table 7 shows the results of regressing the aggregate market portfolio return on the price-dividend ratio in actual and simulated data. The

[29] We examine the sensitivity across a range from 0.6% probability to 5%. Below this 0.6%, the growth sector is extremely small and return moments are unstable.

[30] Lettau and Wachter (2011) show that if a single factor drives risk premiums, then population values of predictive coefficients should be proportional across asset classes.

Table 7
Long-horizon regressions of aggregate market and value-minus-growth returns on the price-dividend ratio and the value spread

	1-year horizon				3-year horizon				5-year horizon			
	Data	NJ	All	Pop.	Data	NJ	All	Pop.	Data	NJ	All	Pop.
Panel A: Market returns on the price-dividend ratio												
Coef.	−0.12	−0.28	−0.17	−0.09	−0.29	−0.67	−0.42	−0.23	−0.41	−0.91	−0.60	−0.35
t-stat	[−2.41]				[−3.37]				[−3.37]			
R^2	0.09	0.15	0.07	0.02	0.22	0.33	0.16	0.05	0.27	0.43	0.22	0.07
Panel B: Market returns on the value spread												
Coef.	−0.50	−0.76	−0.33	−0.06	−1.18	−1.88	−0.86	−0.17	−1.28	−2.61	−1.26	−0.25
t-stat	[−1.86]				[−2.28]				[−3.13]			
R^2	0.05	0.05	0.02	0.001	0.12	0.12	0.06	0.002	0.09	0.16	0.08	0.003
Panel C: Value-minus-growth returns on the price-dividend ratio												
Coef.	0.01	0.11	0.06	0.01	0.05	0.27	0.16	0.02	0.09	0.38	0.24	0.02
t-stat	[0.37]				[0.51]				[0.76]			
R^2	0.00	0.03	0.02	1×10^{-4}	0.01	0.07	0.05	2×10^{-4}	0.02	0.10	0.07	2×10^{-4}
Panel D: Value-minus-growth returns on the value spread												
Coef.	0.46	1.18	0.47	0.01	1.13	2.87	1.23	0.03	1.48	3.95	1.81	0.05
t-stat	[2.52]				[2.44]				[2.37]			
R^2	0.10	0.10	0.03	1×10^{-5}	0.19	0.25	0.08	5×10^{-5}	0.21	0.34	0.12	7×10^{-5}

Coefficients and R^2 statistics from predictive regressions in annual (overlapping) postwar data and in the model. In panels A and B the excess market return is regressed against the price-dividend ratio and the value spread, respectively. Panels C and D repeat this exercise with the value-minus-growth return. The value spread is defined as the log book-to-market ratio of the value sector minus log book-to-market ratio of the aggregate market. For the data coefficients, we report t-statistics constructed using Newey-West standard errors. Population moments (Pop.) are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 100,000 60-year samples and report the average value for each statistic, both from the full set of simulations (All) and for the subset of samples for which no rare events occurred (NJ).

model can reproduce the finding that the price-dividend ratio predicts excess returns. This result arises primarily from the fact that a high value of the disaster probability implies a higher equity premium and a lower price-dividend ratio. It is also the case that a high value of the boom probability implies a lower return in samples that, ex post, have no booms, as well as a higher price-dividend ratio. Coefficients and R^2 statistics are smaller in a sample with rare events than without: this is both because more of the variance of stock returns arises from the greater variance of expected dividend growth during disasters and because the effect of the boom probability reverses (high premiums are associated with high valuations) in the full set of samples. We can see the effect of small-sample bias (Stambaugh 1999) by comparing the population R^2 with the median from the full set of simulations.

In the data, the market return can also be predicted by the value spread, though with substantially smaller t-statistics and R^2 values (panel B of Table 7). The model also captures the sign and the relative magnitude of this predictability. Compared with the price-dividend ratio, the value spread is driven more by the time-varying probability of a boom and less by the probability of a disaster. This explains why risk premiums on the market portfolio, which is mainly driven by the disaster probability, are not captured as well by the value spread.

Panel C of Table 7 shows that, in contrast to the market portfolio, the value-minus-growth return cannot be predicted by the price-dividend ratio. The data coefficient is positive and insignificant. This fact represents a challenge for models that seek to simultaneously explain market returns and returns in the cross-section since the forces that explain time variation in the equity premium also lead to time variation in the value premium (e.g. Lettau and Wachter 2011; Santos and Veronesi 2010); this reasoning would lead the coefficient to be negative.[31] The present model does, however, predict a positive coefficient. A high value of the price-dividend ratio on the market mostly reflects a low probability of a disaster, but also in part a high probability of a boom. If a boom was expected but did not occur, then the average realized returns on value will be high relative to growth (compare the left and right columns of panel B, Figure 3).[32]

One might think that the reason that the value-minus-growth return cannot be predicted by the price-dividend ratio is that it is not very predictable. This, however, is not the case. panel D of Table 7 shows that, as in the data, the value spread predicts the value-minus-growth return with a positive sign in samples without jumps. The median R^2 value at a one-year horizon is 10%, compared with a data value of 10%. At a five-year horizon, the value in the model is 34%, it is 21% in the data. The intuition is the same as for the value-minus-growth regression on the price-dividend ratio. A high value spread indicates a high probability of a boom. If we isolate such periods that, ex post, do not have rare events, growth will have a low return relative to value. The R^2 values are much higher than for the price-dividend ratio because, unlike the price-dividend ratio, the value spread is primarily driven by the probability of a boom.[33]

To summarize, the joint predictive properties of the price-dividend ratio and the value spread would be quite difficult to explain with a model in which a single factor drives risk premiums; they therefore constitute independent evidence of a multiple-factor structure of the kind presented here.

[31] Roussanov (2014) also notes that the conditional mean of the value-minus-growth portfolio does not vary in the way that univariate models of time-varying risk aversion would predict.

[32] The coefficient is positive in population as well, though it is very small. This reflects trade-off between the effect of the boom probability which gives a negative coefficient (compare the left and right columns of panel A in Figure 3), and the effect of disaster probability, which predicts a positive one (same comparison, but in Figure 2). Value has a slightly lower exposure to disaster probability because of its shorter duration.

[33] In population, the effect works in the opposite direction because high values of the boom probability predict low returns on value relative to growth. The resultant R^2 coefficients are very small. For the set of all simulations, the mean coefficient is again positive because of small-sample bias.

Table 8
Data on portfolios formed on the book-to-market ratio

	All stocks					Top size quintile				
	G 1	2	3	4	V 5	G 1	2	3	4	V 5
Panel A: 1947–2010										
$E[R^i - R^f]$	6.08	6.71	8.04	8.61	10.36	6.22	6.46	7.26	7.32	8.36
se	(0.59)	(0.55)	(0.52)	(0.54)	(0.62)	(0.57)	(0.54)	(0.52)	(0.55)	(0.62)
$\sigma[R^i - R^f]$	16.26	15.12	14.47	14.96	17.20	15.85	14.94	14.48	15.16	17.32
β_i	1.06	1.02	0.95	0.96	1.05	1.02	0.99	0.91	0.93	0.99
se	(0.02)	(0.01)	(0.02)	(0.02)	(0.03)	(0.02)	(0.01)	(0.02)	(0.02)	(0.03)
% of total market	40.84	22.24	17.03	12.22	7.66	33.85	16.86	12.37	8.01	4.33
Panel B: 1995–2000										
$E[R^i - R^f]$	16.15	15.07	14.63	15.50	13.00	17.84	16.13	14.80	16.09	12.03
se	(2.04)	(1.82)	(1.77)	(1.60)	(1.40)	(2.01)	(1.88)	(1.93)	(1.91)	(1.73)
$\sigma[R^i - R^f]$	17.27	15.41	15.03	13.60	11.88	17.04	15.98	16.36	16.18	14.68
β_i	1.09	1.10	1.04	0.89	0.78	1.09	1.12	1.10	0.95	0.85
se	(0.08)	(0.04)	(0.06)	(0.05)	(0.04)	(0.08)	(0.03)	(0.06)	(0.09)	(0.07)
% of total market	47.53	21.66	14.69	9.33	6.80	40.21	16.75	10.31	5.76	3.92
Panel C: 1929–1932										
$E[R^i - R^f]$	−21.79	−20.99	−28.65	−21.52	−26.35	−21.48	−19.26	−28.72	−29.79	−16.94
se	(5.77)	(5.71)	(7.14)	(8.82)	(9.12)	(5.77)	(5.70)	(7.47)	(8.79)	(8.74)
$\sigma[R^i - R^f]$	39.98	39.58	49.44	61.08	63.21	39.99	39.48	51.79	60.89	60.58
β_i	0.95	0.94	1.15	1.38	1.44	0.95	0.93	1.19	1.36	1.23
se	(0.04)	(0.03)	(0.06)	(0.13)	(0.12)	(0.04)	(0.03)	(0.07)	(0.11)	(0.17)
% of total market	46.62	24.56	18.22	8.40	2.20	42.72	20.30	14.50	6.01	0.89
Panel D: January 2008–June 2009										
$E[R^i - R^f]$	−19.57	−26.92	−29.31	−35.29	−23.61	−19.25	−27.88	−32.22	−39.58	−20.84
se	(5.69)	(5.03)	(6.28)	(6.64)	(7.64)	(5.49)	(4.59)	(6.15)	(6.71)	(7.23)
$\sigma[R^i - R^f]$	24.13	21.36	26.64	28.17	32.40	23.29	19.49	26.07	28.47	30.66
β_i	0.91	0.82	1.01	1.07	1.20	0.87	0.73	0.97	1.06	1.10
se	(0.06)	(0.03)	(0.07)	(0.05)	(0.12)	(0.06)	(0.04)	(0.09)	(0.06)	(0.13)
%	37.02	26.05	16.32	13.82	6.79	30.19	20.63	12.17	10.13	3.96

Statistics for portfolio excess returns formed by sorting stocks by the ratio of book equity to market equity during various subperiods of the data. The panel reports means and βs with respect to to the value-weighted CRSP market portfolio. Data are at a monthly frequency. We multiply excess returns by 1,200 to obtain annual percent returns. Excess returns are measured relative to the 30-day Treasury bill. The left panel reports results from the full set of equities, and the right panel looks only at the top size quintile.

3. Further Implications

3.1 When does growth outperform value?

Our model predicts that in samples when booms are not realized, value stocks will on average exhibit greater returns than will growth stocks. The model also predicts that there will be periods when growth will outperform value, namely, when booms are realized or when there are positive shocks to the probability of a boom. Both would be expected to occur during times of substantial technological innovation. How does this prediction fare in the data? In this section we examine the performance of value and growth during a period that is indisputably characterized by these shocks, namely, the late 1990s.

Table 8 shows statistics for portfolios formed by sorting stocks into quintiles on the basis of the book-to-market ratio. We examine results for the full CRSP universe, as well as for the top size quintile. As is well known, value outperforms growth by a substantial margin over the postwar sample. However, from 1995-2000, the greatest performance belongs to the lowest book-to-market quintile. Not only do growth stocks exhibit higher returns during this period, they also have even higher betas than usual. This is evidence in favor of a the principal mechanism in the model: that growth stocks are more exposed to boom risk than are value stocks.

This exercise naturally raises the question of the differential performance of value and growth during disasters, and during periods when the probability of a disaster increases. To keep this paper to a manageable length, we have not introduced differential exposure of value and growth dividends to disasters. Considering varying exposure to these shocks, however, would be within the spirit of the model. We therefore look at the differential exposure and performance of value and growth during disaster periods.

Panels C and D of Table 8 report the expected returns and betas to value and growth portfolios during the Great Depression and the financial crisis of 2008, respectively.[34] The table shows that growth has a much lower beta than value during these periods, indicating that value stocks are more exposed to crises than growth stocks. This is consistent with a rare-events model in which value stocks had greater crisis exposure.[35]

3.2 Downside and upside risk

Ang, Chen, and Xing (2006) define downside β as the covariance divided by the variance, where these moments are computed using only those observations at which the market return is below its mean. Likewise, upside β is the measure when the covariance and variance are computed for observations when the market return is above its mean. They show that stocks with higher downside βs have higher returns. Lettau, Maggiori, and Weber (2014) show that this finding also holds across asset classes.

Ang, Chen, and Xing (2006) also define relative upside and relative downside β to be the one-sided β measure minus the traditional β. They find that stocks with high relative downside β have higher mean returns, whereas stocks with high relative upside β have lower mean returns. Because there are many sources of heterogeneity in stocks that are not captured in the present study, these relative β results seem most relevant.

That there would be a relation between observed one-sided risk and rare events is not obvious. Disasters and large booms represent extreme one-sided

[34] Our dates are determined from the NBER peak-to-trough measure of the Great Recession.

[35] There are problems with differential exposure to crises as the main explanation of the value premium. First, if value stocks are more exposed to disasters than are growth stocks, they will also have greater exposure to the probabilities of disasters, and hence counterfactually high betas. This would be less of a problem if this mechanism were paired with a mechanism like the one we emphasize that lowers the betas on value stocks. Second, as Table 8 shows, the actual performance of value stocks, outside of the bottom quintile, was not particularly poor during these periods.

Table 9
Upside and downside betas in the model

	$E[R-R^b]$ (%)	relative β^+	relative β^-
Value	6.06	−0.0024	0.0062
Growth	3.37	0.0686	−0.0634

This table reports the relative upside and downside betas for the value and growth portfolios in the model, along with the average returns on these portfolios. The first column reports the average annual excess returns for value and growth, and the second row reports the relative upside betas, $\beta_i^+ - \beta_i$, where $\beta_i^+ = \frac{\text{cov}(R^i, R^m | R^m > \mu_m)}{\text{var}(R^m | R^m > \mu_m)}$, μ_m is the average return on the aggregate market within each simulation, and β_i is the regular CAPM beta for value and growth portfolio. The third column reports the relative downside betas, $\beta_i^- - \beta$, where $\beta_i^- = \frac{\text{cov}(R^i, R^m | R^m < \mu_m)}{\text{var}(R^m | R^m < \mu_m)}$. We simulate 100,000 60-year samples and report the median values from the subset of samples for which no rare events occurred.

risks that are realized only rarely. In contrast, the studies mentioned above focus on differential exposure to up- and down-moves during normal market conditions. However, in the present model there is a connection because exposure to rare events drives normal-times variation. An asset that is more exposed to disaster risk will also tend to fall during market downturns, because a downturn over a finite time interval is more likely to be caused by an increase in the probability of a disaster than a decrease in the probability of a boom. This effect arises from the link between the volatility of the rare-event probabilities and their magnitudes. Moreover, while changes in boom probabilities in general have a smaller effect on the market return than disaster probabilities, upturns will be disproportionately caused by changes in the boom probability.

Table 9 shows the model's prediction for the relation between one-sided betas and expected returns. In the model, value stocks have low relative upside betas, but high relative downside betas while growth stocks have the opposite. Thus, low relative upside betas and high relative downside betas are associated with high returns, just as in the data.

3.3 Skewness in the time series and cross-section

Our model predicts that positive skewness should be associated with low future returns, both in the time series and in the cross-section. Several recent papers argue that proxies for disaster risk predict future returns (Kelly and Jiang 2014; Manela and Moreira 2013). Colacito, Ghysels, and Meng (2013) shows that skewness in analysts' forecasts, which takes on both negative and positive values, predicts returns with a negative sign. Like other papers on disaster risk, this paper predicts that a greater risk of disaster should be associated with a higher equity premium. It also predicts, consistent with Colacito, Ghysels, and Meng (2013), that a higher chance of a boom will be associated with a lower premium, in a sample in which booms do not occur. Figure 8 shows the skewness conditional on the probability of disasters and booms. Indeed, skewness is decreasing in the disaster probability and increasing in the boom probability. Expected returns (in a sample without rare events) go in the opposite direction.

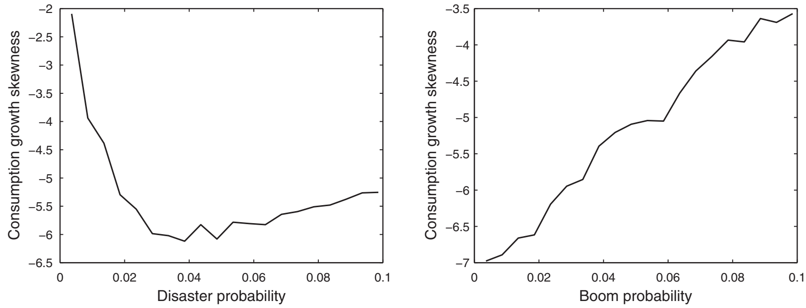


Figure 8
Consumption growth skewness
This figure shows consumption growth skewness conditional on different rare-event probabilities. The left plots the skewness at different disaster probability λ_1, with the boom probability equals to its mean. The right plots the skewness at different boom probability λ_2, with the disaster probability equals to its mean. We calculate skewness by simulating 500,000 years of consumption growth at a monthly frequency for each value of the rare event probabilities. We aggregate the monthly consumption growth to an annual frequency and calculate the skewness over the sample.

Measuring skewness, whether in the time series or the cross-section, is a challenge. Several recent papers, however, are able to calculate ex ante return skewness using option prices on individual stocks (Conrad, Dittmar, and Ghysels 2013; Chang, Christoffersen, and Jacobs 2013). These papers show that higher skewness is associated with lower returns in the cross-section, another prediction of this paper. Conrad, Dittmar, and Ghysels (2013) also finds that stocks with higher valuation ratios have higher skewness, again consistent with the results in this paper. Conrad, Dittmar, and Ghysels (2013) show that stocks with higher book-to-market ratios have lower coskewness, which is the relevant measure in this paper because booms are market-wide. They also report that stocks with higher overall skewness have higher price-to-earnings ratios. These facts are consistent with the finding in this paper that high valuations are tied to the small probability of very high returns.

4. Conclusion

This paper has addressed the question of how growth stocks can have both low returns and high risk, as measured by variance and covariance with the market portfolio. It does so within a framework that is also consistent with what we know about the aggregate market portfolio; namely, the high equity premium, high stock market volatility, and time-variation in the equity premium. The problem can be broken into two parts: why is the expected return on growth lower, and why is the abnormal return relative to the CAPM negative? This latter question is important, because one does not want to increase expected return through a counterfactual mechanism.

This paper answers the first of these questions as follows: growth stocks have, in population, a slightly higher expected return. In finite samples, however, this return may be measured as lower. The answer to the second question is different, because the abnormal return relative to the CAPM appears both in population and small samples without rare events. The abnormal return result arises because risk premiums are determined by two sources of risk, each of which is priced very differently by the representative agent. Covariance during disasters, and covariance with the changing disaster probability is assigned a high price by the representative agent because marginal utility is low in these states. However, growth stock returns are highly influenced by booms, and by the time-varying probability of booms. Because marginal utility is low in boom states, the representative agent requires little compensation for holding this risk. This two-factor structure is also successful in accounting for the joint predictive properties of the market portfolio and of the value-minus-growth return.

A number of extensions of the present framework are possible. In this paper, we have specified the growth and the value claim in a stark manner. Extending our results to a setting with richer firm dynamics would allow one to answer a broader set of questions. Further, we have chosen a relatively simple specification for the latent variables driving the economy. An open question is how the specification of these variables affects the observable quantities. Finally, we abstract from differential exposure to disaster risk. We leave these interesting topics to future research.

Appendix

A. Required Conditions on the Parameters

Assumption 1.

$$\left(\kappa_{\lambda_j}+\beta\right)^2 \geq 2\sigma^2_{\lambda_j} E_{\nu_j}\left[e^{b_{\mu_j} Z_j}-1\right] \qquad j=1,2.$$

Assumption 2.

$$(b_{\lambda_2}\sigma^2_{\lambda_2}-\kappa_{\lambda_2})^2 \geq 2\sigma^2_{\lambda_2} E_{\nu_2}\left[e^{b_{\mu_2} Z_2}\left(e^{\frac{\phi-1}{\kappa_{\mu_2}} Z_2}-1\right)\right].$$

Assumption 3.

$$\bar{\mu}_D-\bar{\mu}_C-\beta+\gamma\sigma^2(1-\phi)-\sum_{j=1,2}\frac{\kappa_{\lambda_j}\bar{\lambda}_j}{\sigma^2_{\lambda_j}}\left(\zeta_{\phi_j}-\kappa_{\lambda_j}+b_{\lambda_j}\sigma^2_{\lambda_j}\right)<0,$$

where

$$\zeta_{\phi_j}=\sqrt{(b_{\lambda_j}\sigma^2_{\lambda_j}-\kappa_{\lambda_j})^2-2E_{\nu_j}\left[e^{b_{\mu_j} Z_j}\left(e^{\frac{\phi-1}{\kappa_{\mu_j}} Z_j}-1\right)\right]\sigma^2_{\lambda_j}}.$$

Assumption 1 is required for the solution for the value function to exist. This restriction rules out parameters that can lead to infinitely negative utility. Note that for booms (assuming relative risk aversion is greater than 1), this restriction is satisfied automatically since the right-hand side is negative.

Assumptions 2 and 3 guarantee convergence of prices, which are given by integrating expected dividends into the infinite future. Assumption 2 ensures that $b_{\phi\lambda_2}(\tau)$ converges as τ approaches infinity, namely, that the effect of the boom on future dividends cannot explode as the horizon increases.[36] Assumption 3 states that the asymptotic slope of $a_\phi(\tau)$ is negative. This is the dynamic analog of the condition that the growth rate be less than the discount rate in the static Gordon growth model. This condition pertains to the market portfolio. If it is satisfied, the analogous condition for the value claim is satisfied automatically.[37]

B. Proofs of Theorems

This Appendix contains a detailed solution of the model. Appendix B.1 describes notation. Appendix B.2 describes the derivation of the value function and thus the state-price density under Assumption 1. Appendix B.3 contains the general results on pricing equities in a model with jumps. This section does not require parametric assumptions on the processes for dividends or state prices. Appendix B.4 and (most of) B.5 give results under the parametric assumptions in the main text.

B.1 Notation

- Definition of jump notation $\mathcal{J}(\cdot)$:
  Let X_t be a pure diffusion process, and let μ_{jt}, $j=1,2$ be defined as in (3). Consider a scalar function $h(\mu_{1t},\mu_{2t},X_t)$. Define
  $$\mathcal{J}_1(h(\mu_{1t},\mu_{2t},X_t)) = h(\mu_1+Z_1,\mu_2,X_t)-h(\mu_1,\mu_2,X_t),$$
  $$\mathcal{J}_2(h(\mu_{1t},\mu_{2t},X_t)) = h(\mu_1,\mu_2+Z_2,X_t)-h(\mu_1,\mu_2,X_t).$$
  Further, define
  $$\bar{\mathcal{J}}_j(h(\mu_{1t},\mu_{2t},X_t)) = E_{\nu_j}\mathcal{J}_j(h(\mu_{1t},\mu_{2t},X_t))$$
  for $j=1,2$, and
  $$\bar{\mathcal{J}}(h(\mu_{1t},\mu_{2t},X_t)) = \left[\bar{\mathcal{J}}_1(h(\mu_{1t},\mu_{2t},X_t)),\bar{\mathcal{J}}_2(h(\mu_{1t},\mu_{2t},X_t)\right]^\top.$$
- We use the notation κ_μ to denote the column vector $[\kappa_{\mu_1},\kappa_{\mu_2}]^\top$, and similarly for κ_λ, σ_λ, and $\bar{\lambda}$. Recall that we have already defined $\lambda_t=[\lambda_{1t},\lambda_{2t}]^\top$, $\mu_t=[\mu_{1t},\mu_{2t}]^\top$, $B_{\lambda t}=[B_{\lambda_1 t},B_{\lambda_2 t}]^\top$, and $B_t=[B_{Ct},B_{\lambda t}^\top]^\top$.
- We use x^2 notation for a vector x to denote the square of each element in x. For example, σ_λ^2 will denote the vector $[\sigma_{\lambda_1}^2,\sigma_{\lambda_2}^2]^\top$.

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36 Note that no extra assumptions are required for the convergence of $b_{\phi\lambda_1}(\tau)$ because $Z_1<0$ and hence $e^{\frac{\phi-1}{\kappa_{\mu_1}}Z_1}<1$. Also no extra assumptions are required for the value function expression $b^v_{\phi\lambda_2}(\tau)$ to converge since this condition replaces $e^{\frac{\phi-1}{\kappa_{\mu_2}}Z_2}$ with $e^{-\frac{1}{\kappa_{\mu_2}}Z_2}$, which is less than one.

37 Specifically, define
$$\zeta^v_{\phi_2}=\sqrt{(b_{\lambda_2}\sigma^2_{\lambda_2}-\kappa_{\lambda_2})^2-2E_{\nu_2}\left[e^{b_{\mu_2}Z_2}\left(e^{-\frac{1}{\kappa_{\mu_2}}Z_2}-1\right)\right]\sigma^2_{\lambda_2}}$$
Then $\zeta^v_{\phi_2}>\zeta_{\phi_2}$. It is also the case that $\zeta^v_{\phi_1}=\zeta_{\phi_1}$.

- We use the notation $*$ to denote element-by-element multiplication for vectors of the same dimensionality.
- Partial derivatives with respect to a vector will be assumed to be row vectors. That is, given a function $h(\mu)$, $\partial h/\partial\mu=[\partial h/\partial\mu_1, \partial h/\partial\mu_2]$, and similarly for λ.
- Because the processes λ are independent, cross-partial derivatives do not enter into the pricing equations. Thus, given a function $h(\lambda)$, we use the notation $\partial^2 h/\partial\lambda^2$ to denote the row vector $[\partial^2 h/\partial\lambda_1^2, \partial^2 h/\partial\lambda_2^2]$.

B.2 The State-Price Density

As a first step to computing the state-price density, we compute continuation value V_t as a function of aggregate wealth W_t and the state variables. As is often the case in models involving square root processes, there are technically two possible solutions corresponding to the choice of a solution to a quadratic equation. We select the solution such that, when there is no rare event risk, utility is equivalent to the case when there are in fact no rare events. Further discussion of this selection procedure is contained in Wachter (2013).

Lemma B.1. In equilibrium, continuation value $V_t = J(W_t, \mu_t, \lambda_t)$, where W_t is the wealth of the representative agent, and J is given as follows:

$$J(W_t,\mu_t,\lambda_t)=\frac{W_t^{1-\gamma}}{1-\gamma}I(\mu_t,\lambda_t), \tag{B.1}$$

where

$$I(\mu_t,\lambda_t)=\exp\left\{a+b_\mu^\top\mu_t+b_\lambda^\top\lambda_t\right\}, \tag{B.2}$$

for vectors $b_\mu=[b_{\mu_1}, b_{\mu_2}]^\top$ and $b_\lambda=[b_{\lambda_1}, b_{\lambda_2}]^\top$. The coefficients a, b_{μ_j}, and b_{λ_j} for $j=1,2$ take the following form:

$$a=\frac{1-\gamma}{\beta}\left(\bar{\mu}_C-\frac{1}{2}\gamma\sigma^2\right)+(1-\gamma)\log\beta+\frac{1}{\beta}b_\lambda^\top(\kappa_\lambda*\bar{\lambda}),$$

$$b_{\mu_j}=\frac{1-\gamma}{\kappa_{\mu_j}+\beta},$$

$$b_{\lambda_j}=\frac{1}{\sigma_{\lambda_j}^2}\left(\beta+\kappa_{\lambda_j}-\sqrt{\left(\beta+\kappa_{\lambda_j}\right)^2-2E_{\nu_j}\left[e^{b_{\mu_j}Z_{jt}}-1\right]\sigma_{\lambda_j}^2}\right).$$

Furthermore,

$$\frac{W_t}{C_t}=\beta^{-1}, \tag{B.3}$$

where C_t is aggregate consumption.

Proof. Let S_t denote the value of a claim to aggregate consumption, and conjecture that the price-dividend ratio for the consumption claim is constant:

$$\frac{S_t}{C_t}=l,$$

for some l. This relation implies that S_t satisfies

$$\frac{dS_t}{S_t}=\frac{dC_t}{C_t}=\mu_{Ct}\,dt+\sigma\,dB_{Ct}. \tag{B.4}$$

Consider an agent who allocates wealth between the claim to aggregate consumption and the risk-free asset. Let α_t be the fraction of wealth in the consumption claim, and let c_t be the agent's consumption. The wealth process is then given by

$$dW_t = \left(W_t\alpha_t\left(\mu_{Ct} - r_t + l^{-1}\right) + W_t r_t - c_t\right)dt + W_t\alpha_t\sigma\, dB_{ct},$$

where r_t denotes the instantaneous risk-free rate. Optimal consumption and portfolio choice must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{\alpha_t, c_t}\left\{\frac{\partial J}{\partial W}\left(W_t\alpha_t\left(\mu_{Ct} - r_t + l^{-1}\right) + W_t r_t - c_t\right) + \frac{\partial J}{\partial \lambda}\left(\kappa_\lambda * \left(\bar{\lambda} - \lambda_t\right)\right) - \frac{\partial J}{\partial \mu}\left(\kappa_\mu * \mu_t\right)\right.$$
$$\left. + \frac{1}{2}\frac{\partial^2 J}{\partial W^2}W_t^2\alpha_t^2\sigma^2 + \frac{1}{2}\left(\frac{\partial^2 J}{\partial \lambda^2}\right)(\sigma_\lambda^2 * \lambda_t) + \lambda_t^\top \bar{\mathcal{J}}(J(W_t, \mu_t, \lambda_t)) + f(c_t, J_t)\right\} = 0. \quad \text{(B.5)}$$

In equilibrium, $\alpha_t = 1$ and $c_t = C_t = W_t l^{-1}$. Substituting these policy functions into (B.5) implies

$$\frac{\partial J}{\partial W}W_t\mu_{Ct} + \frac{\partial J}{\partial \lambda}\left(\kappa_\lambda * \left(\bar{\lambda} - \lambda_t\right)\right) - \frac{\partial J}{\partial \mu}\left(\kappa_\mu * \mu_t\right) + \frac{1}{2}\frac{\partial^2 J}{\partial W^2}W_t^2\sigma^2$$
$$+ \frac{1}{2}\left(\frac{\partial^2 J}{\partial \lambda^2}\right)(\sigma_\lambda^2 * \lambda_t) + \lambda_t^\top \bar{\mathcal{J}}(J(W_t, \mu_t, \lambda_t)) + f(C_t, J_t) = 0. \quad \text{(B.6)}$$

From the envelope condition $\partial f / \partial C = \partial J / \partial W$, we obtain $\beta = l^{-1}$, and prove (B.3). Given that the consumption-wealth ratio equals β^{-1}, it follows that

$$f(C_t, V_t) = f\left(W_t l^{-1}, J(W_t, \mu_t, \lambda_t)\right)$$
$$= \beta W_t^{1-\gamma} I(\mu_t, \lambda_t)\left(\log\beta - \frac{\log I(\mu_t, \lambda_t)}{1-\gamma}\right), \quad \text{(B.7)}$$

where, to derive (B.7), we conjecture (B.1). Substituting (B.7) and (B.1) into (B.6), we find

$$\mu_{Ct} + (1-\gamma)^{-1}I^{-1}\frac{\partial I}{\partial \lambda}\left(\kappa_\lambda * \left(\bar{\lambda} - \lambda_t\right)\right) - (1-\gamma)^{-1}I^{-1}\frac{\partial I}{\partial \mu}\left(\kappa_\mu * \mu_t\right) - \frac{1}{2}\gamma\sigma^2$$
$$+ \frac{1}{2}(1-\gamma)^{-1}I^{-1}\left(\frac{\partial^2 I}{\partial \lambda^2}\right)(\sigma_\lambda^2 * \lambda_t) + (1-\gamma)^{-1}\lambda_t^\top I^{-1}\bar{\mathcal{J}}(I(\mu_t, \lambda_t))$$
$$+ \beta\left(\log\beta - \frac{\log I(\mu_t, \lambda_t)}{1-\gamma}\right) = 0.$$

Note that $\mu_{Ct} = \bar{\mu}_C + \mu_{1t} + \mu_{2t}$. Collecting coefficients on μ_{jt} results in linear equations for b_{μ_j} for $j = 1, 2$. Solving these equations yields

$$b_{\mu_j} = \frac{1-\gamma}{\kappa_{\mu_j} + \beta}, \quad j = 1, 2.$$

Collecting coefficients on λ_{jt} yields a quadratic equation for b_{μ_j} for $j=1,2$. Given the root selection procedure described at the start of this Appendix, the solutions are

$$b_{\lambda_j}=\frac{\beta+\kappa_{\lambda_j}}{\sigma^2_{\lambda_j}}-\sqrt{\left(\frac{\beta+\kappa_{\lambda_j}}{\sigma^2_{\lambda_j}}\right)^2-\frac{2E_{\nu_j}\left[e^{b_{\mu_j}Z_{jt}}-1\right]}{\sigma^2_{\lambda_j}}}.$$

Collecting the constant terms implies

$$a=\frac{1-\gamma}{\beta}\left(\bar{\mu}_C-\frac{1}{2}\gamma\sigma^2\right)+(1-\gamma)\log\beta+\sum_{j=1,2}b_{\lambda_j}\frac{\kappa_{\lambda_j}}{\beta}\bar{\lambda}_j.$$

This verifies our conjecture on the form of I. ■

Lemma B.2. State prices can be characterized as follows:

$$\pi_t=\exp\left\{\int_0^t\frac{\partial}{\partial V}f(C_s,V_s)ds\right\}\beta^\gamma C_t^{-\gamma}I(\mu_t,\lambda_t) \tag{B.8}$$

for $I(\mu_t,\lambda_t)$ given in Lemma B.1.

Proof. Duffie and Skiadas (1994) show that the state-price density π_t equals

$$\pi_t=\exp\left\{\int_0^t\frac{\partial}{\partial V}f(C_s,V_s)ds\right\}\frac{\partial}{\partial C}f(C_t,V_t). \tag{B.9}$$

From (6), we obtain

$$\frac{\partial}{\partial C}f(C_t,V_t)=\beta(1-\gamma)\frac{V_t}{C_t}.$$

Lemma B.1 shows that, in equilibrium, $V_t=J\left(\beta^{-1}C_t,\mu_t,\lambda_t\right)$, where the form of J is given in (B.1). It follows that

$$\frac{\partial}{\partial C}f(C_t,V_t)=\beta^\gamma C_t^{-\gamma}I(\mu_t,\lambda_t). \tag{B.10}$$

Equation B.8 then follows from (B.9). ■

Proof of Theorem 1. We apply Ito's lemma to (B.8) to find

$$\frac{d\pi_t}{\pi_{t-}}=\mu_{\pi t}dt+\sigma_{\pi t}dB_t+\sum_{j=1,2}\frac{\mathcal{J}_j(\pi_t)}{\pi_{t-}}dN_{jt},$$

where

$$\sigma_{\pi t}=\left[-\gamma\sigma,\;\; b_{\lambda_1}\sigma_{\lambda_1}\sqrt{\lambda_{1t}},\;\; b_{\lambda_2}\sigma_{\lambda_2}\sqrt{\lambda_{2t}}\right], \tag{B.11}$$

and

$$\frac{\mathcal{J}_j(\pi_t)}{\pi_t}=e^{b_{\mu_j}Z_{jt}}-1,\quad \text{for } j=1,2. \tag{B.12}$$

Moreover, from Lemma B.1, it follows that a, b_{μ_j}, and b_{λ_j} take the form described. It follows immediately that $b_{\mu_j}<0$. Because $Z_1<0$ and $b_{\mu_1}<0$, $E_{\nu_1}\left[e^{b_{\mu_1}Z_{1t}}-1\right]>0$. Therefore,

$$\sqrt{\left(\beta+\kappa_{\lambda_1}\right)^2-2E_{\nu_1}\left[e^{b_{\mu_1}Z_{1t}}-1\right]\sigma^2_{\lambda_1}}<\beta+\kappa_{\lambda_1}.$$

It follows that $b_{\lambda_1}>0$. Because $Z_2>0$ and $b_{\mu_2}<0$, $E_{\nu_2}\left[e^{b_{\mu_2}Z_{2t}}-1\right]<0$. Therefore,

$$\sqrt{\left(\beta+\kappa_{\lambda_2}\right)^2-2E_{\nu_2}\left[e^{b_{\mu_2}Z_{2t}}-1\right]\sigma^2_{\lambda_2}}>\beta+\kappa_{\lambda_2}$$

and $b_{\lambda_2}<0$. ■

Proof of Corollary 2. The risk-free rate is obtained by taking the derivative of (B.5) with respect to α_t, evaluating at the equilibrium value of $\alpha_t=1$ and setting it equal to 0. ■

Lemma B.3. The drift of the state-price density is given by

$$\mu_{\pi t}=-r_t-\lambda_t^\top \frac{\bar{\mathcal{J}}(\pi_t)}{\pi_t} \tag{B.13}$$

$$=-\beta-\mu_{Ct}+\gamma\sigma^2-\sum_{j=1,2}\lambda_{jt}E_{\nu_j}\left[e^{b\mu_j Z_{jt}}-1\right]. \tag{B.14}$$

Proof. Statement (B.13) follows from the absence of arbitrage. Statement (B.14) follows from Theorem 1, specifically,

$$\lambda_t^\top \frac{\bar{\mathcal{J}}(\pi_t)}{\pi_t}=\sum_{j=1,2}\lambda_{jt}E_{\nu_j}\left[e^{b\mu_j Z_{jt}}-1\right],$$

and from the expression for r_t in Corollary 2. ■

B.3 General Equity Pricing Results

We first establish a no-arbitrage restriction on the price today of a single dividend payment in τ periods.

Lemma B.4. Let $H_t=H(D_t,\mu_t,\lambda_t,\tau)$ denote the time-t price of a single future dividend payment at time $t+\tau$:

$$H(D_t,\mu_t,\lambda_t,\tau)=E_t\left[\frac{\pi_{t+\tau}}{\pi_t}D_{t+\tau}\right]. \tag{B.15}$$

Then,

$$\frac{dH_t}{H_t}=\mu_{H(\tau),t}dt+\sigma_{H(\tau),t}dB_t+\sum_{j=1,2}\frac{\mathcal{J}_j(H_t)}{H_t}dN_{jt} \tag{B.16}$$

for scalar processes $\mu_{H(\tau),t}$ and (row) vector processes $\sigma_{H(\tau),t}$. Moreover,

$$\mu_{\pi t}+\mu_{H(\tau),t}+\sigma_{\pi t}\sigma_{H(\tau),t}^\top+\frac{1}{\pi_t H_t}\lambda_t^\top\bar{\mathcal{J}}(\pi_t H_t)=0. \tag{B.17}$$

Proof. Equation B.16 is a result of Ito's lemma and the Markov property of the state variables and dividends (which implies that H is indeed a function as stated). No-arbitrage implies

$$\pi_t H(D_t,\lambda_t,\mu_t,T-t)=E_t\left[\pi_s H(D_s,\lambda_s,\mu_s,T-s)\right]\quad \text{for } s>t. \tag{B.18}$$

For the remainder of the argument, we simplify the notation by writing $\mu_{Ht}=\mu_{H(\tau),t}$ and $\sigma_{Ht}=\sigma_{H(\tau),t}$. Ito's lemma applied to $\pi_t H_t$ implies

$$\pi_t H_t=\pi_0 H_0+\int_0^t \pi_s H_s\left(\mu_{Hs}+\mu_{\pi s}+\sigma_{\pi s}\sigma_{Hs}^\top\right)ds+\int_0^t \pi_s H_s(\sigma_{Hs}+\sigma_{\pi s})dB_s$$

$$+\sum_{j=1,2}\sum_{0<s_{ij}\le t}\left(\pi_{s_{ij}}H_{s_{ij}}-\pi_{s_{ij}^-}H_{s_{ij}^-}\right), \tag{B.19}$$

where $s_{ij} = \inf\{s : N_{js} = i\}$ (namely, the time that the ith type-j jump occurs). Adding and subtracting the jump compensation term from (B.19) yields:

$$\pi_t H_t = \pi_0 H_0 + \underbrace{\int_0^t \pi_s H_s \left(\mu_{Hs} + \mu_{\pi s} + \sigma_{\pi s} \sigma_{Hs}^\top + \sum_{j=1,2} \lambda_j \frac{\bar{\mathcal{J}}_j(\pi_s H_s)}{\pi_s H_s} \right) ds}_{(1)}$$

$$+ \underbrace{\int_0^t \pi_s H_s (\sigma_{Hs} + \sigma_{\pi s}) dB_s}_{(2)}$$

$$+ \underbrace{\sum_{j=1,2} \left(\sum_{0 < s_{ij} \le t} \left(\pi_{s_{ij}} H_{s_{ij}} - \pi_{s_{ij}^-} H_{s_{ij}^-} \right) - \int_0^t \pi_s H_s \lambda_j \bar{\mathcal{J}}_j(\pi_s H_s) ds \right)}_{(3)}. \quad \text{(B.20)}$$

Equation B.18 implies that $\pi_t H_t$ is a martingale. Moreover, the terms labeled (2) and (3) on the right-hand side of (B.20) have zero expectation. Therefore, the term labeled (1) must also equal zero in expectation. The process $\pi_t H_t$ is strictly positive, and given that the equation must hold for all t, the integrand must equal zero. ■

We now prove an extension of Lemma B.4 that holds for the integral of functions $H(\cdot, \tau)$ over τ.

Lemma B.5. Define

$$F(D_t, \mu_t, \lambda_t) = \int_0^\infty H(D_t, \mu_t, \lambda_t, \tau) d\tau,$$

assuming this indefinite integral exists. Then

$$\frac{dF_t}{F_{t^-}} = \mu_{Ft} dt + \sigma_{Ft} dB_t + \sum_{j=1,2} \frac{\mathcal{J}_j(F_t)}{F_{t^-}} dN_{jt}, \quad \text{(B.21)}$$

for a scalar process μ_{Ft} and a 1×3 vector process σ_{Ft}, satisfying

$$\mu_{\pi t} + \mu_{Ft} + \frac{D_t}{F_t} + \sigma_{\pi t} \sigma_{Ft}^\top + \lambda_t^\top \frac{\bar{\mathcal{J}}(\pi_t F_t)}{\pi_t F_t} = 0. \quad \text{(B.22)}$$

Proof. First note that (B.21) follows from Ito's lemma. We will show (B.22) using the corresponding result for H, Lemma B.4.[38] By applying Ito's lemma applied to both F and H, we find

$$F(D_t, \mu_t, \lambda_t) \sigma_{Ft} = \int_0^\infty H(D_t, \mu_t, \lambda_t, \tau) \sigma_{H(\tau),t} d\tau,$$

and

$$\bar{\mathcal{J}}(\pi_t F(D_t, \mu_t, \lambda_t)) = \int_0^\infty \bar{\mathcal{J}}(\pi_t H(D_t, \mu_t, \lambda_t, \tau)) d\tau.$$

[38] It is also possible to show (B.22) directly along the same lines as Lemma B.4.

Moreover, because H is a function of τ, but F is not,

$$F(D_t,\mu_t,\lambda_t)\mu_{Ft}=\int_0^\infty \left(H(D_t,\lambda_t,\mu_t,\tau)\mu_{H(\tau),t}-\frac{\partial}{\partial\tau}H(D_t,\mu_t\lambda_t,\tau)\right)d\tau. \quad \text{(B.23)}$$

Equation B.23 can be rigorously derived by applying Ito's lemma to F, and differentiating under the integral sign. Namely,

$$F(D_t,\mu_t,\lambda_t)\mu_{Ft}=\int_0^\infty \left(H_t(\tau)\mu_{Dt}+\sum_{j=1,2}\frac{\partial}{\partial\lambda_j}H_t(\tau)(\bar{\lambda}_j-\lambda_j)\right.$$

$$\left.+\sum_{j=1,2}\frac{\partial}{\partial\mu_j}H_t(\tau)\mu_j+\frac{1}{2}\sum_{j=1,2}\frac{\partial^2}{\partial\lambda_j^2}H_t(\tau)\right)d\tau,$$

where $H_t(\tau)=H(D_t,\mu_t,\lambda_t,\tau)$. We then observe that $\mu_{H(\tau),t}$ is equal to the integrand, plus $\frac{\partial}{\partial\tau}H(D_t,\mu_t\lambda_t,\tau)$.

Finally,

$$-\int_0^\infty \frac{\partial}{\partial\tau}H(D_t,\mu_t,\lambda_t,\tau)d\tau=H(D_t,\mu_t,\lambda_t,0)=D_t.$$

The first equality holds because $H_t(\tau)$ must equal zero in the limit as τ approaches infinity; otherwise, the indefinite integral would not exist as assumed. The second equality follows from no-arbitrage. Equation B.22 then follows from Lemma B.4. ■

Consider the instantaneous expected return described in Section 1.3. Because we only need to assume a Markov structure, we will use the superscript D to denote the fact that we are describing returns on a general dividend claim, as opposed to, say, the market portfolio. The following lemma contains a convenient and general characterization of the risk premium in economies with jumps.

Lemma B.6. Let r_t^D denote the instantaneous expected return

$$r_t^D=\mu_{Ft}+\frac{D_t}{F_t}+\lambda_t^\top\frac{\bar{\mathcal{J}}(F_t)}{F_t}. \quad \text{(B.24)}$$

Then

$$r_t^D-r_t=-\sigma_{\pi t}\sigma_{Ft}^\top-\sum_{j=1,2}\lambda_{jt}E_{\nu_j}\left[\frac{\mathcal{J}_j(F_t)}{F_t}\frac{\mathcal{J}_j(\pi_t)}{\pi_t}\right]. \quad \text{(B.25)}$$

Proof. We start with the result in Lemma B.5, substituting in for $\mu_{\pi t}$ using (B.13):

$$\underbrace{-r_t-\lambda_t^\top\frac{\bar{\mathcal{J}}(\pi_t)}{\pi_t}}_{\mu_{\pi t}}+\mu_{Ft}+\frac{D_t}{F_t}+\sigma_{\pi t}\sigma_{Ft}^\top+\lambda_t^\top\frac{\bar{\mathcal{J}}(\pi_t F_t)}{\pi_t F_t}=0.$$

We then substitute in for $\mu_{Ft}+\frac{D_t}{F_t}$ using the definition of r_t^D, (B.24), and add and subtract $\lambda_t^\top\frac{\bar{\mathcal{J}}(F_t)}{F_t}$ to find

$$r_t^D-r_t+\sigma_{\pi t}\sigma_{Ft}^\top-\lambda_t^\top\left(\frac{\bar{\mathcal{J}}(\pi_t)}{\pi_t}+\frac{\bar{\mathcal{J}}(F_t)}{F_t}-\frac{\bar{\mathcal{J}}(\pi_t F_t)}{\pi_t F_t}\right)=0. \quad \text{(B.26)}$$

Finally note that by definition of $\mathcal{J}$,

$$E_{\nu_j}\left[\frac{\mathcal{J}_j(F_t)}{F_t}\frac{\mathcal{J}_j(\pi_t)}{\pi_t}\right]=\frac{\bar{\mathcal{J}}_j(F_t\pi_t)}{F_t\pi_t}-\frac{\bar{\mathcal{J}}_j(F_t)}{F_t}-\frac{\bar{\mathcal{J}}_j(\pi_t)}{\pi_t}, \text{for } j=1,2.$$

The result follows. ■

B.4 Pricing the Aggregate Market

We now specialize to the case of the market portfolio as defined in the main text.

Lemma B.7. Assume dividends follow the process (13) with the state-price density described in Appendix B.2. The function H, defined in (B.15), takes an exponential form:

$$H(D_t,\mu_t,\lambda_t,\tau)=D_t\exp\left\{a_\phi(\tau)+b_{\phi\mu}(\tau)^\top\mu_t+b_{\phi\lambda}(\tau)^\top\lambda_t\right\}, \tag{B.27}$$

where $b_{\phi\mu}=[b_{\phi\mu_1},b_{\phi\mu_2}]^\top$ and $b_{\phi\lambda}=[b_{\phi\lambda_1},b_{\phi\lambda_2}]^\top$ and

$$b_{\phi\mu_j}(\tau)=\frac{\phi-1}{\kappa_{\mu_j}}(1-e^{-\kappa_{\mu_j}\tau}), \tag{B.28}$$

$$\frac{db_{\phi\lambda_j}}{d\tau}=\frac{1}{2}\sigma_{\lambda_j}^2 b_{\phi\lambda_j}(\tau)^2+\left(b_{\lambda_j}\sigma_{\lambda_j}^2-\kappa_{\lambda_j}\right)b_{\phi\lambda_j}(\tau)+E_{\nu_j}\left[e^{b_{\mu_j}Z_{jt}}\left(e^{b_{\phi\mu_j}(\tau)Z_{jt}}-1\right)\right], \tag{B.29}$$

$$\frac{da_\phi}{d\tau}=\bar{\mu}_D-\bar{\mu}_C-\beta+\gamma\sigma^2(1-\phi)+b_{\phi\lambda}(\tau)^\top(\kappa_\lambda*\bar{\lambda}). \tag{B.30}$$

The boundary conditions are $b_{\phi\lambda_j}(0)=a_\phi(0)=0$.

Proof. First note that the boundary conditions follow from $H(D_t,\mu_t,\lambda_t,0)=D_t$. As earlier, we write $H_t=H(D_t,\mu_t,\lambda_t,\tau)$ to simplify notation. To prove the remaining statements, we conjecture the exponential form (B.27). Recall that

$$\frac{dH_t}{H_t}=\mu_{H(\tau),t}dt+\sigma_{H(\tau),t}dB_t+\sum_{j=1,2}\frac{\mathcal{J}_j(H_t)}{H_t}dN_{jt}$$

as in Lemma B.4. Ito's lemma applied to (B.27) implies

$$\frac{\bar{\mathcal{J}}_j(\pi_t H_t)}{\pi_t H_t}=E_{\nu_j}\left[e^{\left(b_{\mu_j}+b_{\mu_j\phi}(\tau)\right)Z_{jt}}-1\right], \tag{B.31}$$

$$\begin{aligned}\mu_{H(\tau),t}&=\frac{1}{H}\left(\frac{\partial H}{\partial D}\mu_{Dt}D_t+\frac{\partial H}{\partial\lambda}(\kappa_\lambda*(\bar{\lambda}-\lambda_t))-\frac{\partial H}{\partial\mu}(\kappa_\mu*\mu_t)-\frac{\partial H}{\partial\tau}+\frac{1}{2}\left(\frac{\partial^2 H}{\partial\lambda^2}\right)(\sigma_\lambda^2*\lambda_t)\right)\\&=\mu_{Dt}+b_{\phi\lambda}(\tau)^\top\left(\kappa_\lambda*(\bar{\lambda}-\lambda_t)\right)-b_{\phi\mu}(\tau)^\top\left(\kappa_\mu*\mu_t\right)\\&\quad-\left(\frac{da_\phi}{d\tau}+\lambda_t^\top\frac{db_{\phi\lambda}}{d\tau}+\mu_t^\top\frac{db_{\phi\mu}}{d\tau}\right)+\frac{1}{2}\left(b_{\phi\lambda}(\tau)^2\right)^\top\left(\sigma_\lambda^2*\lambda_t\right),\end{aligned} \tag{B.32}$$

and

$$\begin{aligned}\sigma_{H(\tau),t}&=\frac{1}{H}\left(\frac{\partial H}{\partial D}\left[D_t\phi\sigma,\ 0,\ 0\right]+\frac{\partial H}{\partial\lambda_1}\left[0,\ \sigma_{\lambda_1}\sqrt{\lambda_{1t}},\ 0\right]+\frac{\partial H}{\partial\lambda_2}\left[0,\ 0,\ \sigma_{\lambda_2}\sqrt{\lambda_{2t}}\right]\right)\\&=\left[\phi\sigma,\ b_{\phi\lambda_1}(\tau)\sigma_{\lambda_1}\sqrt{\lambda_{1t}},\ b_{\phi\lambda_2}(\tau)\sigma_{\lambda_2}\sqrt{\lambda_{2t}}\right].\end{aligned} \tag{B.33}$$

We now apply Lemma B.4, substituting (B.31–B.33), along with state-price density expressions (8) and (B.14) into the no-arbitrage condition (B.17) to find

$$\begin{aligned}&\mu_{Dt}+b_{\phi\lambda}(\tau)^\top\left(\kappa_\lambda*(\bar{\lambda}-\lambda_t)\right)-b_{\phi\mu}(\tau)^\top\left(\kappa_\mu*\mu_t\right)+\frac{1}{2}\left(b_{\phi\lambda}(\tau)^2\right)^\top\left(\sigma_\lambda^2*\lambda_t\right)\\&+b_{\phi\lambda}(\tau)^\top\left(b_\lambda*\sigma_\lambda^2*\lambda_t\right)-\beta-\mu_{Ct}+\gamma\sigma^2(1-\phi)+\sum_{j=1,2}\lambda_{jt}E_{\nu_j}\left[e^{\left(b_{\mu_j}+b_{\phi\mu_j}(\tau)\right)Z_{jt}}-e^{b_{\mu_j}Z_{jt}}\right]\\&\qquad-\left(\frac{da_\phi}{d\tau}+\lambda_t^\top\frac{db_{\phi\lambda}}{d\tau}+\mu_t^\top\frac{db_{\phi\mu}}{d\tau}\right)=0.\end{aligned}$$

Matching the terms multiplying μ_j implies

$$\frac{db_{\phi\mu_j}}{d\tau} = -\kappa_{\mu_j} b_{\phi_j\mu} + (\phi - 1);$$

Equation B.28 then follows from the boundary condition. Matching the terms multiplying λ_j implies (B.29) and matching the constant terms implies (B.30). This also verifies the conjectured form for H. ■

Under our assumptions, the solutions $b_{\phi\lambda_j}(\tau)$ have finite limits.

Lemma B.8.

$$\lim_{\tau\to\infty} b_{\phi\lambda_j}(\tau) = -\frac{1}{\sigma^2_{\lambda_j}}\left(\zeta_{\phi_j} - \kappa_{\lambda_j} + b_{\lambda_j}\sigma^2_{\lambda_j}\right), \tag{B.34}$$

where

$$\zeta_{\phi_j} = \sqrt{(b_{\lambda_j}\sigma^2_{\lambda_j} - \kappa_{\lambda_j})^2 - 2E_{\nu_j}\left[e^{\left(b_{\mu_j} + \frac{\phi-1}{\kappa_{\mu_j}}\right)Z_j} - e^{b_{\mu_j}Z_j}\right]\sigma^2_{\lambda_j}}. \tag{B.35}$$

Moreover, $\lim_{\tau\to\infty} b_{\phi\lambda_1}(\tau) < 0$ and $\lim_{\tau\to\infty} b_{\phi\lambda_2}(\tau) > 0$.

Proof. Let $\bar{b}_{\phi\lambda_j}$ denote the limit, should it exist. In the limit, small changes in τ do not change $b_{\phi\lambda_j}(\tau)$. Taking the limit of both sides of (17) implies that $\bar{b}_{\phi\lambda_j}$ must satisfy the quadratic equation

$$0 = \frac{1}{2}\sigma^2_{\lambda_j}\bar{b}^2_{\phi\lambda_j} + (b_{\lambda_j}\sigma^2_{\lambda_j} - \kappa_{\lambda_j})\bar{b}_{\phi\lambda_j} + E_{\nu_j}\left[e^{\left(b_{\mu_j} + \frac{\phi-1}{\kappa_{\mu_j}}\right)Z_{jt}} - e^{b_{\mu_j}Z_{jt}}\right].$$

Equation B.34 gives a solution to this equation.[39]

To prove that the limits have the signs given in the lemma, note that $Z_1 < 0$ implies that

$$E_{\nu_1}\left[e^{\left(b_{\mu_1} + \frac{\phi-1}{\kappa_{\mu_1}}\right)Z_1} - e^{b_{\mu_1}Z_1}\right] < 0.$$

Therefore,

$$\zeta_{\phi_1} > |b_{\lambda_1}\sigma^2_{\lambda_1} - \kappa_{\lambda_1}|.$$

Therefore, by (B.34) for $j = 1$, $\lim_{\tau\to\infty} b_{\phi\lambda_1}(\tau) < 0$.

Now, note that $Z_2 > 0$ implies that

$$E_{\nu_2}\left[e^{\left(b_{\mu_2} + \frac{\phi-1}{\kappa_{\mu_2}}\right)Z_2} - e^{b_{\mu_2}Z_2}\right] > 0.$$

Therefore,

$$\zeta_{\phi_2} < |b_{\lambda_2}\sigma^2_{\lambda_2} - \kappa_{\lambda_2}|.$$

(Assumption 2 implies that ζ_{ϕ_2} is real-valued.) Moreover, $b_{\lambda_2} < 0$ (Theorem 1), so

$$0 < \zeta_{\phi_2} < \kappa_{\lambda_2} - b_{\lambda_2}\sigma^2_{\lambda_2}.$$

Therefore, by (B.34) for $j = 2$, $\lim_{\tau\to\infty} b_{\phi\lambda_2}(\tau) > 0$. ■

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39 We use the same procedure to select which of the two solutions of the quadratic to use as we used in Appendix B.2. That is, we chose the one in which rare events do not affect prices if there are in fact no rare events. We have verified that (B.34) does indeed correspond to the limit when the ordinary differential equation (17) is solved numerically.

While we cannot solve for $b_{\phi\lambda_j}(\tau)$ in closed form, we can obtain closed-form solutions for the related problem of instantaneous disasters. The difference between the equations is that the last term in (B.29) becomes a constant, rather than a decreasing function of τ. We can use what we know about the instantaneous-disaster case to put bounds on the solution for our current case. These bounds allow us to establish monotonicity.

Lemma B.9. Fix $\tau^* > 0$. Let $b_{\phi\lambda_j}{}^*(\tau)$ be the solution to

$$\frac{db^*_{\phi\lambda_j}}{d\tau} = \frac{1}{2}\sigma^2_{\lambda_j} b^*_{\phi\lambda_j}(\tau)^2 + \left(b_{\lambda_j}\sigma^2_{\lambda_j} - \kappa_{\lambda_j}\right) b^*_{\phi\lambda_j}(\tau) + E_{\nu_j}\left[e^{b_{\mu_j} Z_{jt}}\left(e^{b_{\phi\mu_j}(\tau^*) Z_{jt}} - 1\right)\right] \quad \text{(B.36)}$$

with boundary condition $b_{\phi\lambda_j}{}^*(0) = 0$. Then $b^*_{\phi\lambda_1}(\tau) < b_{\phi\lambda_1}(\tau) < 0$ and $b^*_{\phi\lambda_2}(\tau) > b_{\phi\lambda_2}(\tau) > 0$ for τ such that $0 < \tau < \tau^*$.

Proof. Consider $j = 1$. It follows from (B.29) and the boundary condition that

$$\left.\frac{db_{\phi\lambda_1}}{d\tau}\right|_{\tau=0} = 0,$$

$$\left.\frac{d^2 b_{\phi\lambda_1}}{d\tau^2}\right|_{\tau=0} < 0.$$

It follows from (B.36) that

$$\left.\frac{db^*_{\phi\lambda_1}}{d\tau}\right|_{\tau=0} < 0.$$

Therefore, because $b_{\phi\lambda_1}(0) = b^*_{\phi\lambda_1}(0) = 0$, there exists $\underline{\tau} > 0$ but sufficiently small such that $b^*_{\phi\lambda_1}(\tau) < b_{\phi\lambda_1}(\tau) < 0$ for $\tau < \underline{\tau}$.

Assume, by contradiction, that there exists a $\tau < \tau^*$ such that $b_{\phi\lambda_1}(\tau) \le b^*_{\phi\lambda_1}(\tau)$. Because $b_{\phi\lambda_1}(\tau)$ and $b^*_{\phi\lambda_1}(\tau)$ are smooth functions, there exists a $\hat{\tau} < \tau^*$ such that $b^*_{\phi\lambda_1}(\hat{\tau}) = b_{\phi\lambda_1}(\hat{\tau})$ and such that $b^*_{\phi\lambda_1}(\tau) - b_{\phi\lambda_1}(\tau)$ is increasing in a neighborhood of $\hat{\tau}$. However, because $b_{\phi\mu_1}(\tau^*) Z_{1t} < b_{\phi\mu_1}(\hat{\tau}) Z_{1t}$, it follows from (B.36) and (B.29) that

$$\left.\frac{db^*_{\phi\lambda_1}}{d\tau}\right|_{\tau=\hat{\tau}} < \left.\frac{db_{\phi\lambda_1}}{d\tau}\right|_{\tau=\hat{\tau}},$$

which is a contradiction. Similar reasoning shows that $b_{\phi\lambda_1}(\tau) < 0$ and the results for $j = 2$.[40] ■

Corollary B.10. The function $b_{\phi\lambda_1}(\tau)$ is negative and decreasing. The function $b_{\phi\lambda_2}(\tau)$ is positive and increasing.

Proof. The results $b_{\phi\lambda_1}(\tau) < 0$ and $b_{\phi\lambda_2}(\tau) > 0$ are shown in the lemma above. Consider $j = 1$ and fix $\tau^* > 0$. Define $b^*_{\phi\lambda_1}(\tau)$ as in Lemma B.9. Because $db^*_{\phi\lambda_1}/d\tau < 0$ (see Wachter 2013), it follows from (B.36) that

$$\frac{1}{2}\sigma^2_{\lambda_1} b^*_{\phi\lambda_1}(\tau)^2 + \left(b_{\lambda_1}\sigma^2_{\lambda_1} - \kappa_{\lambda_1}\right) b^*_{\phi\lambda_1}(\tau) + E_{\nu_1}\left[e^{b_{\mu_1} Z_{1t}}\left(e^{b_{\phi\mu_1}(\tau^*) Z_{1t}} - 1\right)\right] < 0. \quad \text{(B.37)}$$

40 For $j = 2$, we need Assumption 2 to guarantee that the function exists and is well behaved.

This inequality must hold for the entire range of values $b^*_{\phi\lambda_1}(\tau)$ for τ between 0 and infinity. That is,

$$\frac{1}{2}\sigma^2_{\lambda_1}x^2+\left(b_{\lambda_1}\sigma^2_{\lambda_1}-\kappa_{\lambda_1}\right)x+E_{\nu_1}\left[e^{b_{\mu_1}Z_{1t}}\left(e^{b_{\phi\mu_1}(\tau^*)Z_{1t}}-1\right)\right]<0 \quad x\in(\inf b^*_{\phi\lambda_1}(\tau),0).$$

Moreover, by Lemma B.9,

$$\inf b^*_{\phi\lambda_1}(\tau)<b^*_{\phi\lambda_1}(\tau^*)<b_{\phi\lambda_1}(\tau^*).$$

Thus,

$$\frac{1}{2}\sigma^2_{\lambda_1}b^*_{\phi\lambda_1}(\tau^*)^2+\left(b_{\lambda_1}\sigma^2_{\lambda_1}-\kappa_{\lambda_1}\right)b^*_{\phi\lambda_1}(\tau^*)+E_{\nu_1}\left[e^{b_{\mu_1}Z_{1t}}\left(e^{b_{\phi\mu_1}(\tau^*)Z_{1t}}-1\right)\right]<0.$$

It follows from (B.29) that the left-hand side is equal to $db_{\phi\lambda_1}/d\tau\big|_{\tau=\tau^*}$, and therefore that $b_{\phi\lambda_1}(\tau)$ is a decreasing function of τ. The result follows because τ^* is arbitrary. The same reasoning works for $j=2$. ■

The following lemma shows that the integral determining the price of the market converges. The logic is similar to that for convergence of a geometric sum.

Lemma B.11.

$$\lim_{\bar{\tau}\to\infty}\int_0^{\bar{\tau}}H(D,\mu,\lambda,\tau)d\tau<\infty$$

for all $D>0$, vectors λ with both elements ≥ 0 and all μ_{jt}.

Proof. Define $\bar{b}_{\phi\lambda_2}$ to be the limit defined in Lemma B.8, and $\bar{b}_{\phi\mu_2}$ to be the limit of $b_{\phi\mu_2}(\tau)$ (the existence of this limit follows immediately from the form of the function). By Corollary B.10,

$$H(D,\mu,\lambda,\tau)\leq De^{\bar{b}_{\phi\lambda_2}\lambda_{2t}+\bar{b}_{\phi\mu_2}\mu_{2t}}e^{a_\phi(\tau)},$$

and thus it suffices to show that

$$\lim_{\bar{\tau}\to\infty}\int_0^{\bar{\tau}}e^{a_\phi(\tau)}d\tau<\infty.$$

Note that by (18),

$$a_\phi(\tau)=\left(\bar{\mu}_D-\bar{\mu}_C-\beta+\gamma\sigma^2(1-\phi)\right)\tau+\sum_{j=1,2}\kappa_{\lambda_j}\bar{\lambda}_j\int_0^\tau b_{\phi\lambda_j}(u)du.$$

Now define

$$\bar{a}_\phi=\bar{\mu}_D-\bar{\mu}_C-\beta+\gamma\sigma^2(1-\phi)+\bar{b}_{\phi\lambda_2}\kappa_{\lambda_2}\bar{\lambda}_2.$$

It follows from the fact that $b_{\phi\lambda_2}(\tau)>0$ and is increasing (Corollary B.10) that

$$a_\phi(\tau)\leq\bar{a}_\phi\tau+\kappa_{\lambda_1}\bar{\lambda}_1\int_0^\tau b_{\phi\lambda_1}(u)du.$$

Because $b_{\phi\lambda_1}(\tau)$ is decreasing, by Assumption 3 and by the equations for the limits in Lemma B.8, there exists a τ_0 such that

$$\bar{a}_\phi+b_{\phi\lambda_1}(\tau_0)\kappa_{\lambda_1}\bar{\lambda}_1<0.$$

Write

$$\begin{aligned}\int_0^{\bar{\tau}}e^{a_\phi(\tau)}d\tau&\leq\int_0^{\bar{\tau}}e^{\bar{a}_\phi\tau+\int_0^\tau b_{\phi\lambda_1}(u)\kappa_{\lambda_1}\bar{\lambda}_1du}d\tau\\&=e^{\bar{a}_\phi\tau_0+\int_0^{\tau_0}b_{\phi\lambda_1}(u)\kappa_{\lambda_1}\bar{\lambda}_1du}\int_0^{\bar{\tau}}e^{\bar{a}_\phi(\tau-\tau_0)+\kappa_{\lambda_1}\bar{\lambda}_1\int_{\tau_0}^\tau b_{\phi\lambda_1}(u)du}d\tau\\&\leq e^{\bar{a}_\phi\tau_0+\int_0^{\tau_0}b_{\phi\lambda_1}(u)\kappa_{\lambda_1}\bar{\lambda}_1du}\int_0^{\bar{\tau}}e^{\bar{a}_\phi(\tau-\tau_0)+\kappa_{\lambda_1}\bar{\lambda}_1b_{\phi\lambda_1}(\tau_0)(\tau-\tau_0)}d\tau,\end{aligned}$$

where the last line follows from the monotonicity of $b_{\phi\lambda_1}(\tau)$. Convergence as $\bar{\tau}\to\infty$ follows from the properties of the exponential function. ■

Proof of Theorem 3. Note that, by no-arbitrage,

$$F(D_t,\mu_t,\lambda_t)=E_t\int_t^\infty \frac{\pi_s}{\pi_t}D_s\,ds,$$

assuming the right-hand side is well-defined. It follows from Lemma B.7 that $H(D_t,\mu_t,\lambda_t,\tau)= E_t\left[\frac{\pi_{t+\tau}}{\pi_t}D_\tau\right]$ takes the required form. Lemma B.11 shows that the indefinite integral of these terms exists, and therefore we can write

$$F(D_t,\mu_t,\lambda_t)=\int_0^\infty H(D_t,\mu_t,\lambda_t,\tau)\,d\tau.$$

The equation for the price-dividend ratio follows immediately from dividing this equation by D_t.■

Proof of Theorem 5. We use the general formula for the risk premium, given in Lemma B.6:

$$r_t^m-r_t=-\sigma_{\pi t}\sigma_{Ft}^\top-\sum_{j=1,2}\lambda_{jt}E_{\nu_j}\left[\frac{\mathcal{J}_j(F_t)}{F_t}\frac{\mathcal{J}_j(\pi_t)}{\pi_t}\right].$$

It follows from $F(D_t,\mu_t,\lambda_t)=D_tG(\mu_t,\lambda_t)$ and Theorem 3 that

$$\sigma_{Ft}=\left[\phi\sigma,\ \ \frac{1}{G}\frac{\partial G}{\partial\lambda_1}\sigma_{\lambda_1}\sqrt{\lambda_{1t}},\ \ \frac{1}{G}\frac{\partial G}{\partial\lambda_2}\sigma_{\lambda_2}\sqrt{\lambda_{2t}}\right].$$

Because dividends do not jump,

$$\frac{\mathcal{J}_j(F_t)}{F_{t^-}}=\frac{\mathcal{J}_j(G_t)}{G_{t^-}}.$$

Substituting in from (B.11) and (B.12) leads to the equation for the equity premium:

$$r_t^m-r_t=\phi\gamma\sigma^2-\sum_{j=1,2}\lambda_{jt}E_{\nu_j}\left[\left(e^{b_{\mu_j}Z_{jt}}-1\right)\frac{\mathcal{J}_j(G_t)}{G_t}\right]-\sum_{j=1,2}\lambda_{jt}\frac{1}{G_t}\frac{\partial G}{\partial\lambda_j}b_{\lambda_j}\sigma_{\lambda_j}^2.$$

■

B.5 Pricing Results for Value and Growth

We first show that the price-dividend ratio for value lies below that of the market. This result does not depend on our specific assumptions about the dividend process.

Proof of Corollary 9. Consider an asset with dividend stream given by

$$\frac{dD_s^v}{D_s^v}=\mu_{Ds}^v dt+\phi\sigma\,dB_{Cs},$$

and $\mu_{Ds}^v=\bar{\mu}_D+\phi\mu_{1s}$, for $s\geq t$, and normalize D_t^v to D_t, the dividend on the market. Comparing the evolution of D_s^v with D_s (given in (13)), it follows that

$$D_s^v\leq D_s\quad \text{for } s\geq t, \tag{B.38}$$

with a positive probability that the inequality is strict. Therefore, because state prices are strictly positive,

$$E_t\int_t^\infty\frac{\pi_s}{\pi_t}D_s^v\,ds<E_t\int_t^\infty\frac{\pi_s}{\pi_t}D_s\,ds.$$

Dividing by D_t implies

$$E_t\int_t^\infty\frac{\pi_s}{\pi_t}\frac{D_s^v}{D_t}ds<E\int_t^\infty\frac{\pi_s}{\pi_t}\frac{D_s}{D_t}ds.$$

Because $D_t^v=D_t$, the left-hand side is the price-dividend ratio on the value claim. By the same reason, the right-hand side is the price-dividend ratio on the market. ■

This result does not require that we define the value claim at time t as an asset having the same dividend as the market. Normalizing the dividends to be equal is a technical step within the proof, not an assumption. The price-dividend ratio on the value claim is determined only by future growth in dividends and in the state-price density, not by the level of current dividends.

It is convenient to state the pricing equations for the value claim. This result follows from an argument directly analogous to that of Theorem 3.

Corollary B.12. Let $F^v(D_t,\mu_t,\lambda_t)$ denote the price of the value sector, namely, the claim to cash flows satisfying (25), where D_t is the current dividend.

$$F^v(D_t,\mu_t,\lambda_t)=\int_0^\infty H^v(D_t,\mu_t,\lambda_t,\tau)d\tau, \tag{B.39}$$

where

$$H^v(D_t,\mu_t,\lambda_t,\tau)=D_t\exp\left\{a^v_\phi(\tau)+b^v_{\phi\mu}(\tau)^\top\mu_t+b^v_{\phi\lambda}(\tau)^\top\lambda_t\right\}, \tag{B.40}$$

$$b^v_{\phi\mu_1}(\tau)=\frac{\phi-1}{\kappa_{\mu_1}}\left(1-e^{-\kappa_{\mu_1}\tau}\right), \tag{B.41}$$

$$b^v_{\phi\mu_2}(\tau)=-\frac{1}{\kappa_{\mu_2}}\left(1-e^{-\kappa_{\mu_2}\tau}\right), \tag{B.42}$$

while $b^v_{\phi\lambda_j}(\tau)$ (for $j=1,2$) and $a_\phi(\tau)$ satisfy

$$\frac{db^v_{\phi\lambda_j}}{d\tau}=\frac{1}{2}\sigma^2_{\lambda_j}b^v_{\phi\lambda_j}(\tau)^2+\left(b_{\lambda_j}\sigma^2_{\lambda_j}-\kappa_{\lambda_j}\right)b^v_{\phi\lambda_j}(\tau)+E_{\nu_j}\left[e^{b_{\mu_j}Z_{jt}}\left(e^{b^v_{\phi\mu_j}(\tau)Z_{jt}}-1\right)\right], \tag{B.43}$$

$$\frac{da^v_\phi}{d\tau}=\bar{\mu}_D-\bar{\mu}_C-\beta+\gamma\sigma^2(1-\phi)+b^v_{\phi\lambda_j}(\tau)^\top(\kappa_\lambda*\bar{\lambda}) \tag{B.44}$$

with boundary conditions $b^v_{\phi\lambda_j}(0)=a^v_\phi(0)=0$. Furthermore, the price-dividend ratio on the value sector is given by

$$G^v(\mu_t,\lambda_t)=\int_0^\infty \exp\left(a^v_\phi(\tau)+b^v_{\phi\mu}(\tau)^\top\mu_t+b^v_{\phi_\lambda}(\tau)^\top\lambda_t\right)d\tau. \tag{B.45}$$

Corollary B.13. The functions $b^v_{\phi\lambda_j}(\tau)$ are negative and decreasing.

Proof. See Lemma B.9 and Corollary B.10. ■

We now show that price of the growth claim is increasing in μ_1 and μ_2, decreasing in λ_1 and increasing in λ_2 (Corollary 14 in the main text), using the corresponding results for the market and the value sector.

Proof of Corollary 14. It follows from the definition of the growth sector price,

$$F^g_t=F_t-F^v_t,$$

that

$$\frac{\partial F^g_t}{\partial\mu_{jt}}=\frac{\partial F_t}{\partial\mu_{jt}}-\frac{\partial F^v_t}{\partial\mu_{jt}} \tag{B.46}$$

and

$$\frac{\partial F^g_t}{\partial\lambda_{jt}}=\frac{\partial F_t}{\partial\lambda_{jt}}-\frac{\partial F^v_t}{\partial\lambda_{jt}}.$$

It follows from Corollary 4 that $\frac{\partial F_t}{\partial\mu_{2t}}>0$ and from Corollary 10 that $\frac{\partial F^v_t}{\partial\mu_{2t}}<0$, and similarly for derivatives with respect to λ_{2t}. Therefore, the price of the growth sector is increasing in μ_{2t} and λ_{2t}.

Now consider $j=1$. Theorem 3 and Corollary B.12 imply that we can rewrite (B.46) as

$$\frac{\partial F_t^g}{\partial \mu_{1t}} = \int_0^\infty b_{\phi\mu_1}(\tau) H(D_t, \mu_t, \lambda_t, \tau) d\tau - \int_0^\infty b_{\phi\mu_1}^v(\tau) H^v(D_t^v, \mu_t, \lambda_t, \tau) d\tau.$$

Moreover, $b_{\phi\mu_1}(\tau) = b_{\phi\mu_1}^v(\tau)$. Because the dividend at time $t+\tau$ on value is less than or equal to the dividend on the market, and strictly less with positive probability, $H(D_t, \mu_t, \lambda_t, \tau) > H^v(D_t^v, \mu_t, \lambda_t, \tau)$. The result follows. The same argument works for the derivative with respect to λ_{1t}. ■

C. Return Simulation

This section describes how we simulate returns on the market and the value and growth sectors, given a simulated series of state variables and one-period observations on dividend growth. We simulate at a monthly frequency, which is sufficiently fine to capture the joint distribution of consumption, dividends, and the state variables.

Consider an asset with price process F_t and dividend D_t. The realized return between time t and $t+\Delta t$ is given by

$$R_{t,t+\Delta t} = \frac{F_{t+\Delta t} + \int_t^{t+\Delta t} D_s \, ds}{F_t}$$

(see Duffie 2001, Chapter 6.L). Consider an asset that pays a positive dividend at each point in time (namely, the market or value claim). Using the approximation $D_{t+\Delta t} \Delta t \approx \int_t^{t+\Delta t} D_s \, ds$, we can compute returns as

$$\begin{aligned} R_{t,t+\Delta t} &\approx \frac{F_{t+\Delta t} + D_{t+\Delta t} \Delta t}{F_t} \\ &= \frac{\frac{F_{t+\Delta t}}{D_{t+\Delta t}} + \Delta t}{\frac{F_t}{D_t}} \frac{D_{t+\Delta t}}{D_t} \\ &= \frac{G(\mu_{t+\Delta t}, \lambda_{t+\Delta t}) + \Delta t}{G(\mu_t, \lambda_t)} \frac{D_{t+\Delta t}}{D_t}, \end{aligned} \quad \text{(C.1)}$$

where G is the price-dividend ratio on the asset. The price-dividend ratio series is obtained from the simulated series of state variables using (19) for the market and (28) for value. Note that this definition of the return makes it unnecessary to have information on either the level of dividends or the level of prices. Thus, this definition is invariant to whether value is redefined to have the same dividend at time t as the market.

Because the growth sector does not pay a dividend every period, we cannot use (C.1) to compute its return. However, its return is implied by the fact that the market is a portfolio of value and growth. Let $R^m_{t,t+\Delta t}$ denote the market return and $R^v_{t,t+\Delta t}$ denote the return on the value sector, each computed using the relevant quantities in (C.1). Furthermore, recall that F_t^v is the price of the value sector at time t and F_t is the price of the market. Then,

$$R^m_{t,t+\Delta t} \approx \frac{F_t^v}{F_t} R^v_{t,t+\Delta t} + \left(1 - \frac{F_t^v}{F_t}\right) R^g_{t,t+\Delta t}. \quad \text{(C.2)}$$

This equation is approximate rather than exact because the model is formulated in continuous time. Solving for the growth return in (C.2) yields

$$R^g_{t,t+\Delta t} = \frac{1}{1 - \frac{F_t^v}{F_t}} \left(R^m_{t,t+\Delta t} - \frac{F_t^v}{F_t} R^v_{t,t+\Delta t} \right). \quad \text{(C.3)}$$

We now assume that the time-t dividend on value and on the market are the same. It follows that

$$\frac{F_t^v}{F_t} = \frac{G^v(\mu_t, \lambda_t)}{G(\mu_t, \lambda_t)}. \quad \text{(C.4)}$$

Growth returns can then be computed using the state variable series and returns on value and the market using (C.3) and (C.4).

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