

**FUNDING INVESTMENTS**  
**FINANCE 238/738, Spring 2008, Prof. Musto**  
**Class 5 – Review of Option Pricing**

- I. Put-Call Parity
- II. One-Period Binomial Option Pricing
- III. Adding Periods to the Binomial Model
- IV. Black-Scholes

**I. Put-Call Parity**

Suppose you have a stock, selling for  $S$ , and two options on that stock

- Option to buy that stock for  $X$  on a given day in the future
  - This is a *call* option, and it costs  $C$
  - $X$  is the *strike price*
  - The given day in the future is the *expiration date*
  - Because the option applies *only* to the expiration date, this is called a *European* option. If it were usable before the expiration date, it would be called an *American* option.
- Option to sell that stock for  $X$  on a given day in the future
  - This is a *put* option, and it costs  $P$
  - Note: same strike price and expiration date as the call

You can also buy a STRIP maturing on the expiration date with face value  $X$

- We will denote its current price as  $PV(X)$  (i.e. the *Present Value* of getting  $X$  on the expiration date)

On the expiration date, the stock will be some price, call it  $S^*$ . Note that

- On that date, the option to *buy* the stock for  $X$  will be
  - worthless if  $S^* < X$  (better just to buy it at the market price)
  - worth  $S^* - X$  if  $S^* > X$  (buy the stock for  $X$ , sell it for  $S^*$ )
- The option to *sell* the stock for  $X$  will be
  - worth  $X - S^*$  if  $S^* < X$  (buy the stock for  $S^*$ , sell it for  $X$ )
  - worthless if  $S^* > X$  (better just to sell it at the market price)

Put-Call Parity refers to a necessary relation between the prices  $C$ ,  $P$ ,  $S$ , and  $PV(X)$ : it must be true that

$$C + PV(X) = S + P$$

or else there's an easy and foolproof arbitrage. Compare the expiration-date values of two portfolios:

- Portfolio #1: Buy the call and the STRIP
- Portfolio #2: Buy the stock and the put

Portfolio #1:

On the expiration day,  $S^*$  will be either higher or lower than  $X$ .

	$\frac{S^* < X}{}$	$\frac{S^* > X}{}$
Value of call:	0	$S^* - X$
Value of STRIP:	$X$	$X$
TOTAL:	$X$	$S^*$

Portfolio #2:

	$\frac{S^* < X}{}$	$\frac{S^* > X}{}$
Value of put:	$X - S^*$	0
Value of stock:	$S^*$	$S^*$
TOTAL:	$X$	$S^*$

So the two portfolios have *exactly* the same value on the expiration date

- And neither pays anything *before* the expiration date
- So their payoffs are identical, so their current prices should be identical
  - If they aren't, buy the cheap one and sell the rich one
- Portfolio #1 costs  $C + PV(X)$
- Portfolio #2 costs  $P + S$

So there it is.

One implication of put-call parity is that (in the context of European options) you only have to know how to calculate a call price, because you just plug that price into the put-call parity relation to get the put price.

## II. One-Period Binomial Option Pricing

The basis of all option pricing everywhere is the *binomial option pricing model*, sometimes abbreviated BOPM. What to notice about this approach is that it is a simple arbitrage argument, repeated over and over.

Suppose there are two dates – which we'll call now and a year from now, and there are two securities you could buy

- **A risk-free bond** paying the risk-free rate  $r_F$ 
  - Pay  $1/(1+r_F)$  today, get 1 in a year, *or*
  - *Get*  $1/(1+r_F)$  today, *pay* 1 in a year, *i.e. you can borrow or lend at  $r_F$*
  - For example,  $r_F=4.04\%$ , so
    - pay  $1/1.0404=0.9612$  today, get 1 in a year, *or*
    - get 0.9612 today, pay 1 in a year
- **A risky stock** that costs  $S$  today, and will either
  - Go *up* to  $S_{up}$ , or
  - Go *down* to  $S_{down}$
  - This is the big simplification of this initial model, that there are only two values the risky asset can have in a year
  - For example, a stock that costs 50 today, and will either go up to 80 or down to 30
- Notice, I'm not saying anything about the probability of going up vs. going down.

We want to know the value  $C$  of a call option with strike price of  $X$  and expiration date of next year

- In our example, a call with strike price 55

*Step 1:* Figure out what the call would be worth if the stock went *up*, and if the stock went *down*. Call these numbers  $C_{up}$  and  $C_{down}$ .

- Stock goes up: Call is worth  $\max\{0, S_{up}-X\} = C_{up}$ 
  - If the stock goes to 80, the option to buy for 55 is worth  $80-55=25$
- Stock goes down: Call is worth  $\max\{0, S_{down}-X\} = C_{down}$ 
  - If the stock goes to 30, the option to buy for 55 is worth 0

So we don't yet know what the call is worth today, but we *do* know what it *will* be worth in each scenario *next year*.

*Step 2:* Calculate the portfolio of the stock and the bond that will be worth  $C_{up}$  if the stock goes up, and  $C_{down}$  if the stock goes down.

*Simple Math Problem:* Suppose you

- Buy  $n$  shares of the stock, which costs  $nS$ , and will be worth
  - $nS_{up}$  if the stock goes up
  - $nS_{down}$  if the stock goes down
- Lend  $B$  at the risk-free rate, which will be worth
  - $B(1+r_F)$ , no matter what

This stock will be worth  $C_{up}$  if the stock goes up and  $C_{down}$  if the stock goes down, *if*

$$\begin{aligned} nS_{up} + B(1+r_F) &= C_{up} \\ nS_{down} + B(1+r_F) &= C_{down} \end{aligned}$$

So we have two equations, and two –  $n$ , and  $B$  – unknowns.

The solution:

$$\begin{aligned} n &= (C_{up}-C_{down})/(S_{up}-S_{down}) \\ B &= (1/(1+r_F))[(C_{down}S_{up}-C_{up}S_{down})/(S_{up}-S_{down})] \end{aligned}$$

So the price of the call is the cost of buying  $n$  shares and lending  $B$  dollars, i.e.  $C = nS+B$

Going back to our example numbers, we get

$$\begin{aligned} n &= (25-0)/(80-30) = 0.5 \\ B &= (1/1.0404)[((0)(80)-(25)(30))/(80-30)] = (1/1.0404)[(-750)/50] = -14.4175 \end{aligned}$$

So we would buy half a share, and lend  $-14.4175$ , or in other words *borrow* 14.4175 so that we would owe 15 next year. The total effect is

- Stock goes to 80:  $\frac{1}{2}$  share is worth 40, owe 15, so 25 net, *same as  $C_{up}$*
- Stock goes to 30:  $\frac{1}{2}$  share is worth 15, owe 15, so 0, *same as  $C_{down}$*

*Exactly the same payoff as the Call, no matter what happens!*

*Step 3:* Price of the call option = price of the portfolio,  $nS+B$

- In our example,  $(0.5)(50) + (-14.4175) = 10.58$

### III. Adding Periods to the Binomial Model

The one-period model gives the exact, arbitrage-free call price, but it is not directly useful for pricing a real-world option

- In the real world, there are many possible future prices, not just two
- We can address this shortcoming, while retaining the arbitrage logic, by subdividing the period
- In each subperiod, the stock goes up to one price or down to another

Let's go back to the example from section II, and divide the year in half. The stock still starts at 50, but in *half* a year it will either

- Go up to 65, or
- Go down to 40

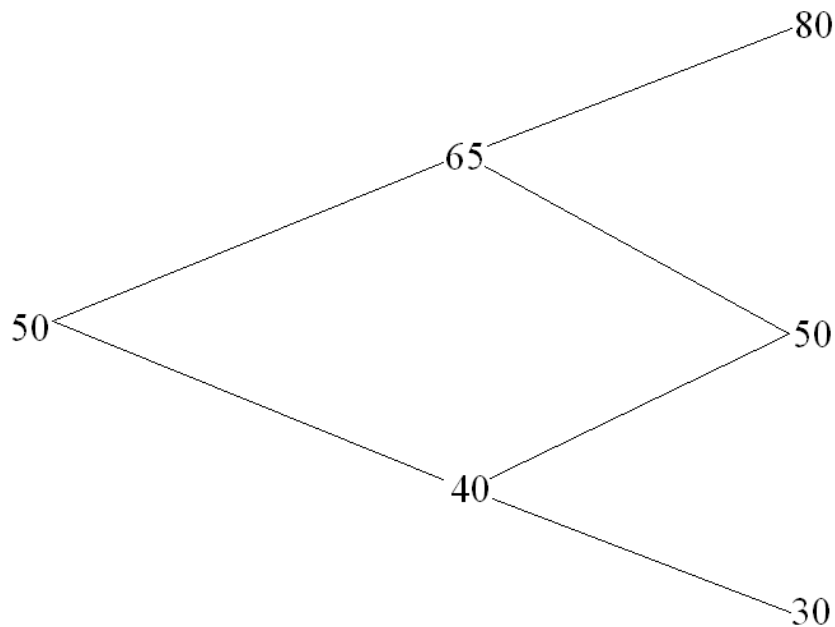
*If* it goes to 65 in half a year, then over the next half-year it will either

- Go up to 80, or
- Go down to 50

*If* it goes to 40 in half a year, then over the next half-year it will either

- Go up to 50, or
- Go down to 30

We can represent the potential outcomes with a *binomial tree*:



If you invest 1 in the bond, you get 1.02 in half a year

- Compounded for a year, that's  $1.02^2 = 1.0404$ , so same  $r_F$  as before

As before, we want the price today of a call option with strike price 55, expiring in one year. We'll get it by applying the arbitrage logic, working *backwards* through the tree.

*Step 1:* Determine the option's value **at expiration** for each possible outcome.

- Easy – **at expiration, the option is either exercised or worthless**
- In this case, the only possible outcome where the call is exercised is where the stock is worth 80, in which case the call is worth 25

*Step 2:* Move back from the last period to the previous period, and solve for the portfolio of the stock and bond that replicates the option over the final subperiod

- **Once you've calculated the values at expiration, from the perspective of the period just *before* expiration, you're looking at the exact same one-period problem we just solved**
  - The stock goes either up or down
  - You know what the call is worth in both cases
  - You know the bond's payoff
  - Solve two equations with two unknowns

In our example, the previous period is half a year, when the stock is either 65 or 40

*Stock is 65:* Over the last half-year,

- Stock goes to 80 and call goes to 25, *or*
- Stock goes to 50 and call goes to 0
- Bond pays 1.02

So we plug in  $S_{up}=80$ ,  $S_{down}=50$ ,  $C_{up}=25$ ,  $C_{down}=0$  and get

$$n = (25-0)/(80-50) = 5/6$$

$$B = 1/(1.02)[((0)(80)-(25)(50))/(80-50)] = -41.33/1.02 = -40.85$$

So *if* the stock goes to 65 in the *first* half-year, then the portfolio that replicates the call over the *second* half-year is

- Buy 5/6 shares, which costs 54.17
- Borrow 40.85, which brings in 40.85
- Net cost is  $54.17 - 40.85 = 13.32$

If the stock goes to 40 in the first half year, the call will be worthless no matter what the stock does next, so it is worth 0

*Step 3:* Go back another subperiod, and do the same thing. Repeat until you're done

The key insight of multiperiod option pricing: Once we know what the option is worth at every node of period  $t$ , we can solve for the period  $t-1$  values with the one-period model. And once we've done that, we can move back to  $t-2$ , then to  $t-3$ , etc. We repeat this folding-back until we're done.

In our example, we have established that, after half a year the call is worth

- 13.32 if the stock goes to 65
  - The portfolio that pays the same as the call, *given* that the stock went to 65, costs 13.32
- 0 if the stock goes to 40
  - No hope of a positive future value

So, at the beginning, the call is worth the same as a portfolio of the stock and bond which is worth

- 13.32 if the stock goes to 65
- 0 if it goes to 40

So we plug  $S_{up}=65$ ,  $S_{down}=40$ ,  $C_{up}=13.32$ ,  $C_{down}=0$  and get

$$n = (13.32 - 0) / (65 - 40) = 0.5328$$

$$B = 1 / (1.02) [((0)(65) - (13.32)(40)) / (65 - 40)] = -20.89$$

$$\text{Cost is } (0.5328)(50) - 20.89 = \mathbf{5.75}$$

The call is worth 5.75 because for 5.75 we can implement a *dynamic* trading strategy

- *We don't just buy and hold, we rebalance every subperiod*

And this strategy is *self-financing*

- *Amount of cash we need at each point is exactly the current value of the portfolio*

and will pay off *exactly* what the call pays off on the expiration date

Convince yourself that this is an arbitrage argument: in the simple world of this model, the dynamic trading strategy can not fail to have the same payoff as the call option.

#### IV. Black-Scholes Formula

The famous Black-Scholes formula takes the binomial model to its logical extreme. As the number of subperiods goes to infinity, or in other words the length of each subperiod goes to zero, the value of the option converges to the number produced by the B-S formula

To put it another way, if you could rebalance your portfolio of the stock and the bond

- continuously
- day and night
- with no transactions costs

and also if the stock price moves smoothly, then the B-S formula gives you the arbitrage-free value of the call.

Of course these modeling assumptions are not literally true; the operative question is whether they are close enough. Here's the formula:

$$C = S N(d_1) - PV(X) N(d_2)$$

where:

$$d_1 = \frac{\log(S/X) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma T^{1/2}$$

and:  $C$  = Call Price  
 $S$  = Stock Price  
 $X$  = Strike Price  
 $r$  = Annual interest rate  
 $T$  = Time to maturity  
 $\sigma^2$  = Variance of the stock return  
 $N()$  = cdf of the standard normal distribution

Notice that you need an estimate of the variance of the stock to price the option. Or in other words, an option price imputes a variance estimate, known as the *implied volatility*.

You are not responsible for this formula for this class, but I will use it for some illustrations

## AN EXAMPLE: AMAZON

<HELP> for explanation. EquityCALL

**CALL SUMMARY PAGE** Hit 99 <go> for option model defaults

AMZN US AMAZON.COM INC 09:37 Hedge: M

81.04 +.84 Bid 81.05 Ask 81.08 Days 184 Fin Rate 3.09 Volat: 46.09

ZQN Jul 08	LAST TRADE INFO			IMPL.VOLAT		HEDGE			*THEORETICAL	
	Bid	Ask	Last Chnge	Bid	Ask	Del	Gam	Vega	Price	Diff
1) GJ 50	33.30	34.10	33.25 unch	58.88	67.39	909	004	.10	32.38	-1.32
2) GK 55	29.35	30.10	29.15 unch	59.05	65.26	871	006	.12	28.02	-1.70
3) GL 60	25.60	26.25	25.35 unch	58.32	62.77	829	007	.15	23.97	-1.96
4) GM 65	22.10	22.70	20.85 unch	57.31	60.83	780	009	.17	20.27	-2.13
5) GN 70	18.90	19.45	18.25 unch	56.33	59.19	726	010	.19	16.97	-2.20
6) GD 72.5	17.45	17.90	17.20 unch	56.02	58.25	697	011	.20	15.47	-2.20
7) GO 75	16.00	16.50	15.80 unch	55.33	57.72	667	011	.21	14.07	-2.18
8) GP 80	13.40	13.85	13.60 +.40	54.32	56.36	606	012	.22	11.57	-2.05
9) GQ 85	11.05	11.50	10.50 unch	53.12	55.09	543	013	.23	9.44	-1.83
10) GR 90	9.05	9.45	8.90 unch	52.19	53.94	481	013	.23	7.65	-1.60
11) GT 95	7.35	7.70	7.00 unch	51.39	52.95	420	013	.23	6.17	-1.36
12) GY 100	5.95	6.20	6.00 +.40	50.82	51.98	363	013	.22	4.94	-1.13

QZN Jul 08 Days 184 Fin Rate 3.09 Volat: 46.09

13) GA 105	4.75	5.00	4.65 unch	50.14	51.37	310	012	.20	3.95	-.93
14) GB 110	3.75	3.90	3.60 unch	49.45	50.26	261	011	.19	3.14	-.69
15) GC 115	2.92	3.10	2.73 unch	48.73	49.79	218	010	.17	2.49	-.52
16) GD 120	2.23	2.35	2.26 +.16	47.90	48.70	177	009	.15	1.97	-.32

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000  
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2008 Bloomberg Finance L.P. 6576-335-0 16-Jan-08 9:52:43

This shows calls expiring July 19, 2008 (FWIW, the standard option expiration dates are the Saturdays after the third Fridays of the indicated months)

Look at row 8, marked GP 80. If you buy this option at the Ask of 13.85, you have the right to buy a share of Amazon for \$80 at any time up to 7/19/08.

- Amazon shares currently trading at 81.04
- 184 days to expiration
- Interest rate assumed to be 3.09%
- Volatility *implied* by the ask of 13.85 is 56.36%
  - Above the recent historical volatility of 46.09%
- This would be considered the *at-the-money* July call, since the strike of 80 is closest to the share price of 81.04
- Dividends can complicate the relation between American options (such as this) and European options (what B-S actually prices) but Amazon doesn't pay dividends.

*Main point to bear in mind about B-S:* The formula gives the arbitrage-free price in the case where it is feasible to replicate the option with dynamic trading. If you *can't* replicate the option with dynamic trading, then the Black-Scholes price is just a guide for thinking about the option's value.

- Is there a liquid market for the underlying security?
- Is there a liquid market for something *correlated* with the underlying security?
  - In dotcom days, many people owned options on companies that hadn't gone public yet, and were not highly correlated with stocks they could trade. The B-S formula does not really apply in these situations
  - That is, the option holder *can't* lay off his risk by undertaking the dynamic hedging strategy, so he must bear some risk to hold the options, so their value to him depends on his attitude toward risk
    - But people use the formula anyhow, as a rough estimate