Forward contracts on financial assets

Financial Derivatives

Finance 206/717

Philip Bond

Wharton, University of Pennsylvania

Themes of course

- Two big themes:
  1. pricing derivatives
  2. using derivatives
- These notes:
  - concentrate on pricing forwards, without distractions of particular institutional trading arrangements in futures markets
- Next set of notes:
  - using forwards/futures
  - institutional details of trading

Derivative prices and the absence of arbitrage

- The key technique for pricing derivatives is the *no arbitrage principle*
- **Definition:**
  - A (pure) **arbitrage** is a set of trades that:
    - does not require any initial funds
    - never loses any money
    - produces strictly positive cash flows with strictly positive probability
- The no arbitrage principle is that:
  - asset prices must be such that no arbitrage exists
- Why is this a reasonable principle?

Synthetic securities

- Synthetic securities:
- Example:
  - Suppose a 3-month T-Bill trades at $\$98$ today, and pays out $\$100$ in 3 months
  - This is a (simple) example of a financial security
  - Now, suppose that I produce a security that pays out $\$100$ in 3 months
  - The cash flows are exactly the same as the government-issued T-Bill
  - We can say that my security is a **synthetic** T-Bill
- In general, given any financial security, a synthetic security is one that has the same cash flows

Synthetic securities and no arbitrage

- Absent transaction costs, the no arbitrage principle implies that the price of a synthetic security must equal the price of the original security
  - Back to example:
    - Suppose that my synthetic T-Bill trades at $\$99$. What would you do?
    - Suppose that my synthetic T-Bill trades at $\$97$. What would you do?
    - At what price is no arbitrage possible?

The forward price

- We begin our examination of derivative prices by looking at the determinants of the forward price.
- Basic case: financial asset, no dividends
- Why is this the basic case?
  - financial assets have no storage costs
  - financial assets are held only for the cash flows they produce
- Forwards vs. futures
  - forwards and futures are basically the same thing
  - forward contracts are arranged between pairs of individuals/firms
  - futures contracts are *exchange traded*
    - whereas forward contracts are often said to be "OTC"
    - i.e. over the counter

The cash flows of a forward contract

- We will price a forward contract by no arbitrage.
- We start by carefully writing down the cash flows:
- Consider a forward contract for a share of Apple stock
  - enter into at time t
  - delivery at time T
  - forward price is $F_{t,T}$
  - the cash flows are:

	t	T
Forward contract		

Synthetic forwards

- Next, we build a synthetic forward.
- Consider the following trading strategy
  - To match the date T cash flow of S_T: buy a share of Apple stock at time t, sell at time T
    - What are the cash flows?
  - To match the date T cash flow of $-F_{t,T}$: borrow and incur a date T obligation of $-F_{t,T}$
- Cash flows:

	t	T
Buy stock		
Borrow		
Total		

- No arbitrage implies that ...

Synthetic T-Bills

- Instead of building a synthetic forward from the stock and risk-free bonds,
- we can build a synthetic T-bill (risk-free bond) out of the stock and forward
- Suppose we invest $\$1$ in T-bills today:
  - the date T cash flow is $e^{r(T-t)}$
- We can create a synthetic T-Bill by:
  - buying a share of Apple stock
  - and hedging the price risk by taking a short position in a forward contract
  - the combined date T cash flow is:
- No arbitrage implies that:

The cash-and-carry arbitrage

- Next, let's check to see if an arbitrage exists if the forward price is different from predicted.
- Suppose the forward price is more than predicted.
- What would you do?

- What does this do to the forward price?
- Note: the cash-and-carry produces a synthetic T-Bill

The reverse cash-and-carry arbitrage

- Suppose the forward price is less than predicted.
- What would you do?

- What does this do to the price?
- Note: the reverse cash-and-carry produces a synthetic short position in a T-Bill

Example

- The spot price of a share in XYZ Corp. is $\$23$
- The six month T-Bill rate is 3% (APR, twice yearly compounding).
- What is the forward price for delivery in 3 months?
  - The continuously compounded interest rate is

$$2 \times \ln(1 + \frac{3\%}{2}) = 2.98\%$$

  - The forward price for an XYZ share with delivery in 3 months is:

$$\$23 \times e^{.25 \times .0298} = \$23.172$$

Using forwards for asset reallocation

- Suppose you own Apple stock, but think its performance over the next six months will be poor.
  - You could sell the stock now, and invest the proceeds in six-month T-Bills
- Alternatively, you could take a short position in a six-month forward contract.
- Value of investment today is S_t
- Value of investment after six months is

$$(F_{t,T} - S_T) + S_T = F_{t,T} = S_t e^{r(T-t)}$$

Asset reallocation, continued

- You have effectively transferred your investments from Apple stock to T-Bills
  - You are investing in T-Bills without holding T-Bills!
- Why do it this way?
  - transactions costs
  - taxes
- Converting a position in one asset into a position in another asset using a forward/futures contracts is called a *futures overlay*
  - Here, we have converted a stock position into a bond position
- Question to think about at home:
  - What would would you do to convert a T-Bill investment into a stock investment?

Selling forwards

- You work for a bank
- A client wants to take a long position in a forward contract.
  - If you supply the forward contract, you will have a short position.
  - What could you do to hedge the risk created?

Forward price = expected spot price?

- What is the relationship between the forward price and the expected spot price?
- For example, what would the forward price for 1 share of IBM for delivery in 1 year tell us about the spot price of IBM in 1 year?
- One possible guess:

$$F_{t,T} = E[S_T]$$

  - this says: the forward price is the market's best guess as to what the price of IBM will be in one year
  - Empirically, this is NOT the case.

Forward price = expected spot price?

- And economically it doesn't make any sense:
  - Consider a long position in the forward, combined with investing $F_{t,T}e^{-r(T-t)}$ in T-Bills
  - What is the expected payoff on this position?
  - What is the expected gross rate of return if $F_{t,T} = E[S_T]$?
  - Would this ever be the case?

The actual relation

- To determine the actual relationship between $E[S_T]$ and $F_{t,T}$ we use the CAPM:
- First, we need to define some notation. Let
  - $E[r^{IBM}] =$ the expected return on IBM over the time period t to T
  - $E[r^{mkt}] =$ the expected return on the market over the time period t to T
  - $r =$ the risk-free rate over the time period t to T
  - $\beta_{IBM} =$ the beta of IBM
- What is the forward price?
- What is the expected spot price?

Observations

- If beta is greater than zero, the expected spot exceeds the forward price.
- So someone holding a long position in the forward expects to earn a positive return.
- Since forward contracts are zero-sum, the person holding the short position expects to earn a negative return.
  - So why is anyone willing to hold a short position?

Pricing with Coupon Payments or Dividends

- If the spot commodity is a stock or a bond then the investor will receive a dividend or coupon payment while carrying the spot commodity.
- So far we have ignored this complication.
- Recall that when we ignored dividends, no arbitrage implies that a synthetic T-Bill produced by a cash-and-carry (long stock, short forward)
- produces the same return as an actual T-Bill
- To see how dividends affect the forward price consider the following modified cash-and-carry:
  - date t: take short position in forward, buy the spot
  - date t': receive the coupon or dividend
  - date T: forward expires and delivery occurs

- The cash flows are:

	t	t'	T
buy spot			
short forward			
receive dividend			
invest dividend			
Total			

- The position resembles a T-Bill.

- To avoid arbitrage possibilities, we need the rate of return on the cash-and-carry to be the same as the rate of return on a T-Bill:
- i.e.

$$F_{t,T} = S_t e^{r(T-t)} - De^{r\left(T-t'\right)}$$

- Note that in this case the forward price may or may not be greater than the spot price.

Continuous dividends

- It is often convenient to assume dividends are paid **continuously**
- How would we do the cash-and-carry with continuous dividends?
  - we need the dividend yield
- Aside on dividend yields:
  - Ford Motor Co. pays a $\$0.30$/share dividend quarterly.
  - If the current share price is $\$30$, as a percent this is 1% per quarter
    - this is the dividend yield
    - i.e. an APR of 4% with quarterly compounding
  - As a continuous yield, this is

$$4 \times \ln(1.01) = 3.98\%$$

  - We will generally denote the dividend yield in continuous form by δ.

Continuous dividends

- In many cases in this class we will be treating dividends as if they are paid continuously.
- This is the standard approach in this area.
- It is a “modeling convention” which makes the formulas and intuition easier to understand and to work with.
- If you buy one share of stock which pays dividends continuously,
  - and you reinvest those dividends when they are paid,
  - then at the end of the period t to T you will have $1 \times e^{\delta(T-t)}$ shares
  - (δ is the continuously compounded dividend yield.)
- Example:
  - You buy one share Ford stock and hold it for 2 years.
  - The dividends are paid at continuously compounded rate of 3.98% and you reinvest all the dividends when they are paid.
  - How many shares do you have at the end of two years?

Back to the forward price

- The cash-and-carry with a continuous dividend yield is done slightly differently.
- At delivery of the forward you need to have one share of stock.
- So at time t buy just enough stock so that when you reinvest the dividends you have accumulated exactly one share by T.
  - (“tailing the position”)
- If you buy $1 \times e^{-\delta(T-t)}$ shares at time t, then at time T you will have

$$1 \times e^{-\delta(T-t)} \times e^{\delta(T-t)} = 1 \text{ share}$$

- The cash flows are:

	t	T
buy the spot		
short the forward		
Total		

- The remaining steps should be familiar by now ...
- the cash-and-carry has produced a synthetic T-Bill
- For no arbitrage, we need:

- i.e.

$$
\begin{aligned}
F_{t,T} &= S_t e^{r(T-t)} e^{-\delta(T-t)} \\
&= S_t e^{(r-\delta)(T-t)}
\end{aligned}
$$

Example

- Ford Motor Co.'s current share price is $30/share.
- The continuously compounded dividend yield is 3.98%
- If you wanted to create a synthetic 3 month T-bill using Ford stock, what would you do?

- And if the forward price is $\$29.90$, what is the yield on the synthetic T-Bill?

Pricing with Transactions Costs

- There are a number of transactions costs associated with cash-and-carry positions.
  - bid-ask spreads on both the forward and the spot good
  - brokerage commissions
  - differential borrowing and lending rates
- How do these transactions costs affect forward pricing?
- Terminology:
  - the **ask** price is the price you buy at
    - (it is the price the seller is "asking")
  - the **bid** price is the price you sell at
    - (it is the price the buyer is "bidding")
  - in general, the bid price is lower than the ask price
  - the difference is often called the **bid-ask spread**

- Notation:
  - $S_t^A =$ the spot ask price
  - $S_t^B =$ the spot bid price
  - $TF =$ future value of any transactions fees
  - $r^b =$ borrowing rate
  - $r^l =$ lending rate
- As before, forward prices are determined by a no arbitrage condition.
  - First, we consider a strategy with a short position in the forward
  - Second, we consider a strategy with a long position in the forward

Short position

- Strategy: take a short position in the forward, hedge by buying stock, and borrow to finance

	t	T
buy the spot, short forward		
borrow the spot price		
pay the transactions fee		
total		

- There is an arbitrage opportunity if:

$$F_{t,T} - TF - S_t^A e^{r^b (T-t)} > 0$$

- i.e. if

$$F_{t,T} > S_t^A e^{r^b (T-t)} + TF$$

  - short position is profitable if forward price high enough

Long position

- Strategy: take a long position in the forward, hedge by shorting stock, and invest short proceeds

	t	T
buy the forward, short the spot		
invest the short proceeds		
pay the transactions fee		
total		

- There is an arbitrage opportunity if:

$$-F_{t,T} - TF + S_t^B e^{r^l(T-t)} > 0$$

- i.e. if

$$F_{t,T} < S_t^B e^{r^l(T-t)} - TF$$

  - long position is profitable if forward price low enough

No arbitrage bounds

- Summary:
  - If $F_{t,T} > S_t^A e^{r^b(T-t)} + TF$, then an arbitrage is possible.
    - So to eliminate arbitrage possibilities, we must have

$$F_{t,T} \leq S_t^A e^{r^b(T-t)} + TF$$

    - This is an upper bound on the forward price
  - If $F_{t,T} < S_t^B e^{r^l(T-t)} - TF$ then an arbitrage is possible.
    - So to eliminate arbitrage possibilities, we must have

$$F_{t,T} \geq S_t^B e^{r^l(T-t)} - TF$$

    - This is an lower bound on the forward price

Topics covered

- The forward price of a financial asset
  - without dividends
  - with dividends
- No arbitrage pricing
- Synthetic positions
- Using derivatives for asset reallocation
- What the forward price tells us about future spot prices.
- Pricing with transactions costs