Continuous compounding

Financial Derivatives

Finance 206/717

Philip Bond

Wharton, University of Pennsylvania

Discrete compounding

- Recall that the effective interest rate depends on the number of compounding periods.
- Interest rates are often quoted in APR form.
- Examples:
  - $r = 6\%$, annual compounding
    - $\$100$ invested for 1 year yields $\$106$
    - the effective interest rate is just $r_{\text{eff}} = 6\%$
  - $r = 6\%$, monthly compounding
    - this means that the monthly interest rate is $6\%/12 = 0.5\%$
    - So $\$100$ invested for 1 year yields $\$100 \times (1.005)^{12} = \$106.17$
    - the effective interest rate is $r_{\text{eff}} = 6.17\%$

- Examples — continued
  - $r = 6\%$, daily compounding
    - this means that the daily interest rate is $6\%/365 = 0.016438\%$
    - So $\$100$ invested for 1 year yields $\$100 \times (1 + 0.016438\%)^{365} = \$106.18$
    - So $r_{eff} = 6.18\%$
- In general, the effective interest rate is given by

$$1 + r_{eff} = \left(1 + \frac{r}{k}\right)^k$$

where k is the number of compounding periods and r is the APR

Continuous compounding

- Continuous compounding is when k is infinite:
  - write $k \to \infty$
  - mathematical fact:

$$\left(1 + \frac{r}{k}\right)^k \to e^r \text{ as } k \to \infty$$

    - e is the Euler number, and is equal to $2.71828\ldots$
    - it can be found on your calculator
    - it is common to write $\exp(x)$ in place of e^x
    - in Excel, use the exp function
  - So with continuous compounding,

$$1 + r_{\text{eff}} = e^r = \exp(r)$$

- Example: interest rate 6%, continuous compounding
  - effective rate is $e^{0.06} - 1 = 6.18\%$

Facts about e

- $e^x e^y = e^{x+y}$
- $(e^x)^y = e^{xy}$
- $e^{\ln x} = \ln(e^x) = x$

More time periods

- Suppose the continuously compounded rate of interest is 6%.
- What is the final balance on $\$100$ invested over 2 years?
  - The effective annual rate of interest is 6.184%.
  - So the ending balance is $\$100 \times 1.06184 \times 1.06184 = \$112.75$
- Equivalently, the ending balance is:
  - $\$100 \times e^{.06} \times e^{.06} = \$100 \times e^{.06+.06} = \$100 \times e^{.12} = \$112.75$
  - That is, the gross interest over the two years is e^{2r}
- In general, if r is the continuously compounded interest rate, then the gross interest over period t is simply

$$e^{rt}$$

Fluctuating interest rates

- Suppose the continuously compounded interest rate this year is 6%, and next year it will be 7%.
- What is the balance after 2 years of $\$100$ invested today?
- $\$100 \times e^{.06} \times e^{.07} = \$100 \times e^{.06+.07} = \$100 \times e^{.13} = \$113.88$
- In general, if the continuously compounded interest rate is r_1 between now and t_1, and then r_2 between t_1 and $t_1 + t_2$, then the total gross return is

$$e^{t_1 r_1 + t_2 r_2}$$

Converting to continuously compounded rates

- Continuously compounded rates are easy to manipulate
- (that is why we use them)
- however, rates are not usually quoted in continuously compounded form.
- so we want to be able to convert quoted rates into continuously compounded rates
- Example: the APR is 6%, annual compounding. What is the continuously compounded rate?
  - We need to find r such that $e^r = 1 + 6\%$.
  - Take logs: $\ln\left(e^r\right) = \ln 1.06$
  - So $r = 5.83\%$

- Example: the APR is 6%, monthly compounding. What is the continuously compounded rate?
  - The monthly rate is $.5\%$.
  - So we need to find r such that $e^{\frac{r}{12}} = 1 + .5\%$
  - So $r/12 = \ln 1.005$
  - So $r = 5.99\%$
- In general, if the APR is r_{APR} with k compounding periods a year, then the continuously compounded rate is

$$r = k \ln \left(1 + \frac{r_{APR}}{k}\right)$$

- As a special case, note that

$$r = \ln \left(1 + r_{\text{eff}}\right)$$