THE USE OF 'ALPHAS' TO IMPROVE INVESTMENT PERFORMANCE

by

Marshall E. Blume

Working Paper No. 14-82

RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

The Wharton School
University of Pennsylvania
Philadelphia, Pennsylvania 19104

ABSTRACT

For reasons of transaction or estimation costs, an investor would typically not hold the market portfolio even if he followed the mean-variance precepts of Markowitz. The paper shows that a variant of the one-parameter performance measure of "alpha" coefficient, originally proposed by Jensen does provide in some cases an inexpensive tool to screen a large number of potential additions or deletions to an existing portfolio. This use of the alpha coefficient as a tool in portfolio selection does not assume the descriptive validity of the Capital Asset Pricing Model.

I. Introduction

Originally proposed by Jensen (1968, 1969), the one-parameter performance measures that are now widely known as "alpha" coefficients have usually been developed within the context of the Sharpe-Lintner capital asset pricing model. Recently, Roll (1977) has questioned the empirical usefulness of this model since as a practical matter it is impossible to measure the return on the market portfolio of all risky assets, and the market portfolio is an integral part of this theory. Moreover, he has devised a numerical example which shows that the use of a proxy to approximate the market returns, even if highly correlated with the market return itself, may not preserve, even as a rough approximation, the empirical usefulness of the model. If correct, this argument of Roll would appear, at least on the surface, to destroy the usefulness of these "alpha" coefficients.[1]

The purpose of this paper is to show that these one-parameter performance measures, the so-called alpha coefficients, do provide in some situations useful information to an investor who wishes to follow the mean-variance precepts of Markowitz, even if the capital asset pricing model ultimately proves to be empirically valueless as a description of equilibrium. The next section describes the investor's decision, and the following sections show how an investor can employ one-parameter performance measures to improve his overall portfolio.

II. The Investor's Decision

The usual development of the capital asset pricing model makes the assumptions (a) that there are no transaction costs and (b) that all investors know the true joint probability distribution of the future returns of all

---

[1] Cf. Roll (1978, 1980).

assets--in short, have homogeneous expectations. In such a world, performance measurement would be a meaningless exercise in that every investor would be holding the best possible portfolio of risky assets, namely the market portfolio.

In reality, there are transaction costs and also estimation costs. Recognizing the existence of these types of costs, Bloomfield, Leftwich, and Long (1977) have examined the effectiveness of various portfolio strategies which an investor might as a practical matter have to follow. The problem to be examined in this paper is similarly a practical problem. An investor for the reasons discussed in Bloomfield et al (1977) or for other reasons would typically hold a portfolio different from the market and might want to know how he could alter his current portfolio so as to obtain a better overall position. The traditional one-parameter performance measures, properly calculated, may often provide a relatively inexpensive screening device to identify potential additions or deletions to an existing portfolio.

To be more precise, assume that the investor currently holds portfolio P, has assessed a predictive distribution of the returns on this portfolio with mean μ_P and standard deviation σ_P, and finally is willing to utilize a mean-variance one-period model to evaluate his alternatives. If the investor could borrow or lend at the risk-free rate r_f, he could obtain any position along the ray emanating from the risk-free rate through portfolio P, as shown in Figure 1. The investor's assumed optimal position has been designated with the superscript '*'. As drawn, the investor's final portfolio would involve a positive position in the risk-free asset; but more importantly, portfolio P need not be, and most likely is not, on the true efficient set.

An investor who actually held P^* would, of course, want to make any changes which would improve his position by, in some sense, moving his portfolio of risky assets towards the efficient set. Such changes could involve increasing or decreasing the levels of some of his current holdings and possibly adding additional assets. His position would obviously be improved if the change allowed him a greater expected return at the same level of risk as before. As an example, if he could modify his risky portfolio P so as to obtain portfolio Q as in Figure 2, he could, with appropriate adjustments in his investment in the risk-free portfolio, obtain portfolio Q^* for a gain in expected return of P^*Q^* with no change in risk. Of course, with the new opportunities given by Q, he would probably want, in addition, to alter his risk level.

In this paper, we shall say that portfolio Q is <u>relatively more efficient</u> than portfolio P in that Q allows a greater expected return than P at any level of risk as measured by standard deviation.[2] It should be noted that this concept of relative efficiency does not require that either portfolio be efficient in an absolute sense.

To summarize, a question often faced by investors is how to change an existing portfolio so as to obtain a relatively more efficient portfolio. It is precisely this question which the traditional one-parameter performance measures of investment performance answer.

III. The Interpretation of Alpha

Consider an investor who currently holds portfolio P as his portfolio of risky assets and is considering combining this portfolio with another portfolio, say A. Based upon his assessments of the expected returns, variances, and covariances of these two portfolios, he would reach one of the following three decisions as to his risky portfolio: (1) put nothing in A,

---

[2]Cf. Blume (1980) for an application of this concept.

(2) invest something in A through a reduction of the amount invested in P, or (3) sell A short and invest the proceeds of the short sale in P. Of course, institutional constraints may preclude this last course of action.

Without attempting to be exhaustive, several possible configurations are shown in Figure 3 as illustrative of these possible decisions. In the first row, portfolio P is relatively more efficient than any other combination of P and a non-zero investment in A. In the second row, the investor can move to the relatively more efficient portfolio Q by selling some of P and investing in A. In the third row, the investor can move to the relatively more efficient portfolio Q by selling A short and investing the proceeds in P. It turns out that the value of the alpha coefficient as defined below does indicate the appropriate type of position to take in the alternative portfolio A. Specially, an alpha of zero indicates that an investor should take no position in portfolio A (first row); a positive alpha indicates that he should take a long position in A (second row); and a negative alpha indicates that he should take a short position in A (third row).

In order to define alpha and demonstrate its informational content, define the following terms:

r_i the realized return on portfolio i

μ_i the expected return on portfolio i

r_f the riskfree rate

σ_{ij} the covariance of returns between portfolios i and j

σ_{ii} the variance of returns for portfolio i

β_{ij} the beta coefficient of portfolio i calculated with respect to portfolio j, defined as the ratio of σ_{ij} to σ_{jj}

α_{ij} the alpha coefficient of portfolio i, defined relative to portfolio j

The expected returns and the other statistics can be viewed as subjective assessments of the underlying statistics or as their true values. In terms of these symbols, the alpha associated with portfolio A, defined relative to portfolio P, α_{AP}, will be given by

$$\alpha_{AP} = (\mu_A - r_f) - \beta_{AP}(\mu_P - r_f) \tag{1}$$

Although this paper is concerned with the alpha coefficient merely as a tool of portfolio analysis, the alpha coefficient has usually been interpreted in the context of the capital asset pricing model, particularly in conjunction with tests of the efficient market theory. For this reason, it may be useful to comment explicitly on the correspondence between (1) and the traditional capital asset pricing model.

In this traditional interpretation, P would be the market portfolio of all risky assets. The product $\beta_{AP}(\mu_P - r_f)$ would be the expected risk premium of asset or portfolio A that is consistent with equilibrium. If the actual expected risk premium $(\mu_A - r_f)$ were equal to this equilibrium value, the alpha would be zero. A positive or negative alpha would indicate that asset or portfolio A had an expected return that was inconsistent with equilibrium. Although the alpha coefficient can be and has been interpreted as a measure of disequilibrium in a macro setting, this paper will only interpret this coefficient as a tool of portfolio analysis, where P is the portfolio of risky assets actually held by an investor.

An investor who followed the mean-variance precepts of Markowitz would want to minimize the variance of the return on his overall portfolio for any given level of expected return. If the investor wished to reallocate his wealth over a riskfree asset, his current portfolio of risky assets P and an alternative portfolio A, he would select the proportions to place in each, denoted respectively x_f, x_P, and x_A, so as to

$$\min \quad x_A^2\sigma_{AA} + x_P^2\sigma_{PP} + 2x_Ax_P\sigma_{AP} \tag{2a}$$

$$\text{s.t.} \quad x_A\mu_A + x_P\mu_P + x_fr_f = \mu^* \tag{2b}$$

$$x_A + x_P + x_f = 1 \tag{2c}$$

where μ^* is some fixed level of expected return. The entire efficient set would be obtained by varying μ^* over the relevant range. Although it is theoretically possible that either μ_A or μ_P could be less than r_f, it is probably more likely that both μ_A and μ_P would be greater than r_f, and this more probable case will be assumed in the following.

Except in one special and uninteresting case,[3] the optimal values of x_A and x_P for problem (2) will exist and are[4]

$$x_A = k[(\mu_A - r_f)\sigma_{PP} - (\mu_P - r_f)\sigma_{AP}] \tag{3a}$$

$$x_P = k[(\mu_P - r_f)\sigma_{AA} - (\mu_A - r_f)\sigma_{AP}] , \tag{3b}$$

where

$$k = \frac{\mu^* - r_f}{(\mu_A - r_f)^2\sigma_{PP} + (\mu_P - r_f)^2\sigma_{AA} - 2(\mu_A - r_f)(\mu_P - r_f)\sigma_{AP}} \tag{3c}$$

For subsequent reference, it should be noted that as long as a solution exists to (2), k will be positive. The numerator is clearly positive for μ^* in excess of r_f, and it can be shown that the denominator is positive.[5]

---

[3] A solution will exist to (2) if the denominator of (3c) is non-zero. The denominator may be recognized as a quadratic form and will be positive with the exception of the one case in which the correlation between the returns of A and P is 1.0 and the ratio of $(\mu_A - r_f)$ to $(\mu_P - r_f)$ is the square root of the ratio of σ_{PP} to σ_{AA}.

[4] This solution can be obtained by substituting x_f from (2c) into (2b) and expressing the resulting problem in Lagrangian form. Solving the first-order conditions for x_A and x_P yields (3a) and (3b).

[5] Cf. footnote 3.

Equation (3a) can be rewritten as

$$x_A = (k\sigma_{PP})\left[(\mu_A - r_f) - \frac{\sigma_{AP}}{\sigma_{PP}}(\mu_P - r_f)\right] \quad (4a)$$

$$= (k\sigma_{PP})\left[(\mu_A - r_f) - \beta_{AP}(\mu_P - r_f)\right] \quad (4b)$$

$$= (k\sigma_{PP})(\alpha_{AP}) \quad (4c)$$

Thus, since $(k\sigma_{PP})$ is positive, x_A will be zero if α_{AP} is zero, x_A will be positive if α_{AP} is positive, and x_A will be negative if α_{AP} is negative.

In sum, an investor who currently holds portfolio P as his portfolio of risky assets and is considering combining this portfolio with another portfolio A can use the information contained in the alpha coefficient of portfolio A, calculated with respect to portfolio P, to determine whether he should place nothing in A, take a long position, or take a short position.

It should be noted however that a non-zero alpha coefficient itself does not provide enough information to determine the magnitude of the position to be taken.

IV. A More General Interpretation

Consider an investor who currently holds an optimal combination of a well diversified equity, say P, and some other portfolio of risky assets, say O, in conjunction with a positive, zero, or negative position in the riskfree asset. The other portfolio of risky assets might, for instance, be a bond portfolio. By well diversified is meant that all major segments of the equity market are represented. Thus, any increment in performance would be due to asset selection within sectors and not switches from one sector to another.

Now suppose the investor asked an investment advisor to propose another well diversified portfolio of equities, say A, as a full or partial alternative for his existing portfolio. Since both equity portfolios are well diversified, the returns on the alternative portfolio would be highly correlated with the returns of the current equity portfolio. Without additional information as to the composition of the alternative portfolio, the investor might view the return on this alternative as a linear function of the return on the existing equity portfolio plus some firm specific effects. If one measures returns as deviations from their expectations, this view takes the form

$$r_A - \mu_A = \beta_{AP}(r_P - \mu_P) + \varepsilon \tag{5}$$

where ε is a mean-zero disturbance independent of both r_P and r_0. The assumption of the independence of ε from both r_P and r_0 implies that the lack of perfect correlation between the returns of portfolios A and P is due solely to firm specific effects.

In this situation, it turns out that if the alpha coefficient of portfolio A, defined relative to portfolio P, is zero, nothing should be placed in A. If the alpha coefficient is non-zero, the sign of the alpha coefficient will indicate whether a positive or negative position should be taken in portfolio A. Thus, the alpha coefficient, even though it is <u>not</u> calculated with respect to the investor's entire risky portfolio, has the same interpretation as in the prior sections. The proof is similar to the one in Section III.

Specifically, if an investor wished to reallocate his wealth over a riskfree asset, the current equity portfolio P, the alternative portfolio A, and the portfolio of other risky assets 0, he would select the proportions to place in each, denoted respectively x_f, x_P, x_A, and x_0, so as to

$$\min \quad x_A^2\sigma_{AA} + x_P^2\sigma_{PP} + x_0^2\sigma_{00} + 2x_Ax_P\sigma_{AP} + 2x_Ax_0\sigma_{A0} + 2x_Px_0\sigma_{P0} \tag{6a}$$

$$\text{s.t.} \quad x_A\mu_A + x_P\mu_P + x_0\mu_0 + x_fr_f = \mu^* \tag{6b}$$

$$x_A + x_P + x_0 + x_f = 1 \tag{6c}$$

where μ^* is some fixed level of expected return. The entire efficient set could be obtained by varying μ^* over the relevant range. As in the prior section, it will be assumed that the expected returns of the three portfolios of risky assets exceed the riskfree rate.

To analyze the interpretation of the alpha coefficient of portfolio A, calculated with respect to portfolio P, α_{AP}, the objective function and constraints of problem (6) will be reexpressed in an alternative way. Turning first to the constraint set, substitute x_f from (6c) into (6b) to obtain

$$x_A(\mu_A - r_f) + x_P(\mu_P - r_f) + x_0(\mu_0 - r_f) = \mu^* - r_f \ . \tag{6b1}$$

Solving (1) for μ_A and substituting into (6b1) yields

$$x_A[\alpha_{AP} + \beta_{AP}(\mu_P - r_f)] + x_P(\mu_P - r_f) + x_0(\mu_0 - r_f) = \mu^* - r_f$$

or

$$x_A\alpha_{AP} + (x_A\beta_{AP} + x_P)(\mu_P - r_f) + x_0(\mu_0 - r_f) = \mu^* - r_f \ . \tag{6b2}$$

If $(x_A\beta_{AP} + x_P)$ is defined as y, (6b2) can be rewritten as

$$x_A\alpha_{AP} + y(\mu_P - r_f) + x_0(\mu_0 - r_f) = \mu^* - r_f \ . \tag{6b3}$$

In words, (6b3) suggests that the expected return of the investor's entire portfolio in excess of the riskfree rate can be viewed as resulting from three sources: the excess return on the portfolio of other risky assets, the return driven by the current equity portfolio including the return of portfolio A explainable by the existing portfolio P, and the return attributable to α_{AP}. Thus, the total expected return in excess of the riskfree rate is related to the expected returns on the original portfolios, P and 0, and the extent to which the investor augments the expected return by taking a position in portfolio A.

On the basis of the stochastic relationship given by (5), the objective function (6a) can be reexpressed as

$$x_A^2(\beta_{AP}^2\sigma_{PP} + \sigma_{\varepsilon\varepsilon}) + x_P^2\sigma_{PP} + x_0^2\sigma_{00} + 2x_A x_P\beta_{AP}\sigma_{PP} + 2x_A x_0\beta_{AP}\sigma_{P0} + 2x_P x_0\sigma_{P0} \qquad (6a1)$$

or, after collecting terms,

$$x_A^2\sigma_{\varepsilon\varepsilon} + (x_A^2\beta_{AP}^2 + 2x_A x_P\beta_{AP} + x_P^2)\sigma_{PP} + x_0^2\sigma_{00} + 2(x_A\beta_{AP} + x_P)x_0\sigma_{P0} \; . \qquad (6a2)$$

Recalling that $(x_A\beta_{AP} + x_P)$ was defined as y, one can rewrite (6a2) as

$$x_A^2\sigma_{\varepsilon\varepsilon} + y^2\sigma_{PP} + x_0^2\sigma_{00} + 2yx_0\sigma_{P0} \; . \qquad (6a3)$$

Similar to the expected return constraint, this rewritten objective function views the total variance as being due to the variability of the original portfolios, P and 0, plus the additional variability due to the additional risk inherent in portfolio A.

The revised portfolio problem is to minimize (6a3) subject to (6b3). The first order conditions for this problem are

$$x_A\sigma_{\varepsilon\varepsilon} = \lambda\alpha_{AP} \qquad (7a)$$

$$y\sigma_{PP} + x_0\sigma_{P0} = \lambda(\mu_P - r_f) \qquad (7b)$$

$$y\sigma_{P0} + x_0\sigma_{00} = \lambda(\mu_0 - r_f) \qquad (7c)$$

and, in addition, constraint (6b3). The symbol λ is a Lagrange multiplier and is shown in the appendix to be positive. Since $\sigma_{\varepsilon\varepsilon}$ is positive, x_A will be zero if α_{AP} is zero; otherwise, x_A will have the same sign as α_{AP}.

A closer examination of (7) suggests that an investor could conceptually construct an optimal portfolio in a series of steps. First, the investor could determine the value of x_A up to the constant of proportionality λ as the ratio of α_{AP} to $\sigma_{\varepsilon\varepsilon}$. Second, the investor could calculate the values of y and x_0 up to the constant of proportionality λ.

On the assumption that the current equity portfolio is an index fund, the first step is associated with the portion of the total portfolio devoted to obtaining superior returns through security selection--sometimes termed the active portfolio.[6] The second step deals with the proper amount of diversification as between portfolios P and 0 by fixing the ratio of y to x_0--sometimes termed the core portfolio. It should be noted that this ratio does not depend upon α_{AP}.

Third, the investor should determine how much of his money should be placed in the risky portfolio in conjunction with a position in the riskfree asset by varying λ over all possible values and evaluating the resulting portfolios. Fourth, knowing the value of x_A and y, the investor can finally determine how much of portfolio P to hold.

VI. The S & P Composite Index

In the previous sections, the beta coefficient of an alternative portfolio was calculated with respect to the investor's existing portfolio or, in a more general case, with respect to a portion of his portfolio. The value

---

[6] Treynor and Black (1973) used this terminology in their Journal of Business article. This author does not know whether they were the first to use this term, but he does know that they were not the last to use it.

of the corresponding alpha coefficient had a direct and unambiguous implication as to whether the investor should take a long, zero, or short position in this alternative portfolio. However, most professional evaluation services reverse the role of these two portfolios and calculate the alpha coefficient of the investor's existing portfolio with respect to the S & P's Composite Index of 500 stocks.

This section will show that these professionally calculated alphas, even though calculated with respect to the wrong portfolio, do in some circumstances contain useful information. Consider an investor who had his existing portfolio, say P, evaluated with respect to the S & P index, say S, and assume that the returns on these two portfolios are positively but not perfectly correlated. The professionally calculated alpha, α_{PS}, would be positive, negative, or insignificantly different from zero. If α_{PS} is negative or even zero, the alpha of the S & P Index calculated with respect to the investor's existing portfolio, α_{SP}, will be strictly positive, implying that the investor could move to a relatively more efficient portfolio by shifting some of his existing portfolio into the S & P Index Fund. If α_{PS} is positive, the sign of α_{SP} is ambiguous without additional analysis.

First, consider the case in which the professionally calculated alpha, α_{PS}, is negative or zero. Thus,

$$\alpha_{PS} = (\mu_P - r_f) - \beta_{PS}(\mu_S - r_f) \leq 0 \tag{8}$$

Rearranging (8) yields

$$(\mu_S - r_f) - \frac{1}{\beta_{PS}} (\mu_S - r_f) \geq 0 \tag{9}$$

since β_{PS} is by assumption positive. Equivalently, (9) can be rewritten as

$$(\mu_S - r_f) - \frac{1}{\rho^2_{SP}} \beta_{SP}(\mu_P - r_f) \geq 0 \tag{10}$$

where ρ_{SP} is the correlation between the returns on portfolios S and P. If μ_P exceeds r_f and ρ_{SP} is positive but less than one, α_{SP} is strictly greater than the left hand side of (10), which establishes the result.

Second, consider the case in which the professionally calculated alpha, α_{PS}, is non-zero and positive. Then, the left hand side of (10) will be strictly less than zero, but this fact alone does not imply that α_{SP} is negative. It is easy to construct numerical examples in which α_{SP} is positive and yet the left hand side of (10) is negative.[7] Thus, the finding that α_{PS} is positive does not necessarily imply that α_{SP} is negative.

Whether α_{SP} is negative or not when α_{PS} is positive depends upon the expected risk premia on the Standard & Poor's Index and the investor's existing portfolio, the correlation between the two portfolios, and the value of α_{PS}. For reasonably well diversified portfolios P and values of α_{PS} which are of sufficient size to be statistically significant, a positive value of the professionally calculated α_{PS} will probably be associated with a negative value of α_{SP}. For example, if the expected return in excess of the riskfree rate was 6 percent on the S & P index and 8 percent on the existing portfolio and β_{SP} was 1.0, the alpha calculated with respect to the existing portfolio would be 2 percent. Now, if the correlation between these two portfolios was in excess of 0.87, which would usually be the case for well-diversified institutional equity portfolios, the professionally calculated alpha, α_{PS}, would be negative. Specifically, in this case, β_{PS} would be 0.9025 and α_{PS} would be -1.415.

---

[7]As one example, let $\mu_S - r_f$ be 8 percent, $\mu_P - r_f$ be 4 percent and β_{SP} be 1.5, implying that α_{SP} is 2 percent. If ρ^2_{SP} is less than .75, the left hand side of 10 will be negative.

VII. Conclusion

This paper used the concept of relative efficiency to examine the informational content of traditional one-parameter performance measures, so-called alpha coefficients. If a potential portfolio has a positive performance measure with respect to an investor's existing porfolio, the investor can improve the relative efficiency of his existing portfolio by spreading his investment in risky assets over both portfolios.

If the performance measure is negative, the investor should sell short the potential portfolio and use the proceeds to increase the amount invested in his existing portfolio. To the extent that there are some restrictions on short sales, this recommended action may not be feasible. If zero, the investor's existing portfolio provides the greatest level of relative efficiency possible and no investment should be placed in the potential portfolio.

The paper then considered the case in which the alpha of a potential portfolio was calculated with respect to a portfolio comprising only a portion of an investor's existing portfolio of risky assets. As an example, the investor might currently hold both an S & P Index Fund and a portfolio of bonds, and the alpha might have been calculated with respect to the S & P Index. If the potential portfolio can be viewed as a well diversified portfolio of those stocks in the S & P Composite Index, there is some justification for interpreting the alpha in the same way as if the alpha were calculated with respect to the investor's total risky portfolio. The exact conditions were specified.

Although alpha coefficients should be calculated with respect to the investor's existing portfolio, most professional evaluation services calculate these measures with respect to the Standard & Poor's Composite Index. If these coefficients are zero or negative,[8] the paper showed that alpha calculated with respect to the existing portfolio will be positive, indicating that the investor could move to a relatively more efficient position by shifting some assets from his existing portfolio to an S & P Index Fund. It should be noted that this interpretation of a zero coefficient differs from the usual interpretation by professional investors. If positive, the implied action was ambiguous and further analysis would be required.

[8] The argument assumed that the correlation of the existing portfolio and the S & P Index was not one or minus one—a reasonable assumption.

Appendix

Let an investor's current portfolio of risky assets be decomposed into two portfolios, say P and O. Let A represent some other portfolio which an investor is considering adding to his portfolio of risky assets with either a long or short position. This appendix will show that the alpha of portfolio A calculated with respect to portfolio P will indicate whether a positive, zero, or negative position should be taken in portfolio A if the following conditions hold

$$\mu_A, \mu_P, \mu_O, \mu^* > r_f \tag{A1a}$$

$$r_A - \mu_A = \beta_{AP}(r_P - \mu_P) + \varepsilon \tag{A1b}$$

and the variance-covariance matrix of A, P, and O is positive definite. Because ε was assumed to be a mean-zero disturbance independent of r_P and r_O, (A1b) implies that

$$\sigma_{AO} = \beta_{AP}\sigma_{PO} \tag{A1c}$$

If x_k represents the proportion invested in portfolio k, the investor would select the x's so as to

$$\min \quad x_A^2\sigma_{AA} + x_P^2\sigma_{PP} + x_O^2\sigma_{OO} + 2x_Ax_P\sigma_{AP} + 2x_Ax_O\sigma_{AO} + 2x_Px_O\sigma_{PO} \tag{A2a}$$

$$\text{s.t.} \quad x_A\mu_A + x_P\mu_P + x_O\mu_O + x_fr_f = \mu^* \tag{A2b}$$

$$x_A + x_P + x_O + x_f = 1 \tag{A2c}$$

The optimal value of x_A for a desired μ^* is

$$x_A = (\mu^* - r_f)[(\mu_A - r_f)(\sigma_{PP}\sigma_{OO} - \sigma_{PO}^2) - (\mu_P - r_f)(\sigma_{AP}\sigma_{OO} - \sigma_{AO}\sigma_{PO}) + (\mu_O - r_f)(\sigma_{AP}\sigma_{PO} - \sigma_{PP}\sigma_{AO})]/K \tag{A3}$$

where K is

$$
-\det \begin{vmatrix} \sigma_{AA} & \sigma_{AP} & \sigma_{AO} & (\mu_A - r_f) \\ \sigma_{PA} & \sigma_{PP} & \sigma_{PO} & (\mu_P - r_f) \\ \sigma_{OA} & \sigma_{OP} & \sigma_{OO} & (\mu_O - r_f) \\ (\mu_A - r_f) & (\mu_P - r_f) & (\mu_O - r_f) & 0 \end{vmatrix}
$$

The substitution of (A1c) into (A3) and the replacement of σ_{AP} by $\beta_{AP}\sigma_{PP}$ yields

$$
x_A(\mu^* - r_f)[(\mu_A - r_f)(\sigma_{PP}\sigma_{OO} - \sigma_{PO}^2) - (\mu_P - r_f)(\beta_{AP}\sigma_{PP}\sigma_{OO} - \beta_{AP}\sigma_{PO}^2)
$$

$$
+ (\mu_O - r_f)(\beta_{AP}\sigma_{PP}\sigma_{PO} - \sigma_{PP}\beta_{AP}\sigma_{PO})]/K . \tag{A4}
$$

Recognizing that the third term in the brackets is zero, one can factor out $(\sigma_{PP}\sigma_{OO} - \sigma_{PO}^2)$ to obtain

$$
\begin{aligned}
x_A &= (\mu^* - r_f)(\sigma_{PP}\sigma_{OO} - \sigma_{PO}^2)[(\mu_A - r_f) - \beta_{AP}(\mu_P - r_f)]/K \\
&= (\mu^* - r_f)(\sigma_{PP}\sigma_{OO} - \sigma_{OP}^2)(\alpha_{AP})/K .
\end{aligned} \tag{A5}
$$

If α_{AP} is zero, the optimal value of x_A will be zero. If α_{AP} is non-zero and if all the terms multiplying or dividing α_{AP} in (A5) are positive, x_A will have the same sign as α_{AP}. The terms multiplying α_{AP} constitute in fact the precise definition of the Lagrange multiplier λ of equation set (7) in the text.

All the terms multiplying α_{AP} are indeed positive, which establishes the result. By assumption, $(\mu^* - r_f)$ is positive. The second term $(\sigma_{PP}\sigma_{OO} - \sigma_{PO}^2)$ is the determinant of a positive definite matrix and thus positive. After tedious manipulation, K can be expressed as

$$\begin{pmatrix} \mu_A - r_f \\ \mu_P - r_f \\ \mu_O - r_f \end{pmatrix}' \begin{pmatrix} \Lambda_{AA} & \Lambda_{AP} & \Lambda_{AO} \\ \Lambda_{PA} & \Lambda_{PP} & \Lambda_{PO} \\ \Lambda_{OA} & \Lambda_{OP} & \Lambda_{OO} \end{pmatrix} \begin{pmatrix} \mu_A - r_f \\ \mu_P - r_f \\ \mu_O - r_f \end{pmatrix} \tag{A6}$$

where the prime indicates transposed and Λ_{ij} is the cofactor of row i and column j of the original variance-covariance matrix.

Because of the symmetry of the original variance-covariance matrix, the matrix of cofactors in (A6) will be the same as its transpose, which in turn is proportional to the inverse matrix of the variance-covariance matrix. Because the variance-covariance matrix has been assumed positive definite, the constant of proportionality will be positive. Since the inverse matrix of a symmetric positive definite matrix is itself positive definite,[1] the quadratic form given by (A6) will be non-zero and positive.

---

[1]Johnston (1972) contains a proof of this statement.

References

Bloomfield, T., R. Leftwich and J. Long, Jr., 1977, Portfolio strategies and performance, Journal of Financial Economics 5, 201-218.

Blume, M.E., 1980, The relative efficiency of various portfolios: some further evidence, Journal of Finance 35, 269-281.

Jensen, M., 1968, The performance of mutual funds in the period 1945-1964, Journal of Finance 23, 386-416.

Jensen, M., 1969, Risk, the pricing of capital assets, and the evaluation of investment portfolios, Journal of Business 42, 167-247.

Johnston, J., 1972, Econometric methods, New York: McGraw-Hill Book Co.

Markowitz, H.M., 1959, Portfolio selection: efficient diversification of investments, New York: John Wiley & Sons, Inc.

Roll, R., 1977, A critique of the asset pricing theory's tests, Journal of Financial Economics 4, 126-176.

Roll, R., 1978, Ambiguity when performance is measured by the securities market line, Journal of Finance 33, 1051-1070.

Roll, R., 1980, The expected return benchmark and portfolio performance evaluation, Journal of Portfolio Management 7, 5-19.

Treynor, J.L. and F. Black, 1973, How to use security analysis to improve portfolio selection, Journal of Business 46, 66-86.

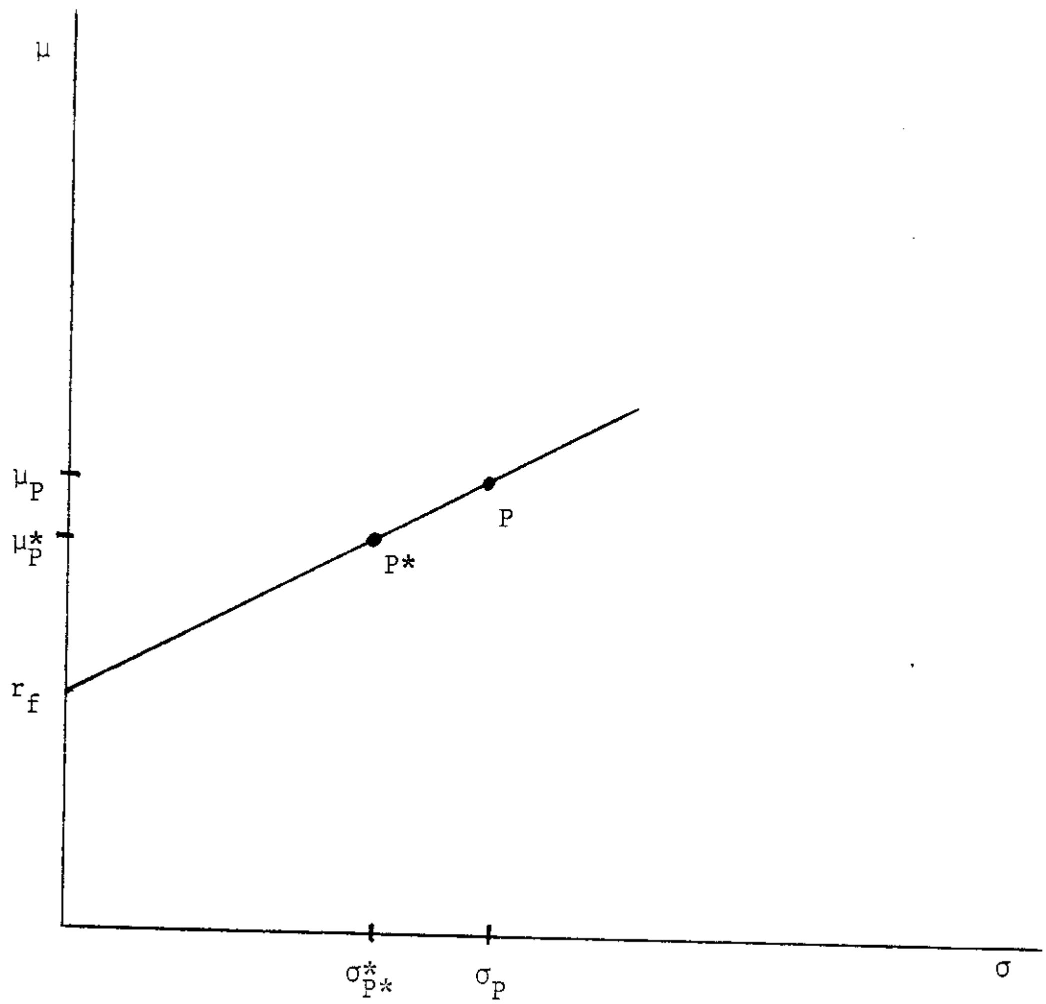

Figure 1: The Investor's Opportunities With Risky Portfolio P.

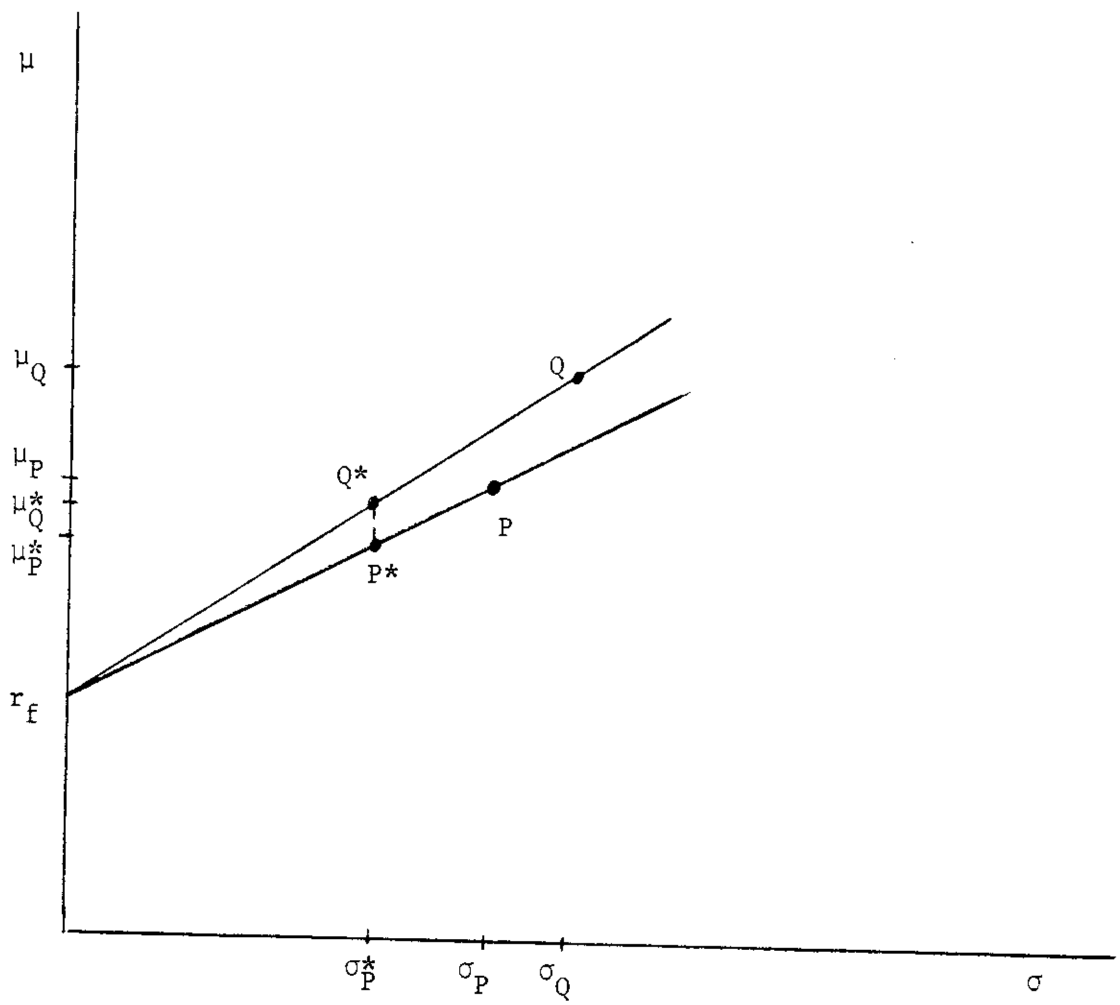

Figure 2: The Gain in Relative Efficiency From Moving From Portfolio P to Portfolio Q.

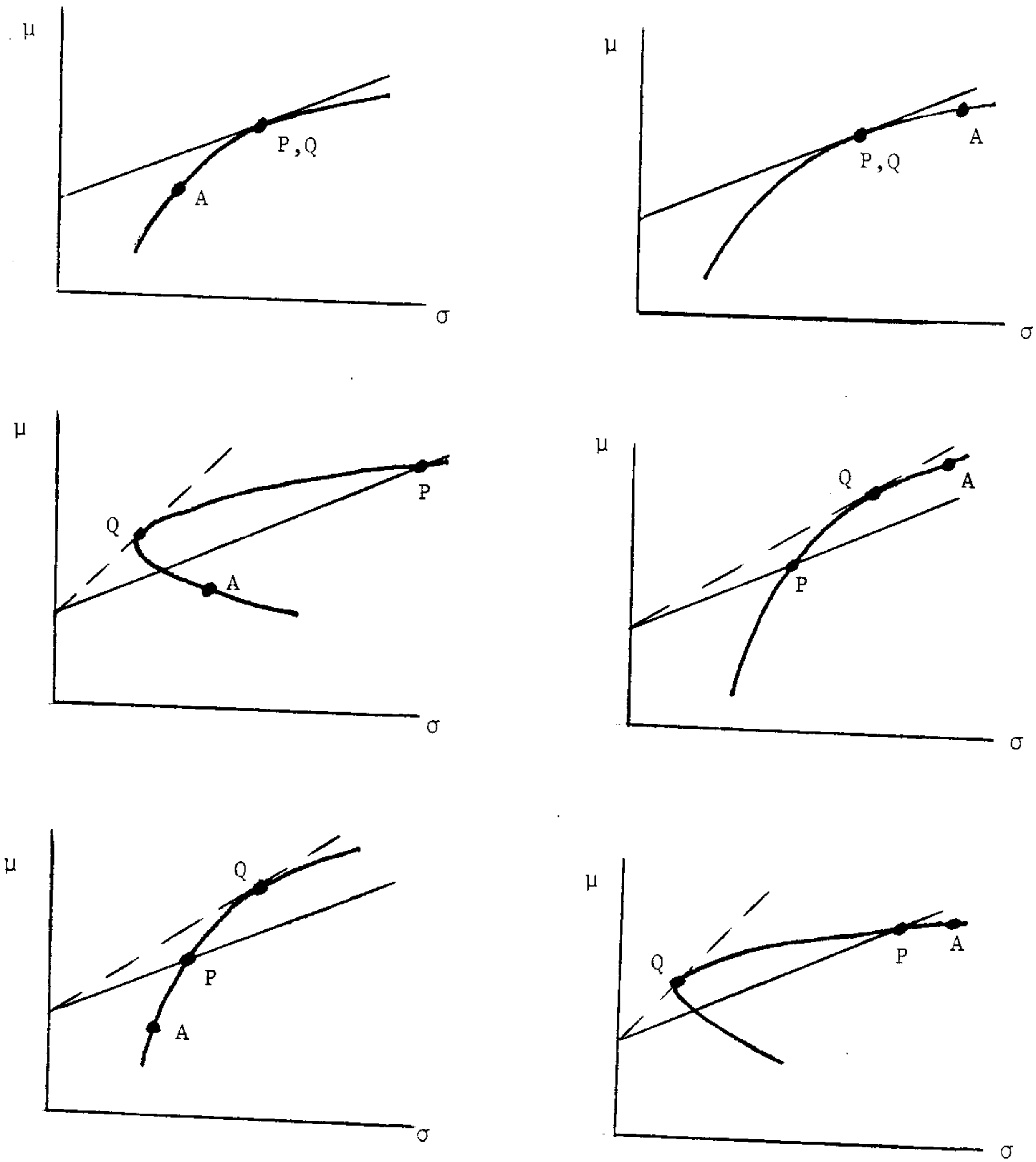

Figure 3: Various Combinations of A and P to Obtain the Relatively More Efficient Portfolio Q.