The Majoritarian Compromise is Majoritarian-Optimal and Subgame-Perfect Implementable*

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Abstract

It is shown that the Majoritarian Compromise of Sertel (1986) is *subgame-perfect implementable* on the domain of strict preference profiles, although it fails to be Maskin-monotonic and is hence not implementable in Nash equilibrium. The Majoritarian Compromise is Pareto-optimal and obeys **SNIP** (strong no imposition power), i.e. never chooses a strict majority's worst candidate. In fact, it is "majoritarian approving" i.e. it always picks "what's good for a majority" (alternatives which some majority regards as among the better "effective" half of the available alternatives). Thus, being Pareto-optimal and majoritarian approving, it is *majoritarian-optimal*.

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Finally, the Majoritarian Compromise is measured against various criteria, such as consistency and Condorcet-consistency.

1. Introduction

The purpose of this paper is to show the majoritarian optimality and subgame-perfect implementability of the social choice rule (scr) called the "Majoritarian Compromise" (Sertel, 1986). In our opinion, accomplishing this task performs two important services. First, the Majoritarian Compromise is a highly recommendable scr to implement in so many important real-world social choice problems where it is important *not to choose an alternative which is regarded as worst by a strict majority* of agents forming society.[1] For the Majoritarian Compromise is actually *majoritarian-optimal*, i.e. not just (weakly) Pareto-optimal, but also guaranteed to choose only those alternatives which some majority in fact ranks among the top "effective half[2]" of the candidates. Its implementability in some self-enforcing, non-cooperative manner, such as through subgame-perfect equilibrium, is therefore of real significance.

Second, the literature has been short of a good example of a scr which is subgame-perfect implementable but not Nash Implementable, and the Majoritarian Compromise forms such an example. For although we will see that it fails Maskin-monotonicity, a well-known requisite for Nash implementability, we will establish that it is subgame-perfect implementable.

To gain an intuitive motivation for the Majoritarian Compromise, imagine a social choice situation where an alternative - for instance, a president - has to be chosen, by some assembly, among many candidates. If we were to apply the plurality rule, we would look for the alternatives which are regarded as best by the largest number of assembly members. A well-known problem with the plurality rule, however, is that, so long as there are more than two candidates, a president chosen in this manner may lack the support of any majority, and in fact may be opposed by a strict majority of agents each of whom even regards the chosen candidate as the worst candidate available.[3] In contrast to the plurality

[1] By a majority (resp., strict majority) of an assembly we will mean any subset of agents which contains at least half (resp., more than half) of the assembly.

[2] By the effective half of an integer m we mean the smallest integer $< \frac{m}{2} >$ which is no less than $\frac{m}{2}$.

[3] To illustrate the importance of this malady, it suffices to indicate to what we may call the "Hitler syndrome", recalling the manner in which the NSDAP (Nazionalsozialistische Deutsche

rule, the Majoritarian Compromise unambiguously selects candidates who have the support of a majority in the best degree possible, and certainly never chooses candidates who are regarded as worst by any strict majority.

A very important property to seek in a scr is that it be non-imposing, or that it give to no agent (or minority) the power to impose its will against the strong disapproval of fellow agents. The weakest form of this class of conditions of "*no imposition power*" requires that an alternative considered to be worst by all but one agent should not be chosen. A stronger version which we call "strong no imposition power" (**SNIP**) is the condition that no alternative is chosen which is regarded by a strict majority of the agents to be worst.[4] In fact, the Majoritarian Compromise satisfies even a stronger version of **SNIP**, namely that of being *majoritarian approving*, i.e. being a refinement of the scr we call Majoritarian Approval, defined as choosing those alternatives which are regarded by some majority to be among the better effective half of the candidates. In particular, the Majoritarian Compromise is *majoritarian-optimal*, by which we mean the conjunction of majoritarian approving and (weak) Pareto-optimality. Thus, the Majoritarian Compromise is a (weakly) Pareto-optimal refinement of the Majoritarian Approval, selecting all the candidates who have the support of a largest majority in the best degree possible.

It is well-known (see, e.g., Maskin, 1986) that Nash implementability of a scr requires its Maskin-monotonicity.[5] On the other hand, Maskin-monotonicity turns out to be unnecessary for a scr to be subgame-perfect implementable. In fact, from Moulin (1979, 1980, 1984), Mueller (1978), Crawford (1977, 1979) and Dutta (1980) we know that with multi-stage mechanisms and imposing the requirement that off-the-equilibrium-strategies must be credible, a larger set of scrs can be so implemented. However, these results are limited to specific environments and specific social choice functions. Moore and Repullo (1988) were the first to generalize this approach and find a necessary condition and a couple of sufficient conditions (depending on the number of agents) for subgame-perfect implementability. With an analogue of Maskin's theorem (for Nash implementability), Abreu and Sen (1990) extended Moore and Repullo's results and almost closed the gap between their necessary and sufficient conditions. Although Nash im-

Arbeiterpartei), i.e. the Nazis of Adolf Hitler, came to power in Germany, 1932, having obtained not even a third of the vote.

[4] Thus, **SNIP** is satisfied by any scr which never chooses a Condorcet loser.

[5] Danilov (1992) actually characterized Nash implementability for societies of at least three agents in terms of "essential monotonicity", a condition stronger than Maskin-monotonicity.

plementability strictly implies subgame-perfect implementability, good examples of a scr, defined on a universal domain, which is subgame-perfect implementable but not Nash implementable, exhibiting the strictness of the implication, have been relatively scarce.[6] Here we take a step toward making up for this deficiency, showing that the Majoritarian Compromise, while failing Maskin-monotonicity, nevertheless is subgame-perfect implementable.[7]

On the other hand, the Majoritarian Compromise defies certain well-known criteria. Although it satisfies "ceteris paribus" monotonicity[8], it violates consistency[9]. Furthermore, as we will see in Section 4, not only might it not choose the Condorcet (1785) winner, but also it fails the Condorcet loser criterion, a condition stronger than **SNIP**.

Now for a preview of the rest of the paper: The next section records some formal definitions and basic facts, after which Section 3 includes our implementation result. Section 4 discusses other properties of the Majoritarian Compromise in a comparative manner. Section 5 collects our concluding remarks.

2. Definitions and Basic Facts

Given a non-empty finite set X of m alternatives, as our permissible "preferences" on X we will take the space P of all linear orders on X, i.e. the set of all functions $p : X \to 2^X$ where $p(x)$ at any $x \in X$ is the "lower counter set" consisting of all alternatives which are worse than or equivalent to x while p obeys (1) *completeness*: for all $x, y \in X$, we have $y \in p(x)$ or $x \in p(y)$; (2) *transitivity*: $y \in p(x)$ & $z \in p(y) \Rightarrow z \in p(x)$; (3) *antisymmetry*: $y \in p(x)$ & $x \in p(y) \Rightarrow x = y$. For any positive integer n, we will denote $N = \{1, 2, ..., n\}$, for the set of n agents, and $\mathbf{P} = P^N$ will stand for the space of all permissible "preference profiles" on X. For any $\mathbf{p} \in \mathbf{P}$, (with slight abuse of notation) we will write $\mathbf{p}(x) = \prod_{i \in N} p_i(x)$ as isomorphic to $\mathbf{p}(x) = \{p_i(x)\}_{i \in N}$. By a social choice rule (scr) we mean any function $\mathbf{F} : \mathbf{P} \to A$, where $A = 2^X \backslash \{\emptyset\}$.

[6] See Palfrey and Srivastava (1991).

[7] In fact, in an other work (see Sertel and Yılmaz, 1993) we show that a large class of scrs (refinements of Approval Voting with a Plurality Floor) which includes the Majoritarian Compromise is subgame-perfect implementable.

[8] We thank an anonymous referee for baptising this condition which was used earlier by Moulin (1983).

[9] In the sense of Smith (1973) or Young (1975).

Every $p \in P$ also determines a (ordinal) "utility" $\pi : X \to \{1, ..., |X|\}$ representing p through $\pi(x) = |p(x)|$ at each alternative $x \in X$. For each "coalition" $K \subset N$, at each $\mathbf{p}_K = \{\mathbf{p}_i\}_{i \in K} \in P^K$ we also define a (ordinal) "welfare" $\pi_K : X \to \{1, ..., |X|\}$ through $\pi_K(x) = \text{Min}\{\pi_i(x)|i \in K\}$, where π_i is the utility representing the preference p_i $(i \in K)$. Thus, we associate the "welfare" of a coalition at an alternative x with the "utility" of the coalition member whose utility at x is least in that coalition. We say that an alternative "gains kth degree approval or support" from a coalition $K \subset N$ with (coalitional) preference profile $\mathbf{p}_K \in P^K$ iff $\pi_K(x) \geq m-k+1$. A coalition $K \subset N$ is called a majority (in N) iff $|K| \geq |N \backslash K|$, and we denote $\mathcal{M}$ for the family of majorities in N. At any profile $\mathbf{p} \in \mathbf{P}$, we write $\overline{\pi} = Max\{\pi_K(x)|K \in \mathcal{M}, x \in X\}$ for the highest majority welfare achievable (by suitable chioce of x) at $\mathbf{p}$, and we define $\overline{\mathbf{M}}(\mathbf{p}) = \{x \in X|\pi_i(x) = \overline{\pi}\}$ as the set of alternatives giving this maximal (majority) welfare. At any $\mathbf{p} \in \mathbf{P}$ and any $x \in X$, we also define $K(x;\mathbf{p}) = \{i \in N|\pi_i(x) \geq \overline{\pi}\}$ as the (largest) set of agents enjoying at least $\overline{\pi}$ utility at x. Now we are ready to define the Majoritarian Compromise.

Definition 1. *The Majoritarian Compromise is the scr* $\mathbf{M} : \mathbf{P} \to A$ *defined by* $\mathbf{M}(\mathbf{p}) = \{x \in \overline{\mathbf{M}}(\mathbf{p})|y \in \overline{\mathbf{M}}(\mathbf{p}) \Rightarrow |K(y;\mathbf{p})| \leq |K(x;\mathbf{p})|\}$.

Thus, $\mathbf{M}(\mathbf{p})$ picks, among the alternatives giving maximal possible majority welfare $\overline{\pi}$, those which give this welfare to a largest majority. This definition will receive some discussion and further attention in our next section.

A scr $\mathbf{F}$ is said to be *Maskin-monotonic* iff for any two profiles $\mathbf{p}$ and $\mathbf{q} \in \mathbf{P}$, if $x \in \mathbf{F}(\mathbf{p})$ and $\mathbf{p}(x) \subset \mathbf{q}(x)$, then $x \in \mathbf{F}(\mathbf{q})$. A scr $\mathbf{F}$ is *ceteris paribus monotonic* iff, for any two profiles $\mathbf{p}$ and $\mathbf{q} \in \mathbf{P}$, any alternative x in $\mathbf{F}(\mathbf{p})$ will also be in $\mathbf{F}(\mathbf{q})$ and $\mathbf{F}(\mathbf{q}) \subset \mathbf{F}(p)$ will obtain (thus, $x \in \mathbf{F}(\mathbf{q}) \subset \mathbf{F}(\mathbf{p})$), as long as $\mathbf{p}(x) \subset \mathbf{q}(x)$ and $\mathbf{q}(y) \subset \mathbf{p}(y)$ for all y in $X \backslash \{x\}$. (In other words, if a winner were to be ranked higher by some voters, all else unchanged, then this candidate would continue to be a winner and there would be no new winners.) Clearly, Maskin-monotonicity implies ceteris paribus monotonicity.

A scr $\mathbf{F}$ is said to be Pareto-optimal if it picks only (weakly) Pareto-optimal alternatives, so that $y \in \mathbf{F}(\mathbf{p})$ implies that there is no $x \in X$ which every agent finds strictly superior to y (i.e. there is no $x \in X$ with $x \notin p_i(y)$ for all $i \in N$). A scr $\mathbf{F}$ is *majoritarian approving* iff $x \in \mathbf{F}(\mathbf{p})$ implies that x is considered by some majority to be among the better effective half of the alternatives. A Pareto-optimal and majoritarian approving scr $\mathbf{F}$ is said to be *majoritarian-optimal.*

By a (normal form) mechanism we mean any function $\Gamma : S \to X$ with $S = \prod_{i \in N} S_i$ the product of "message spaces" S_i, one for each agent $i \in N$. Thus, the combination $(\Gamma, \mathbf{p})$ of a mechanism Γ with any profile $\mathbf{p} \in \mathbf{P}$ determines a normal form game.[10] For a given mechanism Γ and a game-theoretic equilibrium concept σ, $\Gamma(\sigma(\Gamma, \mathbf{p}))$ thus gives the set of all σ-outcomes (in X) of this mechanism at $\mathbf{p}$. We say that a scr $\mathbf{F}$ is σ-implementable iff there exists a mechanism Γ such that $\mathbf{F}(\mathbf{p}) = \Gamma(\sigma(\Gamma, \mathbf{p}))$ for all $\mathbf{p} \in \mathbf{P}$.[11]

The condition of **WNVP** (weak no veto power) for any scr $\mathbf{F}$ forbids any agent's excluding from $\mathbf{F}(\mathbf{p})$ any alternative which all others agree (according to $\mathbf{p}$) to be best in X. (WNVP : For any $i \in N$, if $\mathbf{p}_j(x) = X$ for all $j \in N \backslash \{i\}$, then $x \in \mathbf{F}(\mathbf{p})$.)

Maskin-monotonicity is necessary for Nash implementability at any "population size" n, and its conjunction with **WNVP** is sufficient for Nash implementability when $n \geq 3$ (Maskin, 1986). Although **WNVP** may seem a mild condition, it is well-known to be too strong to be necessary for Nash implementability.

We use Abreu and Sen's (1990) theorem that the conjunction of WNVP and Condition $\boldsymbol{\alpha}$, below, w.r.t. X implies subgame-perfect implementability.

Condition α : A scr $\mathbf{F} : \mathbf{P} \to A$ satisfies Condition $\boldsymbol{\alpha}$ w.r.t. a subset $B \subset X$ iff

1. $\mathbf{F}(\mathbf{p}) \subset B$ for every $\mathbf{p} \in \mathbf{P}$; and

2. whenever $\mathbf{p}$, $\mathbf{p}' \in \mathbf{P}$ and $x \in \mathbf{F}(\mathbf{p}) \backslash \mathbf{F}(\mathbf{p}')$, then there exits a sequence $a_0 = x, a_1, ..., a_{h+1}$ of alternatives in B and a sequence $i(0), i(1), ..., i(h)$ of agents such that

(2.1) $a_{j+1} \in p_{i(j)}(a_j)$ for every $j \in \{0, 1, ..., h\}$,
(2.2) $a_h \in p'_{i(h)}(a_{h+1})$ but $a_{h+1} \notin p'_{i(h)}(a_h)$
(2.3) $B \not\subset p'_{i(j)}(a_j)$ for each $j \in \{0, 1, ..., h\}$, and
(2.4) if $B \subset p_{i(j)}(a_{h+1})$ for every $j \in \{0, 1, ..., h-1\}$, then either $h = 0$ or $i(h-1) \neq i(h)$.

[10] Specifically, $(\Gamma, \mathbf{p})$ denotes the normal form game $\{(S_i, p_i^\Gamma)\}_{i \in N}$, whose joint strategy space is S and preference profile is $p^\Gamma = (p_i^\Gamma)_{i \in N}$, where $p_i^\Gamma : S \to 2^S$ is the preference on S determined by $p_i^\Gamma(s) = \{t \in S | \Gamma(t) \in p_i(\Gamma(s))\} = \Gamma^{-1}(p_i(\Gamma(s)))$ $(i \in N)$. Similarly, an extensive form mechanism Γ^e specifies a game tree, the information and choice partitions and a player for each non-terminal node. Consequently, every path in the game tree is a sequence of "messages" by players. Thus, an extensive form mechanism Γ^e assigns an outcome, $x \in X$, to each path of the game tree. Then the combination $(\Gamma^e, \mathbf{p}$) determines an extensive form game.

[11] For a more detailed discussion see Moore and Repullo (1988) and Abreu and Sen (1990).

3. The Majoritarian Compromise

We defined the Majoritarian Compromise (**M**) on the "quasi-universal" domain of all profiles of strict preferences on sets of alternatives. Specifically, given any set X of alternatives, for any profile $\mathbf{p}$ of strict preferences (linear orders) on X, **M** picks that subset $\mathbf{M}(\mathbf{p})$ of alternatives in X which gain the largest number of agents' $k^*(\mathbf{p})^{th}$ degree approvals, where $k^*(\mathbf{p})$, the *critical degree* of majority approval at $\mathbf{p}$, is the smallest integer k for which some alternative is commonly regarded as k^{th} best or better by at least half of the n agents whose preferences are recorded by the profile $\mathbf{p}$.

It is clear that a winner so picked is Pareto-optimal.[12] (For suppose some other candidate is deemed by every member of the assembly as better than our winner. Then this candidate would gain majority approval of critical degree k^*-1, contradicting the minimality of k^*.) Therefore, to prove that **M** is majoritarian-optimal it suffices to show that **M** is majoritarian approving, and this follows directly from the lemma below.

Lemma 1. *The critical degree of majority approval k^* never exceeds the effective half $< \frac{m}{2} >$ of the number of available alternatives.*

<u>Proof</u>: Given n agents , the total number of approvals of degree k^* or better received by the available alternatives is nk^*, so the average alternative receives $\frac{1}{m}nk^*$ such approvals. Now whenever $k^* \geq < \frac{m}{2} >$, since $< \frac{m}{2} > \geq \frac{m}{2}$, the mean number of approvals of degree k^* or better which an alternative receives is $\frac{1}{m}nk^* \geq \frac{1}{m}n\frac{m}{2} = \frac{n}{2}$, i.e. no less than majority. Since this average is met or exceeded by some alternative, some alternative must receive the k^{*th} or a better degree of approval of a majority. Therefore, the critical degree of approval, k^*, by its minimality, cannot exceed $< \frac{m}{2} >$.◇

Perhaps it would help intuition to run through a computational algorithm to determine $\mathbf{M}(\mathbf{p})$ at any $\mathbf{p} \in \mathbf{P}$. To find $\mathbf{M}(\mathbf{p})$ at any $\mathbf{p} \in \mathbf{P}$, we first imagine a table whose i^{th} column represents the i^{th} preference ($i \in N$) as a list of the candidates from best at the top, worsening monotonically to worst at the bottom. First we check whether there is a candidate which occurs in half or more of the entries in the top row. If there is, then we pick the candidates with this "majority support of the first degree". (This will give one winner if there is a candidate listed

[12] Of course, the Majoritarian Compromise may easily violate the stronger version of Pareto-optimality when we relax the assumpton of strict preferences, permitting non-singleton indifference classes.

$\mathbf{p}_1$	$\mathbf{p}_2$	$\mathbf{p}_3$
x	c	b
b	x	c
c	b	x
$x \in \mathbf{M}(\mathbf{p}) = \{x, b, c\}$		

$\mathbf{p}'_1$	$\mathbf{p}'_2$	$\mathbf{p}'_3$
x	c	b
c	x	c
b	b	x
$x \notin \mathbf{M}(\mathbf{p}') = \{c\}$		

Table 3.1: The Majoritarian Compromise is not Maskin-monotonic.

at the top by more than half the agents. It will give two if there are two candidates each of which is listed at the top by an exact half of the agent population.) If the top row avails of no such candidate, then we lower our hurdle and ask whether there are any candidates which are listed in the top two rows by a majority. If there are, then we pick among these candidates with this "second degree majority approval" those which are listed in the top two rows for a maximal number times. Otherwise, we lower our hurdle to the third row, and so on. Our lemma above says that we will never need to lower our hurdle below row $< \frac{m}{2} >$, for $\mathbf{M}(\mathbf{p})$ will be filled at the end of our examination of the first $k^*(\mathbf{p})$ rows, since $k^*(\mathbf{p})$ is never bigger than $< \frac{m}{2} >$. Of course, this also says that the chosen alternatives are each regarded by some majority of candidates to be among the top $k^*(\mathbf{p}) \leq < \frac{m}{2} >$ of the m candidates available, i.e. that $\mathbf{M}$ is majoritarian approving.

The main fact we wish to demonstrate now is that, although not Maskin-monotonic, $\mathbf{M}$ is subgame-perfect implementable. To see that $\mathbf{M}$ is not Maskin-monotonic, it suffices to consider two profiles $\mathbf{p}, \mathbf{p}' \in \mathbf{P}$, taking $X = \{x, b, c\}$ tripleton, as displayed in Table 3.1, where an alternative x regarded as better than an alternative y is placed above y. This example shows that $\mathbf{M}$ fails to be Maskin-monotonic. For in passing from $\mathbf{p}$ to $\mathbf{p}'$ the position of x has not deteriorated in any agent's preference (i.e., $\mathbf{p}(x) \subset \mathbf{p}'(x)$) but x is absent in $\mathbf{M}(\mathbf{p}')$ although present in $\mathbf{M}(\mathbf{p})$.

Now we agree on some notation and then prove a lemma which we will use in establishing our theorem below, that $\mathbf{M}$ is subgame-perfect implementable.

For any $x \in X$, $k \in \{1, ..., m\}$ and $\mathbf{p} \in \mathbf{P}$, denote $N(x, k, \mathbf{p}) = \{i \in N | \, |\mathbf{p}_i(x)| \geq m - k + 1\}$ for the set of agents who find x to be k^{th} best or better according to the preference profile $\mathbf{p}$, and denote $\underline{N}(x, k, \mathbf{p}) = \{i \in N | \, |\mathbf{p}_i(x)| = m - k + 1\}$ for the set of agents who find x to be exactly k^{th} best accordingly.

Lemma 2. *If x, $y \in X$ are distinct alternatives and, according to some preference profile $\mathbf{p} \in \mathbf{P}$, each receives less than 50% approval of degree $k - 1$ (i.e.,* *Max*$\{|N(x,k-1,\mathbf{p})|, |N(y,k-1,\mathbf{p})|\} < \frac{n}{2}$*), then not both alternatives can gain full (100%) approval of degree* k *according to* $\mathbf{p}$ *(i.e., Min*$\{|N(x,k,\mathbf{p})|, |N(y,k,\mathbf{p})|\} < n$*).*

<u>Proof</u>: As each agent's preference is strict, hence antisymmetric, each (in particular its k^{th}) indifference class is singleton. Note that $|N(x,k,\mathbf{p})| = |N(x,k-1,\mathbf{p})| + |\underline{N}(x,k,\mathbf{p})|$ and $|N(y,k,\mathbf{p})| = |N(y,k-1,\mathbf{p})| + |\underline{N}(y,k,\mathbf{p})|$, while $|\underline{N}(x,k,\mathbf{p})| + |\underline{N}(y,k,\mathbf{p})| \leq n$. Thus, if $\text{Max}\{|N(x,k-1,\mathbf{p})|, |N(y,k-1,\mathbf{p})|\} < \frac{n}{2}$ and $|N(x,k,\mathbf{p})| = n$ then $|\underline{N}(x,k,\mathbf{p})| > \frac{n}{2}$, so $|\underline{N}(y,k,\mathbf{p})| < \frac{n}{2}$ and therefore $|N(y,k,\mathbf{p})| < \frac{n}{2} + \frac{n}{2} = n$, as to be shown.$\diamond$

Theorem 1. *Assuming that* $n \geq 3$*, the Majoritarian Compromise* $\mathbf{M}$ *is subgame-perfect implementable.*

<u>Proof</u>: It is clear that $\mathbf{M}$ obeys **WNVP**. Thus, by Abreu and Sen's Theorem (1990), is suffices to show that $\mathbf{M}$ fulfills Condition $\boldsymbol{\alpha}$ w.r.t. $B = X$. To that end, set $B = X$ and take any two (strict) preference profiles $\mathbf{p}, \mathbf{p}' \in \mathbf{P}$ with $x \in \mathbf{M}(\mathbf{p}) \setminus \mathbf{M}(\mathbf{p}')$ for some $x \in X$. We assume that $\mathbf{p}(x) \subset \mathbf{p}'(x)$, for otherwise there would be an agent $l \in N$ and an alternative $z \in \mathbf{p}_l(x) \setminus \mathbf{p}'_l(x)$, so that the sequence $< x, z >$ of alternatives and the sequence $< l >$ of agents would satisfy $\boldsymbol{\alpha}$ directly. Since x was chosen by $\mathbf{M}$ at profile $\mathbf{p}$ but is no longer chosen at $\mathbf{p}'$, it follows that $k^*(\mathbf{p}) > 1$ and that there exists some $c \in X$ which gains a majority approval of some degree $k' < k^*(\mathbf{p})$ or a larger majority approval of degree $k^*(\mathbf{p})$ than x does under profile $\mathbf{p}'$, although c gains no such approval under $\mathbf{p}$. But given that the position of x has not deteriorated from $\mathbf{p}$ to $\mathbf{p}'$ [i.e., $\mathbf{p}(x) \subset \mathbf{p}'(x)$], there must then also exist some alternative $b \neq x$ which some agent j regarded as better than c under $\mathbf{p}$ [that is $b \notin \mathbf{p}_j(c)$] but worse than c under $\mathbf{p}'$ [that is $c \notin \mathbf{p}'_j(b)$]. Now defining $\tilde{N} = \{i \in N \mid X \neq \mathbf{p}'_i(x)\}$, the set of agents who do not find x to be best under $\mathbf{p}'$, we see that $\tilde{N}$ is of course non-empty, since otherwise x would still have to be chosen by $\mathbf{M}$ at $\mathbf{p}'$. If b is considered to be worse than x according to $\mathbf{p}$ by some agent in $\tilde{N}$, then we could readily satisfy $\boldsymbol{\alpha}$ with the sequence $< x, b, c >$ of alternatives, so we henceforth adopt the <u>assumption</u> **A** that $x \in \mathbf{p}_i(b)$ for every agent $i \in \tilde{N}$. There must be some agent, say i, who finds x to be k^{th} best under $\mathbf{p}$ for some $k \in \{2, ..., k^*(\mathbf{p})\}$ and who does not find x to be best under $\mathbf{p}'$ (for otherwise x would clearly be chosen by $\mathbf{M}$ at $\mathbf{p}'$ as well). We **<u>claim</u>** that $b \in \mathbf{p}_h(y)$ and $X \neq \mathbf{p}'_h(y)$ for some $y \in \mathbf{p}_i(x)$ and some agent h. If this claim is true, then the sequence $< x, y, b, c >$ of alternatives and the sequence $< i, h, j >$ of agents satisfy $\boldsymbol{\alpha}$, for we have $c \in \mathbf{p}_j(b)$, $b \in \mathbf{p}_h(y)$ and $y \in \mathbf{p}_i(x)$, but

$b \in \mathbf{p}'_j(c)$, while x, y, b fail to be best under $\mathbf{p}'$ for the agents i, h, j, respectively. Thus, it remains only to prove this claim.

<u>Proof of claim</u>: By **Lemma 2**, above, we know that b does not gain full (100%) approval of degree $k^*(\mathbf{p})$ under $\mathbf{p}$, for then the chosen x itself would have to receive at least as much such approval under $\mathbf{p}$, while clearly neither x nor b received majority approval of degree $k^*(\mathbf{p}) - 1$ under $\mathbf{p}$, by minimality of $k^*(\mathbf{p})$. So there is some agent, say h, with $|\mathbf{p}_h(b)| < m - k^*(\mathbf{p}) + 1 \leq |\mathbf{p}_i(x)|$. Therefore h finds at least one alternative $y \in \mathbf{p}_i(x)$ to be better than b under $\mathbf{p}$ [Note that $y \neq x$ due to assumption **A**, for otherwise $\boldsymbol{\alpha}$ is already satisfied as discussed above.]. If this y fails to be best for h under $\mathbf{p}'$, then we are through, so suppose not. Then x cannot be best for h under $\mathbf{p}'$, so $h \in \tilde{N}$ and our assumption **A** above gives $x \in \mathbf{p}_h(b)$. Hence, according to the preference $\mathbf{p}_h$, the agent h strictly prefers b to at most $m - k^*(\mathbf{p}) - 2$ alternatives distinct from x. According to $\mathbf{p}_h$, therefore, h strictly prefers at least two alternatives in $\mathbf{p}_i(x)$ to b. Not both of these elements of $\mathbf{p}_i(x)$ can be best for h under $\mathbf{p}'$, so at least one of them, say y, will serve to fulfill our claim: $b \in \mathbf{p}_h(y)$ and $X \neq \mathbf{p}'_h(y)$. This proves our claim and completes the proof of our theorem.$\diamond$

While **M** fails Maskin-monotonicity, one easily sees that it satisfies ceteris paribus monotonicity. To see that **M** is ceteris paribus monotonic, observe that an alternative $x \in \mathbf{M}(\mathbf{p})$ will gain at least as many approvals of degree $k^*(\mathbf{p})$ at any profile $\mathbf{q}$ with $\mathbf{p}(x) \subset \mathbf{q}(x)$ as at the profile $\mathbf{p}$ itself. Thus, if no other alternative has improved its position according to any agent's view in passing from $\mathbf{p}$ to $\mathbf{q}$, then x will continue to be picked by **M** according to $\mathbf{q}$, while no new winners will arise at $\mathbf{q}$.

4. Further Notions[13]

In order to discuss **M** and compare it with known other scrs, we record three further criteria relevant to social choice, namely consistency, admitting the no-show paradox, and consistency with Condorcet's rule.

To formulate consistency, given any two preference profiles $\mathbf{p}' : N' \to P$ with $N' = \{1, ..., n'\}$ and $\mathbf{p}'' : N'' \to P$ with $N'' = \{1, ..., n''\}$ first we need to define the profile $\mathbf{p} = \mathbf{p}' + \mathbf{p}'' : N \to P$ on $N = \{1, ..., n = n' + n''\}$ through $\mathbf{p} = (\mathbf{p}'_1, .., \mathbf{p}'_{n'}, \mathbf{p}''_1, ..., \mathbf{p}''_{n''})$, i.e. $\mathbf{p}_i = \mathbf{p}'_i$ for $i \in N'$ and $\mathbf{p}_{n+i} = \mathbf{p}''_i$ for $i \in N''$.

[13]We are thankful to an anonymous referee to whom we owe our inclusion of the present section. Berg and Nurmi (1988) offer a rich introduction to the relevant literature.

$\mathbf{p}'_1$	$\mathbf{p}'_2$	$\mathbf{p}'_3$	$\mathbf{p}'_4$	$\mathbf{p}'_5$	$\mathbf{p}'_6$	$\mathbf{p}'_7$	$\mathbf{p}''_1$
a	a	a	b	b	b	c	x
x	x	d	d	c	c	d	a
b	c	x	x	x	x	x	d
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Table 4.1: The Majoritarian Compromise violates consistency.

$\mathbf{p}'_1$	$\mathbf{p}'_2$	$\mathbf{p}'_3$	$\mathbf{p}'_4$	$\mathbf{p}''_1$
a	a	b	x	b
x	x	x	b	a
b	c	c	c	c
c	b	a	a	x

Table 4.2: The Majoritarian Compromise admits the no-show paradox.

We say that a scr $\mathbf{F}$ is *consistent* iff $\mathbf{F}(\mathbf{p}' + \mathbf{p}'') = \mathbf{F}(\mathbf{p}') \cap F(\mathbf{p}'')$ at any two profiles $\mathbf{p}'$ and $\mathbf{p}''$ whenever this intersection $[\mathbf{F}(\mathbf{p}') \cap F(\mathbf{p}'')]$ is non-empty. Now $\mathbf{M}$ is far from being consistent as we can see from Table 4.1, where the addition of a preference which regards x as best changes the winner from x to a: $\mathbf{M}(\mathbf{p}') = \mathbf{M}(\mathbf{p}'') = \{x\} \neq \{a\} = \mathbf{M}(\mathbf{p}' + \mathbf{p}'')$.

A related matter is that of the "no-show paradox", where a scr eliminates a candidate, say x, until we append to the society individuals (or an individual) who rank(s) this candidate as worst, whereupon x is chosen. $\mathbf{M}$ also admits the *no-show paradox* as we see in Table 4.2.

Another criterion for scrs we wish to consider is consistency with Condorcet's method. A scr is said to be *Condorcet consistent* iff it picks the singleton consisting of the "*Condorcet winner*" (i.e. the candidate which for each other alternative x, some strict majority $K(x)$ strictly prefers to x) if such a candidate exists. It is also noteworthy for a scr not to choose the "*Condorcet loser*" (i.e. the candidate which for each other alternative x, some strict majority $K(x)$ finds strictly inferior to x).[14]

[14]Note that in the absence of a Condorcet winner a Condorcet consistent scr might very well pick the Condorcet loser.

$\mathbf{p}_1$	$\mathbf{p}_2$	$\mathbf{p}_3$	$\mathbf{p}_4$	$\mathbf{p}_5$	$\mathbf{p}_6$	$\mathbf{p}_7$
a	b	c	d	a	a	b
x	x	x	x	c	d	c
b	a	a	a	b	c	a
c	c	b	b	d	b	d
d	d	d	c	x	x	x

Table 4.3: The Majoritarian Compromise may choose a Condorcet loser even in the presence of a Condorcet winner.

It is well-known that consistency and not admitting the no-show paradox are each incompatible with Condorcet consistency.[15] Since we just saw that **M** fails to be consistent and that it does admit the no-show paradox, however, it becomes natural to ask whether **M** is Condorcet consistent. In fact, **M** is so far from being Condorcet consistent that it can pick uniquely the Condorcet loser even when a Condorcet winner exists. To see this, we look at the profile **p** displayed in Table 4.3, where a is the Condorcet winner but **M** chooses x, the Condorcet loser.

The disagreement between the Majoritarian Compromise and Condorcet consistency should be no surprise, as the underlying philosophy of the Majoritarian Compromise is quite orthogonal to that of rules such as Condorcet consistent scrs which determine their choice via pairwise comparisons by majorities, depending on the concurrence of a possibly different majority at each pair of alternatives. The idea underlying the Majoritarian Compromise is one of *compromise*. By a "social compromise" we could mean (as in Sertel and Kalaycıoğlu, 1994) a scr which maximizes π_N, i.e. one which picks the set of alternatives which are missing in the largest number of rows from the bottom up when we make a table displaying our preference profile (as in our tables so far and as detailed in the paragraph following Lemma 1). But when preference profiles are sufficiently "rich", incorporating sufficiently disparate preferences in the same profile, so that each candidate is ranked as worst by at least one agent, then a social compromise is very non-selective, choosing the entire set X of alternatives. If we settle for maximizing the welfare π_K of smaller majorities K than N itself, then we can arrive at the notion of a "majoritarian compromise", rather than a "social compromise". We

[15] We know from Young (1975) that no anonymous and neutral Condorcet consistent scr can be consistent. Furthermore, Moulin (1988) shows that Condorcet consistency implies the no-show paradox.

wish to avoid rendering any strict majority excessively unhappy, e.g. as when we pick a candidate which is regarded as worst by some strict majority, or when we pick a candidate which is ranked worse than $< \frac{m}{2} >^{th}$by a strict majority. The Majoritarian Compromise fits our specification as it finds the best rank k^* at which there exists a candidate approved at that degree by a majority, and among such candidates it picks those which muster the greatest size of a majority giving it k^{*th} degree of support.

A natural question to ask is how some other scrs perform in comparison with the Majoritarian Compromise. Sertel and Yılmaz (1995) analyzed the properties of six other well-known scrs, namely Plurality with a Runoff, the Single Transferable Vote, and then Borda's, Nanson's, the Venetian, and Condorcet's Practical scr.[16] It turns out that all but Condorcet's Practical scr of these are Pareto-optimal. Alas, we return empty-handed from our quest for another majoritarian-optimal scr among them, since none of them is majoritarian approving (although the Single Transferable Vote, Borda's, Nanson's and the Venetian scr all satisfy **SNIP**). Borda's scr and Condorcet's Practical scr are the only two which satisfy ceteris paribus monotonicity. All but Nanson's scr violate both subgame-perfect implementability and Condorcet consistency among them. The Single Transferable Vote and Plurality with a Runoff are the only two failing the Condorcet loser criterion. Finally, all but Borda's scr violate consistency and admit the no-show paradox.

5. Closing Remarks

Regarding the implementability of the Majoritarian Compromise via solution concepts other than Nash or subgame-perfect Nash equilibrium, we can point out right away that **M** is implementable in undominated Nash equilibrium, since every subgame-perfect implementable scr is so (Palfrey and Srivastava, 1991). On the other hand, from Sertel (1995)[17] we know that it is not implementable in undominated Nash equilibrium via a bounded mechanism (Jackson, Palfrey and Srivastava, 1994). It can be checked that the extension of **M** to the domain of all profiles of complete pre-orders will be non-implementable via subgame-perfect equilibrium, in fact even via undominated Nash equilibrium.

[16] Some of these scrs were partially investigated earlier by Fishburn (1977), Fishburn and Brams (1983) and Berg and Nurmi (1988).

[17] Hurwicz and Sertel (1995) also produce Sertel's (1995) counter-example establishing this non-implementability in undominated Nash equilibrium via bounded mechanisms.

As to the question of what would actually happen (get implemented) if a selection[18] $\mathbf{M}'$ from $\mathbf{M}$ were instituted as the outcome function $\mathbf{M}' : \mathbf{S} \to X$ of a direct mechanism (where $S = \mathbf{P}$), from Sertel and Sanver (1997) we know that when $n \geq 3$ the result would be the trivial scr of all alternatives under the Nash equilibrium while it would give the weak Condorcet winners[19] (where they exist) under the strong Nash equilibrium.

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[18] i.e. a singleton-valued function on the same domain whose graph lies within that of $\mathbf{M}$.

[19] An alternative $x \in X$ is a *weak* Condorcet winner iff, for each $y \in X$ there is a majority $K \subset N$ such that $y \in p_i(x)$ for every $i \in K$.

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