ARBITRAGE, SHORT SALES, AND FINANCIAL INNOVATION

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We describe a model of general equilibrium with incomplete markets in which firms can innovate by issuing arbitrary, costly securities. When short sales are prohibited, firms behave competitively and equilibrium is efficient. When short sales are allowed, these classical properties may fail. If unlimited short sales are allowed, imperfect competition may persist, even when the number of potential innovators is large. If limited short sales are allowed, perfect competition may obtain in the limit but equilibrium can be inefficient because of the presence of an externality: the private benefits of innovation for firms differ from the social benefits.

KEYWORDS: Arbitrage, innovation, short sales, efficiency, competition.

1. INTRODUCTION

The Arrow-Debreu-McKenzie model of general equilibrium assumes that markets are complete so that all contingent commodities can be traded. The theory has now been extended to deal with the more realistic case of incomplete markets, where some commodities cannot be traded. However, even this literature typically assumes a given market structure. The assumption of a given market structure is quite restrictive, as one can see from recent developments in financial markets. The unprecedented rate of financial innovation makes it clear that market structure is neither constant nor exogenous.

In this paper we consider some of the implications of endogenizing market structure by allowing firms to issue new securities. We begin by defining an equilibrium in an economy with a finite number of firms and a continuum of investors. Then we study the behavior of the equilibrium as the number of firms becomes unboundedly large. When the number of firms is finite, one expects the economy to be imperfectly competitive; but as the number of firms becomes large, one expects price-taking competition to obtain. Under certain conditions, that may be the case here, but only if short sales are strictly limited. If

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2 The seminal contributions were made by Diamond (1967), Radner (1972), and Hart (1975), Cass (1988) surveys later developments by himself, Balasko, Duffie, Geanakoplos, Magill, Mas-Colell, Polemarchakis, Shaffer, and others. (See references.)

3 Exceptions are the recent work by Allen and Gale (1988) and Duffie and Jackson (1989).

4 There are finite market games in which agents behave like price-takers in equilibrium, for example, Bertrand pricing games. However, in the present model, a large number of firms is a necessary condition for competitive behavior.
investors are allowed to undertake unlimited short sales, then we can show that:
(I) Firms do not necessarily become price-takers in the limit.
(II) Financial innovation is not necessarily efficient.
(III) Even when the cost of issuing new securities is negligible, markets may not be complete.

These results are disturbing because the assumption of unlimited short sales plays a crucial role in financial theory. A number of important classical results rely on spanning arguments that require agents to take short positions. Some examples are the use of Arrow securities (Arrow (1964)), the Modigliani-Miller theorem (Modigliani-Miller (1958); Hellwig (1981)), the arbitrage pricing theory of Ross (1976), and the options pricing theory of Black-Scholes (1973). But when financial innovation is allowed, it appears that short sales are incompatible with efficiency and price-taking behavior. So the attractive properties of the classical theory appear to be threatened by the possibility of financial innovation.

Although our formal analysis is concerned with the impact of short sales on innovation, it has a wider relevance. In an empirical study of financial innovation by investment banks, Tufano (1988) has found that the first firm to innovate bears substantially higher costs than subsequent imitators. Moreover, the first-mover does not charge a higher price for his product during the brief period of monopoly before imitative products appear. His data show that there is a problem for innovators in recouping their costs. (He suggests the incentive to innovate comes from increased market share.) In our model, all firms move simultaneously and bear the same costs. However, the short-sellers play a role analogous to the imitators in Tufano's study. This suggests that similar techniques might be used to study the effects of imitation, with similar results.\(^5\)

In Section 2 we describe the basic model. There is a finite number of firms with a random future income. Each firm is owned by an entrepreneur who wants to maximize his current consumption. He does this by selling claims against his future income to investors. When no short sales are allowed, the incentive to issue new securities is straightforward. Because it is costly to issue claims, markets will be incomplete. Then different types of investors will typically put different values on the same security. In order to maximize the market value of the firm, an entrepreneur will issue a variety of claims against the future returns of the firm. In effect, he is "splitting" the firm and marketing each piece to the clientele that values it most.

When unlimited short sales are allowed, things are more complicated. For example, it is well known from the literature on stock market economies that a firm's choice of production plan can have a large impact on the economy when

\(^5\) Another empirical question is how important are short sales quantitatively? It appears that on the NYSE, short sale volume is only 8% of the volume of regular trading and most of that is technical trading (Pollack (1986)). However, the enormous volume of trading in derivative securities may to some extent be a substitute for short sales. In an empirical analysis, Figlewski and Webb (1990) argue that options trading is a substitute for short sales. To the extent that our analysis applies to this case, it may be quantitatively very important. See Allen and Gale (1990) for a discussion of the impact of derivative security markets.
unlimited short sales are allowed. By introducing a new production plan, the firm can change the dimension of the space spanned by investors' portfolios (see, for example, Kreps (1979, Essay II)). Similarly, in the present context, the introduction of a new security by a single firm can have a large impact on the economy if it increases the span of the securities in the economy. But if each security is issued by a large number of firms, it is not clear that perfect competition is incompatible with short sales. One of the central points of this paper is that competition is necessarily "imperfect" if there is to be any incentive to innovate.

In Section 3, we study an example that illustrates this idea. In the example, firms are assumed to have identical returns. Then arbitrage ensures that in equilibrium all firms have the same market value. This means that ex post, after the entrepreneurs have made their decisions, there is no incentive to innovate. An entrepreneur who has issued innovative securities is worse off than one who does not, since he has incurred a cost of innovation and yet realizes the same market value. Nonetheless, there may be an incentive to innovate ex ante, before the entrepreneurs make their decisions. By introducing innovative securities, an entrepreneur may influence the prices that prevail ex post and hence change the market values of all firms, his own along with others'. Furthermore, this effect may persist even when the number of entrepreneurs and assets becomes unboundedly large.

The crucial assumption of the example studied in Section 3 is that the future returns of firms are perfectly correlated. In Section 4 we argue that similar conclusions should hold in the general case, when returns exhibit idiosyncratic risk. Without the assumption of perfect correlation, arbitrage is risky, the market values of firms may not be equalized, and there may be an ex post incentive to innovate. But if a large number of entrepreneurs choose the same financial structure (issue the same types of securities), the idiosyncratic risk can be diversified away. Then the market value of firms with the same financial structure must be equalized and the ex post incentive to innovate disappears for those firms. There are two ways that the incentive to innovate can be preserved. First, if only a small number of entrepreneurs issues a particular security, each of the entrepreneurs may perceive that he has a large impact on the equilibrium price. Second, because the returns to individual firms exhibit idiosyncratic risk, arbitrage may be unprofitable when the number of innovating firms is small and so there may remain an ex post incentive to innovate. In either case, the incentive to innovate is preserved only if there is imperfect competition and this conclusion should hold even in the limit, when the number of firms becomes unboundedly large.

Section 5 contains some further examples of the limiting behavior of equilibrium as the economy becomes unboundedly large. Those examples show that non-price-taking behavior and inefficiency persist in the limit when short sales are unconstrained and there is idiosyncratic risk.

In Section 6 we study the case where short sales are strictly limited. Under certain conditions, we can obtain a competitive limit theorem, that is, agents behave as price-takers asymptotically. However, because short sales create a
“pecuniary externality,” equilibrium need not be efficient. The existence of this externality was first recognized by Makowski (1983), in the context of a stock market economy. When there are no short sales, the increase in the value of the firm coincides with the social gain from the innovation. When there are limited short sales, there is an additional source of social benefit that is not taken into account by the firm’s owner. The short sales allow greater risk sharing; but the value of this risk sharing is not captured by the increase in the firm’s value. Roughly speaking, it is the change in the value of the open interest rather than the change in the value of the firm that measures the social benefit of an innovation. Because of the divergence between social and private benefit, innovation may be inefficient, even if entrepreneurs are price-takers.

The results obtained here are in stark contrast to those obtained in an earlier paper (Allen and Gale (1988); hereafter AG). In that paper, we studied a model of financial innovation which differed from the model discussed here in two respects. First, in AG we assumed either that short sales were not allowed or that they were effectively limited by costs. Second, we assumed a continuum of firms, comprising a finite number of types. In contrast to the results obtained here, the model studied in AG exhibited the following properties:

(I) By definition, firms behaved as price-takers.
(II) Financial innovation was efficient.\footnote{More precisely, we showed that the equilibrium was constrained efficient. An allocation was defined to be constrained efficient if a planner who was subject to the same cost of issuing securities could not make everyone better off by making transfers of goods and securities at the first date.}
(III) When the costs of creating new securities were negligible, markets were complete.

Also, as we pointed out in AG, if unlimited short sales were allowed, equilibrium typically did not exist. Incentives for innovation exist in a price-taking equilibrium only if the value of the firm is perceived to depend on its financial structure. But, as we have seen, the dependence of market value on financial structure gives rise to an arbitrage opportunity which is inconsistent with equilibrium. Evidently, the concept of equilibrium used in AG is inappropriate when short sales are allowed.

2. A MODEL OF FINANCIAL EQUILIBRIUM

Time is divided into two periods or dates indexed by $\tau = 1, 2$. At each date there is a single consumption good, which can be thought of as “income” or “money.” Asset returns and prices are measured in terms of this numeraire good.

Economic agents are assumed to be of two kinds, entrepreneurs and investors. Entrepreneurs are assumed to be risk neutral and are only interested in consumption at the first date.\footnote{More precisely, we assume that entrepreneurs are not around at date 2 and so cannot engage in short sales.} Each entrepreneur owns an asset which produces

\footnote{We are grateful to Joe Ostroy for pointing out the connection between our work and Makowski’s.}
a random return at the second date. In order to maximize consumption at the first date, the entrepreneurs will want to sell their assets to the investors. They do this by issuing claims against the returns of the assets; this provides the motivation for financial innovation. Unlike entrepreneurs, investors are risk averse and are interested in consumption at both dates. In equilibrium, they purchase the claims issued by the entrepreneurs and consume the returns at the second date.

Equilibrium at the first date is achieved in two stages. In the first stage, entrepreneurs decide what kinds of claims to issue against the assets they own. At the second stage, these claims are traded on a competitive auction market. At date 2 the asset returns are realized and the claims issued at date 1 are paid off.

We begin by describing an economy with a finite number \( N \) of entrepreneurs and a continuum of investors. Later, we shall be interested in the asymptotic behavior of the economy as \( N \) becomes unboundedly large.

**Asset Returns**

There is a countable set of assets indexed by \( n = 1, 2, \ldots, \infty \). Each asset \( n \) is represented by a random variable \( Z_n \) which gives the asset's return at date 2 in terms of the consumption good. Let \( G_N \) denote the joint probability distribution of the random variables \( \{Z_n\}_{n \in N} \). That is, for any \( z = (z_1, \ldots, z_n) \in \mathbb{R}^N \), \( G_N(z) = \text{Prob}\{Z_n \leq z_n : n \in N\} \). The asset returns are assumed to satisfy the following conditions.

**Assumption 1:** (a) \( Z_n \) is nonnegative with probability 1 and has finite expected value; (b) \( G_N \) is symmetric, i.e., for any permutation \( \pi : N \to N \) and any \( z \in \mathbb{R}^N \),

\[
G_N(z_1, \ldots, z_N) = G_N(z_{\pi(1)}, \ldots, z_{\pi(N)}).
\]

Symmetry does not imply that assets are perfect substitutes, although it is consistent with perfect substitutability. Typically, we will want to assume that assets exhibit some idiosyncratic risk, in which case they will not be perfect substitutes. On the other hand, symmetry does imply that assets make similar contributions to a well diversified portfolio.\(^10\)

**Financial Structures**

When markets are incomplete, it may be possible to increase the value of an asset by "splitting" it, that is, by issuing more than one type of claim against it. A claim is a measurable function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \). If the underlying asset shows a

\(^9\) We adopt the usual abuse of notation and let \( N \) stand for the set \( \{1, \ldots, N\} \).

\(^10\) In this sense there is a single type of asset. The results of this paper could easily be extended to a finite number of types. Assumption 1 simplifies the analysis without significantly reducing the insight we obtain from it.
return of $z$ units, the claim entitles the bearer to $f(z)$ units of the numeraire good at date 2. The collection of claims issued by an entrepreneur is called a financial structure. There is a finite number of financial structures indexed by $i = 1, \ldots, I$. The set of claims associated with the $i$th structure is denoted by $F_i$. $F_i$ is assumed to be finite. The claims associated with the financial structure are assumed to exhaust the returns to the asset. Thus, for any $i$ in $I$,

$$\sum_{f \in F_i} f(z) = z, \quad \forall z \in \mathbb{R}_+.$$  

Issuing claims is assumed to be costly. The cost of the $i$th financial structure is $C_i \geq 0$ units of the numeraire good at date 1.

To illustrate these ideas, suppose the asset owned by the entrepreneur is a "firm" and that the entrepreneur wants to issue debt and equity. The random variable $Z_n$ represents the future income of the "firm." The financial structure chosen by the entrepreneur will consist of two claims, $F_0 = \{f_0, f_1\}$, where $f_0$ corresponds to debt and $f_1$ corresponds to equity. Suppose that $R$ is the face value of the debt. Then $f_0(z) = \min\{R, z\}$ and $f_1(z) = \max\{z - R, 0\}$, for any $z \geq 0$.

**Investors**

There is a continuum of investors divided into a finite set $K$ of investor types. An investor of type $k$ has a von Neumann-Morgenstern utility function $U_k: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.\(^{11}\) If $c_\tau$ denotes consumption at date $\tau$, the associated utility is $U_k(c_1, c_2)$. Preferences are assumed to satisfy the following standard conditions.

**Assumption 2:** $U_k$ is concave, strictly increasing, and bounded above, for all $k \in K$.

We also impose the following nonstandard condition.

**Assumption 3:** Let $\{(c_n, W_n)\}$ be a fixed but arbitrary sequence where, for each $n$, $c_n$ is a real number and $W_n$ is a random variable. Suppose that for any $\epsilon > 0$ and $M > 0$, $\text{Prob} \{W_n \leq -M\} \geq \epsilon$ for infinitely many values of $n$. Then $\lim \inf E[U_k(c_n, W_n)] = -\infty$.

What Assumption 3 says is that expected utility becomes very negative if there is a small amount of mass very far down the tail of the distribution of

\(^{11}\) Note that consumption is not assumed to be bounded below. When short sales are allowed, the assumption that consumption be nonnegative with probability one is excessively strong. On the other hand, we want to avoid the difficulties of modelling bankruptcy. We allow negative consumption and attach high disutility to low values of consumption as a proxy for costly bankruptcy. (See Assumption 3.)
second period consumption. Investors are assumed to have a zero endowment of the consumption good at each date.

The cross-sectional distribution of investors' types is denoted by \( \alpha = (\alpha_1, \ldots, \alpha_K) \), where \( \alpha_k > 0 \) for every \( k \) in \( K \) and \( \sum_{k \in K} \alpha_k = 1 \). \( \alpha_k \) denotes the proportion of investors who are of type \( k \). In an economy of size \( N \), the total measure of investors is assumed to be \( N \) so the measure of investors of type \( k \) is \( \alpha_k N \).

Equilibrium is defined in two stages. First, we define equilibrium for the auction market, given the claims to be traded. Then we describe how claims are issued at the first stage, in anticipation of the auction market equilibrium at the second stage. This two-stage concept of equilibrium is fairly standard, having been used in similar contexts by Hart (1979) and Jones (1987), for example. It is the natural way to model the interaction among a finite number of oligopolistic sellers and a continuum of competitive buyers.

**Exchange Equilibrium**

Equilibrium in the auction market is assumed to be *symmetric*. The symmetry of equilibrium restricts equilibrium in three ways. First, investors of the same type exhibit the same equilibrium behavior. Second, if two entrepreneurs choose the same financial structure, their claims are priced in the same way. Third, if two entrepreneurs choose the same financial structure, investors demand the same amount of the corresponding claims from each entrepreneur. This last restriction is consistent with optimality in view of the symmetry of asset returns. The assumption of symmetry considerably simplifies the asymptotic analysis in the sequel. Symmetry is not needed for existence but for reasons of space we have decided to restrict attention to symmetric equilibria from the outset.

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12 Assumption 3 is implied by Assumption 2 if, in addition, the utility function is additively separable. This property is not shared by some nonadditively separable utility functions, however. A counterexample is given by \( U_k(c_1, c_2) = -\exp(-c_1 - c_2) \).

13 This is simply a normalization in view of the fact that consumption is not bounded below.

14 Suppose that two entrepreneurs \( n_0 \) and \( n_1 \) choose the \( i \)th financial structure. Let \( i_n \) denote the financial structure chosen by entrepreneur \( n \), let \( p(f, n) \) denote the price, and let \( d(f, n) \) denote the amount of claim \( f \in F_i \) (where \( i = i_n \)) purchased from entrepreneur \( n \). Suppose that \( d(f, n_0) \neq d(f, n_1) \) for some \( f \in F_i \). Then define a new portfolio \( d'(f, n) \) by putting

\[
\begin{align*}
  d'(f, n) = \begin{cases} 
    d(f, n) & \forall f \in F_{i_n}, \forall n \in N \setminus \{n_0, n_1\}, \\
    d(f, n_1) & \forall f \in F_{i_n}, n = n_1, \\
    d(f, n_0) & \forall f \in F_{i_n}, n = n_0.
  \end{cases}
\end{align*}
\]

In defining \( d' \) we have simply transposed the quantities of claims purchased from the two entrepreneurs. From Assumption 2 it follows that the probability distribution of the returns to the portfolio \( d \) is the same as the probability distribution of the returns from the portfolio \( d' \). Certainly the cost is the same since \( p(f, n_0) = p(f, n_1) \) for all \( f \in F_i \) in a symmetric equilibrium. We can define a third portfolio by putting \( d'' = (d + d')/2 \). Then \( d'' \) has the same cost as the other two portfolios and must be weakly preferred by Assumption 1.
The cross-sectional distribution of financial structures is denoted by $\beta \in \Delta_N(I)$, where $\Delta_N(I)$ denotes the set of vectors $\beta = (\beta_1, \ldots, \beta_I)$ satisfying

$$\sum_{i \in I} \beta_i = 1 \quad \text{and} \quad \forall i \in I, \ \beta_i = n/N \quad \text{for some integer } n \geq 0.$$  

$\beta_i$ is the proportion of entrepreneurs choosing financial structure $i$ and $\beta_i N$ is the number choosing financial structure $i$. It is assumed that the equilibrium in the auction market depends only on the cross-sectional distribution of financial structures; this is an additional symmetry property imposed on the equilibrium.

The claims traded in the auction market are indexed by $A = \{(f, i) : f \in F_i\}$. Also, let $A_i$ denote the set of claims associated with $F_i$.\(^\text{15}\) Let $A(\beta)$ denote the set of available claims, where $A(\beta)$ is defined by

$$A(\beta) = \{(f, i) \in A | \beta_i > 0\}.$$  

A portfolio is a function $d: A \rightarrow \mathbb{R}_+$, where $d(a)$ is the investor’s position in claims of type $a$. Note that $d(a)$ is the investor’s total position in claims of type $a$: if three entrepreneurs issue this type of claim, the investor buys $d(a)/3$ units from each of them.\(^\text{16}\) The set of portfolios is denoted by $D$ and the set of available portfolios by $D(\beta)$, where

$$D(\beta) = \{d \in D | d(a) = 0 \text{ if } a \notin A(\beta)\}.$$  

In the sequel, it is sometimes assumed that investors can undertake limited short sales. Let $L$ denote the number of units of each claim issued by entrepreneurs that can be sold short by investors. For most purposes, we are interested in the cases $L = 0$ and $L = \infty$. Each investor is assumed to receive a fraction $1/N$ of the total ration, or $L/N$ units per entrepreneur. Then if $\beta_i N$ entrepreneurs choose financial structure $F_i$, the maximum size of an investor’s short position in some claim $f$ in $F_i$ is $(L/N) \beta_i N = L \beta_i$. The short sale constraint can thus be written $d(a) \geq -L \beta_i$ for each $a \in A_i$. In what follows, we let $D(\beta, L)$ denote the set of available portfolios that satisfy the short sale constraint.\(^\text{17}\)

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\(^\text{15}\) The argument in footnote 13 shows that it is enough to distinguish claims by their payoff function and by the financial structure to which they belong.

\(^\text{16}\) As usual, a negative value of $d(a)$ represents a short position. In that case, the investor shorts $d(a)/3$ units of the claim issued by each entrepreneur.

\(^\text{17}\) The crucial point here is that the constraint limits the total amount of short selling. The constraint can be motivated as follows. Under current practice on the NYSE, an investor wanting to take a short position must borrow shares from another investor. Thus, the total size of the short positions must be limited by the amount of the underlying stock available. If only stock held on margin can be borrowed, the total short position will be less than the amount of stock. If borrowed stock can be re-lent, the total short positions may exceed the amount of the underlying stock.

The constraint in the model does not fit this story exactly. It assumes that each investor receives a fixed quota independently of his desire to take a short position. This is only one rationing mechanism among many and not necessarily the most plausible. For example, it does not allow the total amount of short selling to be re-allocated among investors wanting to take short positions. But it has the advantage of being simple and allows us to make our point. Alternative rationing schemes which endogenize the ration of individual investors can be handled using the methods of Drèze (1975) but only at the cost of greatly complicating the model and without any significant gain in insight.
For any cross-sectional distribution of financial structures \( \beta \in \Delta_N(I) \) and any portfolio \( d \in D(\beta) \), let \( W_n(d, \beta) \) denote the total return to the portfolio at date 2. \( W_N(d, \beta) \) is defined as follows. Let \( (i_1, \ldots, i_N) \) denote any assignment of financial structures that has the cross-sectional distribution \( \beta \). Let \( N_i = \{n \in N | i_n = i\} \) denote the set of entrepreneurs with the \( i \)th financial structure. Then

\[
W_n(d, \beta) = \sum_{i \in I} \sum_{f \in F_i} \sum_{n \in N_i} \frac{d(f, i) f(Z_n)}{\beta_i N}.
\]

The right-hand side of (2.5) is independent of the particular assignment \( (i_1, \ldots, i_N) \) by virtue of Assumption 1, so \( W_n(\cdot) \) is well defined. Using (2.5), define an indirect utility function \( u^N_k : \mathbb{R} \times D(\beta) \times \Delta_N(I) \rightarrow \mathbb{R} \) by putting

\[
u_k(c, d, \beta) = E[U_k(c, W_N(d, \beta))], \quad \forall k \in K, \forall (c, d, \beta) \in \mathbb{R} \times D(\beta) \times \Delta_N(I).
\]

\( u^N_k(c, d, \beta) \) is the expected utility derived by an investor of type \( k \) from first-period consumption \( c \) and the portfolio \( d \) when the cross-sectional distribution of financial structures is \( \beta \).

A price system is a function \( p : A \rightarrow \mathbb{R}_+ \). An allocation is an array \( x = (x_k) = ((c_k, d_k)) \in (\mathbb{R} \times D)^K \). Let \( P \) denote the set of price systems and \( X \) the set of allocations. An allocation is attainable relative to \( \beta \) if \( (c_k, d_k) \in \mathbb{R} \times D(\beta) \), for every \( k \), and

\[
\sum_{k \in K} d_k(f, i) \alpha_k = \beta_i, \quad \forall (f, i) \in A.
\]

Now we can define a symmetric exchange equilibrium (SEE) relative to \( \beta \) to be a price system \( p \) and an attainable allocation \( x \) relative to \( \beta \), such that

\[
\forall k \in K, \quad x_k \text{ maximizes } u^N_k(c, d, \beta)
\]
subject to \( c + p \cdot d = 0 \) and \( d \in D(\beta, L) \).

**Theorem 1**: If Assumptions 1 through 3 are satisfied, there exists a SEE relative to \( \beta \) for every \( \beta \in \Delta_N(I) \).

Standard arguments suffice to prove the existence of equilibrium when \( L < \infty \). To show existence in the case \( L = \infty \), we consider a sequence of economies in which \( L \) grows without bound. If certain unnecessary “wash trades” are eliminated, Assumption 3 implies that for \( L \) sufficiently large the short sale constraint is not binding. Essentially, there is enough risk aversion so that short positions do not grow unboundedly large. Then we have the desired equilibrium. Since the arguments are standard apart from this one part, the proof is omitted. (See Allen and Gale (1989) for details.)
Financial Equilibrium of the Full Model

Now we analyze the choice of financial structures at the first stage. An equilibrium selection is a function $\phi: \Delta_N(I) \rightarrow P \times X$ associating a SEE relative to $\beta$ with every cross-sectional distribution $\beta$ in $\Delta_N(I)$. For any equilibrium selection $\phi$, define a finite symmetric game $\Gamma(\phi)$ as follows. Let $N$ be the set of players and let $\Delta(I)$ be the set of mixed strategies. For any $n \in N$ and $\beta_{-n} \in \Delta_{N-1}(I)$, let $(i, \beta_{-n})$ denote the cross-sectional distribution of financial structures when player $n$ chooses $i$ and the choices of the rest are given by the cross-sectional distribution $\beta_{-n}$. The second-stage equilibrium resulting from the choice of $(i, \beta_{-n})$ is $(p, x) = \phi(i, \beta_{-n})$ and the payoff to player $n$ is given by

$$v(i, \beta_{-n}) = \sum_{a \in A_i} p(a) - C_i.$$  

Suppose that player $n$ chooses a mixed strategy $\sigma \in \Delta(I)$ and the other players choose $\sigma^* \in \Delta(I)$. The payoff to player $n$ will be given by

$$v(\sigma, \sigma^*; \phi) = \sum_{i \in I} \sum_{\beta_{-n} \in \Delta_{N-1}(I)} \text{Prob}(\beta_{-n} | \sigma^*) \sigma_i v(i, \beta_{-n}).$$

The payoff function $v$ is evidently symmetric, i.e., independent of $n$, by virtue of the symmetry of exchange equilibrium. Define the game $\Gamma(\phi)$ by putting $\Gamma(\phi) = (N, \Delta(I), v)$. A symmetric Nash equilibrium (SNE) is defined to be a mixed strategy $\sigma^* \in \Delta(I)$ satisfying

$$\sigma^* \in \arg \max \{ v(\sigma, \sigma^*; \phi) : \sigma \in \Delta(I) \}.$$

By a variant of the standard argument, we can show the existence of a mixed strategy satisfying (2.11).

We define a symmetric financial equilibrium (SFE) to be a mixed strategy $\sigma$ and an equilibrium selection $\phi$ such that $\sigma$ is a SNE of $\Gamma(\phi)$.

THEOREM 2: If Assumptions 1 through 3 are satisfied, there exists a SFE.

PROOF: Immediate from Theorem 1 and the preceding discussion. Q.E.D.

The crucial properties of the model that ensure existence are (i) the finiteness of the entrepreneurs' strategy sets and (ii) the use of mixed strategies. The question of the existence of equilibrium has arisen in similar settings (Hart (1979), Jones (1987)). However, these authors focused on pure strategy equilibria, without providing an existence theorem.

3. THE INCENTIVE TO INNOVATE

In this section we study an extended example to illustrate the impact of short sales on the incentive to innovate. We assume there are two states of nature $\Omega = \{\omega_1, \omega_2\}$ and that all asset returns are perfectly correlated. Then asset
returns can be represented by an ordered pair \((z_1, z_2)\), that is,

\[
(3.1) \quad Z_j(\omega_i) = z_i \quad \text{for} \quad i = 1, 2 \text{ and } j = 1, \ldots, N.
\]

Entrepreneurs have a choice of two financial structures, \(F_1\) and \(F_2\). The first consists of equity alone. The second allows the entrepreneur to issue two securities, one giving a claim to the entire return to the asset in state \(\omega_1\) and the other giving a claim to the return in state \(\omega_2\). Formally,

\[
(3.2) \quad F_i = \begin{cases} 
\{f_0\} = \{(z_1, z_2)\} & \text{if } i = 1, \\
\{(f_1, f_2)\} = \{(z_1, 0), (0, z_2)\} & \text{if } i = 2,
\end{cases}
\]

where again we identify a claim with an ordered pair indicating the payoff in each state. Issuing one claim is assumed to be costless (we can think of this as the status quo). Issuing two claims has a positive cost \(\gamma\). Formally, the cost function is defined by

\[
(3.3) \quad C_i = \begin{cases} 
0 & \text{if } i = 1, \\
\gamma & \text{if } i = 2.
\end{cases}
\]

There are two types of investors \(k = 1, 2\). Their preferences are described by the von Neumann-Morgenstern utility functions \(U_k\), where

\[
(3.4) \quad U_k(c_1, c_2) = c_1 + V_k(c_2), \quad \text{for } k = 1, 2.
\]

Note the assumption of "transferable utility." We assume that there are equal measures of investors of each type.

Suppose that unlimited short sales are allowed, that is, \(L = \infty\). Then there is a unique SEE for each cross-sectional distribution of financial structures. In fact, there are only two possible SEE whatever the cross-sectional distribution of financial structures. To see this, consider two cases. First, suppose that all entrepreneurs choose the financial structure \(F_1\). In that case, investors can only trade equity: the definition of equilibrium does not allow them to trade short or long in the securities \(f_1\) and \(f_2\), which have not been issued. As a result, markets are incomplete: there are two states but only one security. Since investors' preferences satisfy "transferable utility," the SEE is unique and the market value of the firm will be given by the price of equity \(p(f_0)\).

Next, suppose that at least one entrepreneur chooses the more complex financial structure \(F_2 = (f_1, f_2)\). The claims offered in this structure are indistinguishable from Arrow securities. Investors can span the entire space of contingent commodities by trading in these securities and markets are effectively complete. As in the first case, transferable utility implies that there is a unique SEE relative to this choice of financial structures. The completeness of markets has two important implications for the nature of the SEE. The first is that the SEE is independent of the number of entrepreneurs who choose \(F_2\). As long as \(F_2\) is chosen by at least one entrepreneur, investors can split the other assets themselves by taking appropriate positions in the claims \(f_1\) and \(f_2\). The second
implication is that the market value of all firms must be the same. That is,

\[(3.5) \quad p(f_0) = p(f_1) + p(f_2)\]

in equilibrium. Otherwise, investors could achieve a riskless arbitrage profit. Thus, the entrepreneur who chooses to split his asset is no better off ex post than those entrepreneurs who choose not to split. In fact, he is worse off to the extent that he has born the cost \( \gamma > 0 \) of issuing the new securities.

**Example 1:** To illustrate the two SEE, we have calculated a numerical example. Suppose that the asset returns are \((z_1, z_2) = (0.5, 2.5)\) and the investors' preferences are given by

\[(3.6a) \quad U_1(c_1, c_2) = 5 + c_1 - \exp(-ac_2), \quad \text{where} \quad a = 10,\]

\[(3.6b) \quad U_2(c_1, c_2) = 5 + c_1 + \ln(c_2).\]

If no one innovates, the equilibrium values are given in Table I. We refer to this equilibrium as the *incomplete markets SEE*.

Because investors' preferences satisfy transferable utility, it is easy to calculate the marginal value of a security to an investor. Let \(\mu_k(\omega)\) denote the marginal utility of consumption at the second date to an investor of type \(k\) in state \(\omega\) and let \(f(\omega)\) denote the return to one unit of claim \(f\) in state \(\omega\). Then the marginal value of claim \(f\) to a type \(k\) investor in equilibrium must be

\[\mu_k(\omega_1)f(\omega_1) + \mu_k(\omega_2)f(\omega_2).\]

If the claim is actually issued in equilibrium, then the price must be equal to the marginal value for all types \(k\).

Using the data in Table I, we see that the marginal valuation of claim \(f_0\) is the same for both types and is equal to the equilibrium price \(p(f_0) = 0.58583\). On the other hand, the two types of investors value the unissued securities \(f_1\) and \(f_2\) differently. For example, a type 1 investor would place a value of 0.57761 on the claim \((0.5, 0)\) which is greater than the value placed on it by a type 2 investor. A type 2 investor would place a value of 0.29291 on the claim \((0, 2.5)\), which is higher than the value placed on it by the type 1 investor. In other words, different clienteles value the claims differently. These differences in marginal valuation stem from the incompleteness of markets and the resulting differences in marginal rates of substitution across states.

<table>
<thead>
<tr>
<th>Investor Type</th>
<th>Equilibrium Demand for Equity</th>
<th>Consumption State (\omega_1)</th>
<th>Consumption State (\omega_2)</th>
<th>Marginal Utility of Consumption State (\omega_1)</th>
<th>Marginal Utility of Consumption State (\omega_2)</th>
<th>Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.29303</td>
<td>0.14652</td>
<td>0.73258</td>
<td>2.31042</td>
<td>0.00658</td>
<td>4.71248</td>
</tr>
<tr>
<td>2</td>
<td>1.70697</td>
<td>0.85348</td>
<td>4.26742</td>
<td>1.17167</td>
<td>0.23433</td>
<td>4.64629</td>
</tr>
<tr>
<td>Market Value</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.58583</td>
</tr>
</tbody>
</table>
If one or more entrepreneurs chooses to innovate, the equilibrium values are given in Table II. We refer to this as the complete markets SEE. In this case, marginal rates of substitution across states are equalized between the two types of investor. As a result, the marginal value of all claims is equalized between types of investor and the market value of assets is independent of financial structure.

As we have seen, the SEE can be extremely sensitive to the choice of financial structure by a single entrepreneur. If every entrepreneur chooses $F_1$, markets are incomplete; but if a single entrepreneur chooses $F_2$, markets are complete. This property of the model is exactly analogous to the property of stock market economies noted by Kreps (1979), who showed that the change of a single firm's production plan could have a large impact on the economy when short sales are allowed. However, this observation is not a statement about equilibrium. It does not imply that entrepreneurs will have a large impact, at the margin, in equilibrium. We will now show that in our model entrepreneurs must have a large impact, at the margin, in equilibrium, in order to have an incentive to innovate.

Whenever one of the entrepreneurs chooses to innovate by selecting the financial structure $F_2$, the market value of the asset ex post is independent of the financial structure. We describe this state of affairs by saying that there is no ex post incentive to innovate. Why would an entrepreneur ever innovate in these circumstances? The answer is that there may be an ex ante incentive to innovate even if there is no incentive ex post. The ex ante incentive comes from the ability of the entrepreneur to use his choice of financial structure to influence the prices in the SEE. In other words, it comes from imperfect competition. In Example 1, the market value of the “firm” is higher in the complete markets SEE than in the incomplete markets SEE. Even after taking account of the cost of innovating, an entrepreneur may be better off in the SEE with complete markets than in the SEE with incomplete markets. By innovating, the entrepreneur ensures that he will end up in the preferred equilibrium.

Suppose, for example, that there are two entrepreneurs ($N = 2$). Let $MV_i$ denote the market value of the asset when markets are incomplete and let $MV_C$ denote the market value of the asset when markets are complete. If no one innovates, the market value is $MV_i$; if one or more innovates, the market value
is $MV_C$. By innovating, a single entrepreneur can raise market value by $MV_C - MV_I$. Of course, each entrepreneur would prefer the other to bear the costs of issuing the second claim, since they achieve the same market value ex post. In a mixed strategy equilibrium, the incentive to innovate comes from the possibility that the other entrepreneur will not innovate. If $\sigma$ is the probability that an entrepreneur innovates, then an entrepreneur will be indifferent between innovating and not innovating if and only if $\sigma MV_C + (1 - \sigma)MV_I = MV_C - \gamma$. Thus, in a mixed strategy SFE $\sigma$ must satisfy

$$\sigma = \frac{MV_C - MV_I - \gamma}{MV_C - MV_I}.$$

In Example 1, $MV_C - MV_I = 0.00020$ so neither entrepreneur will issue two claims unless $\gamma < 0.00020$. If, say, $\gamma = 0.00005$, then $\sigma = 0.25$. As $\gamma$ becomes smaller, the probability with which each entrepreneur innovates becomes larger. In the limit as $\gamma$ approaches 0, they each innovate with probability 1.

Now suppose that the number of entrepreneurs is an arbitrary integer $N$. All entrepreneurs randomize, choosing $F_2$ with probability $\sigma$. In equilibrium, entrepreneurs must be indifferent between the two financial structures, that is,

$$[1 - (1 - \sigma)^{N-1}]MV_C + (1 - \sigma)^{N-1}MV_I = MV_C - \gamma.$$  

Solving for the equilibrium value of $\sigma$ we find that

$$\sigma = 1 - (\gamma / (MV_C - MV_I))^{1/(N-1)}.$$  

Clearly, $\sigma$ approaches 0 as $N$ approaches $\infty$. Moreover, using L'Hôpital's rule, it can be shown that $\sigma N \to \ln((MV_C - MV_I)/\gamma)$. The average number of innovators is finite even when the potential number is infinite. For example, when $\gamma = 0.00005$ the average number who innovate is 1.38629 in the limit. Note also that the probability that no one innovates is bounded away from zero. In fact, from (3.8) we see that $(1 - \sigma)^N$ converges to $\gamma/(MV_C - MV_I)$.

Several aspects of this example are noteworthy.

(i) First, arbitrage among the different claims issued guarantees that in equilibrium all assets have the same market value, independently of financial structure.

(ii) Even when $N$ is very large, entrepreneurs do not become price-takers. By changing his choice of financial structure, for example, choosing $F_2$ with probability 1, an entrepreneur can have a significant effect on the probability distribution of prices in the SEE.

(iii) Hence, even though there is no ex post incentive to innovate, there is an ex ante incentive to innovate. An entrepreneur can alter prices by innovating, so as to increase the market value of all assets.

(iv) Markets do not become complete in the limit. The probability of no innovation is bounded away from zero.
4. ARBITRAGE PRICING IN THE GENERAL CASE

An essential property of the example studied in Section 3 is the fact that asset returns are perfectly correlated. Arbitrage between assets with different financial structures is riskless so the market value of an asset must be independent of the financial structure. Otherwise investors could make an arbitrage profit. When market value is independent of financial structure, there is no ex post incentive for innovation. The entrepreneur who innovates is worse off to the extent that he bears the cost of the more costly financial structure. To have an ex ante incentive to innovate, we need imperfect competition, that is, the choice of an individual entrepreneur must change the SEE prices.

When asset returns exhibit idiosyncratic risk, the argument is not so simple. Since all arbitrage is risky, it is not clear that arbitrage will equalize market values and eliminate the ex post incentive to innovate. However, under certain conditions the earlier conclusion will obtain, that is, imperfect competition must persist in order to provide an incentive to innovate. The argument goes as follows. Suppose that some entrepreneurs innovate, choosing a costly financial structure \( F_i \), say, while the rest choose the status quo structure, \( F_h \). If the number of entrepreneurs choosing \( F_i \) is large, then diversification among the claims issued by the innovating entrepreneurs can eliminate the idiosyncratic risk element. As a result, arbitrage implies that the market values of these assets must be the same as the market values of the assets with the structure \( F_h \). That is, there is no ex post incentive to innovate. The implication then is that in order to have an (ex post or ex ante) incentive to innovate we need imperfect competition. The incentive can arise in two ways: the number of innovators may be small, so that there is an ex post incentive, or an individual entrepreneur's decision to innovate may affect prices, so that there is an ex ante incentive, or both incentives may operate.

We begin by introducing an assumption to ensure that, when the number of claims of a given type is large, it is possible to diversify away all idiosyncratic risk.

**Assumption 4:** There exists a random variable \( Z_w \) such that, for any sequence of sets \( \{J_N\} \) such that \( J_N \subset \{1, \ldots, N\} \) and \( |J_N| \to \infty \) and for any \( f \in \bigcup F_i \), \( \sum_{j \in J_N} |J_N|^{-1} f(Z_j) \) converges to \( E[f(Z_j)]Z_w \) in probability.

Assumption 4 is satisfied if \( \{Z_n\} \) is a family of i.i.d. random variables, for example. A more interesting case is given by

\[
(4.1) \quad Z_n = Z_w + \varepsilon_n,
\]

where \( \{\varepsilon_n\} \) are i.i.d. and \( Z_w \) is an arbitrary random variable.

Consider a sequence of economies increasing in size. For each economy of size \( N \), let \( \beta^N \) be a cross-sectional distribution of financial structures and let \( (p^N, x^N) \) be a SEE relative to \( \beta^N \). Let \( I^* \subset I \) denote the set of structures \( i \in I \) such that \( \beta_i^N N \to \infty \) as \( N \to \infty \). For every \( N \) define the market value of an asset
with financial structure \( i \in I \) by the equation

\[
(4.2) \quad MV_i^N = \sum_{f \in F_i} p^N(f, i).
\]

We are interested in the behavior of the market values as the size of the economy becomes very large.

**Theorem 3:** Suppose that Assumptions 1 through 4 are satisfied and suppose that \( (p^N, x^N) \) is a convergent sequence. Then there exists a constant \( \bar{m} \) such that \( MV_i^N \to \bar{m} \) for every \( i \in I^* \), where \( I^* \) is the set of structures such that \( \beta_{i}^{N}N \to \infty \) as \( N \to \infty \).

**Proof:** See the Appendix. \( Q.E.D. \)

The theorem says that all assets with financial structures in \( I^* \) have the same market value in the limit. The intuition behind this result is straightforward. If it were not true there would exist, in the limit, two infinite sets of similar assets, one of which had uniformly higher market values than the other. Since these two sets of assets are infinite, Assumption 4 implies that diversification eliminates idiosyncratic risk. By shorting the overvalued set and buying the undervalued set it is possible to construct an arbitrage portfolio which, in the limit, is riskless and yields a positive profit. The existence of such a portfolio is inconsistent with equilibrium.

**Remark 1:** It is tempting to interpret this result in the spirit of the Modigliani-Miller Theorem (MMT). The MMT states that under certain conditions corporate financial structure is irrelevant. More specifically, it implies that a firm’s market value will be independent of its financial structure, where financial structure is identified with the debt-equity ratio.\(^{18}\) On the surface, this sounds like Theorem 3 but there are important differences. In the first place, the MMT is a statement about the effect of the firm’s choices. It says that whatever debt-equity ratio the firm chooses, the market value of the firm will be the same. Theorem 3, on the other hand, is a statement about the financial structures chosen in equilibrium. It says that in equilibrium the market value of most assets will be the same. It does not assert the irrelevance of financial structure. In particular, it does not say that the market value would be unaffected if the entrepreneur were to choose a different financial structure, one which is not observed in equilibrium.

A second and more subtle difference is that Theorem 3 leaves out of account any subset of assets that is finite and hence negligible in the limit. By “negligible in the limit” we mean that the fraction of assets in this subset converges to zero.

\(^{18}\) The argument is that as the firm increases (decreases) the amount of its debt outstanding, an investor can exactly offset the effect on his own portfolio by increasing (decreasing) the amount of debt he holds. An investor should therefore be indifferent to the firm’s debt-equity ratio and so the market value of the firm should not be affected by it either.
as $N$ approaches infinity. A set which is negligible in this sense may not be negligible in terms of its importance for risk-sharing in the economy. In particular, because of the possibility of short sales, the open interest in the claims on a negligible set of assets may be nonnegligible in the limit.\footnote{This possibility would seem to be important in the context of the Arbitrage Pricing Theory of Ross (1976). His pricing formula holds for all but a negligible set of assets. More precisely, for any $\epsilon > 0$, the expected returns are $\epsilon$-close to the predicted value for all but a finite number of assets. However, a set of assets which has negligible cardinality need not be negligible in any other sense. In particular, it may provide important risk sharing opportunities.} In the next section, we see that this is precisely what happens.

5. FURTHER EXAMPLES OF SYMMETRIC FINANCIAL EQUILIBRIUM

In this section, we again use examples to investigate the asymptotic properties of equilibrium when there are unlimited short sales. In Example 2 asset returns exhibit purely idiosyncratic risk. In the limit, as $N$ becomes large, there is no "innovation." "Innovation" is not needed because there is no aggregate uncertainty. This is an example where, even with unlimited short sales, there is price-taking in the limit.

Example 3 extends the preceding example by introducing systematic, economy-wide risk. The conclusions here are similar to those obtained in Example 1, thus showing that at least some of the conclusions of Section 3 are robust to the introduction of idiosyncratic risk.

In Examples 1, 2, and 3 there is no ex post incentive to innovate since market values of assets are equalized independently of financial structure. Example 4 shows that there can be an ex post incentive to innovate.

Example 1 showed that there might be too little incentive to innovate. Example 5 shows that, on the contrary, there may be too much. That is, entrepreneurs may innovate when it is socially undesirable for them to do so.

\textbf{Example 2:} As before we assume that there are only two possible levels of asset returns, 0.5 and 2.5. But now asset returns are assumed to be i.i.d., taking each value with equal probability. The other assumptions are the same as in Example 1.

With $N$ assets there are now $2^N$ states of nature. Even if every entrepreneur chooses to innovate and issue two claims, markets will be incomplete. The reason is simply that claims are functions of the returns to the underlying asset. Nevertheless, this example is straightforward to analyze. Since each asset has only two possible returns, it is always possible to construct a portfolio with a constant payoff across states if at least one entrepreneur issues two nontrivial claims.\footnote{Let $D$ denote a constant random variable. A claim is called nontrivial if it is not zero or the identity map. Then for any nontrivial claim $f$ and any asset $Z_n$, there exist numbers $r_1$ and $r_2$ such that $D = r_1(Z_n - f(Z_n)) + r_2 f(Z_n)$. A portfolio consisting of $r_2$ units of $f$ and $r_1$ units of the residual claim is effectively debt. Once debt is available, any claim can be created out of debt and equity. Let $f$ be any claim. Then there exist numbers $r_1$ and $r_2$ such that $f = r_1 D + r_2 Z_n$, that is, a portfolio consisting of $r_1$ units of debt and $r_2$ units of equity is equivalent to $f$.} This portfolio is effectively debt. Since any claim can be artificially
created out of debt and equity, the MMT holds. As a result, all assets have the same market value, regardless of their financial structure. (This is only true when asset returns have two values, of course.)

From the preceding discussion, we see that there are only two possible SEE. One of them results if all entrepreneurs choose the financial structure \( F_1 \); the other results if one or more choose the financial structure \( F_2 \) and the remainder choose \( F_1 \). Let the market values of an asset in these SEE be denoted by \( MV_1 \) and \( MV_D \) respectively. The analysis of the SFE is similar to the example in Section 4. Suppose to begin with that \( N = 2 \). Entrepreneurs will randomize over financial structures, choosing \( F_2 \) with probability \( \sigma \). In equilibrium, \( \sigma = (MV_D - MV_I - \gamma)/(MV_D - MV_I) \). It can be shown that \( MV_I = 0.56987 \) and \( MV_D = 0.57406 \) so that \( MV_D - MV_I = 0.00419 \). Entrepreneurs will issue two claims only if \( \gamma < 0.00419 \). For example, if \( \gamma = 0.00209 \), then \( \sigma = 0.5 \).

Now consider an arbitrary economy of size \( N \). As \( N \) becomes large, investors are able to diversify away most of the idiosyncratic risk simply by holding equity. The return from holding a representative portfolio is 1.5 per asset. The value of the asset is 0.56149 independently of financial structure. In the limit all risk is eliminated and there is no need for financial innovation. It is not worthwhile for an entrepreneur to issue two claims since it does not alter the second-stage equilibrium in any way.

This is an example in which entrepreneurs do become price-takers in the limit. The reason is, of course, that the claims in the second financial structure \( F_2 \) are eventually redundant. This example also shows that the amount of innovation may decline as the size of the market increases.

**Example 3:** The next example differs from the preceding one in two respects. First, there are assumed to be two “macrostates,” Good and Bad, each of which occurs with probability 0.5. In the good state, an asset’s return is 0.5 with probability 0.25 and 2.5 with probability 0.75. In the bad state, these probabilities are reversed: the return is 0.5 with probability 0.75 and 2.5 with probability 0.25. Second, there are assumed to be \( N \) entrepreneurs and \( 2N \) investors.

The case where there are two entrepreneurs can be analyzed in the same way as the preceding example. The main difference is that as \( N \) increases, the idiosyncratic risk associated with each asset is diversified away but there remains the risk attributable to the macrostate. A representative portfolio yields a return of 2 in the good state and 1 in the bad state. In this case, even as \( N \) becomes very large, there is an incentive to innovate and some entrepreneurs will issue two claims.

It is again relatively simple to calculate equilibrium in the limit. By the earlier argument, if anyone innovates, a portfolio equivalent to risk-free debt can be constructed and the MMT holds. There are only two possible SEE at the second stage. One of them occurs if no one innovates; the other occurs if at least one entrepreneur innovates, in which case the equilibrium is the same as if debt and equity were traded.
It follows that, in the limit, if anyone innovates it will be possible to construct portfolios equivalent to contingent commodities on the macrostates. It can be shown that $MV_C = 1.21340$ and $MV_I = 1.20845$ so $MV_C - MV_I = 0.00495$ in the limit. As in the perfectly correlated case (Example 1), $\sigma N$ converges to $\ln((MV_C - MV_I)/\gamma)$. For example, if $\gamma = 0.00124$, then the average number of entrepreneurs who innovate in the limit is 1.38428. Again, the probability that no one innovates is bounded away from zero.

This example shows that the characteristics of Example 1 are not attributable to the assumption of perfectly correlated asset returns. Also, unlike Example 2, which had purely idiosyncratic risk, there is an incentive to innovate in the limit, when all idiosyncratic risk has been diversified away.

**Example 4:** All of the examples so far have had the very special property that in the second-stage SEE all assets have the same market value independently of financial structure. Ex post, there is no incentive to innovate. The incentive to innovate has come exclusively from the ex ante expectation that the innovation will change the equilibrium. In this section, we consider an example where, ex post, the entrepreneur who innovates achieves a higher market value than the entrepreneur who does not.

There are three levels of returns: 1, 2 and 3. All three are equally likely. As in the first examples, the number of investors and entrepreneurs is the same. There is no cost if the entrepreneur wants to issue equity only. Thus, $F_1 = \{(1, 2, 3)\}$ and $C_1 = 0$. The other financial structures have the form

\begin{equation}
F_i = \{(a, b, c), (1, 2, 3) - (a, b, c)\}
\end{equation}

where $a = 0, 1, \ b = 0, 1, 2,$ and $c = 0, 1, 2, 3$. Investors have the usual preferences. It can be shown that when there are two entrepreneurs ($N = 2$) there exists a SFE in which all entrepreneurs randomize between $F_i$ and the financial structure $F_2 = \{(0, 1, 1), (1, 1, 2)\}$. Let $\sigma$ denote the probability that they choose the structure with two claims. If both entrepreneurs innovate, the market value of their assets is 0.55305 in the SEE. If neither innovates the market value is 0.55298. When one innovates and the other does not, the innovator's market value is 0.55340 and the non-innovators' market value is 0.55283.

It is weakly optimal to innovate if and only if

\begin{equation}
0.55305\sigma + 0.55340(1 - \sigma) - \gamma \geq 0.55283\sigma + 0.55298(1 - \sigma)
\end{equation}

or, rearranging, $\sigma \leq (0.00042 - \gamma)/0.00020$. It can be seen that for $\gamma \leq 0.00022$ there exists a pure strategy equilibrium in which both entrepreneurs innovate. In the previous examples, the only symmetric pure strategy equilibria were those in which no one innovated. When there is no ex post incentive to innovate, innovation will occur in a pure strategy equilibrium only if it is asymmetric. Specifically, at most one entrepreneur will innovate. For values of $\gamma$ above 0.00022, mixed strategies are used in equilibrium.
EXAMPLE 5: We have seen that innovation may not occur when it is socially desirable. More surprisingly, it may occur when it is socially undesirable.

Consider Example 1 but change the ratio of entrepreneurs to investors to be 0.25:1. In this case, completion of the market reduces the investors’ utilities by 0.00262 for type \( k = 1 \) and 0.02128 for type \( k = 2 \). The market value of assets increases by 0.05617 so that the overall increase in the sum of utilities (ignoring the cost of the innovation) is 0.03227. It can be seen that for \( 0.03227 < \gamma < 0.05617 \) entrepreneurs will have an incentive to innovate—and will innovate with positive probability in a SFE—but welfare will be reduced.

The possibility of making unlimited short sales does increase the opportunities for risk sharing. However, as a result of the inefficient incentives for innovation, everybody may be worse off. In Example 6 (below), we take the data from Example 1 and consider what would happen if no short sales were allowed. It turns out that there is some innovation if \( 0.00020 < \gamma < 0.28468 \) and short sales are not allowed, but none if they are. Everyone is better off when no short sales are allowed, even if there are no transfers at the first date. In other words, the constrained SFE Pareto dominates the unconstrained SFE.

6. EFFICIENCY OF COMPETITIVE EQUILIBRIUM

In this section we examine the asymptotic properties of equilibrium under the assumption that investors can only make limited short sales \( (L < \infty) \). In Allen and Gale (1989) we showed that, under certain conditions, the SFE of a sequence of finite economies converge to a competitive (price-taking) equilibrium in the limit. In what follows we simply assume, without stating formal conditions, that a SFE converges in the limit to a competitive, price-taking equilibrium as the size of the economy becomes unboundedly large.\(^{21}\)

We define a large economy as follows. There is a large number of entrepreneurs and investors. (Formally, we want a countably infinite number of entrepreneurs and a continuum of investors. In what follows, equilibrium is defined in terms of distributions of types and structures, so the set of agents is not mentioned explicitly.) As usual, each entrepreneur \( n \) is characterized by an asset with a random return \( Z_n \). There is a finite number of financial structures \( \{ F_i \} \) with cost function \( \{ C_i \} \). The asset returns are assumed to satisfy (A.1) and (A.4). If \( \beta_i > 0 \) the number of entrepreneurs choosing \( F_i \) is infinite. Then it must be possible to diversify away all idiosyncratic risk from the claims \( a \in A_i \).

\(^{21}\) Strictly speaking, we proved this for the case \( L = 0 \) but the proof is identical for \( 0 < L < \infty \). There are two important assumptions needed for this result. One is a continuity assumption that says, roughly speaking, that the SEE depend continuously on the cross-sectional distributions of financial structures. The second is an assumption about asset returns. It ensures that idiosyncratic risk can be eliminated through diversification. This assumption was introduced as Assumption 4 in Section 4. In a large economy, the law of large numbers ensures that the cross-sectional distribution of financial structures is nonstochastic and equal to the mixed strategy \( \sigma \). As a result, the SEE resulting from the entrepreneurs’ choices is nonstochastic and independent of the choice of any individual entrepreneur (here we need the continuity assumption). Entrepreneurs can take the prices in the auction market as given when they make their choices, so we are dealing essentially with a competitive price-taking equilibrium.
If $\beta_i = 0$, then investors will not be able to purchase any of the claims $a \in A_i$ in equilibrium. In either case, we can define the return to a (well-diversified) unit of claim $a$ to be

$$R_a(Z_w) = E[f(Z_n)|Z_w], \quad \forall a = (f, i) \in A.$$ 

There is a finite number of investor types $k \in K$, each characterized by the von Neumann-Morgenstern utility function $U_k$. The proportion (measure) of investors of type $k$ is $\alpha_k$. (A.2) and (A.3) are satisfied for each $k$. Define an indirect utility function by putting

$$W(d) = \sum_{a \in A} d(a) R_a(Z_w) \quad \forall d \in D \quad \text{and}$$

$$u_k(c, d) = E[U_k(c, W(d))] \quad \forall k \in K, \forall (c, d) \in \mathbb{R} \times D.$$ 

Let $D(\beta, L)$ denote the set of portfolios available to investors, where

$$D(\beta, L) = \{d \in D|d(a) \geq -\beta_i L, \forall a \in A_i, \forall i \in I\}$$

and $\beta$ is the cross-sectional distribution of financial structures. For any $p \in P$, let $MV_i(p)$ denote the market value of an asset with financial structure $F_i$, that is, $MV_i(p) = \sum_{a \in A_i} P(a)$. Then a competitive equilibrium for a large economy is an array $(\sigma, p, x)$ in $\Delta(I) \times P \times X$ such that $\sigma$ is profit-maximizing at the prevailing prices, that is,

$$(6.1) \quad \sigma_i > 0 \text{ implies that } i \text{ maximizes } MV_i(p) - C_i,$$

and $(p, x)$ is a SEE relative to $\sigma$, that is, $x$ is an attainable allocation relative to $\beta = \sigma$ and

$$(6.2) \quad \forall k \in K, \quad x_k = (c_k, d_k) \text{ maximizes } u_k(c, d) \text{ subject to } c + p \cdot d = 0 \text{ and } d \in D(\beta, L).$$

Note that the definition of $D(\beta, L)$ used here differs from the one used in Section 2. In Section 2 investors were forbidden to trade short or long in any security that was not issued by some firm. In other words, if a particular security was not issued by an entrepreneur, the market for that security remained closed. Here we allow investors to trade in any security subject only to the short sale constraint. This means that all markets are open, whether or not the corresponding security has been issued by any entrepreneur. Of course, if $\beta_i = 0$ then the market-clearing condition and the short sale constraint together imply that there is no trade in the claims belonging to $A_i$. However, the fact that markets in these claims are open means that there is a market-clearing price for each claim $a \in A_i$. This notion of equilibrium, which is used in AG, is therefore stronger than the one used in Section 2.\footnote{It is not obvious that this stronger notion of competitive equilibrium will be the limit of a sequence of SFE as the number of assets becomes unboundedly large. Nonetheless, we have shown in Allen and Gale (1989) that, if a certain continuity condition is satisfied and $L < \infty$, then the limit of a sequence of SFE as $N \to \infty$ is indeed a competitive equilibrium in this stronger sense. The same would not be true if $L = \infty$.}
The question we want to answer is whether an equilibrium of this kind is efficient in some sense. Since markets are incomplete there is no reason to expect that the equilibrium is Pareto efficient. Neither is this concept of efficiency necessarily appropriate. Instead, we use a notion of constrained efficiency. An equilibrium \((\sigma, p, x)\) is said to be constrained efficient if there does not exist an allocation \((\sigma', x')\) such that \(x'\) is attainable for \(\sigma'\), every investor is better off, and \((\sigma', x')\) uses less resources at the first date:

\[
\sum_{k \in K} c_k^i \alpha_k + \sum_{i \in I} C_i \sigma_i' < \sum_{k \in K} c_k^i \alpha_k + \sum_{i \in I} C_i \sigma_i.
\]

This last condition ensures that entrepreneurs can be made better off.

If no short sales are allowed, the economy is isomorphic to an Arrow-Debreu economy in which the only traded commodities are current consumption and claims. Not surprisingly, the equilibrium inherits the efficiency properties of the Arrow-Debreu model.

**Theorem 4:** If \(L = 0\) any equilibrium of a large economy is constrained efficient.

The proof, which is omitted, uses standard “revealed preference” arguments. (See Allen and Gale (1989) for details.)

The more surprising result is that, even in this competitive equilibrium, there can be inefficiencies if short sales are allowed.

**Theorem 5:** Let \((\sigma, p, x)\) be a fixed but arbitrary equilibrium of a large economy. Suppose that \(u_k\) is continuously differentiable for every \(k\) in \(K\) and suppose that for some financial structures \(F_i\) and \(F_j\), the following conditions are satisfied:

(i) \(\sigma = 0 < \sigma_j\) and \(MV_i(p) - C_i < MV_j(p) - C_j\);  
(ii) \(d_k(a) > -L \beta_j\) for all \(a \in A_j\);  
(iii) \((1 + L)MV_i(p) - C_i > (1 + L)MV_j(p) - C_j\).

Then the equilibrium allocation is not constrained efficient.

**Proof:** See the Appendix.

The first condition in the theorem says that structure \(j\) is chosen in equilibrium and is strictly more profitable than \(i\). The second says that the short-sale constraint is not binding for the claims associated with \(F_j\). This means that, at the margin, one unit of a claim \(a \in A_j\) is worth exactly \(p(a)\) units of first-period

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23 The efficiency notion we use is essentially the same as that proposed by Grossman (1977) and Grossman and Hart (1979). A stronger concept is proposed by Geanakoplos and Polemarchakis (1986).
consumption to any investor. In this sense, we can say that the value of one unit of an asset with financial structure \( F_i \) is \( MV_i(p) \).

The meaning of condition (iii) can be understood as follows. Suppose that a small measure \( \epsilon > 0 \) of entrepreneurs switch from \( F_j \) to \( F_i \). Investors’ holdings in claims \( a \in A_j \) must be reduced by \( \epsilon \) (we do not need to worry about the short sale constraint, since it is not binding) and the value of this reduction to them in terms of first-period consumption is \( MV_j(p)\epsilon \). On the other hand, we can increase holdings of claims \( a \in A_i \) by \( \epsilon(1 + L) \), including short sales, and the value of this in terms of first-period consumption is \( MV_i(p)(1 + L)\epsilon \) since each claim can be given to someone whose marginal valuation is equal to the price. To provide the necessary short sales, have every investor go short \( \epsilon L \) units in each claim \( a \in A_i \). The value of this is \( MV_i(p)\epsilon L \), not \( MV_i(p)\epsilon L \), since investors are in effect shorting the whole asset. Then the net effect on investors’ welfare, measured in first-period consumption, is \( MV_i(p)(1 + L)\epsilon - MV_j(p)(1 + L)\epsilon \). The cost of producing the claims has increased by \( (C_i - C_j)\epsilon \). It is possible to make everyone better off by redistributing first-period consumption if the increase in investors’ welfare is greater than the increase in cost, that is,

\[
MV_i(p)(1 + L)\epsilon - MV_j(p)(1 + L)\epsilon > (C_i - C_j)\epsilon.
\]

But this is exactly what condition (iii) claims.

**Example 6:** We can illustrate the theorem using the data from Example 1. (But now we assume there is a large number of entrepreneurs and investors.) Suppose first of all that \( L = 0 \), that is, no short sales are allowed. If \( \gamma \) is sufficientlylarge, there is a unique equilibrium in which no entrepreneur chooses to innovate, that is, \( \sigma_1 = 1 \) and \( \sigma_2 = 0 \). The equilibrium values are given in Table I. (The equilibrium values are the same as in the incomplete markets SEE of Example 1 because in that SEE investors do not want to make short sales.)

The equilibrium price of the single claim issued is 0.58583. If an entrepreneur splits his “firm” by issuing two claims, the claim (0.5, 0) will be purchased only by investors of type 1. Its value to them is 0.57761. Similarly, the claim (0, 2.5) will be purchased only by investors of type 2 at a price of 0.29291. Thus, the market value of the split “firm” will be 0.87052. So it is optimal for the entrepreneur to choose the simpler structure \( F_1 \) if and only if \( \gamma > 0.87052 - 0.58583 = 0.28469 \). In this case the marginal social benefit from financial innovation (splitting) is measured by the increase in the value of the “firm” so entrepreneurs make the efficient decision.

Now suppose that limited short sales are possible without cost. The equilibrium is unchanged as long as \( \gamma > 0.28469 \). However, the social benefit of innovation is now greater than the private benefit to entrepreneurs. In fact the marginal social benefit to splitting is at least \( (1 + L)0.87051 - (1 + L)0.58583 \). If this is greater than \( \gamma \) then efficiency requires that some “firms” split. Note that in the proof of the theorem we assume that short sales are evenly distributed.
across investors. If we had instead allocated these short sales to the investors
who valued them most (valued the claim least), then the social benefit of the
innovation would have been greater. In the example, the marginal social benefit
would be $(1 + 2L)0.87053$.

Remark 2: Example 6 allows us to illustrate the nonexistence problem
pointed out in AG. Suppose that there is a continuum of entrepreneurs and a
continuum of investors. Taking the data from Example 1, we can calculate the
market-clearing prices for a Walrasian equilibrium. As before, there are only
two cases that need to be considered. Either no one innovates and markets are
incomplete (Table I) or a positive measure of entrepreneurs innovates and
markets are complete (Table II). The difference here is that entrepreneurs are
price-takers: they do not take account of the effect of their actions on the
second-stage equilibrium.\footnote{Hart (1975) provides another example of nonexistence of
equilibrium when unbounded short sales are allowed. Hart’s problem differs from ours both in
its source and in its severity. In Hart’s model, the subspace of the commodity space that can be
spanned by assets depends on prices in a discontinuous way. However, the discontinuity is non-
generic (see Duffie and Shafer (1985 and 1986a)). The problem discussed here is generic and seems
to arise from a misspecification of equilibrium.}

Consider first the case where all entrepreneurs choose $F_1$ and markets are
incomplete. As we saw in the discussion of Example 6, in a price-taking
equilibrium, the market value of a “split” asset will be $0.57761 + 0.29291 =
0.87052$. So it would be optimal for an entrepreneur to choose the financial
structure $F_2$ as long as the perceived increase in market value were greater
than the cost; that is, as long as $\gamma < 0.87052 - 0.58583 = 0.28469$.

On the other hand, if a positive measure of entrepreneurs chooses the
financial structure $F_2$, markets are complete and all investors have the same
marginal rates of substitution. The market value of an asset is independent of its
financial structure. In that case, there is no incentive for an entrepreneur to
choose the more expensive financial structure.

Thus, as long as $\gamma < 0.28469$, there cannot exist a price-taking, competitive
equilibrium. Assuming that a positive measure of entrepreneurs chooses $F_2$, we
have shown they would be better off choosing $F_1$. Assuming that they all choose
$F_1$, we have shown they would be better off choosing $F_2$. The problem is that
entrepreneurs are not taking account of the effect of their choices on the
equilibrium prices.\footnote{Imperfect competition is thus seen to be an important ingredient in establishing the existence
of equilibrium by providing entrepreneurs with incentives they could not have in a perfectly
competitive equilibrium. A similar use is made of imperfect competition by Jackson (1988).
Grossman and Stiglitz (1980) argued that rational expectations equilibrium might not exist. If prices
revealed all the relevant information no one would have an incentive to become informed, in which
case prices could not be informative. Jackson shows that if only a few agents become informed, they
will have market power, which may provide them with an incentive to become informed even if
prices reveal all their information. The argument made by Grossman and Stiglitz (1980) depends on
an inappropriate use of the price-taking assumption.} This assumption may not be justified even when there is a
continuum of agents.
7. CONCLUDING REMARKS

This paper has considered a model where the set of traded securities is endogenous and short sales are possible. It has been shown that if firms move first, issuing securities that are subsequently traded on a competitive market, equilibrium exists but may not be efficient. The analysis here has focused on securities issued by firms. Of course, this is not the only kind of financial innovation that occurs. In Allen and Gale (1990) we consider the incentives for third parties to set up exchanges on which derivative securities (options) are traded. In this institutional framework, the innovator can control entry to the exchange and thus capture more of the rents from innovation. However, inefficiency may arise for other reasons. Much remains to be done in the direction of modeling the market microstructure in a detailed way. This is an interesting area for future research.

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APPENDIX

Proof of Theorem 3: The proof is by contradiction. Let \((p^N, x^N)\) denote the limit of the sequence \((p^N, x^N)\) and let \(x^N = (\langle c^N_k, d^N_k \rangle)\). Suppose, contrary to what is to be proved, that there exist structures \(i_0^N\) and \(i_1^N\) in \(I^N\) such that \(MV^N_{i_0^N} \to m_0\) and \(MV^N_{i_1^N} \to m_1\) as \(N \to \infty\), where \(m_1 > m_0\). Define a sequence of portfolios \(\{\delta^N_k\}\) for some fixed but arbitrary investor-type \(k\) as follows. Put

\[
\delta^N_k(a) = \begin{cases} 
    d^N_k(a) & \text{if } a \in A \setminus (A_{i_0} \cup A_{i_1}), \\
    d^N_k(a) - 1 & \text{if } a \in A_{i_1}, \\
    d^N_k(a) + 1 & \text{if } a \in A_{i_0}.
\end{cases}
\]

By Assumption 4, there exists a subsequence, which can be taken to be the original sequence, along which \(W_N(d^N_k, \beta^N_k)\) converges in probability to \(W_\infty\), say. Then by Assumption 4 and the definition of \(W_N(d, \beta)\),

\[
W_N(\delta^N, \beta^N) = W_N(d^N_k, \beta^N) + (\beta^N_{i_0} N)^{-1} \sum_{n \in S^N_{i_0}} Z_n - (\beta^N_{i_1} N)^{-1} \sum_{n \in S^N_{i_1}} Z_n
\]

\[
\to W_\infty + E[Z_n | Z_\infty] - E[Z_n | Z_\infty] = W_\infty
\]

where, for each \(N, S^N_{i_0}\) and \(S^N_{i_1}\) are disjoint subsets of \(\{1, \ldots, N\}\) with the property that \(|S^N_{i_j}| = \beta^N_{i_j} N\) for \(j = 0, 1\). From the budget constraint, however, \((\gamma^N, \delta^N)\) is an affordable choice for the investor of type \(k\), where \(\gamma^N\) is defined by

\[
\gamma^N = -p^N \cdot \delta^N = c^N_{i_0} - MV^N_{i_0} + MV^N_{i_1} \geq c^N_{k} + (m_1 - m_0) - \varepsilon
\]
for $N$ sufficiently large. Therefore,

$$u_N^N(y^N, \delta^N, \beta^N) \geq u_N^N(c_k^N + (m_1 - m_0) - \delta^N, \beta^N)$$

$$= E\left[U_k(c_k^N + (m_1 - m_0) - \delta^N, W_N(\delta^N, \beta^N))\right]$$

$$\rightarrow E\left[U_k(c_k^N + (m_1 - m_0) - \delta^N, W_N)\right].$$

Since $\epsilon > 0$ is arbitrary and $u_N^k(c_k^N, d_k^N, \beta^N) \rightarrow E[U_k(c_k^N, W_N)]$, we must have $u_N^k(y^N, \delta^N, \beta^N) > u_N^k(c_k^N, d_k^N, \beta^N)$ for $N$ sufficiently large, contradicting the definition of equilibrium.

**Proof of Theorem 5:** Let $(\sigma, p, x)$ be a fixed but arbitrary equilibrium. Assume that for every investor of type $k$, the indirect utility function $u_k(c, d)$ is continuously differentiable. In equilibrium, the following Kuhn-Tucker conditions must be satisfied:

$$\frac{\partial u_k(c_k, d_k)}{\partial c_k} = \lambda_k$$

$$\frac{\partial u_k(c_k, d_k)}{\partial d_k} = \lambda_k p,$$

with equality where $d_k(a) > -L\beta$, for any $a$ in $A$. Without loss of generality, we can assume that for every claim $a$ in $A$, there exists some type who is prepared to buy the claim at the margin, that is, $\frac{\partial u_k(c_k, d_k)}{\partial d_k} = \lambda_k p$. (If not, simply reduce prices until the condition is satisfied.)

Suppose there are two financial structures $i$ and $j$ satisfying the conditions (i) to (iii) of the theorem. Now consider a change in the equilibrium allocation that, by construction, satisfies the constraints in the definition of constrained efficiency. First, define $d' = (d'_k)$ by putting:

$$\sigma_h' = \begin{cases} 
\sigma_h & \text{for } h \neq i, j, \\
\sigma_i + \epsilon & \text{for } h = i, \\
\sigma_j - \epsilon & \text{for } h = j. 
\end{cases}$$

A small measure $\epsilon > 0$ of entrepreneurs has switched from structure $i$ to structure $j$. Define a new allocation of portfolios $d' = (d'_k)$ by putting:

$$d'_k(a) = \begin{cases} 
d_k(a) & \text{for } a \in A_i \cup A_j, \\
d_k(a) - \epsilon L + \theta_k(a) \epsilon & \text{for } a \in A_i, \\
d_k(a) - \theta_k(a) \epsilon & \text{for } a \in A_j. 
\end{cases}$$

The investors' holdings are unchanged except for claims associated with the structures $i$ and $j$. All agents are assumed to sell short $\epsilon L$ units of the claims associated with $F_i$ and to buy back a nonnegative quantity $\theta_k(a)$ of each claim $a$ in $A_j$. Furthermore, we assume that $\theta_k(a) = 0$ unless type $k$ is willing to hold the claim at the margin. It is clear from the construction and from our initial hypotheses that the short-sale constraint will be satisfied for every claim. To ensure that the resulting allocation is feasible, we assume that $\sum_{k \in K} \theta_k(a) \alpha_k = 1 + L$, for every $a \in A_i$, and $\sum_{k \in K} \theta_k(a) \alpha_k = -1$, for every $a \in A_j$.

By inspection, it can be seen that the portfolio allocation $d'$ satisfies the conditions in the definition of constrained efficiency. Now define a consumption allocation $c' = (c'_k)$ by putting $c'_k = c_k + y_k \epsilon$, for each $k$ in $K$. That is, each investor of type $k$ receives a transfer of $y_k \epsilon$ units of the consumption good at the first date. From the differentiability assumption, an investor of type $k$ will be better off for $\epsilon > 0$ sufficiently small if

$$\sum_{a \in A_i} \frac{\partial u_k(c_k, d_k)}{\partial c_k} y_k + \sum_{a \in A_i} \frac{\partial u_k(c_k, d_k)}{\partial d_k(a)} (\theta_k(a) - L) + \sum_{a \in A_i} \frac{\partial u_k(c_k, d_k)}{\partial d_k(a)} \theta_k(a) > 0.$$ (A.1)

From the symmetry assumption and the definition of the indirect utility function, note that:

$$\sum_{a \in A_i} \frac{\partial u_k(c_k, d_k)}{\partial d_k(a)} = \sum_{a \in A_i} \frac{\partial u_k(c_k, d_k)}{\partial d_k(a)} \leq \sum_{a \in A_i} \lambda_k p(a).$$
Substituting from this inequality and from the first-order conditions into (A.1), we see that (A.1) will be satisfied if

\[(A.2) \quad \lambda_k \gamma_k + \sum_{a \in A_j} \lambda_k p(a) \theta_k(a) + \sum_{a \in A_j} \lambda_k p_k^e(a) \theta_k(a) - \sum_{a \in A_j} \lambda_k p(a)L > 0.\]

Dividing (A.2) by \(\lambda_k/\alpha_k\) and summing over \(k\) yields

\[
\sum_{k \in K} \alpha_k \gamma_k + \sum_{k \in K} \sum_{a \in A_j} p(a)\alpha_k \theta_k(a) + \sum_{k \in K} \sum_{a \in A_j} p_k^e(a)\alpha_k \theta_k(a) - \sum_{a \in A_j} p(a)L > 0
\]

or

\[
\sum_{k \in K} \gamma_k \alpha_k + MV(p)(1 + L) - MV(p)(1 + L) > 0.
\]

Now in order to show that the equilibrium is not constrained efficient, it is sufficient to show that

\[(A.3) \quad \sum_{k \in K} \alpha_k \gamma_k + C_i - C_j < 0,\]

for some choice of \(\gamma_k\) consistent with (A.2). From inspection of these inequalities, it is clear that we can choose \(\gamma_k\) to satisfy both (A.2) and (A.3) if and only if

\[
MV(p)(1 + L) - MV(p)(1 + L) > C_i - C_j.
\]

Since this is precisely what our initial hypothesis (iii) claims, we have proved the theorem. Q.E.D.

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