International Risk Sharing, Investment Restrictions and Asset Prices*

Issouf Soumaré†    Tan Wang‡

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Abstract

We study the impact investment restrictions on cost of capital, volatilities of asset returns and their welfare implications in a dynamic, two-country, two-good general equilibrium model. We also study the issue of optimal timing of the removal of investment restrictions. Some of the findings of this paper are: when domestic country caps foreign investment in some key industries in the domestic economy, the cost of capital of the protected industry increases, that of the non-protected industry decreases, all else being equal; On the other hand, when imposing restrictions on its residents' foreign investments, the domestic country improves its cost of capital, all else being equal. We also find that by having investment restrictions, countries can reduce financial contagion effects. This result contributes to the debate on why recent crises in international financial markets have had different effects on countries located in same geographical area or having similar economic characteristics. Finally, we show that when the restriction is protective, the welfare of the agents of the country imposing the restriction increases. This result helps us understand why some countries are so reluctant to change their protective financial policies.

Keywords: Investment Restrictions, Financial Contagion, Incomplete Markets, International Asset Pricing.

JEL Classification: D51, D52, F31, F36, G11, G12, G15.

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†Laval University, Faculty of Administrative Sciences, Quebec, QC., Canada G1K 7P4; Tel: 1-418-656-2131; Email: issouf.soumare@fsa.ulaval.ca.

‡CCFR and University of British Columbia, Sauder School of Business, 2053 Main Mall, Vancouver, B.C., Canada V6T 1Z2; Tel: 1-604-822-9414; Email: tan.wang@sauder.ubc.ca.
1 Introduction

The virtues of market liberalization have been lauded extensively in the international finance literature. They range from the benefit of international risk sharing to the positive influence on economic growth.\(^1\) However, despite the claimed advantages, we still see countries reluctant to fully open their domestic capital markets even when they are well-developed. For example, many countries restrict their residents’ investments abroad and/or limit foreign investments in some key sectors of their domestic economies.\(^2\) The ‘rationale’ for these restrictions, often stated by the government of the country involved, is to hold capital inside the country and to prevent foreign investors from taking control of some key production processes. While the merits of the rationale are arguably ambiguous, the implications of these restrictions are nonetheless far from being well understood.

In this paper, we develop a general equilibrium model of international capital markets that allows for endogenous removal of investment restrictions and study the implications of the restrictions. We consider two cases. In the first case, the domestic country caps the share of foreign assets in its residents’ portfolio holding. In the second case, the domestic country caps foreign investment in some key industries in the domestic economy. The issues studied are: what are the factors that influence the decision by the government to remove the investment restrictions, what are the implications of the investment restrictions on international financial markets?

Several interesting results emerge from our study. Firstly, it is likely that a government may delay the removal of investment restrictions for the benefit of its residents. This result may come as a surprise to some people, as they would argue that investment restrictions are frictions in international financial markets and its remove should be beneficial to all residents in the world economy. We show that this is in general not true. The reason is that in the presence of investment restrictions, markets are incomplete. Even if it completes the markets, the removal of the investment restrictions may not be Pareto improving for the residents of the world economy (Milne and Shefrin (1984), Cass and Citanna (1998) and Elul (1995, 1999)). Therefore, it may be in the interest of one country or all countries to keep the restrictions in place and, as the condition of the economy changes, remove them later.

Secondly, investment restrictions have diverse and sometimes opposite effect on asset prices. For example, when foreign investment is capped in some key industries, the cost of capital of the protected industry increases while that of the non-protected industry

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\(^1\)For example, Dumas and Uppal (2001) shows that integration of financial markets can mitigate goods market imperfections. Henry (2003) claims that “Since the cost of capital falls, investment booms, and the growth rate of output per worker increases when countries liberalize the stock market, the increasingly popular view that capital account liberalization brings no real benefits seems untenable.” The view that market liberalization is a worthwhile goal to pursue is not shared by some eminent economists including the 2001 Nobel Prize winner in Economics, Joseph E. Stiglitz, who believes that market liberalization creates instability, increases poverty and economic insecurity in the liberalizing country (Stiglitz (2003)).

\(^2\)Examples of the first type of restriction include Canadian pension fund investment regulations, which stipulate that Canadian pension funds cannot invest more than 30 percent of their assets under management in foreign stocks. Examples of the second type of restriction includes the regulation on foreign holding of the airline and banking industries in Canada.
decreases. On the other hand, when there is restriction on its residents’ foreign investments, the domestic country’s cost of capital decreases. In both cases, the cost of risk free borrowing and lending in the domestic country is lower, but the risk free rate in a foreign country can be higher or lower depending on the restriction. Furthermore, the uncovered interest rate parity may or may not hold depending on the nature of the investment restrictions.

Thirdly, investment restriction may have effect on the co-movement of stock markets in the world. When agents’ investment is restricted, stock markets are affected asymmetrically by shocks. As the results, the volatilities of stock returns could be higher, causing instability. They could also be lower, improving stability. In terms of correlation, the effect is also ambiguous. Our results contribute to the debate on why recent crises in international financial markets have had different effects on countries located in same geographical area or having similar economic characteristics. As stated by Joseph Stiglitz in Time (2003), “... Every major emerging market that had liberalized its capital market had had a crisis; the two major countries that had not, China and India, had not only avoided the East Asian crisis, but managed to grow steadily throughout the period.” The results also contribute to another debate in the international economics and finance literature, that is, whether financial liberalization leads to more or less stable financial market.

Our paper relates to a number of papers in international economics and finance literature. Zapatero (1995) determines the dynamic of the real exchange rate in a dynamically complete international market with two countries and two goods. In this setting financial markets are perfectly integrated and also there is perfect mobility of goods between countries. Dumas (1992) identifies the exchange rate process when there is a transfer cost associated with the mobility of goods between countries. In similar vein, Uppal (1993) in a general equilibrium context analyzes the dynamic of international portfolio choice in presence of shipping cost and perfect integrated financial markets. Dumas and Uppal (2001) and Basak and Croitoru (2003) show how financial markets integration can attenuate the anomalies introduced by imperfection in the good markets. Obstfeld (1994) and Devereux and Saito (1997) study the welfare implications when international risk-sharing are allowed. Pavlova and Rigobon (2003) in a complete market setting study financial market co-movements.

These cited papers, however relies on the assumption of internationally complete financial markets and/or exogenous real exchange rate. Exceptions to these assumptions are Serrat (2001) and Brandt, Cochrane and Santa-Clara (2002) with different objectives from ours. Serrat (2001) studies a two-country exchange economy with heterogenous agents and non-traded goods, and shows how the presence of non-traded goods helps explain the home bias puzzle. Brandt, Cochrane and Santa-Clara (2002) examine the relationship between stock market returns and the real exchange rate, and argue that exchange rate is less volatile than the implied marginal utility growths from stock market returns.

Our objective in this paper and hence the setting are different from those papers. Our objective is to examine the implications of market incompleteness caused by investment restrictions.

The three papers that are closest to ours are Pavlova and Rigobon (2005), Bhamra
(2003) and Sellin and Werner (1993). Using a model similar to ours, Pavlova and Rigobon (2005) also study the implication of investment restrictions on stock price co-movements across the world economy. Their model and ours however differ. In their model, each country produces a single good and there are demand shocks. In our model, each country can produce multiple goods. As a result, the equilibrium asset prices, cost of capital, and the contagion effects we obtain are different from theirs. Also they do not study the removal of investment restrictions. Bhamra (2003) studies the effects of stock market liberalization on equity risk premia, stock return volatilities and the cross-country correlation of stock returns. There is only one good in his model. Therefore the good market channel of transmitting shocks is assumed away. Sellin and Werner (1993) consider restrictions similar to ours but focus on international portfolio holdings and the changes in the risk-free rates. Their model don’t allow them to analyze the implications of investment restrictions on risky securities prices and agents welfare. Also, there is only one good in their model.

The remainder of the paper is structured as follows. Section 2 describes the economy environment. Section 3 gives the results for the benchmark case. Section 4 studies the equilibrium in presence of investment restrictions. Section 5 concludes. The proofs are presented in the appendices.

2 The model

We consider a two-country continuous-time finite horizon economy. One country is called domestic and the other foreign, indexed by $d$ and $f$, respectively. The uncertainty in this economy is generated by four independent Brownian motions $W = (W_1, W_2, W_3, W_4)\top$ in a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \in [0, T])$ is the augmented filtration generated by $\{W(t) \in \mathbb{R}^4; t \in [0, T]\}$. The $\sigma$-algebra $\mathcal{F}_t$ represents the available information set at time $t$ in the economy. All stochastic processes to appear in this paper are adapted to $\mathcal{F}_t$ and all equalities involving random variables are understood to hold $P$-almost surely. We denote by $E_t$ the expectation conditional on the information $\mathcal{F}_t$ at time $t$. We define by $x^+ = \max(x, 0)$ the positive part of $x$. We use $\|x\|$ to designate the norm of vector $x$.

There are two goods and three firms in the global economy. We will follow the standard notation in the literature and use superscript $\ast$ to denote variables associated with the foreign country and good two. Firms one and two reside in the domestic country and produce good one. The output of the two firms are described by

$$d\delta_j(t) = \delta_j(t) \left[ \mu_{\delta_j}(t) dt + \sigma_{\delta_j}(t)\top dW(t) \right], \quad j = 1, 2,$$

respectively. Firm three resides in the foreign country. It produces both good one and good two. Its output of good one is given by

$$d\delta_3(t) = \delta_3(t) \left[ \mu_{\delta_3}(t) dt + \sigma_{\delta_3}(t)\top dW(t) \right],$$

and its output of good two is given by

$$d\delta^\ast(t) = \delta^\ast(t) \left[ \mu_{\delta^\ast}(t) dt + \sigma_{\delta^\ast}(t)\top dW(t) \right].$$
Thus the total supply of consumption good one is \( \delta = \delta_1 + \delta_2 + \delta_3 \), which follows an Ito process,

\[
d\delta(t) = \delta(t) \left[ \mu_\delta(t) dt + \sigma_\delta(t)^T dW(t) \right].
\]

There are three financial securities in the domestic country and two in the foreign country. Domestic country has a risk-free bond and two stocks with claims to the dividends \( \delta_1 \) and \( \delta_2 \), respectively. Foreign country has a risk-free bond and a risky stock with claims to the endowed dividends \( \delta_3 \) and \( \delta^* \). The two bonds are in zero net supply. The total number of shares of each stock is normalized to one. The prices of the financial securities are expressed in unit of good one. Denote the prices of the domestic and foreign bonds by \( B \) and \( B_f \), respectively. The prices of the two domestic stocks and the foreign stock are denoted by \( S_1 \), \( S_2 \) and \( S_f \), respectively. It will be shown that in equilibrium the processes of the securities are diffusion processes,

\[
\begin{align*}
    dB(t) &= r(t)B(t)dt, \\
    dB_f(t) &= B_f(t) \left[ (r^*(t) + \mu_p(t)) dt + \sigma_p(t)^T dW(t) \right], \\
    dS_j(t) + \delta_j(t)dt &= S_j(t) \left[ \mu_j(t)dt + \sigma_j(t)^T dW(t) \right], \quad j = 1, 2, \\
    dS_f(t) + (\delta_3(t) + p(t)\delta^*(t))dt &= S_f(t) \left[ \mu_f(t)dt + \sigma_f(t)^T dW(t) \right],
\end{align*}
\]

where \( r(t) \) and \( r^*(t) \) are the riskfree interest rates in domestic and foreign countries, respectively. Let

\[
\mu = (\mu_1, \mu_2, \mu_f, r^* + \mu_p)^\top
\]

and

\[
\Sigma = (\sigma_1, \sigma_2, \sigma_f, \sigma_p)^\top.
\]

These are the mean vector and the volatility matrix of the risky financial securities. In the world economy, there are five securities and four sources of uncertainty. Therefore the market is complete when the volatility matrix \( \Sigma \) is invertible. Since \( \Sigma \) is endogenous, this property can only be verified in equilibrium.

To focus on the frictions in the financial market, we assume that there are no frictions in the goods markets. The relative price of the two consumption goods is given by the real exchange rate \( p(t) \), units of good one per unit of good two. We will show later that the exchange rate process follows an Ito process:

\[
dp(t) = p(t) \left[ \mu_p(t)dt + \sigma_p(t)^T dW(t) \right]. \quad (1)
\]

Each country is populated by a representative agent, whose preference is represented by a time-additive expected utility function,

\[
U_i = E \left[ \int_0^T e^{-\beta t} \left[ \ln c_i(t) + \ln c_i^*(t) \right] dt \right].
\]

The representative agent in each country is initially endowed with the shares of the stocks of the country.

To close the description of the model, we note that none of the five securities in the world economy is redundant even when there are no market frictions. This is in contrast with Zapatero (1995). In Zapatero (1995), the stock that represents the claim
to all foreign goods is perfectly correlated with the stock that represents the claim to all
domestic goods, even when the production of foreign goods is not perfectly correlated
with that of domestic goods. This is so because, in equilibrium, the exchange rate
responds to the shocks to the production of domestic and foreign goods so that, in terms
of domestic good, the total value of foreign goods is perfectly correlated with that of
domestic goods (Cass and Pavlova (2003), Pavlova and Rigobon (2003) and Zapatero
(1995)). In our model, the stock in the foreign country represents the claim to good two
in the economy as well as part of good one produced in the economy. As a result, the
stock of the foreign country is no longer perfectly correlated with any combination of the
other securities in the financial markets.

3 The Bench-Mark Case – no restriction on assets
holding

We begin with the case where there are no restrictions on the trading of financial assets.
Since the market is complete in this case, there exists a unique pricing kernel process \( \xi \)
in unit of good one,

\[
d\xi(t) = -\xi(t) \left[ r(t)dt + \theta(t)^{\top}dW(t) \right] \quad \text{with} \quad \xi(0) = 1,
\]

where \( \theta(t) \) is the unique market price of risk. It is given by \( \theta = \Sigma^{-1}(\bar{\mu} - r1) \), where \( 1 \) is
the four dimensional vector of 1 when \( \Sigma \) is of full rank. Here \( \xi(t, \omega) \) has the interpretation
as the Arrow-Debreu price of one unit of good one per unit of probability \( P \) in state \( \omega \)
at time \( t \).

Since the markets is complete, the equilibrium prices of the financial securities and the
exchange rate in this economy are identical to that in an aggregate representative agent
economy where the representative agent has the following aggregate utility function:

\[
U(\delta, \delta^*, \lambda) = \max_{c_d, c_f, c_d^*, c_f^*} \left[ \ln c_d + \ln c_d^* \right] + \lambda \left[ \ln c_f + \ln c_f^* \right],
\]

subject to

\[
c_d + c_f = \delta \quad \text{and} \quad c_d^* + c_f^* = \delta^*.
\]

Here \( \lambda \) represents the welfare weight of the foreign agent. The solution of the max-
imization problem in (2) is given by \( c_d(t) = \delta(t)/(1 + \lambda) \), \( c_d^*(t) = \delta^*(t)/(1 + \lambda) \),
\( c_f(t) = \delta(t)/\lambda/(1 + \lambda) \), and \( c_f^*(t) = \delta^*(t)/\lambda/(1 + \lambda) \). The aggregate utility function
take the form of log utility \( U(\delta, \delta^*, \lambda) = (1 + \lambda) \left[ \ln \delta + \ln \delta^* \right] \).

Since the market is complete, \( \lambda \) is constant and the allocation is Pareto optimal. The
equilibrium in the benchmark economy is characterized by Proposition 3.1.

**Proposition 3.1.** The equilibrium exchange rate is given by

\[
p(t) = \frac{\delta(t)}{\delta^*(t)}.
\]

The pricing kernel is

\[
\xi(t) = e^{-\beta t} \left[ \frac{\delta(0)}{\delta(t)} \right].
\]
The equilibrium interest rate and market price of risk are
\[ r(t) = \beta + \mu_\delta(t) - \|\sigma_\delta(t)\|^2, \]
\[ r^*(t) = \beta + \mu_\delta^*(t) - \|\sigma_\delta^*(t)\|^2 \]
\[ \theta(t) = \sigma_\delta(t). \]

The riskfree rates in domestic and foreign countries satisfy the so-called uncovered interest rate parity relationship
\[ r^*(t) - r(t) = -\mu_p(t) + \sigma_\delta(t)\top\sigma_p(t). \]

The risk premia of the three stocks in the economy are given by
\[ \mu_j(t) - r(t) = \sigma_j\sigma_\delta, \quad j = 1, 2, f. \]

The results here are standard, even though the setting of our world economy differs somewhat from the standard setting in the literature. For example, the riskfree rate of the domestic country is decreasing with the volatility of the aggregate output of good one, but increasing with the expected growth rate of aggregate output of good one. The intuition is well understood. When the aggregate consumption is more volatile, agents demand more of the safe security to hedge consumption risk. In equilibrium, the interest rate decreases. The risk premia in this economy satisfy the standard international CAPM.

Of special interest to us is the correlation between the domestic and foreign equity markets. Before the actual calculation of the correlation coefficient, it is useful to examine intuitively what drives the correlation between the two markets. The price of the foreign stock is
\[ S_f(t) = E_t \left[ \int_t^T e^{-\beta(s-t)} \left( \frac{\delta(t)}{\delta(s)} \right) \left[ \delta_3(s) + p(s)\delta^*(s) \right] ds \right] \]
\[ = E_t \left[ \int_t^T e^{-\beta(s-t)} \left( \frac{\delta(t)}{\delta(s)} \right) \left( \delta_1(s) + \delta_2(s) + 2\delta_3(s) \right) ds \right] \]

On the other hand, the total value of the stock market in the domestic country is
\[ S_d(t) = E_t \left[ \int_t^T e^{-\beta(s-t)} \left( \frac{\delta(t)}{\delta(s)} \right) \left( \delta_1(s) + \delta_2(s) \right) ds \right]. \]

These two expressions suggests that it is as if the two stock markets share a common cash flow component \( \delta_1 + \delta_2 \). Then it should come as no surprise that the two markets are correlated, but not perfectly correlated due to the \( \delta_3 \) component. What is happening is that the exchange rate provides a channel so that a shock to \( \delta \) translates to an equal change in value of \( \delta^* \). Intuitively, when a shock, say a positive shock, hits the production of the second commodity in the foreign country, the value of the foreign stock increases. The increase leads to, for example, the need of portfolio re-balancing, which translates to higher demand for domestic stocks, which in turn puts downward pressure on the
exchange rate. Zapatero (1995) and Pavlova and Rigobon (2003) demonstrated this result. It is also the reason why we have the foreign country produce both goods. If each country were to specialize in one good, i.e., domestic firms produce only good one and the foreign firm produces only good two, then \( S_d(t) \equiv S_1(t) + S_2(t) = S_f(t) = (1 - e^{-\beta(T-t)})\delta(t)/\beta \).

Now we demonstrate that the correlation coefficient can be anything between -1 and 1. Let's for simplicity assume \( \mu_{\delta_3/\delta} = 0 \). Then \( S_d(t) = \frac{1-e^{-\beta(T-t)}}{\beta} (\delta(t) - \delta_3(t)) \) and \( S_f(t) = \frac{1-e^{-\beta(T-t)}}{\beta} (\delta(t) + \delta_3(t)) \). Some calculation shows that the equity volatilities are

\[
\sigma_d = \frac{\delta \sigma_{\delta} - \delta_3 \sigma_{\delta_3}}{\delta - \delta_3} \quad \text{and} \quad \sigma_f = \frac{\delta \sigma_{\delta} + \delta_3 \sigma_{\delta_3}}{\delta + \delta_3}.
\]

The correlation between \( S_d(t) \) and \( S_f(t) \) is:

\[
\rho = \frac{\sigma_d^\top \sigma_f}{\| \sigma_d \| \| \sigma_f \|} = \frac{\delta^2 \| \sigma_{\delta} \|^2 - \delta_3^2 \| \sigma_{\delta_3} \|^2}{\| \delta \sigma_{\delta} + \delta_3 \sigma_{\delta_3} \| \| \delta \sigma_{\delta} - \delta_3 \sigma_{\delta_3} \|}.
\]

This correlation coefficient can be any value in \((-1, 1)\).

4 Equilibrium with Investment Restrictions

Now we turn to the main focus of this paper: the effect of investment restrictions on asset prices. We study two cases. In the first case, the domestic investor is restricted in his foreign investment. He cannot invest more than \( \bar{\pi} \) of his wealth in foreign assets (equity and bond). The investment restriction, however, can be removed by the benevolent government of the domestic country at a time \( \tau \) of its choice. We explore this case in section 4.1. In the second case, the second industry of the domestic economy is protected and the investment of the foreign investor in the industry is capped at \( \bar{\pi} \) percent. However, the protection can be removed by the benevolent government of the domestic country at a time \( \tau \) of its choice. This case is analyzed in section 4.2.

4.1 Restriction on domestic residents

The optimization problem of the domestic country can be stated as

\[
\max_{\tau} \max_{c_d, c_d^*} E \left[ \int_0^\tau e^{-\beta t} [\ln c_d(t) + \ln c_d^*(t)] \, dt + \int_\tau^T e^{-\beta t} [\ln c_d(t) + \ln c_d^*(t)] \, dt \right] \tag{3}
\]

under the wealth constraint:

\[
dX_d(t) = X_d(t) \left[ r(t) + \pi_d(t)^\top (\bar{\mu}(t) - r(t)) \right] \, dt - [c_d(t) + p(t)c_d^*(t)] \, dt + X_d(t) \left[ \pi_d(t)^\top \Sigma(t) \right] \, dW(t),
\]

\[
\]
where $\tau$ is the time when the investment restriction is removed, $X_d(t)$ is the domestic investor’s wealth at time $t$, $\pi_d = (\pi_{d1}, \pi_{d2}, \pi_{df}, \pi_{db^*})$ is the vector of portfolio weights in domestic stocks, foreign stock, and foreign bond, respectively.

The inner maximization problem in (3) is the standard utility maximization problem. In this economy, we assume that initially the investment in foreign capital market by the domestic investor is not to exceed $\bar{\pi}$ percent of his wealth, that is, $\pi_{df}(t) + \pi_{db^*}(t) \leq \bar{\pi}$. However, at the discretion of the domestic government, the investment restriction can be removed at a time $\tau$. We assume a benevolent government. The optimization problem is represented by the outer maximization in (3). Clearly, the assumption is not realistic as in the real world the government of a country does not always acts in the interest of its residents. The appropriate modelling of the objective of the government is thus important for a thorough understanding of the rationale behind the removal of investment restrictions. However, our objective in this paper is somewhat limited. The issue we would like to raise and understand is whether the incompleteness of financial markets arising from investment restrictions gives rise to a potential delay in international market integration. While the benevolent government assumption may not be realistic, it allows us to disentangle the effect of market incompleteness from the effect of different objective function of the government and thus to better understand the factors that affect the decision of government to remove investment restrictions.

Since the domestic investor faces investment restriction, the market is incomplete. To characterize the equilibrium of this economy, we use the representative agent approach introduced by Cuoco and He (1994). We construct the representative agent by assigning a stochastic weight to the domestic investor.

$$U(\delta, \delta^*, \lambda) = \max_{c_d, c_f, c_{d^*}, c_{f^*}} \left[ \ln c_f + \ln c_{f^*} \right] + \lambda \left[ \ln c_d + \ln c_{d^*} \right],$$

subject to

$$c_d + c_f = \delta \quad \text{and} \quad c_{d^*} + c_{f^*} = \delta^*.$$

Assuming the existence of equilibrium, its characterization is given in the following proposition.

**Proposition 4.1.** Let $\tau^*$ be the optimal time for removing the foreign asset holding restriction. The world economy is characterized as follows: the $\lambda$ process is a martingale given by, for $t \leq \tau^*$,

$$d\lambda(t) = -\frac{\lambda(t)(1 + \lambda(t))}{\|\Sigma^{-1}(\iota_3 + \iota_4)\|^2} \left[ S_f(t)/S(t) - \bar{\pi} \right]^+ (\iota_3 + \iota_4)^\top (\Sigma^{-1}(t))^\top dW(t),$$

where $\lambda(0) = X_d(0)/X_f(0)$, $\iota_3 = (0, 0, 1, 0)^\top$ and $\iota_4 = (0, 0, 0, 1)^\top$. For $t > \tau^*$, $\lambda(t) = \lambda(\tau^*)$ is a constant. The pricing kernel is

$$\xi(t) = \begin{cases} 
  e^{-\beta t} \frac{1 + \lambda(t)}{1 + \lambda(0)} \frac{\delta(0)}{\delta(t)} & t \leq \tau^* \\
  e^{-\beta t} \frac{1 + \lambda(\tau^*)}{1 + \lambda(0)} \frac{\delta(0)}{\delta(t)} & t \geq \tau^*.
\end{cases}$$
The exchange rate is the same as in the benchmark economy. The equilibrium interest rates, \( r \) and \( r^* \), and the price of risk, \( \theta \), are given by

\[
\begin{align*}
  r(t) &= \begin{cases} 
    \beta + \mu_\delta(t) - ||\sigma_\delta(t)||^2 - \frac{\lambda(t)}{||\Sigma^{-1}(\epsilon_3 + \epsilon_4)||^2} \frac{S_f(t)}{S(t)} [S_f(t)/\bar{S}(t) - \bar{\pi}]^+ & t \leq \tau^* \\
    \beta + \mu_\delta(t) - ||\sigma_\delta(t)||^2 & t \geq \tau^*.
  \end{cases} \\
  r^*(t) &= \begin{cases} 
    \beta + \mu_\delta^*(t) - ||\sigma_\delta^*(t)||^2 + \frac{\lambda(t)}{||\Sigma^{-1}(\epsilon_3 + \epsilon_4)||^2} \frac{S_d(t)}{S(t)} [S_f(t)/\bar{S}(t) - \bar{\pi}]^+ & t \leq \tau^* \\
    \beta + \mu_\delta^*(t) - ||\sigma_\delta^*(t)||^2 & t \geq \tau^*.
  \end{cases} \\
  \theta(t) &= \begin{cases} 
    \frac{\sigma_\delta + \lambda(t)}{||\Sigma^{-1}(\epsilon_3 + \epsilon_4)||^2} [S_f(t)/\bar{S}(t) - \bar{\pi}]^+ \Sigma^{-1}(\epsilon_3 + \epsilon_4) & t \leq \tau^* \\
    \frac{\sigma_\delta}{\sigma_\delta} & t \geq \tau^*.
  \end{cases}
\end{align*}
\]

The risk premia are

\[
\begin{align*}
  \mu_j(t) - r(t) &= \sigma_j \sigma_\delta, \quad j = 1, 2, \\
  \mu_f(t) - r(t) &= \begin{cases} 
    \sigma_f \sigma_\delta + \frac{\lambda(t)}{||\Sigma^{-1}(\epsilon_3 + \epsilon_4)||^2} [S_f(t)/\bar{S}(t) - \bar{\pi}]^+ & t \leq \tau^* \\
    \sigma_f \sigma_\delta & t \geq \tau^*.
  \end{cases} \\
  r^*(t) + \mu_p - r(t) &= \begin{cases} 
    \sigma_p \sigma_\delta + \frac{\lambda(t)}{||\Sigma^{-1}(\epsilon_3 + \epsilon_4)||^2} [S_f(t)/\bar{S}(t) - \bar{\pi}]^+ & t \leq \tau^* \\
    \sigma_\delta \sigma_p & t \geq \tau^*.
  \end{cases}
\end{align*}
\]

Proposition 4.1 identifies several important differences between the benchmark economy and the current economy where the domestic investor is restricted in his investment in foreign capital market. First, as he has to put more money in domestic capital markets, the domestic investor demands more of the domestic bond. The higher demand for domestic bond drives up the price and leads to lower riskfree rate. Moreover, as \( \lambda(t) \) is time-varying, the riskfree rate is more volatile. For example, if the aggregate dividend \( \delta \) follows geometric Brownian process, the riskfree rate in the benchmark economy is constant, while in the current economy it is time-varying.

The risk-free rate in the foreign country, on the other hand, is higher than its benchmark level. The intuition is similar. The investment restriction, when binding, results in less demand for foreign bond and hence leads to higher risk free rate.

It is interesting to observe that the uncovered interest rate parity relationship doesn’t hold in this economy. That is, compared to the benchmark economy, the difference between the foreign and domestic riskfree rates is augmented by an extra term, which is strictly positive when the investment restriction is binding. The reason is that the standard no-arbitrage relation between domestic and foreign interest rates breaks down when the investment restriction is binding. While the foreign may appear too high, the foreign asset holding restriction prevents the domestic investor from buying more of the foreign bond.

Second, while the risk premia of the domestic stocks remain as described in the standard international-CAPM obtained in the benchmark, that of the foreign stock deviates from the international CAPM. The extra term in the premium of the foreign stock is in
fact strictly positive when the investment restriction is binding. The intuition is straightforward. When the domestic agents want to invest more in the foreign stock than they are allowed to, the demand for foreign stock is lower than when there is no investment restriction. This translates to lower stock price and hence higher risk premium. It should be noted that the extra premium is not due entirely to lower risk free rate. Indeed,

$$\mu_f = \beta + \mu_\delta(t) + \|\sigma_\delta(t)\|^2 + \sigma_f \sigma_\delta + \frac{\lambda(t)}{\|\Sigma^{-1}(\ell_3 + \ell_4)\|^2} \frac{S_\delta(t)}{S(t)} [S_f(t)/S(t) - \bar{\pi}]^+.$$ 

Therefore, there is an extra term in the cost of capital which is strictly positive when the investment restriction is binding. It should also be noted that the higher risk premium and higher cost of capital are relative to the standard international CAPM. As $\sigma_f$ is typically different from that in the benchmark economy, there is no easy comparison in absolute terms between the risk premium and cost of capital for the foreign stock and their counterparts in the benchmark economy. See, however, Proposition 4.2 below for an approximate expression for $\sigma_f$.

Third, there is poor risk-sharing. This is reflected in the consumption enjoyed by both the domestic and foreign investors. Indeed, the consumption process of the foreign investor is given by

$$c_f(t) = \frac{e^{-\beta t}}{y_f \xi(t)} \frac{1 + \lambda(0)}{1 + \lambda(t)}$$
$$c^*_f(t) = \frac{e^{-\beta t}}{y_fp(t) \xi(t)} \frac{1 + \lambda(0)}{1 + \lambda(t)},$$

where $y_f$ is the Lagranian multiplier of the budget constraint, which is a constant, and

$$\xi^o = e^{-\beta t} \frac{\delta(0)}{\delta(t)}$$

is the pricing kernel in the benchmark economy. In the benchmark economy, $\lambda$ is a constant. Consumption variation is entirely driven by the aggregate risk in $\delta(t)$ and $\delta^*_o(t) = p(t)\delta(t)$. There is perfect risk sharing. In the current economy, even though the foreign investor faces a complete market and he on the surface can diversify the risks and smooth his consumption, his consumption is still subject to the additional risk coming from the time-varying $\lambda(t)$ which represents the change in investment environment, an equilibrium effect of the market friction faced by the domestic investor. The consumption of the domestic investor is subject to more risk. It is shown in the appendix that

$$c_d(t) = \frac{e^{-\beta t}}{y_d \xi^o(t)} \frac{1 + \lambda(0)}{1 + \lambda(t)} z(t),$$
$$c^*_d(t) = \frac{e^{-\beta t}}{y_dp(t) \xi^o(t)} \frac{1 + \lambda(0)}{1 + \lambda(t)} z(t),$$

where $z(t)$ is a stochastic process whose expression can be found in the appendix. Thus the consumption of the domestic investor is subject to the aggregate risk, the risk coming from the change in investment environment, and the lack of diversification due to the market friction he faces.

Finally, as a prelude to what is coming, we note that the $\lambda(t)$ process is a random walk. As shown in the appendix,

$$\lambda(t) = \frac{X_d(t)}{X_f(t)},$$
Thus the wealth ratio of the two countries follows a random walk. As

$$X_d(t) + X_f(t) = \tilde{S} = \frac{(1 - e^{-\beta T})}{\beta} \delta(t),$$

which is the same as in the benchmark economy, there is back and forth wealth transfer over time. Pavlova and Rigobon (2005) reports similar results on wealth transfers. We will see later that this feature of the wealth ratio process is important in the decision of the domestic government to remove the industry protection.

### 4.1.1 Market volatility and correlation

The potential effect of financial market integration on the stability of the financial market of individual countries and the co-movement of the financial markets across countries is a much debated topic both in the popular press and in the academic literature. As referred in the introduction, Stilitz has argued that financial liberalization, while beneficial for some countries, may lead to instability in some other economies. Zepatero (1995) and Pavlova and Rigobon (2003) show that terms of trade absent of frictions will act a transmitting channel so that the shock experienced in one country is transmitted in other countries in such a way that the financial markets of world are perfect correlated. Pavlova and Rigobon (2005) show, however, that when either there are frictions in one or both of the financial markets or demand shifts, the correlation will be tempered. In this section, we report a similar result. Our result differ, however, in two respects from that of Pavlova and Rigobon (2003, 2005), which we will describe in detail after stating our result.

Proposition 4.2 below provides approximate closed-form expressions for stock prices and volatilities. In general, we cannot solve and obtain explicit closed-form solutions for stock volatilities. Basak and Gallmeyer (2002) derived the Forward Backward Stochastic Differential equation (FBSDE) for the stock price and use numerical methods to obtain their results. We instead use perturbation methods [see Judd (1998)] to obtain approximate closed-form solutions. To achieve that, we first derive the FBSDE of the price processes. Second, we use continuation methods to solve for stock prices. Third, we use the expression of stock prices and solve the simultaneous equations of volatilities. The different steps of this procedure are exposed in details in the Appendix. For a discussion on the precision of the perturbation methods, we refer the interested reader to Zeidler (1986).\(^4\)

Let denote by the superscript “o” quantities from the benchmark economy. For example, \(S^o_j\) is the price of the domestic stock \(j\) in the benchmark economy, and \(\sigma^o_j\) its returns’ volatility. To compute the stock prices and volatilities, we assume \(\mu_{\delta_s/\delta} = \mu_{\delta_s/\delta} = 0\). We also use \(\epsilon = \left[\frac{S^o_f}{\tilde{S} - \bar{\pi}}\right]^+\).

\(^4\)The expression of the variable \(y\) is written as the Taylor series expansion of \(\epsilon\) as follows: \(y = y_0 + \sum_{n=1}^{N} y_n \epsilon^n + O(\epsilon^{N+1})\), where \(O(\epsilon^{N+1})\) means that \(||y - [y_0 + \sum_{n=1}^{N} y_n \epsilon^n]|/||\epsilon||^{N+1} < \infty\). We use the FBSDE satisfied by the price process to solve for the coefficients of the power terms. Once we obtain the parameters estimates, using stochastic calculus, we write the stocks volatilities as a system of simultaneous equations, which we solve.
Proposition 4.2. For $t < \tau^*$, stock prices are

$$S_d = S_d^0 \left[ 1 + \frac{S_f^0 \lambda(t) \left(1 - \frac{\delta_3}{\delta}\right) \left(1 - \ln\left(\frac{\delta}{\delta}\right)\right)}{S_d^0 \|\sigma_{\delta_3/\delta}\|^2 \|\Sigma^o\|^{-1} (t_3 + t_4)^2} \right] + O(\epsilon^2),$$

$$S_f = S_f^0 \left[ 1 - \frac{\lambda \left(1 - \frac{\delta_3}{\delta}\right) \left(1 - \ln\left(\frac{\delta}{\delta}\right)\right)}{\|\sigma_{\delta_3/\delta}\|^2 \|\Sigma^o\|^{-1} (t_3 + t_4)^2} \right] + O(\epsilon^2).$$

Stock volatilities are

$$\sigma_d = \sigma_d^0 + \frac{\lambda}{\|\sigma_{\delta_3/\delta}\|^2 \|\Sigma^o\|^{-1} (t_3 + t_4)^2} \frac{S_f^0}{S_d^0} \left(1 - \frac{\delta_3}{\delta}\right) \left(1 - \ln\left(\frac{\delta}{\delta}\right)\right) (\sigma_f^0 - \sigma_d^0) \epsilon$$

$$- \frac{\lambda}{\|\sigma_{\delta_3/\delta}\|^2 \|\Sigma^o\|^{-1} (t_3 + t_4)^2} \frac{S_f^0}{S_d^0} \left[1 - \frac{\delta_3}{\delta} \ln\left(\frac{\delta}{\delta}\right)\right] (\sigma_{\delta_3} - \sigma_d) \epsilon + O(\epsilon^2),$$

$$\sigma_f = \sigma_f^0 + \frac{\lambda}{\|\sigma_{\delta_3/\delta}\|^2 \|\Sigma^o\|^{-1} (t_3 + t_4)^2} \left[1 - \frac{\delta_3}{\delta} \ln\left(\frac{\delta}{\delta}\right)\right] (\sigma_{\delta_3} - \sigma_d) \epsilon + O(\epsilon^2).$$

Proposition shows that the market volatilities of both domestic and foreign countries can be higher or lower than those in the benchmark economy.

To analyze the implication of the investment restriction for contagion, let's compute the correlation between domestic and foreign equity returns. We can rewrite the stocks’ volatilities as follows:

$$S_d \sigma_d = S_d^0 \sigma_d^0 + B \epsilon + O(\epsilon^2), \quad S_f \sigma_f = S_f^0 \sigma_f^0 - B \epsilon + O(\epsilon^2),$$

with

$$B = \frac{\lambda S_f^0}{\|\sigma_{\delta_3/\delta}\|^2 \|\Sigma^o\|^{-1} (t_3 + t_4)^2} \left\{ \left(1 - \frac{\delta_3}{\delta}\right) \left(1 - \ln\left(\frac{\delta}{\delta}\right)\right) \sigma_f^0 - \left(1 - \frac{\delta_3}{\delta} \ln\left(\frac{\delta}{\delta}\right)\right) (\sigma_{\delta_3} - \sigma_d) \right\}.$$

Using these volatility expressions, the correlation is given by:

$$\rho = \rho^o - \left(\frac{1}{S_d^0 \|\sigma_d^0\|} \right) \left[ \frac{1}{\|\sigma_d^0\| \sigma_d^0 \top} - \frac{1}{\|\sigma_f^0\| \sigma_f^0 \top} \right] B \epsilon + O(\epsilon^2).$$

Table 1 presents simulated correlations between the domestic and foreign equity market returns. With the set of parameters values we used, the correlation is lower than that of the benchmark one ($\rho < \rho^o$), which implies that domestic country market will be less affected by international shocks. Intuitively, there are several forces at work. The exchange rate provides a channel of financial contagion as discussed earlier. The investment restriction, however, puts a valve on the transmission channel. For example, when a positive shock hits the production of good one in the domestic country, the stock price, $S_d$, increases, which would normally lead to a demand for foreign stock due to portfolio re-balancing. However, if the investment restriction is already binding for the domestic investor, he cannot increase his holding of foreign stock. As a result, the positive shock
to domestic stocks does not lead to a sufficient increase in the price of foreign stock.\(^5\)

If there is a negative shock, \(S_d\) falls. If it falls significantly so that, due to portfolio re-balancing, the desired holding of foreign stock drops below the investment restriction, then the reduced demand for foreign stock leads to lower \(S_f\). Thus, while we are not able to prove it, the simulation result is consistent with the intuition that the particular investment restriction considered in this section leads to less correlation across financial markets.

Table 1: **Simulated correlations between domestic and foreign equity returns.**

This table shows the simulated correlations between the domestic and foreign market returns. The values of the parameters used in the simulations are: \(T = 6\) years, \(\delta_1(0) = \delta_2(0) = \delta_3(0) = \delta^*(0) = 100\). In Panel (a), \(\mu_{\delta_1} = 0.03, \mu_{\delta_2} = \mu_{\delta_3} = \mu_{\delta^*} = 0.05, \sigma_{\delta_1} = (0.25, 0, 0, 0)^{\top}, \sigma_{\delta_2} = (0, 0.25, 0, 0)^{\top}, \sigma_{\delta_3} = (0, 0, 0.25, 0)^{\top}, \sigma_{\delta^*} = (0, 0, 0, 0.25)^{\top}\). In Panel (b), \(\mu_{\delta_1} = \mu_{\delta_2} = \mu_{\delta_3} = \mu_{\delta^*} = 0.05, \sigma_{\delta_1} = (0.15, 0, -0.05, -0.05)^{\top}, \sigma_{\delta_2} = (-0.05, 0.15, 0, -0.05)^{\top}, \sigma_{\delta_3} = (-0.05, 0.05, 0.15, -0.10)^{\top}, \sigma_{\delta^*} = (-0.05, -0.05, -0.05, 0.15)^{\top}\).

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<td>0.5663</td>
</tr>
</tbody>
</table>

**4.1.2 Timing of the removal of investment restriction**

So far we have characterized the financial markets of the two countries when in equilibrium. In this section, we will turn to the question of optimal timing of the removal of investment restriction. We will assume that the decision will be made by the benevolent government of the domestic country. The objective of the decision is to maximize the welfare of the domestic investor.

It will useful to decompose the welfare of the domestic investor into two components: \(U_d = U^*_d + \Delta U_d\), where \(U^*_d\) is the welfare in the benchmark economy. The welfare of the foreign investor can be decomposed similarly. Now let \(\tau^*\) denote the optimal time to remove the investment restriction. It can be shown that in general \(P(\tau^* > 0) > 0\). See the appendix for an example of such case.\(^6\)

The following proposition provides the expressions for \(\Delta U_d\) and \(\Delta U_f\).

---

\(^5\)The foreign investor is not constrained. He can re-balance his portfolio. Therefore there will be some upward pressure on foreign stock price. At the same time, the upward pressure on foreign stock comes also from the relatively more supply of good one in the domestic country.

\(^6\)See Lemma B.1 in the appendix.
The welfare variation of the domestic and foreign investors are

$$
\Delta U_d = E \left[ \frac{2(1-e^{-\beta T})}{\beta} \ln X_d(0) X_d^o(0) + E \left[ \int_0^{\tau^*} e^{-\beta t} \int_0^t \ln \left\{ \frac{[S_f/S - \bar{\pi}]^+}{(1 + \lambda)^2} \right\} dsdt \right] \right]
$$

$$
\Delta U_f = E \left[ \frac{2(1-e^{-\beta T})}{\beta} \ln X_f(0) X_f^o(0) + E \left[ \int_0^{\tau^*} e^{-\beta t} \int_0^t \ln \left\{ \frac{[S_f/S - \bar{\pi}]^+}{(1 + \lambda)^2} \right\} dsdt \right] \right].
$$

The welfare variation expression for $\Delta U_d$ has three terms. The first term captures the change in initial wealth, relative to the benchmark economy, when the domestic investor is restricted in his foreign investment. The second term is identical to the second term in $\Delta U_f$ and will be discussed shortly. The third term captures the hedging loss. Indeed, $S_f/\bar{S}$ is the desired portfolio weight on foreign stock when there is no foreign asset holding restriction. If $\bar{\pi}$ is greater than $S_f/\bar{S}$, the investment restriction is non-binding, the hedging loss is zero. Recall that we assume that the domestic investor is endowed with the production firms in his home country. Thus at time zero, his wealth is equal to $X_d(0) = S_1(0) + S_2(0)$. As Proposition 4.2 above illustrates, when there is foreign asset holding restriction, the additional capital held within the domestic country pushes up the price of the domestic stocks. The initial wealth $X_d(0)$ is higher than that in the benchmark economy. Thus both the first and second terms are positive. The third term is negative. Therefore, the sign of $\Delta U_d$ depends on whether the loss due to lack of hedging dominates the other two terms. If $\Delta U_d$ is strictly positive, then the benevolent government will not remove the investment restriction immediately.

The expression of welfare variation for the foreign investor has two terms. The first term is the potential loss in initial wealth. It is readily shown that

$$
X_d(0) + X_f(0) = X_d^o(0) + X_f^o(0) = \bar{S} = 2(1-e^{-\beta T})\delta(0)/\beta.
$$

Thus if $X_d(0)/X_d^o(0) > 1$, then $X_f(0)/X_f^o(0) < 1$. The second term that is common for both $\Delta U_d$ and $\Delta U_f$ is due to change in investment opportunity set. This term arises because when there is investment restriction, equilibrium asset price processes are different from those in the benchmark economy and in a dynamic economy, investment opportunity set is partially determined by the asset price processes. Thus even though the foreign investor faces no investment restriction, the investment opportunity set he faces is also changed due to the change in asset price processes. In a static economy, such term would not arise for the agent facing no frictions in financial markets. The impact of market incompleteness on him would be only through the change in wealth.

The following proposition provides a characterization of the optimal timing decision.

**Proposition 4.4.** Suppose that $\Delta U_d(0) > 0$ and $\tau^*$ is the optimal time of removing the foreign asset holding restriction. Then $S_f^o(\tau^*)/\bar{S}^o(\tau^*) = \bar{\pi}$, where $S_f^o$ and $\bar{S}^o$ are the
counterparts of \( S_f \) and \( \bar{S} \), respectively, in the benchmark economy, and \( \tau^* \) is the first time \( S_f(t)/\bar{S}(t) \) reaches \( \bar{\pi} \) from above, that is, \( \tau^* = \inf \{ t > 0 : S_f(t)/\bar{S}(t) \geq \bar{\pi} \} \).

While there is tradeoff between the wealth effect and the effect of the change in investment opportunity set on the one hand, and hedging loss on the other in the timing decision for removing the foreign asset holding restriction, the proposition suggests that the decision is mainly driven by the wealth effect. This is because, as shown in the appendix, the hedging loss and the effect of the change in investment opportunity set are second order effect. Thus as long as the desired level of holding for foreign assets, which is \( S_f/\bar{S} \), is greater than \( \bar{\pi} \), having the restriction will always create extra demand for domestic assets and depress the demand for foreign assets. The domestic investor holds more domestic assets because his investment in foreign assets is limited. As a result, his wealth is higher than in the benchmark economy. This wealth effect becomes zero when \( S_f/\bar{S} = \bar{\pi} \).

There is a paradoxical aspect of the wealth effect in the timing decision. Suppose that \( \Delta U_d(0) > 0 \) and \( X_d(0)/X^*(0) > 1 \). At time zero, the domestic investor owns only the domestic stocks. So the wealth effect is the biggest at the time. However, the process of the relative wealth process satisfies

\[
\frac{X_d(t)}{X_d(t) + X_f(t)} = \frac{\lambda(t)}{1 + \lambda(t)}.
\]

Using the process of \( \lambda(t) \) given in Proposition 4.2, the drift of the relative wealth process,

\[
E \left[ \frac{d}{dt} \left( \frac{X_d(t)}{X_d(t) + X_f(t)} \right) \right] = -\frac{\lambda^2}{(1 + \lambda)\|\Sigma^{-1}(t_3 + t_4)\|^2} \left( [S_f/\bar{S} - \bar{\pi}]^+ \right)^2
\]

is negative. Thus the initial gain in wealth will dissipate over time. This suggests the immediate removal of the foreign asset holding restriction. On the other hand, there is the wealth effect at time zero precisely because the government is to remove the restriction at a later time.

Combining Proposition 4.4 and Proposition 4.1 allows us to describe the effect of the timing decision on asset prices. At the time when the investment restriction is removed, both the asset expected returns and asset prices themselves will experience a smooth transition. Thus, although the cost of capital will come down for foreign country, for example, it will come down smoothly. In particular, both the domestic and foreign risk free rate will move smoothly through the transition time.

### 4.2 Restriction on foreign investors

In this section, we study the second case where the domestic government imposes a protection restriction on foreign investors. We assume that firm two represents the protected industry. Foreign investors are not allowed to hold more than \( \bar{\pi} \) percent of stock two.

Due to the investment restriction, the market is incomplete from the viewpoint of the foreign agents. The representative investor is constructed similar to equation (2), except that the weighting parameter \( \lambda \) assigned to foreign agent is time varying.

Assuming the equilibrium exists, the proposition below characterizes the equilibrium.
Proposition 4.5. Let $\tau^*$ be the optimal time for removing the industry protection. The world economy is characterized as follows: the $\lambda$ process is a martingale given by, for $t \leq \tau^*$,
\[ d\lambda(t) = -\frac{\lambda(t)(1 + \lambda(t)) S_2(t)}{\| \Sigma^{-1}(t) \|_2^2} \left[ \frac{\lambda(t)(1 + \lambda(t)) - \bar{\pi}^+}{\lambda(t)(1 + \lambda(t))} \right]^{\tau}S(t)^{\tau}dW(t), \]
where $\lambda(0) = X_f(0)/X_d(0)$ and $t \geq (0,1,0,0)^\top$. For $t \geq \tau^*$, $\lambda(t) = \lambda(\tau^*)$ is a constant. The pricing kernel is
\[ \xi(t) = \begin{cases} e^{-\beta t} \frac{1 + \lambda(t)}{1 + \lambda(0)} \delta(0) & t \leq \tau^* \\ \frac{1 + \lambda(\tau^*)}{1 + \lambda(0)} \delta(0) & t \geq \tau^* \end{cases} \]
The exchange rate is the same as in the benchmark economy. The equilibrium interest rates, $r$ and $r^*$, and market price of risk, $\theta$, are
\[ r(t) = \begin{cases} \beta + \mu_\delta(t) - \| \sigma_\delta(t) \|_2^2 - \frac{1 + \lambda(t)}{\| \Sigma^{-1}(t) \xi_2 \|_2^2} \left( \frac{S_2(t)}{S(t)} \right)^2 \left[ \frac{\lambda(t)}{1 + \lambda(t)} - \bar{\pi}^+ \right]^{\tau} & t \leq \tau^* \\ \beta + \mu_\delta(t) - \| \sigma_\delta(t) \|_2^2, & t \geq \tau^* \end{cases} \]
\[ r^*(t) = \begin{cases} \beta + \mu_\delta^*(t) - \| \sigma_\delta^*(t) \|_2^2 - \frac{1 + \lambda(t)}{\| \Sigma^{-1}(t) \xi_2 \|_2^2} \left( \frac{S_2(t)}{S(t)} \right)^2 \left[ \frac{\lambda(t)}{1 + \lambda(t)} - \bar{\pi}^+ \right]^{\tau} & t \leq \tau^* \\ \beta + \mu_\delta^*(t) - \| \sigma_\delta^*(t) \|_2^2, & t \geq \tau^* \end{cases} \]
\[ \theta(t) = \begin{cases} \frac{\sigma_\delta(t) + \frac{1 + \lambda(t)}{\| \Sigma^{-1}(t) \xi_2 \|_2^2} \left[ \frac{\lambda(t)}{1 + \lambda(t)} - \bar{\pi}^+ \right]^{\tau} S_2(t)}{\sigma_\delta(t)} & t \leq \tau^* \\ \frac{\lambda(t)}{\| \Sigma^{-1}(t) \xi_2 \|_2^2} S(t)^{\tau} & t \geq \tau^* \end{cases} \]
The risk premia are:
\[ \mu_j(t) - r(t) = \sigma_j \sigma_\delta, \quad j = 1, f, \]
\[ \mu_2(t) - r(t) = \begin{cases} \sigma_2 \sigma_\delta + \frac{(1 + \lambda(t)) S_2(t)}{\| \Sigma^{-1}(t) \xi_2 \|_2^2} \left[ \frac{\lambda(t)}{1 + \lambda(t)} - \bar{\pi}^+ \right]^{\tau} & t \leq \tau^* \\ \sigma_2 \sigma_\delta, & t \geq \tau^* \end{cases} \]
\[ r^*(t) + \mu_p(t) - r(t) = \sigma_p \sigma_\delta. \]

Several interesting points can be learned from the proposition. First, as in the preceding section, the exchange rate process is independent of the weight variable $\lambda$. It is characterized by the two aggregate output processes. In other words, the financial market friction does not have an effect on the goods market.

Second, as in the case where foreign asset holding restriction is imposed, the risk-free rate, $r$, is the domestic country is lower than that in the benchmark economy. However, unlike the case in the preceding section, the interest rate in the foreign country is also lower than that in the benchmark economy. The intuition is that as foreign investors...
have limited access to the stock two, they will turn their investment to other asset, in particular, the domestic and foreign bonds. The higher demand for bonds drive down the interest rates, \( r(t) \) and \( r^* \). It is interesting to note that the uncovered interest rate parity holds true. This is in contrast to the case where foreign asset holding restriction is imposed on the domestic agents. As \( \lambda(t) \) is time varying, both interest rates are more volatile than in the benchmark economy.

Third, the risk premia deviate from the standard international-CAPM. In particular, the risk premium of stock two has an additional term. All else equal, the cost of capital of the protected industry in the domestic economy is higher than that in the benchmark economy. For other stocks, although on the surface, their risk premia satisfy an international CAPM, the risk premia are in general different from those in the benchmark economy because in the presence of market friction, the market equilibrium is different and hence the stock volatilities can be different from those in the benchmark economy.

Fourth, there is a poor risk sharing. This is reflected in the consumption enjoyed by both the domestic and foreign agents. Indeed, the consumption process of the domestic agent is given by

\[
c_d(t) = \frac{e^{-\beta t}}{y_d \xi(t)} = \frac{e^{-\beta t}}{y_d \xi_o(t)} \frac{1 + \lambda(0)}{1 + \lambda(t)},
\]

\[
c^*_d(t) = \frac{e^{-\beta t}}{y_d p(t) \xi(t)} = \frac{e^{-\beta t}}{y_d p(t) \xi_o(t)} \frac{1 + \lambda(0)}{1 + \lambda(t)},
\]

where \( y_d \) is the Lagrange multiplier of the budget constraint, which is a constant, and

\[
\xi_o = e^{-\beta t} \frac{\delta(0)}{\delta(t)}
\]

is the pricing kernel in the benchmark economy. In the benchmark economy, \( \lambda \) is a constant. The consumption variation is entirely driven by the aggregate risk in \( \delta(t) \) and \( \delta^*(t) = p(t) \delta(t) \). There is perfect risk sharing. In the current economy, even though the domestic agent faces a complete market and he on the surface can diversify the risks and smooth his consumption, his consumption is subject to additional risk coming from the time-varying \( \lambda(t) \) which represents the change in investment environment. This additional risk is due to the market friction faced by the foreign agents. The consumption of the foreign agent is subject to more risk. It is shown in the appendix that

\[
c_f(t) = \frac{e^{-\beta t}}{y_f \xi(t)} \frac{1 + \lambda(0)}{1 + \lambda(t)} z(t),
\]

\[
c^*_f(t) = \frac{e^{-\beta t}}{y_f p(t) \xi(t)} \frac{1 + \lambda(0)}{1 + \lambda(t)} z(t),
\]

where \( z(t) \) is a stochastic process whose expression can be found in the appendix. Thus the consumption of the foreign agent is subject to the aggregate risk, the risk coming

\[^{8}\text{If there are financial derivatives in the market, institutional investors will use them to overcome the restrictions on stock holdings imposed on them. By taking positions on the derivatives markets they can attenuate substantially the effect of the investment restrictions. The existence of financial innovations can be addressed within this framework. For that purpose, stock 2 will be the real asset and the risky security 1 the financial derivative. Depending on the level of correlation between the two securities the constraint will have less or more impact. If the two securities are highly correlated then the constraint effect is negligible since agents will use security 1 to overcome the holdings restrictions on stock 2. This practice is current in international financial markets, examples are the cloned or mirror funds in Canada.}\]
from the change in investment environment, and the lack of diversification due to the market friction he faces.

Finally, the \( \lambda(t) \) process is a random walk. As shown in the appendix,

\[
\lambda(t) = \frac{X_f(t)}{X_d(t)}
\]

Thus the relative wealth of the two countries follows a random walk. As

\[
X_d(t) + X_f(t) = \bar{S} = \frac{(1 - e^{-\beta T})}{\beta} \delta(t),
\]

which is the same as in the benchmark economy, there is back and forth wealth transfer over time. We will see later that this feature of the relative wealth process is important in the decision of the domestic government to remove the industry protection.

4.2.1 Market volatility and correlation

Proposition 4.6 below provides approximate closed form solutions for stock volatilities. As in Section 4.1.1, denote by the superscript “\( \circ \)” quantities from the benchmark economy. When \( \mu_{\delta_j/\delta} = \mu_{\delta_j} - \mu_{\delta} - \sigma_{\delta_j}^T (\sigma_{\delta_j} - \sigma_{\delta}) \) is non stochastic, \( S_\circ(t) = \delta_j(t) \int_t^T \exp\{\int_t^\tau (-\beta + \mu_{\delta_j/\delta}(\tau))d\tau\}d\tau \). To derive the stock prices and volatilities, we assume that \( \mu_{\delta_2/\delta} = \mu_{\delta_3/\delta} = 0 \), then \( S_2(t) = \frac{1-e^{-\beta(T-t)}}{\beta} \delta_2(t), S_3(t) = \frac{1-e^{-\beta(T-t)}}{\beta} (\delta(t) - \delta_3(t)) \) and \( S_\circ(t) = \frac{1-e^{-\beta(T-t)}}{\beta} (\delta(t) + \delta_3(t)) \). We also use \( \epsilon = [\lambda/(1 + \lambda) - \bar{\pi}]^+ / \{\lambda/(1 + \lambda)\} \). In the expression \( [\lambda(t)/(1 + \lambda(t) - \bar{\pi}(t)]^+ \), the term \( \lambda(t)/(1 + \lambda(t)) \) represents the proportion of stock two foreign investors would like hold if there were no holding restrictions. Thus \( \epsilon \) represents the relative deviation from the desired holding of stock two.

**Proposition 4.6. Stock prices are**

\[
S_d(t) = S_\circ(t) - \frac{\lambda(t)\bar{S}(t)}{\|\sigma_{\delta_j/\delta}(t)\|^2 \|\Sigma_{\delta_j^{-1}}(t)\|^2} \left( \frac{S_\circ(t)}{S(t)} \right)^2 \left( 1 + \frac{\delta_3(t)}{\delta(t)} \right) \left( 1 - \ln\left( \frac{\delta_3(t)}{\delta(t)} \right) \right) \epsilon + \mathcal{O}(\epsilon^2),
\]

\[
S_f(t) = S_\circ(t) + \frac{\lambda(t)\bar{S}(t)}{\|\sigma_{\delta_j/\delta}(t)\|^2 \|\Sigma_{\delta_j^{-1}}(t)\|^2} \left( \frac{S_\circ(t)}{S(t)} \right)^2 \left( 1 + \frac{\delta_3(t)}{\delta(t)} \right) \left( 1 - \ln\left( \frac{\delta_3(t)}{\delta(t)} \right) \right) \epsilon + \mathcal{O}(\epsilon^2),
\]

**Stock volatilities are**

\[
\sigma_d = \sigma_\circ - \frac{\lambda}{\|\sigma_{\delta_j/\delta}\|^2 \|\Sigma_{\delta_j^{-1}}(t)\|^2} \bar{S} \left( \frac{S_\circ}{S} \right)^2 \left( 1 + \frac{\delta_3}{\delta} \right) \left( 1 - \ln\left( \frac{\delta_3}{\delta} \right) \right) (2\sigma_{\delta_2} - \sigma_\circ - \sigma_{\delta}) \epsilon
\]

\[
+ \frac{\lambda}{\|\sigma_{\delta_j/\delta}\|^2 \|\Sigma_{\delta_j^{-1}}(t)\|^2} \bar{S} \left( \frac{S_\circ}{S} \right)^2 \left[ 1 + \frac{\delta_3}{\delta} \ln\left( \frac{\delta_3}{\delta} \right) \right] (\sigma_{\delta_3} - \sigma_{\delta}) \epsilon + \mathcal{O}(\epsilon^2),
\]

\[
\sigma_f = \sigma_\circ + \frac{\lambda}{\|\sigma_{\delta_j/\delta}\|^2 \|\Sigma_{\delta_j^{-1}}(t)\|^2} \bar{S} \left( \frac{S_\circ}{S} \right)^2 \left( 1 + \frac{\delta_3}{\delta} \right) \left( 1 - \ln\left( \frac{\delta_3}{\delta} \right) \right) (2\sigma_{\delta_2} - \sigma_\circ - \sigma_{\delta}) \epsilon
\]

\[
- \frac{\lambda}{\|\sigma_{\delta_j/\delta}\|^2 \|\Sigma_{\delta_j^{-1}}(t)\|^2} \bar{S} \left( \frac{S_\circ}{S} \right)^2 \left[ 1 + \frac{\delta_3}{\delta} \ln\left( \frac{\delta_3}{\delta} \right) \right] (\sigma_{\delta_3} - \sigma_{\delta}) \epsilon + \mathcal{O}(\epsilon^2).
\]
The approximate closed-form solutions in the proposition suggest that, at least for small $\epsilon$, the price of the domestic protected stock is lower than its counterpart in the benchmark economy. The changes on stock volatilities are ambiguous. The volatilities of the two market equities move in opposite directions. It is shown in the appendix that the domestic sectors are also affected differently. Thus protection of one sector of the domestic economy reduces the exposure of other sectors to shocks in the international capital market.

To analyze the co-movement of the markets more closely, we compute the correlation between the two equity markets. We can rewrite the stocks’ volatilities as follows:

$$S_d\sigma_d = S_d^0\sigma_d^0 - A\epsilon + O(\epsilon^2), \quad S_f\sigma_f = S_f^0\sigma_f^0 + A\epsilon + O(\epsilon^2),$$

with

$$A = \frac{\lambda \tilde{S}}{\|\sigma_{d/}\|^2 \|\Sigma_{T}^{-1}\|_2^2} \left\{ \frac{S_d^0}{S} \left( 1 + \delta_3 \delta \right) \left( 1 - \ln \left( \frac{\delta_3}{\delta} \right) \right) \left( 2\sigma_2 - \sigma_3 \right) \right\}.$$

Using these volatility expressions,

$$\rho = \rho^0 - \left( \frac{1}{S_d^0 \|\sigma_d^0\|} + \frac{\rho^0}{S_f^0 \|\sigma_f^0\|} \right) \left[ \frac{1}{\|\sigma_f^0\|} \sigma_f^0 A - \frac{1}{\|\sigma_d^0\|} \sigma_d^0 A \right] \epsilon + O(\epsilon^2).$$

Table 2 presents simulated correlations between the domestic and foreign equity market returns. Depending on the initial parameters values, the correlation between the two market returns can either be higher or lower than the benchmark level. For example, in panel (a), $\rho < \rho^0$, and in panel (b), $\rho > \rho^0$. This suggests that, unlike the case in Section 4.1.1, the effect of industry protection on financial market contagion is ambiguous.

### 4.2.2 Timing of the removal of investment restriction

In this section, we analyze the decision by the domestic government to remove the industry protection. The analysis is similar to that in Section 4.1.2. However, it has its difference. We will focus on the difference.

**Proposition 4.7.** Let $\tau^*$ be the optimal time to remove the industry protection. Agents’ welfare variations $\Delta U_i$ are

$$\Delta U_d = \frac{2(1 - e^{-\beta T})}{\beta} \ln \frac{X_d(0)}{X_d^0(0)} + E \left[ \int_0^{\tau^*} \int_0^t e^{-\beta t} \lambda^2 \left\{ \frac{S_d^0}{S} \|\lambda/(1 + \lambda) - \pi \|_2^2 \lambda/(1 + \lambda) \right\}^2 dsdt \right],$$

$$\Delta U_f = \frac{2(1 - e^{-\beta T})}{\beta} \ln \frac{X_f(0)}{X_f^0(0)} + E \left[ \int_0^{\tau^*} \int_0^t e^{-\beta t} \lambda^2 \left\{ \frac{S_f^0}{S} \|\lambda/(1 + \lambda) - \pi \|_2^2 \lambda/(1 + \lambda) \right\}^2 dsdt \right]$$

$$- E \left[ \int_0^{\tau^*} \int_0^t e^{-\beta t} (1 + \lambda)^2 \left\{ \frac{S_d^0}{S} \|\lambda/(1 + \lambda) - \pi \|_2^2 \lambda/(1 + \lambda) \right\}^2 dsdt \right].$$
Table 2: Simulated correlations between domestic and foreign equity returns.

This Table shows the simulated correlations between the domestic and foreign market returns. The values of the parameters used in the simulations are: \( T = 6 \) years, \( \mu_{\delta_1} = \mu_{\delta_2} = \mu_{\delta_3} = \mu_{\delta^*} = 0.05, \delta_1(0) = 200, \delta_2(0) = \delta_3(0) = \delta^*(0) = 100 \). In Panel (a), \( \sigma_{\delta_1} = (0.15, 0, -0.05, -0.10)^T, \sigma_{\delta_2} = (-0.05, 0.15, 0, -0.05)^T, \sigma_{\delta_3} = (-0.05, 0.05, 0.15, -0.10)^T, \sigma_{\delta^*} = (-0.05, -0.05, -0.05, 0.15)^T \). In Panel (b), \( \sigma_{\delta_1} = (0.15, 0, 0, 0)^T, \sigma_{\delta_2} = (0, 0.15, 0, 0)^T, \sigma_{\delta_3} = (0, 0, 0.15, 0)^T, \sigma_{\delta^*} = (0, 0, 0, 0.15)^T \).

Panel (a):

<table>
<thead>
<tr>
<th>Values of ( \rho^o )</th>
<th>Values of ( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\pi} )</td>
<td>( \bar{\pi} )</td>
</tr>
<tr>
<td>T-t</td>
<td>0.10</td>
</tr>
<tr>
<td>5</td>
<td>0.7373</td>
</tr>
<tr>
<td>4</td>
<td>0.7341</td>
</tr>
<tr>
<td>3</td>
<td>0.7348</td>
</tr>
<tr>
<td>2</td>
<td>0.7353</td>
</tr>
<tr>
<td>1</td>
<td>0.7359</td>
</tr>
</tbody>
</table>

Panel (b):

<table>
<thead>
<tr>
<th>Values of ( \rho^o )</th>
<th>Values of ( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\pi} )</td>
<td>( \bar{\pi} )</td>
</tr>
<tr>
<td>T-t</td>
<td>0.10</td>
</tr>
<tr>
<td>5</td>
<td>0.7428</td>
</tr>
<tr>
<td>4</td>
<td>0.7402</td>
</tr>
<tr>
<td>3</td>
<td>0.7409</td>
</tr>
<tr>
<td>2</td>
<td>0.7406</td>
</tr>
<tr>
<td>1</td>
<td>0.7410</td>
</tr>
</tbody>
</table>

Since the industry protection restriction is on foreign investors, \( \Delta U_d \) has two terms. Due to the protection restriction, there is less demand from the foreign investors for stock two of the domestic country. The price of stock two is lower than in the benchmark economy. As a result, the initial wealth of the domestic investor, \( S_1(0) + S_2(0) \), is lower (see, for example, Proposition 4.6). This effect of industry protection is reflected in the first term. As in Section 4.1.2, the second term captures the effect due to change in investment opportunities. The welfare variation for the foreign investor, \( \Delta U_f \), has three terms. The first two terms are similar to those of the domestic investor except for the difference in the sign. The third term captures the loss due to inability to hedge fully.

**Proposition 4.8.** Suppose that \( \Delta U_d(0) > 0 \) and \( \tau^* \) is the optimal time of removing the industry protection. Then \( \bar{\pi} = \lambda(\tau^*)/(1 + \lambda(\tau^*)) \), and \( \tau^* \) is the first time \( \lambda(t)/(1 + \lambda(t)) \) reaches \( \bar{\pi} \) from above, that is, \( \tau^* = \inf \{ t > 0 : \lambda(t)/(1 + \lambda(t)) \geq \bar{\pi} \} \).

The proof of the proposition shows that, similar to Section 4.1.2, the wealth effect dominates the effect of investment opportunity change and hedging gain/loss combined.
Unlike the case of Section 4.1.2, however, the drift of the relative wealth of the domestic agent is positive,

$$E_t \left[ \frac{d}{dt} \left( \frac{X_d(t)}{X_d(t) + X_f(t)} \right) \right] = E_t \left[ \frac{d}{dt} \left( \frac{1}{1 + \lambda(t)} \right) \right] = 1 + \lambda(t) \frac{S_2}{\|\Sigma^{-1}t_2\|^2} \left( \left( \frac{\lambda(t)}{1 + \lambda(t)} - \bar{\pi} \right)^+ \right)^2.
$$

This seems to suggest that if initially it is not optimal for the domestic government to remove the industry protection, it is unlikely it will remove it later. However, the government will remove the restriction for a different reason. As is readily shown,

$$E_t \left[ \frac{d}{dt} \left( \frac{X_f(t)}{X_d(t) + X_f(t)} \right) \right] = E_t \left[ \frac{d}{dt} \left( \frac{\lambda(t)}{1 + \lambda(t)} \right) \right] < 0.
$$

Thus the relative wealth of foreign agent is expected to deteriorate over time. Eventually, it will deteriorate to the point where $\lambda(t)/(1 + \lambda(t)) \leq \pi$ so that even if allowed, the foreign investor will not want to own more than $\bar{\pi}$ percent of industry two in the domestic country. When that happens, the industry protection is irrelevant.

It is interesting to note that even if the domestic government wants to remove the industry immediately, the foreign agent government may actually want the protection in place for a while for the benefit of its residents. The reason is that the foreign residents initially enjoy a wealth gain and that gain dissipates over time. The foreign government would like its residents to enjoy the initial wealth gain and then have the protection removed before the relative wealth of its residents deteriorate too much.

## 5 Conclusion

Recent crises on international capital markets have reopened the debate on whether or not economic integration and market globalization are good for the countries involved. The international economics and finance literature present a mixed view. The goal of this paper is to study the issue from the perspective of market incompleteness. To achieve our goal, we develop a general equilibrium model of international capital markets in the presence of investment restrictions.

Our main findings are as follow. Firstly, when a domestic country caps foreign investment in some key industries of the domestic economy, the cost of capital of the protected industry is higher, all else being equal. On the other hand, when the restriction is on its residents’ foreign investment, the cost of capital the domestic country is lower. Moreover, in both restricted economies, the domestic cost of risk free borrowing and lending is lowered. However, the risk free rate in the foreign country can be higher or lower depending on the restriction. Also, the uncovered interest rate parity relationship is violated. Secondly, by having investment restrictions, countries may reduce the contagion effects on their financial markets. Our results could explain why recent crises in international markets have had different effects on countries located in same geographical area or having similar economic characteristics. Thirdly, when restrictions on investments are imposed, the effects on stock market volatilities are ambiguous. This result suggests that it would be difficult for the debate on whether opening up financial markets is the cause of financial instability to produce conclusive results.
Another important issue that is studied in this paper is, assuming that investment restrictions are already in place, when is optimal to remove them from the perspective the country that imposes the restrictions. We took a real option approach to the problem. Contrary to conventional wisdom that removing frictions in financial market is good, we find that sometimes it is optimal to wait. The reason is that in an incomplete market, making a market complete is not always welfare enhancing for the agents in the economy. To the best of our knowledge, this is the first attempt in the literature to address the issue with a real option approach. We also study the impact of the removal decision on asset prices. We find, as described above, stability and market co-movement can go either way after the removal of the investment restrictions.
A Benchmark case

A.1 Proof of Proposition 3.1

The exchange rate is defined as the marginal rate of substitution of good $\delta^*$ for good $\delta$ and is given by:

$$p(t) = \frac{\partial U(\delta(t), \delta^*(t), \lambda)}{\partial \delta^*} = \frac{\delta(t)}{\delta^*(t)}.$$

The equilibrium pricing kernel is the marginal utility of the representative investor:

$$\xi(t) = e^{-\beta t} \frac{\partial U(\delta(t), \delta^*(t), \lambda)}{\partial \delta} = e^{-\beta t} \left[ \frac{\delta(0)}{\delta(t)} \right].$$

Applying Ito’s formula to $\xi(t)$, we get

$$d\xi(t) = -\xi(t) \left( \beta + \mu_\delta(t) - \|\sigma_\delta(t)\|^2 \right) dt - \xi(t) \sigma_\delta(t)^\top dW(t).$$

The expressions for $r(t)$ and $\theta(t)$ follow. The risk premia are obtained by $\bar{\mu} - r = \Sigma \theta$. From the risk premium of the foreign bond, we obtain the interest rate differential.

B Economy with Restriction on Domestic Investors

B.1 Constrained investor optimization problem

First, we fix a stopping time $\tau$. After the investment restriction is removed at time $\tau$, the world financial market becomes complete. The characterization of the economy is given by Proposition 3.1 for $t \geq \tau$. So we will focus on $t < \tau$.

Suppose that the domestic investor is limited to hold no more than $\bar{\pi}$ of his wealth in foreign securities, which are the foreign bond and stock. To characterize the optimal policies, we use the framework developed by Cvitanic and Karatzas (1992). Let's define the constraint space $K$ of agent $i$:

$$K = \{ \pi \text{ such that } \pi_3 + \pi_4 \leq \bar{\pi} \},$$

where $\pi_j$ is the percentage of wealth investor $i$ invests in security $j$, and $\bar{\pi}$ is the maximum percentage of wealth he can put. We define the support function of $K$, as

$$\psi(x) = \sup_{\pi \in K} (\pi^\top x) = \begin{cases} \bar{\pi}x & \text{if } x_1 = x_2 = 0 \text{ and } x_3 = x_4 = x \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

For each $x$, introduce a new fictitious financial market as follows:

$$dB^x(t) = (r(t) + \psi(x(t)))B^x(t)dt$$

$$dS^x(t) = I_\xi^x(t)(\mu(t) - \xi(t)\sigma_\delta(t)^\top dW(t)$$

where $S^x = (S^x_1, S^x_2, S^x_f, B^x_f)^\top$ is the price vector of the stocks and the foreign bond, and $I_\xi^x = diag(S^x)$. Define the new (fictitious) pricing kernel facing investor $d$ by

$$d\xi_d(t) = -\xi_d(t) \left[ r_d(t)dt + \theta_d(t)^\top dW(t) \right], \quad \text{with } \xi_d(0) = 1,$$
where $r_d = r + \psi(x)$ and $\theta_d = \theta - \Sigma^{-1}x$ are the interest rate and market price for risk facing the constrained investor $d$, and $\theta = \Sigma^{-1}(\mu - r1)$.

Under investment constraint, the constrained optimization problem of investor $d$ can be solved as follows: for each fixed fictitious market, solve the following variational static problem:

$$\max_{c_d(t), c_{d}^*(t), t < \tau} \mathbb{E} \left[ \int_0^\tau e^{-\beta t} [\ln c_d(t) + \ln c_d^*(t)] dt + U_d(X(\tau)) \right],$$

under the budget constraint:

$$\mathbb{E} \left[ \int_0^\tau \xi_d(t) [c_d(t) + p(t)c_d^*(t)] dt + \xi_d(\tau)X_d(\tau) \right] = X_d(0),$$

where $U_f(X_d(\tau))$ is the utility of investor $d$ at time $\tau$ given wealth $X_d(\tau)$ and $X_d(\tau)$ is the remaining wealth of investor $d$ at time $\tau$. Note that the investment restriction faced by investor $d$ is removed at time $\tau$.

From the dual optimization (see Cvitanic and Karatzas (1992) for the technical details), letting $x = x(t_3 + t_4)$,

$$x = \arg\min_{x \geq 0} [2\pi x + \|\theta - x\Sigma^{-1}(t_3 + t_4)\|^2],$$

which implies

$$x = \frac{1}{\|\Sigma^{-1}(t_3 + t_4)\|^2} \left[ \theta^\top\Sigma^{-1}(t_3 + t_4) - \bar{\pi} \right]^+. \quad (4)$$

### B.2 Proof of Proposition 4.1

#### B.2.1 Exchange rate

The representative agent’s utility is $U(\delta, \delta^*, \lambda) = (1 + \lambda) [\ln \delta + \ln \delta^*]$. The exchange rate is defined as the marginal rate of substitution of good $\delta^*$ for good $\delta$ and its process is given by:

$$p(t) = \frac{\partial U(\delta(t), \delta^*(t), \lambda(t))/\partial \delta^*}{\partial U(\delta(t), \delta^*(t), \lambda(t))/\partial \delta} = \frac{\delta(t)}{\delta^*(t)}.$$

Applying Ito’s lemma to the expression of $p$ and comparing the terms to equation (1) gives the instantaneous return $\mu_p$ and volatility $\sigma_p$ of changes in the exchange rate:

$$\mu_p(t) = \mu_\delta(t) - \mu_{\delta^*}(t) + \|\sigma_\delta(t)\|^2 - \sigma_\delta(t)^\top \sigma_{\delta^*}(t), \quad (5)$$

$$\sigma_p(t) = \sigma_\delta(t) - \sigma_{\delta^*}(t). \quad (6)$$

#### B.2.2 The weight process

The result on $\lambda$ is similar as well. Using

$$\lambda(t) = y_f \xi_f(t)/y_d \xi_d(t),$$

where $y_i$ is the lagrange multiplier of investor $i$’s maximization problem, and $\xi_i$ is the minimax pricing kernel of the fictitious market facing investor $i$. Applying Ito’s formula to the expression of $\lambda$ gives

$$d\lambda(t) = \lambda(t)(r_d(t) - r_f(t) + \theta_d^\top(t)(\theta_d(t) - \theta_f(t)))dt + (\theta_d(t) - \theta_f(t))^\top dW(t).$$
Thus
\[ \mu_\lambda(t) = r_d(t) - r_f(t) + \theta_d^T(t)\sigma_\lambda(t) \]
and
\[ \sigma_\lambda(t) = \theta_d(t) - \theta_f(t). \]

Since the domestic investor is constrained, \( r_d = r + \pi x \) and \( \theta_d = \theta - x\Sigma^{-1}(\iota_3 + \iota_4) \). Since foreign investor has no investment restrictions, \( \xi_f = \xi \), so \( r_f = r \), \( \theta_f = \theta \), and
\[
\begin{align*}
\mu_\lambda(t) &= \pi x(t) + \theta_d^T(t)\sigma_\lambda(t), \\
\sigma_\lambda(t) &= \theta_d(t) - \theta(t) = -x(t)\Sigma(t)^{-1}(\iota_3 + \iota_4) \quad \text{or} \quad \theta_d(t) = \theta(t) + \sigma_\lambda(t). \quad (7) \ (8)
\end{align*}
\]

### B.2.3 Pricing kernel and risk free rate

Following the same argument as Cuoco and He (1994), the pricing kernel is obtained as the marginal utility of the representative agent:
\[
\xi(t) = e^{-\beta t} \frac{\partial U(\delta(t), \delta^*(t), \lambda(t))/\partial \delta}{\partial U(\delta(0), \delta^*(0), \lambda(0))/\partial \delta} = e^{-\beta t} \frac{1 + \lambda(t) \delta(0)}{1 + \lambda(0) \delta(t)}.
\]

Applying Ito’s lemma yields
\[
d\xi(t) = -\xi(t) \left[ \left( \beta + \mu_\delta - \|\sigma_\delta\|^2 - \frac{\lambda}{1 + \lambda} \mu_\lambda + \frac{\lambda}{1 + \lambda} \sigma_\delta^T \sigma_\lambda \right) dt + \left( \sigma_\delta - \frac{\lambda}{1 + \lambda} \sigma_\lambda \right) dW(t) \right].
\]

It follows
\[
\begin{align*}
r(t) &= \beta + \mu_\delta(t) - \|\sigma_\delta(t)\|^2 + \frac{\lambda(t)}{1 + \lambda(t)} (1 + \lambda(t)) \left( \sigma_\delta(t)^T \sigma_\lambda(t) - \mu_\lambda(t) \right), \\
\theta(t) &= \sigma_\delta(t) - \frac{\lambda(t)}{1 + \lambda(t)} \sigma_\lambda(t). \quad (9) \ (10)
\end{align*}
\]

Using the expressions of \( \mu_\lambda \) and \( \sigma_\lambda \) from equations (7)-(8), \( r \) and \( \theta \) simplify as follows:
\[
\begin{align*}
r(t) &= \beta + \mu_\delta(t) - \|\sigma_\delta(t)\|^2 - \frac{\lambda(t)}{1 + \lambda(t)} \left( \bar{a}(t)x_2(t) + \frac{\lambda(t)}{1 + \lambda(t)} (x_2(t))^2 \|\Sigma(t)^{-1}x_2\|^2 \right), \\
\theta(t) &= \sigma_\delta(t) + \frac{\lambda(t)}{1 + \lambda(t)} x_2(t) \Sigma(t)^{-1} x_2(t).
\end{align*}
\]

The risk premia are obtained by using \( \bar{\mu} - r = \Sigma \theta \). From the risk premium of the foreign bond, the difference between the two countries risk-free rates is obtained as follows: \( r^* + \mu_p - r = \sigma_p^T \theta \).

### B.2.4 Expression for \( x \)

Using (4) and \( \theta = \sigma_\delta + \frac{\lambda}{1 + \lambda} x \Sigma^{-1}(\iota_3 + \iota_4) \), and
\[
x = x_0 = x_3 = \frac{1 + \lambda}{\|\Sigma^{-1}(\iota_3 + \iota_4)\|^2} \left[ \sigma_\delta^T \Sigma^{-1}(\iota_3 + \iota_4) - \pi^* \right]^+. \]
The volatility of the $\lambda$ process is

$$\sigma_\lambda = -x\Sigma^{-1}(\iota_3 + \iota_4),$$

and it can be shown that $\mu_\lambda = 0$. Using these results, the risk free rate simplifies to

$$r = \beta + \mu_\delta - \|\sigma_\delta\|^2 - \frac{\lambda}{1 + \lambda}x\Sigma^{-1}(\iota_3 + \iota_4).$$

The total wealth available in the world is

$$\bar{S}(t) = S_1 + S_2 + S_f = 2(1 - e^{-\beta(T-t)})\delta(t)/\beta,$$

hence its volatility is the weight average as follows: $\sigma_\delta = (S_1/\bar{S})\sigma_1 + (S_2/\bar{S})\sigma_2 + (S_f/\bar{S})\sigma_f$. Recall $\Sigma = (\sigma_1, \sigma_2, \sigma_f, \sigma_p)^\top$. So $\sigma_\delta^\top\Sigma^{-1}\iota_3 = S_f/\bar{S}$ and $\sigma_\delta^\top\Sigma^{-1}\iota_4 = 0$. Therefore,

\[
x = \frac{1 + \lambda}{\|\Sigma^{-1}(\iota_3 + \iota_4)\|^2} \left[ S_f/\bar{S} - \bar{\pi} \right]^+. \\
\]

\[
r = r_f = \beta + \mu_\delta - \|\sigma_\delta\|^2 - \frac{\lambda}{1 + \lambda}xS_f/\bar{S}. \\
\]

\[
r_d = r + x\bar{\pi}. \\
\]

\[
\theta_d = \sigma_\delta - \frac{x}{1 + \lambda}\Sigma^{-1}(\iota_3 + \iota_4). \\
\]

\[
\theta_f = \theta = \sigma_\delta + \frac{\lambda}{1 + \lambda}x\Sigma^{-1}(\iota_3 + \iota_4). \\
\]

**B.3 Proof of Proposition 4.2**

Stocks prices are computed using perturbation methods. We follow three steps. Step 1: we compute the equity prices $S_f$ and $S_f$. Step 2: we compute the prices $S_1$ and $S_2$ using results from step 1. And lastly in Step 3: we compute the volatilities of the stocks and the volatility of the weight using results from Step 1 and Step 2.

**B.3.1 Step 1**

Discounting future dividends using the pricing kernel, the foreign country stock index is

$$S_f(t) = \delta(t)E_t\left[ \int_t^T e^{-\beta(s-t)} \frac{1 + \lambda(s)}{1 + \lambda(t)} ds \right] + \delta(t)E_t\left[ \int_t^T e^{-\beta(s-t)} \frac{\delta_3(s)}{1 + \lambda(t)} \delta(s) ds \right].$$

Since $\lambda$ is a martingale, it is easy to see that

$$S_f(t) = \frac{1 - e^{-\beta(T-t)}}{\beta} \delta(t) + \delta(t)E_t\left[ \int_t^T e^{-\beta(s-t)} \frac{\delta_3(s)}{\delta(s)} \delta(s) ds \right] + \delta(t)e^{\beta t} \frac{H(\lambda(t), \delta_3(t)/\delta(t), t)}{1 + \lambda(t)},$$

where $H(\lambda(t), \delta_3(t)/\delta(t), t) = E_t\left[ \int_t^T e^{-\beta s} \lambda(s) \frac{\delta_3(s)}{\delta(s)} ds \right]$ and $H(\lambda(T), \delta_3(T)/\delta(T), T) = 0$. In absence of constraints $S_f(t) = S_f^0(t).$
The total market capitalization of the world is \( \bar{S}(t) = S_d(t) + S_f(t) = 2(1-e^{-\beta(T-t)})\delta(t)/\beta \), which implies

\[
S_d(t) = 2(1-e^{-\beta(T-t)})\delta(t)/\beta - S_f(t).
\]

Assuming \( d(\delta_3/\delta) = (\delta_3/\delta)[\mu_{\delta_3/\delta}dt + \sigma_{\delta_3/\delta}dW(t)] \), with \( \mu_{\delta_3/\delta} \) non stochastic, then

\[
S_f(t) = \frac{1-e^{-\beta(T-t)}}{\beta} \delta(t) + \int_t^T e^{\int_t^s (\beta + \mu_{\delta_3/\delta}(\tau))d\tau} ds \frac{\delta_3(t)}{1 + \lambda(t)} + \frac{\delta(t)e^{\beta t}}{1 + \lambda(t)} H(\lambda(t), \delta_3(t)/\delta(t), t).
\]

Applying Ito’s lemma to the expressions of \( S_d(t) \) and \( S_f(t) \) give

\[
\sigma_f(t) = \frac{1-e^{-\beta(T-t)}}{\beta} \delta(t) S_f(t) + \delta(t) \int_t^T e^{\int_t^s (\beta + \mu_{\delta_3/\delta}(\tau))d\tau} ds \frac{\delta_3(t)}{S_f(t)(1 + \lambda(t))} \sigma_\delta(t) + \left( \frac{e^{\beta t} \delta(t) H_\lambda}{S_f(t)(1 + \lambda(t))} - \frac{e^{\beta t} \delta(t) H}{S_f(t)(1 + \lambda(t))} - \frac{\delta_3(t) \int_t^T e^{\int_t^s (\beta + \mu_{\delta_3/\delta}(\tau))d\tau} ds}{S_f(t)(1 + \lambda(t))^2} \right) \lambda(t) \sigma_\lambda(t) + \frac{e^{\beta t} H \delta(t)}{1 + \lambda(t) S_f(t)} \sigma_\delta(t) + \frac{e^{\beta t} \delta_3(t) H_\delta/\delta}{S_f(t)(1 + \lambda(t))} \sigma_{\delta_3/\delta}(t), \tag{11}
\]

\[
\sigma_d(t) = \frac{[2(1-e^{-\beta(T-t)})\delta(t)/\beta] \sigma_\delta(t) - S_f(t) \sigma_f(t)}{2(1-e^{-\beta(T-t)})\delta(t)/\beta} / S_f(t). \tag{12}
\]

We write \( H \) as a series of \( \epsilon \). When \( \epsilon = 0 \) the problem reduces to the benchmark case. We choose \( \epsilon = [S^0_f/S - \bar{\pi}]^+ \). The Taylor expansion of \( H \) with respect to \( \epsilon \) is

\[
H = H^0 + \sum_{n=1}^{N} H^n \epsilon^n + O(\epsilon^{N+1})
\]

where \( H^n = \partial^n H/\partial \epsilon^n (\epsilon = 0) \) for \( n = 1, ..., N \) and \( H^0 = H(\epsilon = 0) \).

Under appropriate regularity conditions, it can be shown that

\[
\xi(t)S_f(t) + \int_0^t \xi(s)(\delta_3(s) + p(s)\delta^*(s))ds
\]

is a martingale under \( \mathcal{P} \) (see Cuoco & He (1994), and Basak & Gallmeyer (2002)). Therefore, the drift must be zero. Hence, \( H \) solves the quasi-linear partial differential equation

\[
(L + \frac{\partial}{\partial \delta})(H(\lambda(t), \delta_3(t)/\delta(t), t) + e^{-\beta t} \lambda(t) \delta_3(t)/\delta(t)) = 0 \text{ with boundary condition } H(\lambda(T), \delta_3(T)/\delta(T), T) = 0.
\]

We solve for

\[
H^0(\lambda(t), \delta_3(t)/\delta(t), t) = \lambda(t) E_t \left[ \int_t^T e^{-\beta s} \frac{\delta_3(s)}{\delta(s)} ds \right] = \lambda(t) \frac{\delta_3(t)}{\delta(t)} e^{-\beta t} \int_t^T e^{\int_t^s (\beta + \mu_{\delta_3/\delta}(\tau))d\tau} ds,
\]

and for \( H^n, n \geq 1, \) by solving the following partial differential equation

\[
\frac{1}{2} \|\sigma_\lambda\|^2 \lambda^2 \sum \frac{\partial^2 H^n}{\partial \lambda^2} \epsilon^n + \frac{1}{2} \|\sigma_{\delta_3/\delta}\|^2 \left( \frac{\delta_3}{\delta} \right)^2 \sum \frac{\partial^2 H^n}{\partial (\delta_3/\delta)^2} \epsilon^n + \sigma_\lambda^T \sigma_{\delta_3/\delta} \lambda \frac{\delta_3}{\delta} \sum \frac{\partial^2 H^n}{\partial (\delta_3/\delta)} \partial \lambda \epsilon^n + \mu_{\delta_3/\delta} \frac{\delta_3}{\delta} \sum \frac{\partial H^n}{\partial (\delta_3/\delta)} \epsilon^n = 0,
\]
with \( H^n(\lambda(T), \delta_3(T)/\delta(T), T) = 0 \). Lets denote by \( g = \delta_3/\delta \). Putting together the \( \epsilon \) terms, the partial differential equation for \( H^1 \) is:

\[
\frac{1}{2} \| \sigma_g \|^2 g^2 H^1_{gg} + \mu_g g H^1_g = \sigma_g \Sigma^{o-1} t_3 \frac{\lambda(1 + \lambda)}{\| \Sigma^{o-1}(t_3 + t_4) \|^2} g^0 \tag{13}
\]

For ease of exposition and also in order to provide intuitive economic interpretations for the derived formulas to follow, we assume \( \mu_g = \mu_{\delta_3/\delta} = 0 \). We solve the PDE (13) to obtain \( H^1 \). Also recall, \( \sigma^o \Sigma^{o-1}(t_3 + t_4) = S_f' / S \) and \( \sigma^o \Sigma^{o-1}(t_3 + t_4) = 1 \).

From \( \sigma^g_\delta = \frac{S}{2S_f} \sigma + \frac{1-e^{-\beta(t-\bar{t})}}{\beta S_f} \delta_3 \delta_3 \), we compute \( \sigma^o_\delta \Sigma^{o-1}(t_3 + t_4) = \frac{S_f'}{2(1-e^{-\beta(t-\bar{t})} \delta_3 / \beta} \).

So, \( \sigma^o_\delta \Sigma^{o-1}(t_3 + t_4) = \frac{S_f'}{2(1-e^{-\beta(t-\bar{t})} \delta_3 / \beta} - \frac{S_f}{S} \). Equitiy prices are

\[
S_f = S_f' - \frac{\lambda S_f'}{\| \sigma_g \|^2 \Sigma^{o-1}(t_3 + t_4) ^2} (1 - g)(1 - \ln(g)) \epsilon,
\]

\[
S_d = S_d' + \frac{\lambda S_d'}{\| \sigma_g \|^2 \Sigma^{o-1}(t_3 + t_4) ^2} (1 - g)(1 - \ln(g)) \epsilon.
\]

Next we compute the individual stock prices in the domestic market.

**B.3.2 Step 2**

The prices of domestic stocks are

\[
S_2(t) = \delta(t) E_t \left[ \int_t^T e^{-\beta(s-t)} \frac{1 + \lambda(s) \delta_2(s)}{1 + \lambda(t) \delta(s)} ds \right] = \frac{S_2'}(t) + \frac{\delta(t) e^{\beta t}}{1 + \lambda(t)} h(\lambda(t), \delta_2(t)/\delta(t), t),
\]

\[
S_1(t) = S_2(t) - S_2(t),
\]

where \( h(\lambda(t), \delta_2(t)/\delta(t), t) = E_t \left[ \int_t^T e^{-\beta s} \lambda(s) \frac{\delta_2(s)}{\delta(s)} ds \right] \), and \( h(\lambda(T), \delta_2(T)/\delta(T), T) = 0 \).

In absence of constraints \( S_j(t) = S^o_j(t) \).

Applying Ito's lemma to the expressions of \( S_j \) gives

\[
\sigma_2(t) = \sigma_2'(t) + e^{\beta t} \delta(t) h \sigma_2(t) + e^{\beta t} \delta(t) \frac{\partial}{\partial \delta} \frac{\delta_2(t)}{\delta(t)} \sigma_2(t) \delta_2(t) \]

\[
+ \left( e^{\beta t} \delta(t) \frac{\partial h}{\partial \lambda} - S_2(t) \right) \lambda(t) \sigma_2(t) \right] / (1 + \lambda(t)) S_2(t),
\]

\[
\sigma_1(t) = \sigma_1(t) + \frac{S_2(t)}{S_1(t) - S_2(t)} (\sigma_1(t) - \sigma_2(t)).
\]

We write \( h \) as the Taylor series expansion of \( \epsilon \):

\[
h = h^0 + \sum_{n=1}^N h^n \epsilon^n + O(\epsilon^{N+1})
\]

where \( h^n = \partial^n h / \partial \epsilon^n (\epsilon = 0) \) for \( n = 1, ..., N \) and \( h^0 = h(\epsilon = 0) \).

Again under appropriate regularity conditions, it can be shown that

\[
\xi(t) S_2(t) + \int_0^t \xi(s) \delta_2(s) ds
\]
is a martingale under $\mathcal{P}$ (see Cuoco & He (1994), and Basak & Gallmeyer (2002)). Therefore, the drift must be zero. Hence, $h$ solves the quasi-linear partial differential equation

\[ (\mathcal{L} + \frac{\partial}{\partial \beta}) h(\lambda, \delta_2/\delta, t) + e^{-\beta t} \lambda(t) \frac{\partial h}{\partial \delta(t)} = 0 \]

with boundary condition $h(\lambda(T), \delta_2(T)/\delta(T), T) = 0$. We solve for

\[ h^0(\lambda(t), \delta_2(t)/\delta(t), t) = e^{-\beta t} \lambda(t) \frac{S_2^0(t)}{\delta(t)}, \]

and for $h^n(\lambda, \delta_2/\delta, t)$ by solving the partial differential equation

\[
\frac{1}{2} \|\sigma\|^{2} \lambda^{2} \sum \frac{\partial^{2} h^{n}}{\partial \lambda^{2}} \epsilon^{n} + \frac{1}{2} \|\sigma_{\delta/\delta}\|^{2} \left( \frac{\delta^{2}}{\delta}\right)^{2} \sum \frac{\partial^{2} h^{n}}{\partial(\delta^{2}/\delta)^{2}} \epsilon^{n} + \\
\sigma_{\lambda} \sigma_{\delta/\delta} \lambda \frac{\delta_{2}}{\delta} \sum \frac{\partial^{2} h^{n}}{\partial \lambda \partial(\delta^{2}/\delta)} \epsilon^{n} + \mu_{\delta_{2}/\delta} \frac{\delta_{2}}{\delta} \sum \frac{\partial h^{n}}{\partial(\delta^{2}/\delta)} \epsilon^{n} = 0,
\]

with $h^n(\lambda(T), \delta_2(T)/\delta(T), T) = 0$.

For ease of exposition and in order to have interpretable approximate expressions, we assume $\mu_{\delta_{2}/\delta} = 0$. We solve the above PDE to obtain $h^1$. We can also show that:

\[(\sigma_{\delta_{2}} - \sigma_{\delta})^{\top} \Sigma^{-1} f_3 = 1 - S_{2}^0 / S. \]

Hence, domestic stock prices are

\[
S_2 = S_2^0 \left[ 1 + \frac{S_f^0}{S} \|\sigma_{\delta_{2}/\delta}\|^2 \Sigma^{-1}(f_3 + f_4)^2 (1 - \ln(\frac{\delta_{2}}{\delta})) \epsilon, \right]
\]

\[
S_1 = S_1^0 + \frac{\lambda S_f^0}{\Sigma^{-1}(f_3 + f_4)^2} \left\{ (1 - \frac{\lambda}{1 + \lambda})(1 - \ln(\frac{\delta_{2}}{\delta})) - \frac{S_2^0 2(1 - \ln(\frac{\delta_{2}}{\delta}))}{S} \right\} \epsilon.
\]

B.3.3 Step 3

The last step consists of using the stock prices to obtain the volatilities. We use 11, 12, 14 and 15 and solve the system of simultaneous equations to get the volatilities of the stocks.

B.4 Proof of Proposition 4.3

The proof is similar to the proof of Proposition 4.7. Therefore we can use the expression of $U_i$ in that proof. The final expressions are obtained by replacing $\Delta r_i(\lambda)$ and $\Delta \theta_i(\lambda)$ by their expressions as follows.

For foreign agent

\[
\Delta r_f(\lambda) = -x \frac{\lambda}{1 + \lambda} \sigma_{\delta}^{\top} \Sigma^{-1}(f_3 + f_4),
\]

\[
\Delta \theta_f(\lambda) = x \frac{\lambda}{1 + \lambda} \Sigma^{-1}(f_3 + f_4),
\]

Using these expressions,

\[
\Delta r_f + \sigma_{\delta}^{\top} \Delta \theta_f + \|\Delta \theta_f\|^2/2 = -x \frac{\lambda}{1 + \lambda} \sigma_{\delta}^{\top} \Sigma^{-1}(f_3 + f_4) + x \frac{\lambda}{1 + \lambda} \sigma_{\delta}^{\top} \Sigma^{-1}(f_3 + f_4)
\]

\[
+ x^2 \|\frac{\lambda}{1 + \lambda} \Sigma^{-1}(f_3 + f_4)\|^2/2
\]

\[
= x^2 \|\frac{\lambda}{1 + \lambda} \Sigma^{-1}(f_3 + f_4)\|^2/2.
\]
For domestic agent

\[
\Delta r_d(\lambda) = -x \frac{\lambda}{1 + \lambda} \sigma_\delta^T \Sigma^{-1}(t_3 + t_4) + \tilde{\pi} x,
\]

\[
\Delta \theta_d(\lambda) = x \frac{\lambda}{1 + \lambda} \Sigma^{-1}(t_3 + t_4) - x \Sigma^{-1}(t_3 + t_4) = -x \frac{1}{1 + \lambda} \Sigma^{-1}(t_3 + t_4).
\]

Using these expressions,

\[
\Delta r_d + \sigma_\delta^T \Delta \theta_d + \|\Delta \theta_d\|^2/2 = -x \frac{\lambda}{1 + \lambda} \sigma_\delta^T \Sigma^{-1}(t_3 + t_4) + \tilde{\pi} x
\]

\[
- x \frac{1}{1 + \lambda} \sigma_\delta^T \Sigma^{-1}(t_3 + t_4) + x^2 \left\{ -x \frac{1}{1 + \lambda} \Sigma^{-1}(t_3 + t_4) \right\}^2 /2,
\]

\[
x^2 \left\{ -x \frac{1}{1 + \lambda} \Sigma^{-1}(t_3 + t_4) \right\}^2 /2 - x \sigma_\delta^T \Sigma^{-1}(t_3 + t_4) + \tilde{\pi} x.
\]

Multiplying the FOC for \(x\) by \(x\) yields

\[
x \sigma_\delta^T \Sigma^{-1}(t_3 + t_4) - \tilde{\pi} x = x^2 \frac{1}{1 + \lambda} \|\Sigma^{-1}(t_3 + t_4)\|^2.
\]

Thus,

\[
\Delta r_d + \sigma_\delta^T \Delta \theta_d + \|\Delta \theta_d\|^2/2 = -x^2 \left\{ -x \frac{1}{1 + \lambda} \|\Sigma^{-1}(t_3 + t_4)\|^2 + x^2 \frac{1}{2(1 + \lambda)^2} \|\Sigma^{-1}(t_3 + t_4)\|^2 \right\}
\]

\[
= -x^2 \frac{1 + 2 \lambda}{2(1 + \lambda)^2} \|\Sigma^{-1}(t_3 + t_4)\|^2.
\]

### B.5 Delay in Removal of Foreign Asset Holding Restriction

**Lemma B.1.** The set of parameters for which \(U_d(0) - U_d^o(0) > 0\) is non-empty.

*Proof.* We prove the lemma by example. That is we show that when \(\tilde{\pi}\) is close to the unconstrained level \(S_f/\bar{S}\), \(U_d(0) - U_d^o(0) > 0\). The proof is based on the approximation results. At time zero, \(X_d(0) = S_d(0)\) and \(X_d^o(0) = S_d^o(0)\). By Proposition 4.2,

\[
\frac{X_d(0)}{X_d^o(0)} = 1 + \frac{S_f(0)}{S_d^o(0)} \frac{\lambda(0)}{\sigma_{\delta_3/\delta}(0)} \left\{ 1 - \ln \left( \frac{\delta_3(0)}{\delta(0)} \right) \right\} \epsilon + \mathcal{O}(\epsilon^2).
\]

It follows that \(\ln \frac{X_d(0)}{X_d^o(0)}\) is approximately the same order of \(\epsilon = \left[ S_f/\bar{S} - \tilde{\pi} \right]^+\), whereas

\[
E \left[ \int_0^T e^{-\beta t} \int_0^t \frac{\left\{ S_f/\bar{S} - \tilde{\pi} \right\}^2}{\|\Sigma^{-1}(t_3 + t_4)\|^2} dsdt \right]
\]

is approximately in the order of \(\epsilon^2\). Thus for small \(\epsilon\), \(U_d(0) - U_d^o(0) > 0\). \(\square\)

**Lemma B.2.** \(U_d^o(0) + U_f^o(0) \neq U_d(0) + U_f(0)\).
Proof. The general expression of utility is

\[
U_i = -\frac{1 - e^{-\beta T}}{\beta} \left[ 2 \ln(2(1 - e^{-\beta T})/\beta X_i(0)) + \ln(\delta(0)/\delta^*(0)) \right] + E \left[ \int_0^T e^{-\beta t} \int_0^t \left( \mu(t) + \mu^*(t) - (\sigma(t)^2 + \sigma^*(t)^2) \right) dt \right] + 2E \left[ \int_0^T e^{-\beta t} \int_0^t \left( \Delta r_i(\lambda(t)) + \sigma_i(t) \Delta \theta_i(\lambda(t)) + \Delta^2 \theta_i(\lambda(t)) \right) dt \right]
\]

By the expressions in Proposition 4.3,

\[
\Delta U_d + \Delta U_f = \frac{2(1 - e^{-\beta t'})}{\beta} \ln \frac{X_d(0)}{X_d^*(0)} - E \left[ \int_0^{t'} e^{-\beta t} \int_0^t (1 + 2\lambda) \left\{ \left[ \frac{S_f/\bar{S} - \bar{\pi}}{\xi^2} \right] ^2 \right\} \frac{dsdt}{(\Sigma^{-1}(\tau_3 + \tau_4))} \right] + \frac{2(1 - e^{-\beta t'})}{\beta} \ln \frac{X_f(0)}{X_f^*(0)} + E \left[ \int_0^{t'} e^{-\beta t} \int_0^t \lambda^2 \left\{ \left[ \frac{S_f/\bar{S} - \bar{\pi}}{\xi^2} \right] ^2 \right\} \frac{dsdt}{(\Sigma^{-1}(\tau_3 + \tau_4))} \right]
\]

\[
= E \left[ \int_0^{t'} e^{-\beta t} \int_0^t (\lambda^2 - 1 - 2\lambda) \left\{ \left[ \frac{S_f/\bar{S} - \bar{\pi}}{\xi^2} \right] ^2 \right\} \frac{dsdt}{(\Sigma^{-1}(\tau_3 + \tau_4))} \right]
\]

B.6 Proof of Proposition 4.4

At any time \( s > t \), the optimal consumption of the domestic investor is

\[
c_d(s) = \frac{e^{-bs}}{y_d(t)\xi_d(s)/\xi_d(t)}, \quad c_d(s) = \frac{e^{-bs}}{y_d(t)p(s)\xi_d(s)/\xi_d(t)},
\]

where \( y_d(t) \) is the Lagrangian multiplier of the budget constraint at time \( t \). The utility of the investor at the optimal consumption is

\[
U_d(t) = E_t \left[ \int_t^T e^{-bs} \left[ \ln c(s) + \ln c^*(s) \right] ds \right]
\]

\[
= E_t \left[ \int_t^T e^{-bs} \left[ -2bs - 2 \ln y_d(t) - 2 \ln \frac{\xi_d(s)}{\xi_d(t)} - \ln p(s) \right] ds \right]
\]

\[
= E_t \left[ \int_t^T e^{-bs} \left[ -2 \ln \frac{2(e^{-bt} - e^{-bT})}{\beta X_d(t)} - \ln p(t) \right] \right]
\]

\[
+ E_t \left[ \int_t^T e^{-bs} \int_t^s \left[ \mu_d + \mu_d^* - \frac{1}{2} \left( \sigma_d^2 + \sigma_d^*^2 \right) \right] dvds \right]
\]

\[
+ E_t \left[ \int_t^T e^{-bs} \int_t^s \left[ 2\Delta r_d + 2\sigma_d \Delta \theta_d + \Delta^2 \theta_d \right] dvds \right]
\]
\[
\frac{e^{-\beta t} - e^{-\beta T}}{\beta} \left[ 2 \ln \frac{2(e^{-\beta t} - e^{-\beta T})}{\beta X_d(t)} + \ln p(t) \right] + E_t \left[ \int_t^T e^{-\beta s} \int_t^s \left[ \mu_\delta + \mu_\delta^* - \frac{1}{2} \left( \|\sigma_\delta\|^2 + \|\sigma_\delta^*\|^2 \right) \right] ds \right] \]

\[
- E_t \left[ \int_t^T e^{-\beta s} \int_t^s \frac{1 + 2\lambda}{\|\Sigma^{-1}(t_3 + t_4)\|^2} \left( \left[ S_f / \bar{S} - \bar{\pi} \right]^+ \right)^2 ds \right],
\]

where \( X_d(t) \) is the wealth of the investor at time \( t \) and \( \tau \) is the time when the foreign asset holding restriction is removed. Let \((a_b(t), a_1(t), a_2(t), a_f(t), a_{b'}(t))\) be the shares held of the stocks and the two bonds. Then

\[
X_d(t) = a_b(t)B(t) + a_1(t)S_1(t) + a_2(t)S_2(t) + a_f(t)S_f(t) + a_{b'}(t)B_f(t).
\]

Let

\[
X_d^o(t) = a_b(t)B(t) + a_1(t)S_1^o(t) + a_2(t)S_2^o(t) + a_f(t)S_f^o(t) + a_{b'}(t)B_f(t),
\]

which is the wealth of the investor in the benchmark economy. With an abuse of notation, we will write the two sums as \( \sum_j a_j(t)S_j(t) \) and \( \sum_j a_j(t)S_j^o(t) \), respectively. Given \( X_d^o(t) \), the utility of the investor in the benchmark economy is

\[
U_d^o(t) = \frac{e^{-\beta t} - e^{-\beta T}}{\beta} \left[ 2 \ln \frac{2(e^{-\beta t} - e^{-\beta T})}{\beta X_d^o(t)} + \ln p(t) \right] + E_t \left[ \int_t^T e^{-\beta s} \int_t^s \left[ \mu_\delta + \mu_\delta^* - \frac{1}{2} \left( \|\sigma_\delta\|^2 + \|\sigma_\delta^*\|^2 \right) \right] ds \right].
\]

Subtracting yields,

\[
U_d(t) - U_d^o(t) = \frac{e^{-\beta t} - e^{-\beta T}}{\beta} \ln \frac{X_d(t)}{X_d^o(t)} - E_t \left[ \int_t^T e^{-\beta s} \int_t^s \frac{1 + 2\lambda}{\|\Sigma^{-1}(t_3 + t_4)\|^2} \left( \left[ S_f / \bar{S} - \bar{\pi} \right]^+ \right)^2 ds \right],
\]

Now suppose that \( \tau^* \) is the optimal time to remove the foreign asset holding restriction. Suppose that instead of removing the restriction at time \( \tau^* \), the benevolent government decides to remove it at time \( \tau^* + \Delta \) with \( \Delta > 0 \). To simplify notation, we will assume without loss of generality \( \tau^* = t \) and set \( \tau = \tau^* + \Delta = t + \Delta \). Since \( \tau^* \) is optimal, \( U_d(t) - U_d^o(t) \) assumes its maximum at \( \Delta = 0 \). In the following we will evaluate the derivative of \( U_d(t) - U_d^o(t) \) with respect to \( \Delta \) at \( \Delta = 0 \) and set it equal to zero, which will give us the result stated in the proposition.

We will first derive an expression which is more convenient for evaluating the derivative. Let \( \xi(t) \) denote the pricing kernel in the benchmark economy.

\[
X_d(t) = E_t \left[ \int_t^{t+\Delta} \frac{\xi_d(s)}{\xi_d(t)} \sum_j a_j(t)\delta_j(s) ds + \int_t^T \frac{\xi_d(t + \Delta)}{\xi_d(t)} \frac{\xi_o(s)}{\xi_o(t + \Delta)} \sum_j a_j(t)\delta_j(s) ds \right]
\]

\[
= X_d^o(t) + E_t \left[ \int_t^{t+\Delta} \left( \frac{\xi_d(s)}{\xi_d(t)} - \frac{\xi_o(s)}{\xi_o(t)} \right) \sum_j a_j(t)\delta_j(s) ds \right] + E_t \left[ \left( \frac{\xi_d(t + \Delta)}{\xi_d(t)} - \frac{\xi_o(t + \Delta)}{\xi_o(t)} \right) \sum_j a_j(t)S_j(t + \Delta) \right].
\]
Since
\[ 
\xi_d(s) = \exp \left\{ - \int_0^s \left( \beta + \mu_d - \|\sigma_d\| + \frac{1}{2}\|\sigma_d\|^2 \right) dv - \int_0^s \sigma_d dW(v) \right\} 
\times \exp \left\{ - \int_0^s \left( \Delta r_d + \sigma_d \Delta \theta_d + \frac{1}{2}\|\Delta \theta_d\|^2 \right) dv - \int_0^s \Delta \theta_d dW(v) \right\} = e^{-\beta s} \delta(0) \eta(s), 
\]
where
\[ 
\eta(s) = \exp \left\{ - \int_0^s \left( \Delta r_d + \sigma_d \Delta \theta_d + \frac{1}{2}\|\Delta \theta_d\|^2 \right) dv - \int_0^s \Delta \theta_d dW(v) \right\}, 
\]
and satisfies
\[ 
d\eta(s) = -\eta(s)(\Delta r_d + \sigma_d \Delta \theta_d)ds - \eta(s)\Delta \theta_d dW(s). 
\]
Since \( \xi^o(s) = e^{-\beta s} \delta(0) \),
\[ 
\frac{\xi_d(s)}{\xi_d(t)} - \frac{\xi^o(s)}{\xi^o(t)} = e^{-\beta(s-t)} \frac{\delta(t)}{\delta(s)} (\eta(s) - \eta(t)) \frac{\eta(t)}{\eta(t)}. 
\]
Applying Ito formula to
\[ 
\left( \frac{\xi_d(s)}{\xi_d(t)} - \frac{\xi^o(s)}{\xi^o(t)} \right) \sum_j a_j(t) S_j^o(s) \]
yields
\[ 
\lim_{\Delta \to 0} \frac{1}{\Delta} \mathbb{E}_t \left[ \left( \frac{\xi_d(t + \Delta)}{\xi_d(t)} - \frac{\xi^o(t + \Delta)}{\xi^o(t)} \right) \sum_j a_j(t) S_j^o(t + \Delta) \right] 
\]
\[ 
= \left[ - (\Delta r_d + \sigma_d \Delta \theta_d) + \sigma_d \Delta \theta_d \right] \sum_j a_j(t) S_j^o(t) - \Delta \theta_d \sum_j a_j(t) S_j^o(t) \sigma_j^o(t). 
\]
From the expression for \( X_d(t) \) derived above, it follows that, at \( \Delta = 0 \),
\[ 
\frac{d}{d\Delta} \ln \frac{X_d(t)}{X_d(t)} = \frac{1}{X_d(t)} \left[ -\Delta r_d(t) \sum_j a_j(t) S_j^o(t) - \Delta \theta_d(t) \sum_j a_j(t) S_j^o(t) \sigma_j^o(t) \right] 
\]
\[ 
= \frac{1}{X_d(t)} \left[ \left( \frac{\lambda(t)}{1 + \lambda(t)} \frac{S_j(t)}{S(t)} - \bar{\pi} \right) (1 + \lambda(t)) \sum_j a_j(t) S_j^o(t) \right. 
\]
\[ 
+ (\iota_3 + \iota_4) \Sigma^{-1} \sum_j a_j(t) S_j^o(t) \sigma_j^o(t) \left] \frac{[S_f/\bar{S} - \bar{\pi}]^+}{\Sigma^{-1}(\iota_3 + \iota_4)\Sigma^{-1}[S_f/\bar{S} - \bar{\pi}]^+}. \right. \]
Therefore, \( d(U_d(t) - U_d^o(t))/d\Delta|_{\Delta=0} = 0 \) is equivalent to either \( [S_f/\bar{S} - \bar{\pi}]^+ = 0 \) or
\[ 
0 = \left( \frac{\lambda(t)}{1 + \lambda(t)} \frac{S_j(t)}{S(t)} - \bar{\pi} \right) (1 + \lambda(t)) \sum_j a_j(t) S_j^o(t) + (\iota_3 + \iota_4) \Sigma^{-1} \sum_j a_j(t) S_j^o(t) \sigma_j^o(t). 
\]
Since stock prices are continuous, their quadratic variations are also continuous, which implies that at time \( t = \tau^* \), \( \Sigma = \Sigma^o \). Thus
\[
\left( \tilde{\pi} - \frac{\lambda(t)}{1 + \lambda(t)} \frac{S_f(t)}{S(t)} \right) (1 + \lambda(t)) X^o_d(t) = a_f(t) S^o_f(t) + a_{b^*}(t) B^o_f(t).
\]
Or
\[
\tilde{\pi} = \frac{\lambda(t)}{1 + \lambda(t)} \frac{S^o_f(t)}{S^o(t)} + \frac{a_f(t) S^o_f(t) + a_{b^*}(t) B^o_f(t)}{(1 + \lambda(t)) X^o_d(t)}.
\]
Note that we have assumed that \( \Delta > 0 \). Thus \( [a_f(t) S^o_f(t) + a_{b^*}(t) B^o_f(t)]/X^o_d(t) = \tilde{\pi} \).
Since asset prices are continuous, as \( \Delta \downarrow 0 \), asset prices converges to their counterparts in the benchmark economy. So \( [a_f(t) S^o_f(t) + a_{b^*}(t) B^o_f(t)]/X^o_d(t) = \tilde{\pi} \) and at optimal time \( t = \tau^* \),
\[
\tilde{\pi} = S^o_f(t)/S^o(t).
\]
This completes the proof. \( \square \)

C Economy with Restriction on Foreign Investors

C.1 Proof of Proposition 4.5

C.1.1 Constrained investor optimization problem

Agent \( f \) is limited to hold no more than \( \tilde{\pi} \) percent of stock 2. To characterize his optimal policies, we use the framework developed by Cvitanic and Karatzas (1992). Define the constraint space \( K \) of investor \( f \):
\[
K = \{ \pi \text{ such that } \pi_2 \leq \bar{a} \},
\]
where \( \pi_2 \) is the percentage of wealth of investor \( f \) invested in stock 2 and \( \bar{a} \) is the maximum percentage on that stock. There is an obvious relationship between \( \bar{\pi} \) and \( \bar{a} \).

Let \( x = (x_1, x_2, x_3, x_4)^\top \) and define the support function of \( K \) as
\[
\psi(x) = \sup_{\pi \in K} (\pi^\top x) = \begin{cases} \bar{a} x_j & \text{if } x_k = 0 \text{ and } x_j \geq 0, \ k \neq j, \\ \infty & \text{otherwise.} \end{cases}
\]

For each \( x \), introduce a new fictitious financial market as follows:
\[
\begin{align*}
\text{d} B^x(t) &= (r(t) + \psi(x(t))) B^x(t) \text{d} t, \\
\text{d} S^x(t) &= I^x(t) (\mu(t) - x(t) + \psi(x(t))) \text{d} t + I^x(t) \Sigma(t) \text{d} W(t)
\end{align*}
\]
where \( S^x = (S_1^x, S_2^x, S_f^x, B_f^x)^\top \) is the price vector of the stocks and the foreign bond, and \( I^x = \text{diag}(S^x) \). Define the new (fictitious) pricing kernel facing investor \( f \) by
\[
\text{d} \xi_f(t) = -\xi_f(t) \left[ r_f(t) dt + \theta_f(t)^\top \text{d} W(t) \right], \quad \text{with } \xi_f(0) = 1,
\]
where \( r_f = r + \psi(x) \) and \( \theta_f = \theta - \Sigma^{-1} x \) are the interest rate and market price for risk facing the constrained investor \( f \), and \( \theta = \Sigma^{-1}(\mu - r 1) \).
Under investment constraint, the constrained optimization problem of investor \( f \) can be solved as follows: for each fixed fictitious market, solve the following variational static problem:

\[
\max_{c_f(t), c_f^*, t < \tau} \mathbb{E} \left[ \int_0^\tau e^{-\beta t} \left[ \ln c_f(t) + \ln c_f^*(t) \right] dt + U_f(X(\tau)) \right],
\]

under the budget constraint:

\[
\mathbb{E} \left[ \int_0^\tau \xi_f(t) \left[ c_f(t) + p(t)c_f^*(t) \right] dt + \xi(\tau)X_f(\tau) \right] = X_f(0),
\]

where \( U_f(X_f(\tau)) \) is the utility of investor \( f \) at time \( \tau \) given wealth \( X_f(\tau) \) and \( X_f(\tau) \) is the remaining wealth of investor \( f \) at time \( \tau \). Note that the investment restriction faced by investor \( f \) is removed at time \( \tau \).

From the dual optimization (see Cvitanic and Karatzas (1992) for the technical details), \( x \) is obtained as follows

\[
x = \arg \min_x [2\psi(x) + \|\theta - \Sigma^{-1}x\|^2].
\]

where the minimization is over \( x = x_{2\ell_2} \), where \( \ell_2 = (0, 1, 0, 0)^T \), \( x_2 \geq 0 \). The optimization is equivalent to

\[
x_2 = \arg \min_{x_2 \geq 0} [2\bar{\pi}x_2 + \|\theta - x_2\Sigma^{-1}\ell_2\|^2].
\]

The FOC of this optimization problem is:

\[
\bar{\pi} + x_2\|\Sigma^{-1}\ell_2\|^2 - \theta^T\Sigma^{-1}\ell_2 = 0,
\]

which implies

\[
x_2 = \frac{1}{\|\Sigma^{-1}\ell_2\|^2} \left[ \theta^T\Sigma^{-1}\ell_2 - \bar{\alpha} \right]^+.
\]

C.1.2 Exchange rate

The exchange rate is defined as the marginal rate of substitution of good \( \delta^* \) for good \( \delta \) and its process is given by:

\[
p(t) = \frac{\partial U(\delta(t), \delta^*(t), \lambda(t))/\partial\delta^*}{\partial U(\delta(t), \delta^*(t), \lambda(t))/\partial\delta} = \frac{\delta(t)}{\delta^*(t)}.
\]

Thus the results of appendix B apply.

C.1.3 The weight process

It is readily shown that the weight \( \lambda \) satisfies

\[
\lambda(t) = y_d\xi_d(t)/y_f\xi_f(t).
\]
where $y_i$ is the lagrange multiplier of investor $i$’s maximization problem, and $\xi_i$ is the minimax pricing kernel of the fictitious market facing investor $i$. Applying Ito’s formula to the expression of $\lambda$ gives

$$d\lambda(t) = \lambda(t) \left[ (r_f(t) - r_d(t) + \theta_f^T(t)(\theta_f(t) - \theta_d(t)))dt + (\theta_f(t) - \theta_d(t))^\top dW(t) \right].$$

Thus

$$\mu_\lambda(t) = r_f(t) - r_d(t) + \theta_f^T(t)\sigma_\lambda(t),$$

and

$$\sigma_\lambda(t) = \theta_f(t) - \theta_d(t).$$

Since foreign investor is constrained in his portfolio holding, from the constrained investor optimization in Appendix (C.1.1), $r_f = r + \bar{a}x_2$ and $\theta_f = \theta - x_2\Sigma^{-1}v_2$, where $\bar{a}$ is the percentage limit of wealth in holding stock 2. Since domestic investor has no investment restrictions, $\xi_d = \xi$, so $r_d = r$ and $\theta_d = \theta$. We can rewrite:

$$\mu_\lambda(t) = \bar{a}(t)x_2(t) + \theta_f^T(t)\sigma_\lambda(t),$$

$$\sigma_\lambda(t) = \theta_f(t) - \theta(t) = -x_2(t)\Sigma^{-1}(t)v_2 \quad \text{or} \quad \theta_f(t) = \theta(t) + \sigma_\lambda(t). \quad (18)$$

The explicit expression for $x_2(t)$ will be provided below.

### C.1.4 Pricing kernel and riskless rate

Following the same argument as Cuoco and He (1994), the pricing kernel is obtained as the marginal utility of the representative agent:

$$\xi(t) = e^{-\beta t} \frac{\partial U(\delta(t), \delta^*(t), \lambda(t))}{\partial \delta} = e^{-\beta t} \frac{1 + \lambda(t) \delta(0)}{1 + \lambda(0)} \delta(t).$$

Applying Ito’s lemma yields

$$d\xi(t) = -\xi(t) \left[ \left( \beta + \mu_\delta - \|\sigma_\delta\|^2 - \frac{\lambda}{1 + \lambda} \mu_\lambda + \frac{\lambda}{1 + \lambda} \sigma_\delta^\top \sigma_\lambda \right) dt + \left( \sigma_\delta - \frac{\lambda}{1 + \lambda} \sigma_\lambda \right) dW(t) \right].$$

It follows

$$r(t) = \beta + \mu_\delta(t) - \|\sigma_\delta(t)\|^2 + \frac{\lambda(t)}{1 + \lambda(t)} \left( \sigma_\delta^\top(t)\sigma_\lambda(t) - \mu_\lambda(t) \right), \quad (19)$$

$$\theta(t) = \sigma_\delta(t) - \frac{\lambda(t)}{1 + \lambda(t)} \sigma_\lambda(t). \quad (20)$$

Using the expressions of $\mu_\lambda$ and $\sigma_\lambda$ from equations (17)-(18), $r$ and $\theta$ simplify as follows:

$$r(t) = \beta + \mu_\delta(t) - \|\sigma_\delta(t)\|^2 \left( \bar{a}(t)x_2(t) + \frac{\lambda(t)}{1 + \lambda(t)}(x_2(t))^2\|\Sigma^{-1}(t)v_2\|^2 \right),$$

$$\theta(t) = \sigma_\delta(t) + \frac{\lambda(t)}{1 + \lambda(t)}x_2\Sigma^{-1}(t)v_2.$$
C.1.5 Expression for \( x_2 \)

From equation (16), \( x_2 = \frac{1}{\|\Sigma^{-1} t_2\|^2} \left[ \theta^T \Sigma^{-1} t_2 - \bar{a} \right]^+ \). Using \( \theta = \sigma_\delta + \frac{\lambda}{1 + \lambda} x_2 \Sigma^{-1} t_2 \) in the expression of \( x_2 \), we get

\[
x_2 = \frac{1 + \lambda}{\|\Sigma^{-1} t_2\|^2} \left[ \sigma_\delta^T \Sigma^{-1} t_2 - \frac{\pi S_2 1 + \lambda}{S} \right]^+.
\]

In the expression of \( x_2 \), \( \bar{a} \) represents the percentage wealth holding threshold in stock 2. We need to establish the equivalence between the percentage stock holdings and the percentage wealth invested in the stock. Suppose \( \bar{a} \) is the maximum percentage of wealth the foreign investor can invest in the domestic stock 2, and \( \bar{\pi} \) the maximum percentage of shares of stock 2, foreign investors can hold. We will show that there is a relationship between \( \bar{a} \) and \( \bar{\pi} \). That allows us to use the results from Appendix C.1.1.

The relationship is as follows: \( \bar{a}X_f = \bar{\pi}S_2 \), and implies \( \bar{a} = \bar{\pi}S_2 / X_f \). Lets denote by \( \bar{S} \) the total wealth of the world, it is given as follows: \( \bar{S} = X_d + X_f = S_1 + S_2 + S_f \). Since \( X_f = 2(1 - e^{-\beta T})/(\beta y_f) \) and \( \lambda(0) = y_d / y_f \), the wealth of foreign investor is \( X_f = \bar{S}\lambda/(1 + \lambda) \) and that of domestic investor is \( X_d = \bar{S}/(1 + \lambda) \). Thus

\[
\bar{a} = \bar{\pi} \frac{S_2}{X_f} = \frac{S_2 1 + \lambda}{S} \frac{\pi}{\lambda}.
\]  

(21)

Using equation (21) in the expression of \( x_2 \), we get

\[
x_2 = \frac{1 + \lambda}{\|\Sigma^{-1} t_2\|^2} \left[ \sigma_\delta^T \Sigma^{-1} t_2 - \frac{\pi S_2 1 + \lambda}{S} \right]^+ = \frac{1 + \lambda}{\|\Sigma^{-1} t_2\|^2} \frac{S_2 [\lambda/(1 + \lambda) - \bar{\pi}]^+}{S} \frac{\lambda/(1 + \lambda)}{\|\Sigma^{-1} t_2\|^2}.
\]

If there is no constraint, the percentage shares of stock 2 foreign investor will hold will be: \( \lambda/(1 + \lambda) \).

Using \( \sigma_\delta^T \Sigma^{-1} t_2 = S_2 / \bar{S} \), it can be verified that \( \mu_\lambda(t) = 0 \) and

\[
\sigma_\lambda(t) = -x_2(t) \Sigma^{-1}(t) t_2.
\]

The expression for risk free rate also simplifies

\[
r = r_d = \beta + \mu_\delta - \|\sigma_\delta\|^2 + \frac{\lambda}{1 + \lambda} \sigma_\lambda^T \sigma_\lambda = \beta + \mu_\delta - \|\sigma_\delta\|^2 - \frac{\lambda}{1 + \lambda} \frac{S_2}{S} x_2.
\]

\[
r_f = r + x_2 \frac{S_2 1 + \lambda}{S} \frac{\lambda}{\lambda} = r + x_2 \frac{S_2 1 + \lambda}{S} \frac{\lambda}{\lambda}.
\]

\[
\theta = \theta_d = \sigma_\delta + \frac{\lambda}{1 + \lambda} x_2 \Sigma^{-1} t_2.
\]

\[
\theta_d = \theta - x_2 \Sigma^{-1} t_2.
\]
C.2 Proof of Proposition 4.6

Stocks prices are computed using perturbation methods. We follow three steps. Step 1: we compute the equity prices $S_d$ and $S_f$. Step 2: we compute the prices $S_1$ and $S_2$ using results from step 1. And lastly in Step 3: we compute the volatilities of the stocks and the volatility of the weight using results from Step 1 and Step 2.

C.2.1 Step 1

Discounting future dividends using the pricing kernel, the foreign country stock index is

$$S_f(t) = \delta(t) E_t \left[ \int_t^T e^{-\beta(s-t)} \frac{1 + \lambda(s)}{1 + \lambda(t)} ds \right] + \delta(t) E_t \left[ \int_t^T e^{-\beta(s-t)} \frac{1 + \lambda(s)}{1 + \lambda(t)} \delta(s) ds \right].$$

Since $\lambda$ is a martingale, it is easy to see that

$$S_f(t) = \frac{1 - e^{-\beta(T-t)}}{\beta} \delta(t) + \frac{\delta(t) E_t \left[ \int_t^T e^{-\beta(s-t)} \delta(s) \beta(t) \right]}{1 + \lambda(t)} + \frac{\delta(t) e^{\beta t}}{1 + \lambda(t)} H(\lambda(t), \delta_3(t)/\delta(t), t),$$

where $H(\lambda(t), \delta_3(t)/\delta(t), t) = E_t \left[ \int_t^T e^{-\beta s} \frac{\delta_3(s)}{\delta(s)} ds \right]$ and $H(\lambda(T), \delta_3(T)/\delta(T), T) = 0.

In absence of constraints $S_f(t) = S^*_f(t)$.

The total market capitalization of the world is $S(t) = S_d(t) + S_f(t) = 2(1 - e^{-\beta(T-t)}) \delta(t)/\beta$, which implies

$$S_d(t) = 2(1 - e^{-\beta(T-t)}) \delta(t)/\beta - S_f(t).$$

Assuming $d(\delta_3/\delta) = (\delta_3/\delta)[\mu_{\delta_3/\delta} dt + \sigma_{\delta_3/\delta} dW(t)]$, with $\mu_{\delta_3/\delta}$ non stochastic, then

$$S_f(t) = \frac{1 - e^{-\beta(T-t)}}{\beta} \delta(t) + \int_t^T e^{\delta(s)} \frac{\delta_3(t)}{1 + \lambda(t)} + \frac{\delta(t) e^{\beta t}}{1 + \lambda(t)} H(\lambda(t), \delta_3(t)/\delta(t), t).$$

Applying Ito’s lemma to the expressions of $S_d(t)$ and $S_f(t)$ give

$$\sigma_f(t) = \frac{1 - e^{-\beta(T-t)}}{\beta} \frac{\delta(t)}{S_f(t)} \sigma_\delta(t) + \frac{\delta_3(t) e^{\delta(t)} \delta_3(t)}{S_f(t)(1 + \lambda(t))} \sigma_\delta_3(t)$$

$$+ \frac{e^{\beta t} \delta(t) H_{\delta}}{S_f(t)(1 + \lambda(t))} - \frac{e^{\beta t} \delta(t) H}{S_f(t)(1 + \lambda(t))^2} - \frac{\delta_3(t) e^{\delta(t)} \delta_3(t)}{S_f(t)(1 + \lambda(t))^2} \lambda(t) \sigma_\lambda(t)$$

$$+ \frac{e^{\beta t} \delta(t)}{1 + \lambda(t)} \frac{\delta(t)}{S_f(t)} \sigma_\delta(t) + \frac{e^{\beta t} \delta_3(t) H_{\delta_3/\delta}}{S_f(t)(1 + \lambda(t))} \sigma_\delta_3(t)/\delta(t),$$

$$\sigma_d(t) = \frac{[2(1 - e^{-\beta(T-t)}) \delta(t)/\beta] \sigma_\delta(t) - S_f(t) \sigma_f(t)}{2(1 - e^{-\beta(T-t)}) \delta(t)/\beta - S_f(t)}.$$}

We write $H$ as a series of $\epsilon$. When $\epsilon = 0$ the problem reduces to the benchmark case. We choose $\epsilon = \frac{[\lambda/(1 + \lambda) - \bar{\pi}]^+}{\lambda/(1 + \lambda)}$. The Taylor expansion of $H$ with respect to $\epsilon$ is

$$H = H^0 + \sum_{n=1}^N H^n \epsilon^n + O(\epsilon^{N+1})$$
where \( H^n = \partial^n H/\partial e^n (\epsilon = 0) \) for \( n = 1, \ldots, N \) and \( H^0 = H(\epsilon = 0) \).

Under appropriate regularity conditions, it can be shown that

\[
\xi(t) S_f(t) + \int_0^t \xi(s) (\delta_3(s) + p(s) \delta^*(s)) ds
\]
is a martingale under \( \mathcal{P} \) (see Cuoco & He (1994), and Basak & Gallmeyer (2002)). Therefore, the drift must be zero. Hence, \( H \) solves the quasi-linear partial differential equation (\( \mathcal{L} + \frac{\partial}{\partial t} \)) \( H(\lambda(t), \delta_3(t)/\delta(t), t) + e^{-\beta t} \lambda(t) \delta_3(t)/\delta(t) = 0 \) with boundary condition \( H(\lambda(T), \delta_3(T)/\delta(T), T) = 0 \). We solve for

\[
H^0(\lambda(t), \delta_3(t)/\delta(t), t) = \lambda(t) E_t \left[ \int_t^T e^{-\beta s} \delta_3(s)/\delta(s) ds \right] = \lambda(t) \delta_3(t)/\delta(t) e^{-\beta t} \int_t^T e^t e^{(\beta - \mu g g/\delta)} ds,
\]
and for \( H^n, n \geq 1 \), by solving the following partial differential equation

\[
\frac{1}{2} \| \sigma \|^2 \lambda^2 \sum \frac{\partial^2 H^n}{\partial \lambda^2} \epsilon^n + \frac{1}{2} \| \sigma_{\delta/\delta} \|^2 \left( \frac{\delta_3}{\delta} \right)^2 \sum \frac{\partial^2 H^n}{\partial (\delta_3/\delta)^2} \epsilon^n + \sigma^\top \sigma_{\delta/\delta} \lambda \delta_3 \sum \frac{\partial^2 H^n}{\partial (\delta_3/\delta)} \delta_3 + \frac{\partial H^n}{\partial (\delta_3/\delta)} \epsilon^n = 0,
\]
with \( H^n(\lambda(T), \delta_3(T)/\delta(T), T) = 0 \). Let's denote by \( g = \delta_3/\delta \). Putting together the \( \epsilon \) terms, the partial differential equation for \( H^1 \) is:

\[
\frac{1}{2} \| \sigma \|^2 g^2 H^1_{gg} + \mu g H^1_g = \sigma^\top \Sigma^{\alpha-1} \lambda (1 + \lambda) \frac{S_0}{\Sigma^{\alpha-1} \lambda} g H^0_{gg}.
\]

For ease of exposition and also in order to provide intuitive economic interpretations for the derived formulas to follow, we assume \( \mu g = \mu_{\delta/\delta} = 0 \). We solve the PDE (24) to obtain \( H^1 \). Also recall, \( \sigma_2^\top \Sigma^{\alpha-1} \lambda = S_2/\bar{S} \) and \( \sigma_1^\top \Sigma^{\alpha-1} \lambda = 0 \). From \( \sigma_1^\top \Sigma^{\alpha-1} \lambda = -\frac{S_2^2}{S_2^2} \bar{S} \), we compute \( \sigma_2^\top \Sigma^{\alpha-1} \lambda = -\frac{S_2^2}{2(1-e^{-\beta(T-t)})\delta_3/\delta} \). So, \( \sigma_2^\top \Sigma^{\alpha-1} \lambda = -\frac{S_2^2}{2(1-e^{-\beta(T-t)})\delta_3/\delta} - \frac{S_2^2}{\bar{S}} \). Hence equity prices are

\[
S_f = S_f^0 + \frac{\lambda \bar{S}}{\| \sigma \|^2 \Sigma^{\alpha-1} \lambda} \left( \frac{S_2^2}{\bar{S}} \right)^2 (1 + g)(1 - \ln(g)) \epsilon,
\]
\[
S_d = S_d^0 - \frac{\lambda \bar{S}}{\| \sigma \|^2 \Sigma^{\alpha-1} \lambda} \left( \frac{S_2^2}{\bar{S}} \right)^2 (1 + g)(1 - \ln(g)) \epsilon.
\]

Next we compute the individual stock prices in the domestic market.

**C.2.2 \ Step 2**

The prices of domestic stocks are

\[
S_2(t) = \delta(t) E_t \left[ \int_t^T e^{-\beta(s-t)} \frac{1 + \lambda(s) \delta_3(s)}{1 + \lambda(t) \delta(s)} ds \right] = \frac{S_2^0(t)}{1 + \lambda(t)} + \frac{\delta(t)e^{\beta t}}{1 + \lambda(t)} h(\lambda(t), \delta_2(t)/\delta(t), t),
\]
\[
S_1(t) = S_d(t) - S_2(t),
\]

39
where \( h(\lambda(t), \delta_2(t)/\delta(t), t) = E_t \left[ \int_t^T e^{-\beta s} \lambda(s) \frac{\delta_2(s)}{\delta(s)} ds \right] \), and \( h(\lambda(T), \delta_2(T)/\delta(T), T) = 0 \).

In absence of constraints \( S_j(t) = S_j^o(t) \).

Applying Ito’s lemma to the expressions of \( S_j \) gives

\[
\sigma_2(t) = \left[ S_2^o(t) \sigma_2^o(t) + e^{\beta \delta}(t) h\sigma_2(t) + e^{\beta \delta}(t) \frac{\partial h}{\partial(\delta_2/\delta)} \delta_2 \right] (1 + \lambda(t)) S_2(t),
\]

\[
(25)
\]

\[
\sigma_1(t) = \sigma_d(t) + \frac{S_2(t)}{S_d(t) - S_2(t)} (\sigma_d(t) - \sigma_2(t)).
\]

We write \( h \) as the Taylor series expansion of \( \epsilon \):

\[
h = h^0 + \sum_{n=1}^{\infty} h^n \epsilon^n + O(\epsilon^{N+1})
\]

where \( h^n = \partial^n h/\partial \epsilon^n (\epsilon = 0) \) for \( n = 1, ..., N \) and \( h^0 = h(\epsilon = 0) \).

Again under appropriate regularity conditions, it can be shown that

\[
\xi(t) S_2(t) + \int_0^t \xi(s) \delta_2(s) ds
\]

is a martingale under \( \mathcal{P} \) (see Cuoco & He (1994), and Basak & Gallmeyer (2002)). Therefore, the drift must be zero. Hence, \( h^0 \) solves the quasi-linear partial differential equation

\[
(\mathcal{L} + \frac{\partial}{\partial t}) h(\lambda, \delta_2/\delta, t) + e^{-\beta \lambda(t)} \frac{\delta_2(t)}{\delta(t)} = 0
\]

with boundary condition \( h(\lambda(T), \delta_2(T)/\delta(T), T) = 0 \). We solve for

\[
h^0(\lambda(t), \delta_2(t)/\delta(t), t) = e^{\beta \lambda(t)} \frac{S_2^o(t)}{\delta(t)},
\]

and for \( h^n(\lambda, \delta_2/\delta, t) \) by solving the partial differential equation

\[
\frac{1}{2} \left\| \sigma_{\lambda} \right\|^2 \lambda^2 \sum \frac{\partial^2 h^n}{\partial \lambda^2} \epsilon^n + \frac{1}{2} \left\| \sigma_{\delta_2/\delta} \right\|^2 \left( \frac{\delta_2}{\delta} \right)^2 \sum \frac{\partial^2 h^n}{\partial (\delta_2/\delta)^2} \epsilon^n +
\]

\[
\sigma_{\lambda}^T \sigma_{\delta_2/\delta} \left( \frac{\delta_2}{\delta} \right) \sum \frac{\partial^2 h^n}{\partial \lambda \partial (\delta_2/\delta)} \epsilon^n + \mu_{\delta_2/\delta} \left( \frac{\delta_2}{\delta} \right) \sum \frac{\partial h^n}{\partial (\delta_2/\delta)} \epsilon^n = 0,
\]

with \( h^n(\lambda(T), \delta_2(T)/\delta(T), T) = 0 \).

For ease of exposition and in order to have interpretable approximate expressions, we assume \( \mu_{\delta_2/\delta} = 0 \). We solve the above PDE to obtain \( h^1 \). We can also show that:

\[
(\sigma_{\delta_2/\delta} - \sigma_{\delta}) \sigma_{\delta_2/\delta} \left( \frac{\delta_2}{\delta} \right) \sum (1 - \frac{S_2^o}{S}) (1 - \ln(\frac{\delta_2}{\delta})) \epsilon
\]

\[
\frac{2(1 - \frac{S_2^o}{S}) (1 - \ln(\frac{\delta_2}{\delta}))}{\left\| \sigma_{\delta_2/\delta} \right\|^2} - \frac{(1 + \frac{\delta_2}{\delta})(1 - \ln(\frac{\delta_2}{\delta}))}{\left\| \sigma_{\delta_2/\delta} \right\|^2} \epsilon.
\]

C.2.3 Step 3

The last step consists of using the stock prices to obtain the volatilities. We use 22, 23, 25 and 26 and solve the system of simultaneous equations to get the volatilities of the stocks.
C.3 Proof of Proposition 4.7

Agent $i$’s welfare is given as follows:

$$U_i = E \left[ \int_0^T e^{-\beta t} \left[ \ln(c_i(t)) + \ln(c_i^*(t)) \right] dt \right],$$

with $c_i(t) = e^{-\beta t} / y_i \xi_i(t)$ and $c_i^*(t) = e^{-\beta t} / y_i \rho(t) \xi_i(t)$ where $y_i = 2(1 - e^{-\beta T}) / \beta X_i(0)$ and $X_i(0)$ is agent $i$’s initial wealth. Agents’ wealths are: $X_d(t) = 2 \left(1 - e^{-\beta(T-t)}\right) e^{-\beta/(\beta y_i \xi_i(t))}$ and $X_f(t) = 2 \left(1 - e^{-\beta(T-t)}\right) e^{-\beta/(\beta y_f \xi_f(t))}$.

The fictitious pricing kernel facing investor $i$ (Appendix (C.1.1)) is

$$\xi_i(t) = \exp \left( - \int_0^t \left[ r_i(\tau) + ||\theta_i(\tau)||^2/2 \right] d\tau - \int_0^t \theta_i(\tau)^T dW(\tau) \right),$$

The welfare function then becomes

$$U_i = E \left[ \int_0^T e^{-\beta t} \left[ -2\beta t + 2 \ln(y_i) - \ln(p(t)) \right] dt \right] - 2E \left[ \int_0^T e^{-\beta t} \ln(\xi_i(t)) dt \right].$$

We can also rewrite $r_i$ and $\theta_i$ as follows: $r_i = \beta + \mu_\delta - ||\sigma_\delta||^2 + \Delta r_i(\lambda)$ and $\theta_i = \sigma_\delta + \Delta \theta_i(\lambda)$, where $\Delta r_i(\lambda)$ and $\Delta \theta_i(\lambda)$ represent the deviation from the benchmark quantities. Then using these expressions and the expressions for $\mu_p$ and $\sigma_p$ (equations (5) and (6)) we get

$$U_i = E \left[ \int_0^T e^{-\beta t} \left( -2 \ln(y_i) - \ln(p(0)) \right) dt \right]$$

$$+ E \left[ \int_0^T e^{-\beta t} \left( \mu_\delta(\tau) + \mu_\delta^*(\tau) - \left(||\sigma_\delta(\tau)||^2 + ||\sigma_\delta^*(\tau)||^2\right)/2 \right) d\tau dt \right]$$

$$+ 2E \left[ \int_0^T e^{-\beta t} \left( \Delta r_i(\lambda(\tau)) + \sigma_\delta(\tau)^T \Delta \theta_i(\lambda(\tau)) + ||\Delta \theta_i(\lambda(\tau))||^2/2 \right) d\tau dt \right],$$

or

$$U_i = -\frac{1 - e^{-\beta T}}{\beta} \left[ 2 \ln(2(1 - e^{-\beta T})/\beta X_i(0)) + \ln(\delta(0)/\delta^*(0)) \right]$$

$$+ E \left[ \int_0^T e^{-\beta t} \left( \mu_\delta(\tau) + \mu_\delta^*(\tau) - \left(||\sigma_\delta(\tau)||^2 + ||\sigma_\delta^*(\tau)||^2\right)/2 \right) d\tau dt \right]$$

$$+ 2E \left[ \int_0^T e^{-\beta t} \left( \Delta r_i(\lambda(\tau)) + \sigma_\delta(\tau)^T \Delta \theta_i(\lambda(\tau)) + ||\Delta \theta_i(\lambda(\tau))||^2/2 \right) d\tau dt \right].$$

To obtain the final expressions, we replace $\Delta r_i(\lambda)$ and $\Delta \theta_i(\lambda)$ by their respective expressions: for domestic residents

$$\Delta r_d(\lambda) = \frac{\lambda(t)}{||\Sigma^{-1}(t)\nu_2||^2} \left[ \frac{S_2(t)}{S(t)} \right]^2 \left[ \frac{\lambda(t)/(1 + \lambda(t)) - \bar{\pi}(t)^+}{\lambda(t)/(1 + \lambda(t))} \right],$$

$$\Delta \theta_d(\lambda) = \frac{\lambda(t)}{||\Sigma^{-1}(t)\nu_2||^2} \left[ \frac{S_2(t)}{S(t)} \right]^2 \left[ \frac{\lambda(t)/(1 + \lambda(t)) - \bar{\pi}(t)^+}{\lambda(t)/(1 + \lambda(t))} \right] \Sigma^{-1}(t)\nu_2.$$

41
and for foreign agent

\[ \Delta r_f(\lambda) = \Delta r_d(\lambda) + \bar{\pi} \left( \frac{1 + \lambda(t)}{\lambda(t)} \frac{S_2(t)}{S(t)} \right) x_2, \]

\[ \Delta \theta_f(\lambda) = - \frac{1}{\|\Sigma^{-1}(t)_{t2}\|^2} \frac{S_2(t) \left[ \lambda(t)/(1 + \lambda(t)) - \bar{\pi}(t) \right]^+}{\lambda(t)/(1 + \lambda(t))} \Sigma^{-1}(t)_{t2}. \]

### C.4 Proof of Proposition 4.8

The proof is very similar to that for Proposition 4.4. We will just provide the crucial steps and follow the same notation as much as possible. The utility of the investor at the optimal consumption is

\[
U_d(t) = E_t \left[ \int_t^T e^{-\beta s} \left[ \ln c(s) + \ln c^*(s) \right] ds \right] 
- \frac{(e^{-\beta t} - e^{-\beta T})}{\beta} \left[ 2 \ln \frac{2(e^{-\beta t} - e^{-\beta T})}{\beta} X_d(t) + \ln p(t) \right]
+ E_t \left[ \int_t^T e^{-\beta s} \int_s^T \left[ \mu_{\delta} + \mu_{\delta^*} - \frac{1}{2} \left( \|\sigma_{\delta}\|^2 + \|\sigma_{\delta^*}\|^2 \right) \right] dvds \right] 
- E_t \left[ \int_t^\tau e^{-\beta s} \int_t^s \lambda^2 \left\{ \frac{S_2}{S} \frac{[\lambda/(1 + \lambda) - \bar{\pi}]^+}{\|\Sigma^{-1}t2\|^2\lambda/(1 + \lambda)} \right\}^2 dvds \right] ,
\]

where \( X_d(t) \) is the wealth of the investor at time \( t \) and \( \tau \) is the time when the industry protection is removed.

\[
U_d(t) - U_d^o(t) = \frac{(e^{-\beta t} - e^{-\beta T})}{\beta} \ln \frac{X_d(t)}{X_d^o(t)} - E_t \left[ \int_t^\tau e^{-\beta s} \int_t^s \lambda^2 \left\{ \frac{S_2}{S} \frac{[\lambda/(1 + \lambda) - \bar{\pi}]^+}{\|\Sigma^{-1}t2\|^2\lambda/(1 + \lambda)} \right\}^2 dvds \right] .
\]

Focusing on the wealth term,

\[
\frac{d}{d\Delta} \ln \frac{X_d(t)}{X_d^o(t)} = -\Delta r(t) \sum_j a_j(t)S_j^o(t) - \Delta \theta(t) \sum_j a_j(t)S_j^o(t)\sigma_j^o(t)
= \left[ \frac{S_2(t)}{S(t)} \sum_j a_j(t)S_j^o(t) - t_2 \sum_j a_j(t)S_j^o(t)\sigma_j^o(t) \right] \left[ \frac{S_2(t)}{S(t)} \frac{[\lambda(t)/(1 + \lambda(t)) - \bar{\pi}]^+}{\Sigma^{-1}(t)_{t2}} \right].
\]

Therefore, \( d(U_d(t) - U_d^o(t))/d\Delta|_{\Delta=0} = 0 \) is equivalent to either \([\lambda(t)/(1 + \lambda(t)) - \bar{\pi}]^+ = 0\) or

\[
0 = \frac{S_2}{S} \sum_j a_j(t)S_j^o(t) - t_2 \sum_j a_j(t)S_j^o(t)\sigma_j^o(t).
\]

Thus

\[
\frac{S_2}{S} X_d^o(t) = a_2(t)S_2^o(t).
\]
Note that we have assumed that $\Delta > 0$. Thus $a_2(t) = \bar{\pi}$. Since asset prices are continuous, as $\Delta \downarrow 0$, asset prices converges to their counterparts in the benchmark economy. So at time $t = \tau^*$,

$$\frac{X_{d}^{o}(t)}{S(t)} = \bar{\pi}.$$ 

On the other hand,

$$\frac{X_{d}^{o}(t)}{S(t)} = \frac{X_{d}^{o}(t)}{X_{d}^{o}(t) + X_{f}^{o}(t)} = \frac{\lambda(t)}{1 + \lambda(t)}.$$ 

This completes the proof.
References


