Definitions of Options

- **Defining options**
  - options are specific opportunities (contractual or implicit) unique to option holder that allow the owner the right, but not obligation, to do something at some time in the future

- **basic features of options**
  - expected payoff (value) of an option is always positive
    - in negative situations just choose not to exercise the option and payoff is zero
  - as a result, options typically involve an up front investment
    - this is called the option *premium* and is paid when the option is acquired (purchased)
      - not really a cost, but an investment to purchase expected future payoff
    - with financial options this is necessary to keep the option a zero NPV project
      - otherwise, no one would want to take the negative NPV side
Option Terminology

- Option terminology
  - strike price or exercise price, \( K \)
    - the pre-set price at which the asset can be bought or sold

- types of options
  - right to buy or sell an asset
    - call option - right to buy or acquire the asset for some price
    - put option - right to sell or discard the asset at some price

- exercise time
  - American options - can be exercised any time through to maturity
  - European options - can be exercised only at maturity

- “monieness” of options
  - an option is said to be in-the-money if it would be profitable to exercise today at the current market price for the asset
    - market price > exercise price for a call option
    - market price < exercise price for a put option
  - an option is said to be out-of-the-money if it would not be profitable to exercise today at the current market price for the asset

Option Markets

- Options can be either private or public
  - options have been privately negotiated for centuries
    - option contracts for use of wine press in ancient Greece
    - today many financial options firms use are privately negotiated, typically with a financial institution in the OTC market
      - little to no public record keeping of these individual transactions
        - just aggregate activity
  - options have traded on exchanges since early 1970s
    - CBOE - Chicago Board Option Exchange opened in 1973
      - many other exchanges that trade standardized financial options
        - standardized by contract size, maturity dates, and strike price
  - exchange traded options on DLM stock (contract = 100 shares)

<table>
<thead>
<tr>
<th>11/7/2008 Options on DLM Expiring Fri, Dec 19, 2008</th>
<th>Stock price on 11/7/08 = 6.08</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calls</td>
<td>Puts</td>
</tr>
<tr>
<td>Symbol</td>
<td>Last</td>
</tr>
<tr>
<td>DLMLA.X</td>
<td>1.95</td>
</tr>
<tr>
<td>DLMLB.X</td>
<td>0.15</td>
</tr>
<tr>
<td>DLMLB.X</td>
<td>0.25</td>
</tr>
<tr>
<td>DLMLB.X</td>
<td>0.15</td>
</tr>
</tbody>
</table>
Simple Option Example

Let's consider an option on XYZ stock

- here are prices of a few XYZ options as of July 2008
  - stock price was $100/share on this day (XYZ pays no dividends)

<table>
<thead>
<tr>
<th>Exercise Date</th>
<th>Strike Price</th>
<th>Price of Calls</th>
<th>Price of Puts</th>
</tr>
</thead>
<tbody>
<tr>
<td>October 2008</td>
<td>$90</td>
<td>$14.73</td>
<td>$3.62</td>
</tr>
<tr>
<td>January 2009</td>
<td>$90</td>
<td>$17.56</td>
<td>$5.36</td>
</tr>
<tr>
<td>January 2009</td>
<td>$105</td>
<td>$10.98</td>
<td>$13.42</td>
</tr>
</tbody>
</table>

- notice some basic pricing features
  - both calls and puts get more expensive as maturity of options gets longer
    - options with more time attached to them are generally more valuable
  - when strike price rises, puts become more expensive and calls become cheaper
    - this is because puts are more in-the-money and calls are more out-of-the-money when the strike price rises

Position Profiles

To better understand options it is useful to graph them

- we will look primarily at payoff profiles
  - these are the cash flows at maturity from investment positions
    - just the cash flows at settlement, ignoring the cost of getting into the positions originally
  - we will look at both long positions and short positions
    - long position – having purchases the assets (you own it) and you receive the cash flows at maturity from selling or exercising the asset
    - short position – you sold the position initially (even if you did not own it) and you must pay the cash flows at maturity by re-buying the asset
  - payoff profiles are different from profit profiles and return profiles
    - profit profiles are like payoff profiles but net the FV (at settlement) of the initial cost of entering the position with the maturity cash flows
    - return profiles show the % return the investor earns on the position

- let's look at settlement cash flows for positions in the January 2009 ($90 strike) call and put option and share of XYZ stock
  - diagram the payoffs of at maturity ($r_f = 2.5% for 6 months)
    - ignores costs of getting into positions
**Long Position Payoff Profiles**

**Payoff from Owning Call Option** (long a call option)
Payoff to Jan09 $90 call option
= max (XYZ Price\textsubscript{Jan09} - 90, 0)
Price of position = $17.56

**Payoff from Owning Put Option** (long a put option)
Payoff to Jan09 $90 put option
= max (90 - XYZ Price\textsubscript{Jan09}, 0)
Price of position = $5.36

**Payoff from Owning a Share of XYZ Stock** (long a share of stock)
Payoff to long XYZ share in Jan09
= XYZ Price\textsubscript{Jan09}
Price of position = $100

**Long option payoffs:**
P = asset price
K = exercise price
call = max (P – K, 0)
put = max(K – P, 0)

**More Option Payoff Profiles**

**Payoff from Writing Call Option** (short a call option)
Payoff to short Jan09
= min (90 - XYZ Price\textsubscript{Jan09}, 0)
Price of position = -$17.56

**Payoff from Writing Put Option** (short a put option)
Payoff of short put option Jan04
= min (XYZ Price\textsubscript{Jan09} - 90, 0)
Price of position = -$5.36

**Payoff from Selling a Share of XYZ Stock** (short a share of stock)
Payoff of short XYZ share in Jan 09
= - XYZ Price\textsubscript{Jan09}
Price of position = -$100

**Short option payoffs:**
P = asset price
K = exercise price
call = min (K - P, 0)
put = min (P - K, 0)
Profit and Return Profiles

Profit from Owning Call Option (long a call option)
- Cost of option today = $17.56
- Profit to Jan09 $90 call option = max (XYZ Price$_{Jan09}$ - 90, 0) - FV$_{Jan09}$(17.56)
- Return to Owning Call Option (long a call option)
  - Cost of call - $17.46
  - Return to Jan09 $90 call = call payoff /17.56 - 1

Profit from Owning Put Option (long a put option)
- Cost of option today = $5.36
- Profit to Jan09 $90 put option = max (90 - XYZ Price$_{Jan09}$, 0) - FV$_{Jan09}$(5.36)
- Return to Owning Put Option (long a put option)
  - Cost of put - $5.36
  - Return to Jan09 $90 put = put payoff / 5.36 - 1

Financial Engineering

- Combining options, stocks and borrowing / lending we can achieve all sorts of payoffs
  - this is the science of financial engineering
    - combining financial instruments along with underlying positions a firm can obtain any desired payoff structure they like
    - they involve different initial investments
  - simple rule of arbitrage
    - if two combinations of instruments provide exactly the same payoffs in all states of the world, then they must have the same initial investment
    - let’s look at some financial engineering examples and learn how call and put options are related
**Investment Strategies**

- buy downside protection (insurance) for share by purchasing a put option
- alternative way to achieve the same payoff profile

<table>
<thead>
<tr>
<th>Stock Price</th>
<th>Payoff</th>
<th>Payoff</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td></td>
<td></td>
<td>90</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Owning a share and put option (K = $90) |
| owning a risk free bond (face value = K = $90) and call option (K = $90) |
| since they have the same payoffs, by arbitrage they must have the same initial investment (cost) |

- this gives us a relation called Put-Call Parity

at today’s prices it must be that

\[ S + P = C + PV(K) \]

- see call option as levered investment plus insurance

  - from put-call parity we see the call option is a levered position in the stock plus insurance against a downside risk (put option)

\[ C = S - PV(K) + P \]

- \( S - PV(K) \) = levered position in the stock
  - you own a share of stock but have paid for some of it with borrowing
- \( P \) = put option that provides downside price protection (insurance)
More with Put-Call Parity

- Put-call parity is also useful for valuing put options
  \[ P = C + PV(K) - S \]
  which means a put option is the same as owning
  » a call option
  » a risk free bond (or bank deposit) paying off strike price K at maturity
  » shorting a share of stock
- so even if put options were not available, we could synthetically create one with a call, a bond and a short share of stock
- if the stock will pay a dividend during the option life, then these two positions are not the same
  » the call option and the bond with FV = K will not get the holder the dividend
    - to receive the dividend you must be an owner of record
  » with a dividend paying stock, Put – Call parity has to be modified so account for the PV of the dividend
  \[ S + P = C + PV(K) + PV(Div) \]

What Determines Option Value

- The value of a call option is bounded by two limits
  » the call option’s value must lie between the asset’s value and the option’s value if exercised immediately (intrinsic value)
  » value of the option prior to maturity will be a smooth path between these
    - the difference between option value and intrinsic value = time value
      » time value driven by volatility around K and discount factor deep in-the-money
        - As call gets deep in-the-money the value of option converges to
          \[ P - PV(K) \]
          time value = K - PV(K)
        - Near K, the time value is most significantly affected by volatility as time gives more chance for stock price to move up

![Diagram showing the value of a call option and its bounds](SAIS 380.760, Lecture 9 Slide 8 14)
Some Basic Features of Option Prices

■ Rule 1:
  - option price moves with asset price
    ► when the asset is worthless, the option is worthless
      - when the asset price is zero, the option price is zero
        » only place where option and asset have same value
      - everywhere else the option is less valuable than the asset, but its value increases with the price of the asset

  "Value of a call option increases with price of underlying asset"

■ Rule 2:
  - when the asset price is high relative to the exercise price, the option price converges to a value of the asset price less the PV(K)
    - when stock price is very high, option will be exercised for sure and its value becomes stock price minus the exercise price, K
    - but you only pay exercise price at maturity so a call option is a way to acquire the asset on credit
      » the “cost” of the exercise price now is only PV(K)
    - this delay in payment is valuable, especially if interest rates are high

  "Value of an option increases with interest rates and time to maturity"

Basic Features of Option Prices

■ Rule 3:
  - the value of an option always exceeds its intrinsic value
    - except when intrinsic value is zero
  - consider stock price = exercise price
    - if exercised today the option has zero value
      » value of option that lets you buy an asset at market price is zero
    - but what if there is one day to maturity
  - assume 50-50 chance of price increase or decrease
    - if asset price decreases option will expire worthless
    - if asset price increases option will have positive value
    - expected value of option with one day to maturity is
      50% (zero value) + 50% (positive value) > 0
  - the amount the option value is above zero is related to how much the stock might move per unit of time (its variability) and the time it has until maturity

  "Value of an option increases with volatility of asset price and time to maturity"
Summary of Call Option Behavior

- Price of a call option, \( C(P,r,T,\sigma,K) \), depends on:
  - increases in variables
    - the asset price, \( P \)
    - the risk free interest rate, \( r \)
    - volatility of the asset price, \( \sigma \)
    - time to expiration, \( T \)
  - lead to an increase in the price of a call option
  - but increases in
    - the exercise price, \( K \)
  - lead to a decrease in the price of a call option
  - other properties of call option prices
    - the option price is always less than the asset price
    - the option price is always positive
      - except when asset price \( P = 0 \) or at maturity when \( P < K \)
    - as the option get very deep in-the-money, the call option price approach the asset price less PV of exercise price
      - \( C = P – PV(K) \)

Options and Corporate Finance

- Consider a firm in financial distress
  - market value of assets = $35m, face value of debt = $50m
    - debt is zero coupon and has to be repaid one year from now
  - firm’s economic balance sheet (market values)
    \[
    \begin{array}{c|c|c}
    \text{Assets} & 35 & \text{Debt} \\
    \text{Value} & 35 & \text{Debt + Equity} \\
    \hline
    \text{30} & \text{5} \\
    \end{array}
    \]
  - since assets < face value of debt, why is the equity = $5?
    - because the debt is not due to be paid now, but in the future
    - the payoff to the shareholders of the firm in one year is the same as a call option on the assets of the firm with strike price = \( D \)
      - equity has positive payoff only if assets are worth more than \( D \)
Equity as Call Option

- Equity is like a call option
  - equity is a call option with \( K = \$50m \)
    - when debt matures, shareholders (SH) get 0 payoff if asset value is below \$50m and positive upward sloping payoff if asset value is above \$50m
  
  - since a call option always has positive value, even if out-of-the-money, this equity will have positive value for the same reasons
    - when using debt, the SHs essentially give the firm’s assets to the creditors but hold a call option to buy back the assets by repaying the debt
  
  - we can consider using option pricing methods to value equity

- we can restate the firm’s economic balance sheet

\[
\begin{align*}
\text{Assets} &= 35 \\
\text{Debt} &= 30 = \text{Asset value - call option on assets} \\
\text{Equity} &= 5 = \text{value of a call option on firm’s assets} \\
\text{Value} &= 35 \\
\text{D} + \text{E} &= 35 = \text{Value}
\end{align*}
\]

- notice that the firm’s \( D = V – E \) = Asset value – value of call option on assets

Options and Corporate Finance

- Now recall the put - call parity condition

\[
\text{Value of share} + \text{Value of put} = \text{value of call} + \text{PV(K)} =
\]

  - redefine some terms: Value of share = value of our firm’s assets, \( V \)
  
  - Value of call = value of our firm’s equity, \( E \)
  
  - \( \text{PV(K)} = \text{PV of the Debt (at risk free rate)} \)

  - now let’s restate put-call parity using our terms

\[
\text{Value of assets} + \text{Value of put} = \text{Value of equity} + \text{PV(Pmts to D @ r_f)}
\]

- since \( D = V – E \)

  - from put-call parity above it is the case that \( V – E \) equals :

\[
\text{Value of assets} – \text{Value of equity} = \text{PV(Pmts to D @ r_f)} - \text{Value of put}
\]

  - so the firm’s risky debt must equal \( PV \) of debt at \( r_i \) – value of a put on assets

\[
D = \text{PV(Pmts to D @ r_i)} - \text{Value of put}
\]

  - the market value of the firm’s risky debt is the value of the debt as if it was risk free less the value of a put option on the assets of the firm

  - the put is the right to sell the assets for face value of the debt
Value of Risky Debt

- \( D = \text{PV(Pmts to } D \ @ \ r_F) \) - Value of put

- so risky debt is like two securities to debtholders (DH)
  - a risk-free bond (guaranteed claim on the promised payments)
  - a short (sold) put option to buy the assets of the firm for the value of the debt

- the holders of the risky debt are short this option because this is the option DH write for the SH giving them the option to exercise limited liability and repay the debt with the assets
  - limited liability gives the SHs the right to repay the debt by giving the DHs the assets of the firm rather than the promised debt payments
  - SH's pay for this option by getting less less they issue their debt then they would if the debt was default free (PV(Pmts to D\( @ r_F \))
  - the price of the risk is the option to default (a put option)

- the DH receive value for the put by reducing the amount they pay for the firm’s risky debt relative to its value if it were risk free

- paying less for the debt today is equivalent to discounting the payments at a rate that is higher than the risk free rate
  - they charge a risk premium on the debt to cover the premium (value) of the default option

- paying less for the debt today is equivalent to discounting the payments at a rate that is higher than the risk free rate

Economic Position of DHs

- Assets = 35
- Debt = 30 = risk free debt - put option on assets
- Equity = 5 = value of a call option on firm’s assets
- Value = 35

- in our example the value of the DHs position is:
  - a risk free version of the debt with value = PV(repmts)
    » thus the interest rate they earn on this will be the risk free rate
  - a short position (sale) of a put option to SHs to default
    » value of this option can be estimated using option pricing techniques

- over time if the the default option increases in value the debt becomes worth less
  - in our example we know this option is worth just under $20 as current market value of D = PV($50) - value of put option = $30
    » if \( r_F = 5\%\), then \( \text{PV($50)} = $47.62 \) so default option is worth $17.62
    » high value of right to default means this means the implied yield to maturity on the debt is very high; \( \text{YTM} = (50/30) - 1 = 66\%\)
  - with info on volatility of asset value, we could directly determine the value of this put option
    » this is what advanced credit analysis does for pricing risk debt
A Model for Valuing Options

- Despite what we knew about option price behavior, discovering a model to price options was difficult
  - the difficulty is that the risk of an option changes with its value
    - price and total risk move in opposite directions
      - options are riskier than the underlying asset
  - in the early 1970s Black and Scholes discovered how to price options using arbitrage
    - replicate option with a portfolio of a share and a risk free bond
      - arbitrage says that investments with same payoff in the future must have same value today
    - in the simple version of the model, just assume the stock price either moves up or down by a given amount over a period
      - a binominal model
        - Up state: Stock Price₁ (P₁^{UP}) = P x (1 + R_{up})
        - Down state: Stock Price₁ (P₁^{DN}) = P x (1 + R_{down})

Binominal Model

- Example: 6 month call option on XYZ (European)
  - K = $90 at maturity in January 2009 (6 month in the future)
  - XYZ price in July 2008 = $100 so option is in-the-money
  - risk-free interest rate is 2.5% for 6 months (periodic rate)
  - for volatility, assume XYZ stock will either rise 30% or fall 20%
    - price will either rise to $130 (R_{up} = 30%) or fall to $80 (R_{down} = -20%)
  - stock and call option payoffs
    - Payoff to XYZ stock = P₁
      - July 2008: XYZ Stock Price = $100
      - January 2009: (Up State) XYZ Stock Price = $130
        - (Down State) XYZ Stock Price = $80
    - Payoff to call option = max(P₁ - K, 0)
      - XYZ January 2009 call option (K = 90) = ??
        - Call option = max($130 - 90, 0) = $40
        - Call option = max($80 - 90, 0) = $0
Replicating Portfolio

Let's consider three instruments:

- a share of XYZ stock,
- a 6 month risk-free bond with face value = K = $90
  - current price of bond is PV of K = $90/1.025 = 87.80
- the 6 month call option with K = $90

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Share of XYZ</td>
<td>$100</td>
<td>$130</td>
<td>$80</td>
</tr>
<tr>
<td>Bond with FV = K</td>
<td>$87.80</td>
<td>$90</td>
<td>$90</td>
</tr>
<tr>
<td>Call option</td>
<td>???</td>
<td>$40</td>
<td>$0</td>
</tr>
</tbody>
</table>

- the price of the call option on XYZ can be found by pricing a portfolio of XYZ stock and the risk free bond that replicate the payoff of the call option in both the up and down states

- this replicating portfolio will consist of an investment in XYZ stock and an investment in the risk free bond (long or short)
- the replicating portfolio will have the same future cash flows as the call option and so, by arbitrage, will have the same price today
  - the price of buying the replicating portfolio is the price of the option

Constructing the Replicating Portfolio

Assume the replicating portfolio will be $\Delta$ shares of XYZ stock and M risk free bonds

- it must be that payoffs to the portfolio = payoffs to option
  - in the Up State: $130 \times \Delta + 90 \times M = 40$
  - in the Down State: $80 \times \Delta + 90 \times M = 0$

- solve the two equations simultaneously,
- take second one $80 \times \Delta + 90 \times M = 0 \implies 90 \times M = -80 \times \Delta$
  - plugging this into the other equation we can solve for $\Delta$
  - $130 \times \Delta - 80 \times \Delta = 40 \implies 50 \times \Delta = 40 \implies \Delta = 0.80$
  - this means that we can replicate the payoffs to the call option by buying 0.8 shares of XYZ stock and buying -0.7111 (borrowing 0.7111) units of the $90 risk-free bond
- the net cost of obtaining this portfolio is the cost of the option
Pricing the Call Option

- Compare final payoffs to replicating portfolio
  - up state: $0.8 \times $130 - 0.71 \times $90 = $40
  - down state: $0.8 \times $80 - 0.71 \times $90 = $0

- Price of creating replicating portfolio today
  - buying $\Delta$ shares of XYZ costs $0.80 \times $100 = $80.00
  - buying M units of $90 FV bond today costs $-0.7111 \times $87.80 = $-62.44
    - provides inflow of cash today of $62.44 today
  - total cost today of creating replicating portfolio
    - $80 - $62.44 = $17.56
  - by arbitrage, the cost of the call option today must be $17.56
    - this is replication portfolio option pricing
      - call option price is: $C = S \times \Delta + M \times PV(K)$
    - if price of option where higher than this, you could arbitrage by selling option and buying replicating portfolio (and vice versa)

Alternative Method for Portfolio

- There is another way to determine the portfolio investments to replicate the call option payoffs
  - we know that the replicating portfolio consists of $\Delta$ shares of stock and some amount of a risk free bond
  - define the size of the investment in the risk free bond as $B$
  - in the binomial case we can determine $\Delta$ and $B$ directly from the option and stock payoffs in the two final states $C_u$ and $C_d$, and $S_u$ and $S_d$
    - $\Delta = (C_u - C_d)/(S_u - S_d)$ and $B = (C_d - S_d \Delta)/(1 + r_f)$
    - then it will be the case that the call price today is $C = S \times \Delta + B$
    - in our example
      - $\Delta = (40 - 0)/(130 - 80) = 0.80$
      - $B = (0 - 64)/(1 + 0.025) = -62.44$
      - $C = $100 \times 0.80 + x $100 - $62.44 = $17.56$
Risk Neutral Valuation

- We can also price options by taking expected values of the option payoffs from the binominal tree

\[
\text{Price Call Option} = \frac{\text{PV(Expected future payoffs)}}{\text{(Up State) Call option = $40}}
\]

\[
\text{PV(Up State) Call option = $40}
\]

\[
\text{PV(Down State) Call option = $0}
\]

- expected value of the future states requires probabilities
  - since XYZ is risky, these are dependent on investor’s risk aversion
  - since the expected value has to hold for any level of risk aversion, we can do it for an imaginary investor who is risk neutral (RN)
    - if an investor is risk neutral, then her required return on XYZ stock is just 2.5% over the next 6 months (required return = risk-free rate)
  - RN investor still assumes that the stock price in 6 months will either fall by 20% ($80) or rise by 30% ($130)
  - but we must determine the risk neutral probabilities of the up and down states that produce a 2.5% expected return for XYZ stock

\[
\text{Price Call Option} = \frac{\Sigma_i p_i x r_i}{r_f}
\]

\[
\text{Solve expected return } E(R) = \Sigma_i p_i x r_i = r_f \text{ for } p_i
\]

\[
E(R_{XYZ}) = Pr_{RN}(UP) \times 30\% + (1 - Pr_{RN}(UP)) \times (-20\%) = 2.5\%
\]

- \(Pr_{RN}(UP) = 45\%\)
  - this is the “risk neutral probability” that XYZ will go up in price given our simple binomial model

- once we have risk-neutral probabilities, we can estimate the expected option payoff in the future
  - \(E(\text{option payoff}) = Pr_{RN}(UP) \times 40 + (1 - Pr_{RN}(UP)) \times 0\)
    - \(= (.45) \times 40 + (.55) \times 0 = \$18\)
    - this is the expected payoff in January 2009
  - \(\text{PV of this expected payoff (@ } r_f = \$18/1.025 = \$17.56\)
    - this is exactly the same price as the replication approach

- 2 ways to value an option
  - find replication portfolio and determine PV of stock and bond parts
  - determine risk neutral probabilities and calculate PV of expected option payoff
Pricing a Put Option

- 6 month put option on XYZ
  - XYZ stock price is 100 today and 6 month interest rate = 2.5%
    - we will continue assumption that stock price in 6 months will be either 80 or 130
    - exercise (strike) price on put is 90
  - payoff on put option in these two states = max((K – P),0)

XYZ January 2009 put option (K of 90) = ??

\[
\begin{align*}
\text{Put option (UP)} &= \max(90 - 130, 0) = 0 \\
\text{Put option (DN)} &= \max(90 - 80, 0) = 10
\end{align*}
\]

- replicating portfolio equations estimating N and M
  - in the Up state: \( 130 \times \Delta + 90 \times M = 0 \)
  - in the Down state: \( 80 \times \Delta + 90 \times M = 10 \)
  - solution \( \Delta = -0.20 \) and \( M = 0.2888 \)
    - buy -0.2 (sell 0.2) shares and buy 0.2888 units of the bond

\[
\text{price of put today} = \text{price of replicating portfolio}
\]
\[
\begin{align*}
\text{Put} &= -0.20 \text{ shares of XYZ stock} + 0.2888 \text{ units of $90 FV bond} \\
&= -0.2 \times 100 + 0.2888 \times 90/1.025 = 5.36
\end{align*}
\]

Risk Neutral Method to Price Put

- We already know risk neutral probabilities for XYZ’s stock price movement at 6 months
  - \( Pr_{RN}(\text{UP}) = 45\% \)

- get expected payoff of put option and take PV
  - using with these risk neutral probabilities
  \[
  E(\text{Payoff to put}_{\text{Jan 09}}) = 45\% \times 0 + 55\% \times 10 = 5.50
  \]
  - take PV of expected payoff
  \[
  PV(E(\text{Payoff to put}_{\text{Jan 09}})) = 5.50/1.025 = 5.36
  \]
  - produces same price for put as other method

- now let’s check put-call parity to see if all is OK
  - 6 month put and call options on XYZ with \( K = 90, P = 100 \)
    - put-call parity
      - value of put = value of call - share price + PV(exercise price)
      \[
      5.36 = 17.56 - 100 + 90/1.025 = 5.36 \quad \text{(yeah!)}
      \]
Black-Scholes Formula

- The trick to option pricing is setting up replicating portfolios and valuing them across uncertainty
  - we have done simple approach of only two possible outcomes over the life of the option
    » this is known as the binomial method
  - we can make this more realistic by breaking the time to maturity into two 3-month periods, each with an up / down price movements to produce 4 price outcomes at month 6
    - but we could make it more realistic by modeling six 1-month periods with $2^6 = 64$ possible stock outcomes at maturity
  - eventually we could move to continuous time specification where at every moment in time the stock price can move up or down
    - this is what Black and Scholes did, they solved the replicating portfolio formula in a continuous time specification
  - at every moment in time the call option value is given by
    $$\Delta x \text{ share price } + M x \text{ bond}$$
    - they just discovered a elegant way to define $\Delta$ and $M$ for continuous time movements in share price

Beautiful Finance

- Their specification looks complicated but is very elegant
  - call option $= \Delta x \text{ share price } + M x \text{ bond}$
    $$= \left[ N(d_1) x P \right] - \left[ N(d_2) x PV(K) \right]$$
  - where
    $$d_1 = \frac{\ln[P / PV(EX)] + \sigma \sqrt{t}}{\sigma \sqrt{t}}$$
    $$d_2 = d_1 - \sigma \sqrt{t}$$
  - $N(d)$ = cumulative normal probability density function
  - $K$ = exercise price of option
  - $PV(K)$ = present value of exercise price at continuously compounded risk free rate $= K/e^{rt}$
    » $e$ = exponential function and $r$ = risk-free APR
  - $t$ = time to option expiration (expressed in years)
  - $P$ = current price of stock
  - $\sigma$ = standard deviation per year of continuously compounded stock price
  - $N(d)$ is the probability that a random variable $x$, with a standard unit normal distribution, will be less than or equal to the value, $d$
Normal Distribution

- The normal distribution is the classic bell shaped distribution in statistics
  - this distribution explains the probability of final outcomes for a large number of identical random events

- a standard normal distribution is just a special distribution with a mean value of zero and a standard deviation of one
  - standard deviation measures the spread of the distribution
    - for a normal distribution 95% of the possible realizations lie within \( \pm 1.965 \) standard deviations of the mean

\[ N(X) \text{ function} \]

- \( N(x) \) = what fraction of the area under the distribution is below (to the left of) \( x \) standard deviations from the mean
  - \( N(\infty) = 1.00, N(1) = .8414, N(0) = .50, N(-1) = .1586, N(-\infty) = 0 \)
  - can obtain values using standard normal distribution tables in back of textbook or Excel function \text{NORMSDIST}(X)
    - when using the tables: \( N(-X) = 1 - N(X) \)
Example Using Black-Scholes Formula

- **Valuing 6 month XYZ call (Jan 2009, K = 90)**
  - $P = 100$, $K = 90$, $t = .5$ (half a year)
  - $\sigma = .39$ (std dev of continuously compounded annual return - assumed)
  - $r = \text{risk free annual rate (APR)} = 5\%$ so $PV(K) = K/e^{rt} = 90/e^{.05 \times .5}$
    - to discount in continuous time use $e^{rt}$ instead of $(1 + r/(360/t))$ or $(1 + r)^t$
  - **B-S formula for value of a call**
    
    \[
    [N(d_1) \times P] - [N(d_2) \times PV(K)]
    \]

  - $d_1 = \left[ \ln\left( \frac{100}{90/e^{.05 \times .5}} \right) \right]/(0.39 \times .51/2) + (0.39 \times .51/2)/2 = .6106$
  - $d_2 = .6106 - (0.39 \times .51/2) = 0.3348$
  - value of call = $[N(.61) \times 100] - [N(.34) \times 90/e^{.05 \times .5}]$
  - **calculate N(x) using tables in appendix or Normsdist in Excel**
    - from tables $N(.61) = .7291$, $N(.34) = .6312$ ($0.6312 + 0.6331)/2$
    - $\text{Normsdist (.6106) = 0.7293, Normsdist(.348) = 0.6311}$
  - call option value = $0.7293 \times 100 - 0.6311 \times 90/e^{.05 \times .5} = 17.53$
  - this almost exactly the quoted price of the call from slide 7
  - indicates that 0.39 is close to the market's estimate of $\sigma$ for XYZ

Summary

- **Options**
  - types and terminology
    - calls and puts, American vs European, exercise price
  - options and corporate finance
    - debt and equity valuation using option theory
  - determinants of option prices
    - asset prices, exercise price, interest rate, time, price volatility
  - valuation of options
    - replication portfolios, risk neutral probabilities, Black Scholes

- also, be sure to look at
  - chapter on real options
  - operating options within investment decisions
  - chapter on warrants and convertibles
    - warrants are another name for stock options
    - convertibles are bonds that can be converted into stock

SAIS 380.760, Lecture 9 Slide 37

SAIS 380.760, Lecture 9 Slide 37