Financial Decision Making

- Finance decision making is about evaluating costs and benefits
  - some complications:
    - measuring cash value of costs and benefits
    - costs and benefits spread out over time
    - uncertainty about the cash value of future costs and benefits
  - financial decision makers depends on other skills to help measuring costs and benefits
    - marketing – revenues based upon market size and advertising
    - economic – price and quantity tradeoffs
    - organizational behavior – impact of structure on productivity
    - strategy – behavior and response of competitors
    - operations – production costs
  - in all cases we try to use market prices to determine cash values of these amounts
    » market value in monetary terms
Adjusting for Values at Different Times

- **Time value of money**
  - to move values of money across time, we adjust for *time value*
    - use an **interest rate factor** to move values across time
    - interest rate factor = \((1 + r_f)\)
      - \(r_f\) is the risk-free interest rate for the period (used for "certain" values)
      - the period for a standard interest rate quote is one year
    - this interest rate factor is like an exchange rate across time periods as it has units of \((\text{value in future period} / \text{value today})\)
    - the interest rate factor is used to compute values in the future

**Example**

*You are offered $1,000 today and the interest rate for one year is 5%; what is the value of this offer in one year?*

interest rate factor = \(1 + 5\% = 1.05\)

\(\$1,000 \text{ today} \cdot (1.05 \text{ value in 1 year/value today}) = $1,050 \text{ in 1 year}\)

- we call $1,050 the **Future Value** of $1,000 today
  
  \[
  \text{Future value} = \text{value today} \cdot \text{interest rate factor}
  \]

Adjusting for Values at Different Times

- We can also convert values in the future to today
  - we do this by dividing future values by the interest rate factor
    - we refer to \(1/(1 + r)\) as the **discount factor**

**Example**

*You are offered $1,000 in 1 year and the interest rate for one year is 5%, what is the value of this offer today?*

discount rate factor = \(1/(1 + 5\%) = (1/1.05) = 0.95238\)

\(\$1,000 \text{ in 1 yr} \cdot (\text{value today/1.05 value in 1 yr}) = $952.38\)

- we call $952.38 the **Present Value** of $1000 in 1 year
  
  \[
  \text{Present value} = \text{value in future} \cdot \text{discount factor}
  \]

- note: future value and present value are related
  - if $1,000 is the present value of $1,050 in one year
  - then $1,050 will be the future value of $1,000 today
Net Present Value

- The main decision rule in finance is known as Net Present Value (NPV)
  - measure all costs and benefits to a project in terms of their PV
    - this is their cash values as of today
    - take difference between PV of benefits and costs
      \[
      \text{Net present value of project} = \text{PV(benefits)} - \text{PV (costs)}
      \]
  - NPV decision rule says to take projects with positive NPV
    - the NPV of a project reflects the value of the project in terms of cash today

Example

You have a project with PV(benefits) = $100 and PV(costs) = $75; then NPV of project = $100 - $75 = $25

taking this project affects your wealth just like getting $25 cash today

- when deciding amongst projects, chose the project with the largest NPV
  - this projects generates the most value (cash equivalent today)

Arbitrage

- Arbitrage is an important force in finance
  - arbitrage is the buying and selling of an item in different markets at the same time to take advantage of different prices
    - by definition arbitrage is a positive NPV activity
  - a well functioning competitive market is one with no arbitrage opportunities => they will be transacted away
    - we call such a market a normal market
  - in a normal market, arbitrage insures that the law of one price will hold
    - prices of equivalent items must trade for the same price across different markets (or differences in price will not last long)
    - this is especially true for investment opportunities
      - investment opportunities traded in markets are called securities
        - these are most commonly stocks, bonds, and derivatives
    - to prevent arbitrage for securities:
      - price of security = PV(all cash flows paid by the security)
No Arbitrage Pricing of Securities

- All securities must trade today for a price equal to the PV(all future cash flows) to prevent arbitrage

**Example**

You have a security that pays $100 a year from now with no risk. The risk-free interest rate for one year is 10%. What is the no arbitrage price of the security today?

Cash flows of security = $0 today + $100 in 1 year

Discount factor with rf = 10% => 1/1.10 = 0.909

PV(cash flows of security) = $0 + $100 · (1/1.10) = $90.90

No Arbitrage price of security today = $90.90

- if the price was not $90.90 there would be an arbitrage opportunity

What if the price of the security is currently $95?

Agents would sell the security today for $95, investing $90.90 today at the 10% interest rate to cover the $100 payment in one year. This would leave the agent with $4.10 (95 – 90.90) today.

NPV of Security Trading

- By trading at arbitrage-free prices, the expected NPV of buying a security is zero

NPV (buying security) = PV (all expected CFs) – Security Price = 0

- the same is true for NPV of selling a security

- the NPV of security trading in a normal market is zero

- from the market’s perspective (expectations) there is no net value transferred between buyer and seller when trading

- this leads to the Separation Principle

Security transactions in normal markets neither create nor destroy value on their own. Thus we can evaluate the NPV of an investment decision separately from the decision of now to finance (raise money for) the investment.

- in other words, the NPV of an investment project is not effected by how it is financed
Risk Aversion & Time Value of Money

- When cash flows are not certain, investors adjust the interest rate they use for determining present or future values

  - investors typically add a **risk premium** to the risk free interest rate when doing time value of money calculations with cash flows that are risky
    - by risky we mean that the cash flows may deviate from expectations so that the security’s rate of return may vary
  - the risk premium is an additional return factor to compensate the investor for this risk
    - the more variable the security’s possible returns, the greater the risk premium

  - thus the interest rate used for risky assets is
    
    \[ r_s = r_f + \text{risk premium for security } s \]

    - interest rate factor for security \( s \) = \( 1 + r_s \)
    - discount factor for security \( s \) = \( 1 / (1 + r_s) \)

Time Value of Money and Time

- **Timelines**
  - when doing TVM problems is it critical that you get used to drawing timelines

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flow ( C_t )</td>
<td>-1,000</td>
<td>2,000</td>
<td>1,000</td>
</tr>
</tbody>
</table>

- time is in terms of periods (typically years)
  - though could be months, semiannual periods, quarters
- cash flows are listed at the time they are assumed to occur
  - a cash flow for period 1 occurs at the end of period 1
    - this is also beginning of period 2
- cash flows that you pay are typically negative and cash flows you receive are typically positive (arbitrary, key is opposite signs)
  - in timeline above, you pay $1000 today and receive $2000 one period from now and receive $1000 two periods from now
Three Rules of Time Travel

- **Rule 1:** only values measured as of the same point in time can be compared or combined
  
  “a dollar today is worth more than a dollar tomorrow”
  
  - you cannot make direct comparison or combine values unless their cash value is measured as of the same point in time

- **Rule 2:** to move a value forward in time you must compound it
  
  - to compound you multiply a value by the interest rate factor \((1 + r)\)
  
  - this is equivalent to determining its future value

- **Rule 3:** to move a value backwards in time you must discount it
  
  - to discount you multiply by the discount factor \(\frac{1}{1 + r}\)
  
  - this is equivalent to determining its present value

---

**FV over Time**

- **Compounding**
  
  - when moving values forward in time we compound
  
  - compounding over multiple periods recognizes interest that is earned (or must be paid) on previous interest

  **Example**

  Determine the FV of $100 in 2 years with interest rate of 8% per year

  
  \[
  \begin{array}{cccc}
  \text{time (from now)} & 0 & 1 & 2 \\
  \text{investment } C_t & 100 & 108 & FV^t \\
  \end{array}
  \]

  - in year 1 the FV is \[100 \cdot (1.08)\] = $108.00
    - this is $8 of simple interest -- the interest on the initial amount
  
  - in year 2 the FV is \[108 \cdot (1.08)\] = $116.64
    - the $8.64 of interest in year 2 is $8 of simple interest and $.64 of compound interest
    - the $.64 is interest on last year’s $8 of interest

  we can do the calculation in one step using powers

  \[
  FV_2 = 100 \cdot (1.08)^2 = 116.64
  \]
Compounding

- General compounding rule for a future value \( n \) periods from now
  \[
  \text{FV}_n = C_0 \cdot (1 + r)^n
  \]
  - where
    - \( C_0 \) is the cash value today
    - \( r \) is the interest rate for the period
    - \( n \) is the number of periods in the future

- Compounding is a powerful force in the world
  - allows anyone to take small amounts of money and turn them into large amounts of money (just need time)

\textbf{Example}

Determine FV of $1,000 today in 50 years at 8% interest rate?
\[
\text{FV}_{50} = 1000 \cdot (1.08)^{50} = 46,901
\]
but add 10 more years \( \text{FV}_{65} = 1000 \cdot (1.08)^{65} = 148,780 \)

- secret to compounding is exponential growth
  - value grows more than linearly

\textbf{Impact of Compounding}

value of $1 in future compounded at various rates

\textbf{“Rule of 72”}

the number of years for an investment to double is approximately the number 72 divided by the interest rate (%)

ex. at 8% interest the value doubles every \( (72/8) = 9 \) years
Discounting

- when moving values backwards in time we discount
  - discounting over multiple periods also becomes exponential
  - except we compound using the discount factor $1/(1+r)$

Example

Determine the PV of $100 payment to be received 2 years from now with an interest rate of 8% per year

<table>
<thead>
<tr>
<th>Time (from now)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment $C_t$</td>
<td>PV?</td>
<td>$100</td>
<td>$100</td>
</tr>
<tr>
<td></td>
<td>85.73</td>
<td>92.59</td>
<td>92.59</td>
</tr>
</tbody>
</table>

in year 1 the PV is $[100 \cdot (1 / 1.08)] = 92.59$

in year 0 the PV is $[92.59 \cdot (1 / 1.08)] = 88.73$

we can do the calculation in one step using powers

$PV_2 = 100 \cdot (1/1.08)^2 = 100/1.08^2 = 85.73$

- compounding rule for a future value $n$ periods from now

$$PV_t = C_{t+n}/(1+r)^n = C_{t+n} \cdot (1+r)^{-n}$$

Multiple Cash Flows

- Suppose we have multiple cash flows
  - an investment produces cash flows $C_0$, $C_1$, $C_2$, ..., $C_N$
    - what is the present value of this stream?
  - we just calculate the PV of each cash flow and sum

$$PV = \sum_{n=0}^{N} \frac{C_n}{(1+r)^n}$$
**Net Present Value of a Stream of CFs**

- The NPV of a stream of cash flows with benefits and costs is just PV(benefits – costs)

  **Example**
  Consider a project with the following cash flows
  
<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flows</td>
<td>-100</td>
<td>20</td>
<td>60</td>
<td>50</td>
</tr>
</tbody>
</table>

  the interest rate is 12% ⇒ discount factor is 1/1.12 = 0.8929

  **What is NPV of this project’s cash flow stream?**
  
<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV</td>
<td>-100</td>
<td>17.86</td>
<td>47.83</td>
<td>35.59</td>
</tr>
</tbody>
</table>

  **or**
  
<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV</td>
<td>-100</td>
<td>20(0.8929)</td>
<td>60(0.8929)</td>
<td>50(0.8929)</td>
</tr>
</tbody>
</table>

  summing PV of both benefits (+) and costs (-) gives NPV
  
  \[
  \text{NPV} = -100 + 17.86 + 47.83 + 35.59 = 1.28
  \]

  ⇒ this project creates small amount of new value

**FV of a Stream of Cash Flows**

- The FV of a stream of cash flows can be determined either directly or from its PV

  **Example**
  Consider the same project as before
  
<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flows</td>
<td>-100</td>
<td>20</td>
<td>60</td>
<td>50</td>
</tr>
</tbody>
</table>
  
  the interest rate is 12% ⇒ interest rate factor is 1.12

  **What is FV of this project’s cash flow stream at end of period 3?**
  
  1. \[ \text{FV}_3 = PV \cdot (1 + r)^n = 1.28 \cdot (1.12)^3 = 1.80 \] or
  2. \[ \text{FV}_3 = -100(1.12)^3 + 20(1.12)^2 + 60(1.12) + 50 = 1.80 \]

  summing FV of both benefits (+) and costs (-) gives total FV
  
  \[
  \text{FV}_3 = -140.49 + 26.09 + 67.20 + 50 = 1.80
  \]
  
  or \[ \text{FV}_3 = PV_0 (1 + r)^3 = 1.28 \cdot (1.12)^3 = 1.80 \]

  ⇒ both approaches provide the same FV
Special Cases for Cash Flow Stream

- PV or FV of a stream can always be determined by the summing of period by period CF
  - however, there are some special cases for which we can determine PV (or FV) in one step
    - perpetuities and annuities

- Perpetuities
  - stream of equal cash flows \( C \) at regular intervals forever
    - commonly used for equities, endowments, etc.
  - for cash flow \( C \) forever, and interest rate \( r \)
    \[
    PV(C \text{ in perpetuity}) = \frac{C}{r}
    \]
  - Example
    Consider a perpetuity of $20 with an interest rate of 8%.
    
    | period | 0  | 1  | 2  | 3  | . . . forever |
    |--------|----|----|----|----|--------------|
    | cash flows | PV? | 20 | 20 | 20 | . . .         |
    | PV = \( \frac{C}{r} \) = \$20 / 0.08 = $250 |
    - note: PV is determined the period before the first cash flow

Annuities

- An annuity is a stream of \( N \) equal cash flows
  - common in consumer loan agreements, lotteries, etc.
    - when CFs occur at the end of each period it is an ordinary annuity
    - when CFs occur at the beginning of each the period it is an immediate annuities or annuity due
  - for annuities we can determine both PV and FV as there are a finite number of cash flows
    
    | period | 0  | 1  | 2  | 3  | . . . |
    |--------|----|----|----|----|------|
    | cash flows | C   | C   | C   | C   |      |
    - solutions 
      \[
      PV = \sum_{n=1}^{N} \frac{C}{(1+r)^n} \quad \text{and} \quad FV_N = \sum_{n=1}^{N} C \cdot (1+r)^{N-n} = PV \cdot (1+r)^N
      \]
      - for ordinary annuities these summations have simple solutions defined by annuity factors (AF)
      \[
      PV = C \cdot PVAF \quad \text{where PV annuity factor is}\quad PVAF = \frac{1-(1+r)^N}{r}
      \]
      \[
      FV = C \cdot FVAF \quad \text{where FV annuity factor is} \quad FVAF = \frac{(1+r)^N-1}{r}
      \]
FV Example

You decide to save $2,500 a year for retirement

- you just turned 28 years old and plan to retire when you turn 70
- you will invest for retirement at the end of each year
- you expect to earn an interest rate of 10%

What is the future value of your retirement savings?

Details

- N = 42 years until retirement
- C = $2,500
- interest rate factor = 1.10

<table>
<thead>
<tr>
<th>period</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>41</th>
<th>42</th>
</tr>
</thead>
<tbody>
<tr>
<td>age</td>
<td>28</td>
<td>29</td>
<td>30</td>
<td>31</td>
<td>...</td>
<td>69</td>
<td>70</td>
</tr>
<tr>
<td>cash flows</td>
<td>2,500</td>
<td>2,500</td>
<td>2,500</td>
<td>2,500</td>
<td>...</td>
<td>2,500</td>
<td>2,500</td>
</tr>
</tbody>
</table>

FV = C · FVAF (it is an ordinary annuity)

FVAF = \(\frac{(1+r)^N - 1}{r}\)

\(FV = 2,500 \cdot 537.64 = 1,344,100\)

Note: $2,500 · 42 payments is only $105,000 ($1,239,100 is interest)

PV Example

You win a $1,000,000 lottery

- the prize is actually 20 annual payments of $50,000
- what is the present value of your award?

Details

- N = 20 annual payments
- C = $50,000
- interest rate factor = 1.10

<table>
<thead>
<tr>
<th>period</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flows</td>
<td>50,000</td>
<td>50,000</td>
<td>50,000</td>
<td>50,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

PV = C · PVAF (it is an ordinary annuity)

PVAF = \(\frac{1- (1+r)^{-N}}{r}\)

\(PV = 50,000 \cdot 8.514 = 425,700\)

Note: $50,000 · 8.514 = $425,700

The present value of the lottery winnings is less than $500,000
Immediate Annuities

- When the payments for the annuities start immediately, we have to adjust the annuity factors
- first cash flow is right now, not at the end of the first period
  - for immediate annuities with N cash flow payments the summations have solutions have to be adjusted one period
  \[
  PV = C \cdot PVAF \quad \text{where} \quad PVAF_{IMMED} = \frac{1 - (1 + r)^N}{r} \cdot (1 + r)
  \]
  \[
  FV = C \cdot FVAF \quad \text{where} \quad FVAF_{IMMED} = \frac{(1 + r)^N - 1}{r} \cdot (1 + r)
  \]
- in the retirement savings example, if you started savings now and still made 42 contributions the FV of the savings is
  \[
  FV = 2,500 \cdot PVAF_{IMMED} = 2,500 \cdot [537.64 \cdot 1.1] = $1,478,510
  \]
  - an increase of $134,410 over the ordinary annuity
- in the lottery example if you received the first of 20 payments now the FV of the winnings is
  \[
  FV = 50,000 \cdot PVAF_{IMMED} = 50,000 \cdot [8.514 \cdot 1.1] = $468,270
  \]
  - an increase of $42,570 over the ordinary annuity

Growing Cash Flows

- Another special case is a growing perpetuity
  - consider a payment stream with the first cash flow \( C_1 \) growing a rate \( g \), forever with an interest rate of \( r \)
  \[
  PV \text{ of growing perpetuity} = \frac{C_1}{r - g}
  \]
  - note it is necessary that \( r > g \) or the PV will be infinite

Example

- Consider a stream of cash flows that starts at $1000 in one year and grows at a constant rate, \( g \), of 5% a year, forever
- years from now: \( 1 \), \( 2 \), \( 3 \), \( 4 \), \( 5 \), \( 6 \), ....
- cash flow: \( 1,000 \), \( 1,050 \), \( 1,102.5 \), \( 1,157.6 \), \( 1,215.5 \), \( 1,276.3 \), ...
- interest rate is 10%
  - the PV of this growing perpetuity will be
  \[
  PV = \frac{1,000}{.10 - .05} = $20,000
  \]
  - to determine a future value \( N \) periods in the future use
  \[
  FV = PV \cdot (1 + r)^N
  \]
Growing Perpetuity Example

- You want to value a share of a firm that you expect to last forever
  - suppose you expect the firm will pay you $1/share next year
  - you expect this payment per share will grow at a rate of 3% per year forever
  - the interest rate for this investment is 10%
- we can determine the value of this stream today using the growing perpetuity formula
  \[ C_1 = \$1, \quad g = 3\%, \quad r = 10\% \]
  \[ PV = \frac{C_1}{r - g} = \frac{\$1}{.10 - .03} = \$14.29 \]
  - the price per share of this firm is $14.29
    - this is how much you should be willing to pay for the share of stock
  - in 10 years time you expect that this share will be worth
  \[ FV_{10} = PV \cdot (1+r)^{10} = \$14.29 \cdot (1+10\%)^{10} = \$37.06 \]

Growing Annuities

- A growing annuity is a finite set of CFs that grow from \( C_1 \) for \( N \) periods at rate \( g \)
  - we can determine the PV of a growing annuity from the difference between two growing perpetuities
    - consider an \( N \) year growing annuity starting with \( C_1 \) and rate \( g \)
      \[ \begin{array}{cccccccc}
         \text{year} & 0 & 1 & 2 & 3 & 4 & \ldots & N-1 & N \\
         \text{Perp1} & C_1 & Cl(1+g) & Cl(1+g)^2 & Cl(1+g)^3 & \ldots & Cl(1+g)^{N-2} & Cl(1+g)^{N-1} & Cl(1+g)^N \\
         \text{-Perp2} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & Cl(1+g)^N \\
         \text{Diff} & C_1 & Cl(1+g) & Cl(1+g)^2 & Cl(1+g)^3 & \ldots & Cl(1+g)^{N-2} & Cl(1+g)^{N-1} & Cl(1+g)^N \\
      \end{array} \]
    - it is just the difference in cash flows of two growing perpetuities
      perpetuity 1 starts with \( C_1 \) in year 1 and payments grows at \( g \) thereafter
      perpetuity 2 starts with \( C_1 \cdot (1+g)^{N} \) in year \( N+1 \) and grows at \( g \) thereafter

Growing Annuities

- The PV of the growing annuity is the difference between the PV of the growing perpetuities
  - growing perpetuity 1 from above has a PV of
    \[ PV = \frac{C_1}{r - g} \]
  - PV of growing perpetuity 2 can be determined in two steps
    1. in year N its PV \( N \) = \( C_{N+1}/(r - g) = C_1 \cdot (1+g)^N/(r - g) \)
      - standard PV growing perpetuity formula applied at period \( N \)
    2. bring this value back to time zero = \[ \frac{C_1 \cdot (1+g)^N/(r - g)}{(1 + r)^N} \]
  - differencing these PV gives us
    \[ PV = \frac{C_1}{r-g} - \frac{C_1(1+ g)^N/(r-g)}{(1 + r)^N} \]
  - rearranging: \[ PV = \frac{C_1}{r-g} \cdot \left[1 - \frac{(1+ g)/(1+ r)}{(r-g)}\right] \]
  - this equation captures the PV of all the other special cases
    - if \( g = 0 \) we have the annuity formula: \( PV = C_1 \cdot \left( \frac{1}{r} \right) \cdot \left[1 - \frac{(1+r)^N}{(r+1)^0}\right] \)
    - if \( N \) goes to infinity we get a growing perpetuity: \( PV = C_1/(r - g) \)
    - if \( g = 0 \) and \( N \) goes to infinity we have a perpetuity: \( PV = C/r \)
  - note \( FV_n \) of growing annuity = \( PV \cdot (1 + r)^n \)

Growing Annuity Example

- You are trying to decide when you can retire
  - in retirement you
    - want your assets to generate income for you of $80,000
      - you will take payment at the end of the year
    - require this payment to grow with the expected inflation rate of 3% to maintain your standard of living
    - expect to live in retirement for 25 years
    - expect to earn an interest rate of 6% on your assets
  - how much do you need to have saved to retire?

<table>
<thead>
<tr>
<th>year</th>
<th>retire</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>…</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>80K</td>
<td>80K(1+3%)</td>
<td>80K(1+3%)²</td>
<td>…</td>
<td>80K(1+3%)²⁴</td>
<td></td>
</tr>
</tbody>
</table>

- PV of this stream = \( C_1/(r-g) \cdot \left[1 - \frac{(1+ g)/(1+ r)}{(r-g)}\right] \)
  - \( PV = \frac{80,000}{(0.06 - 0.03)} \cdot \left[1 - \frac{(1.03/1.06)^{25}}{(1.06/1.03)}\right] \)
  - \( = \frac{2,666,666 \cdot [0.5122]}{2,666,666} = \frac{1,365,866}{2,666,666} \)
  - in order to retire at this income you need to have accumulated a retirement account balance of $1,365,866
Exchange Rates and TVM

- Suppose you have a choice to invest $1,000
  - US$ deposit at 5% per year or a ¥ deposit at 2% per year
    - current exchange rate is $0.01 per yen (¥100 / $)
  - the best choice depends on the future exchange rate (XR)
    - if you invest in US$ you have $1,050 in one year
      - $1,000 \cdot (1 + .05) = $1,050
    - if you invest in ¥ you have ¥102,000 in one year
      - $1,000 in converts into ¥100,000 and is invested at 2%
        \[ \Rightarrow $1000 \cdot ¥100/$ \cdot (1 + .02) = ¥102,000 \]
  - there is a future XR($/ ¥) that makes these equal
    \[ \text{US$1,050} = \text{XR($/ ¥)}_1 \cdot ¥102,000 \]
    \[ \Rightarrow \text{XR($/ ¥)}_1 = 1.050/102,000 = 0.01029 \]
    - $0.01029/ ¥ (or ¥97.14/$) is the break-even exchange rate
      - if you expect future XR($/ ¥) > 0.01029 invest in ¥ deposit
      - if you expect future XR ($/ ¥) < 0.01029 invest in US$ deposit

NPV and Different Currencies

- Fundamental rule of time value with multiple currencies
  - the cash flows and interest rates must be denominated in the same currency
  - example: consider projects in both the US and Japan each with initial investment of $1,000
    - US project pays off $600 per year for 5 years
      - US$ interest rate, r_{US$} = 6% (APR)
    - Japanese project pays off ¥56,000 per year for 5 years
      - Japanese yen interest rate, r_¥ = 3% (APR)
      - current exchange rate S($/ ¥)_0 = $0.01/ ¥ (or ¥100/$)
  - NPV = PV of benefits – PV of costs
    - remember to keep currency of interest rate and currency of cash flows the same when taking PV
    - note also that arbitrage implies \[ \text{PV}_0($) = \text{PV}_0(¥) \cdot S($/ ¥)_0 \]
International NPV

- Which project has higher NPV?
  - US project: NPV = -$1,000 + $600 \cdot PVAF(N=5, r=6\%)
    - NPV\(_0\) = $1,527.42
  - Japan project NPV = -$1,000 + PV of ¥ payments in $
    - PV of Japanese payments in ¥
      - ¥56,000 \cdot PVAF(N=5, r=3\%) \cdot PV(¥) = ¥256,464
    - PV of Japanese payments in US$ = PV(¥) \cdot S(\$/¥)\(_0\)
      - PV(¥) = ¥256,464 \cdot $0.01/¥ = $2,564.64
    - NPV of Japanese project = $2,564.64 - $1,000 = $1,564.64

Japan NPV = $1,564.64; US NPV = $1,527.42

- you would choose Japanese project since NPV is higher
- if you had used US interest rate of 6\% for ¥ cash flows you would have incorrectly chosen US project
- with multiple currencies: ALWAYS use interest rate of the currency of the cash flows when you are discounting

Annual Percentage Rate

- Rates of interest are almost always stated in annual percentage rate (APR) terms
  - this is often referred to as the “annual interest rate”
  - by definition the APR is the product of the rate of return, r, earned over the period times and the number of periods per year, m
    - \[ APR = r \cdot m \]
  - the periodic rate of return is the rate used to calculated interest each interest calculation period
    - when interest is calculated once per year then the APR is the periodic interest rate
  - when interest is calculated more often then once a year we need to know the periodic interest rate to determine the APR or vice versa
    - if the per period rate of return, r, is 1.5\% and it is compounded 12 times a year, then the APR = r \cdot m = 1.5\% \cdot 12 = 18\%
    - if the APR is 8\% and it is compounded quarterly then the per-period interest rate is r = APR/m = 8\% / 4 = 2\%
De-Annualizing APRs

- Adjusting APR for short holding periods
  - $10,000 in a 90-day deposit paying 8% APR
    - actual interest rate earned over the period is a fraction of the annual rate as the holding period is less than one year
    - periodic interest rate earned = APR / (360/n)
      - where n = number of days you earn interest
        » the money market and international markets assume a 360 day year
  - what is the value of your deposit at maturity?
    - assume that you are paid interest only at the end of investment
      - 90 day rate of return = 0.08/(360/90) = 2.0%
      - value of deposit with simple interest
        $10,000 · (1 + 0.02)) = $10,200.00
  - what if they compounded interest monthly (3 times)?
    - monthly rate of return = 0.08/(360/30) = .6667%
    - FV = $10,000 · (1.006667)³ = $10,201.35

Effective Annual Rate

- Making rates comparable: effective annual rate
  - 8% APR compounded monthly produces better annual return than 8% APR compounded annually
  - how to make rates with different compounding comparable?
    - use an Effective Annual Rate (EAR)
      - this is the “equivalent annual interest rate” if interest were calculated only once a year rather than m times a year
        \[ \text{EAR} = (1 + \text{APR}/m)^m - 1 \]
      - example: APR = 8%
        - annual compounding (m=1) => EAR = (1+.08/1)¹ – 1 = 8%
        - quarterly compounding (m = 4) => EAR = (1+.08/4)⁴ – 1 = 8.24%
        - monthly compounding (m =12) => EAR = (1+.08/12)¹² – 1 = 8.30%
        - daily compounding (m = 365) => EAR = (1+.08/365)³⁶⁵ – 1 = 8.327%
        - as m →∞, \[ \text{EAR} = \exp^{\text{APR}} - 1 \] => EAR = \exp^{.08} – 1 = 8.33%
          » exp is the number 2.718; when m →∞ the EAR becomes the continuously compounded rate
  - EAR is the correct rate when using annual periods
Interest Rates and Maturity

- We have been assuming the same interest rate for different periods cash flows
  - the annual interest rate will vary with the maturity
    - this relation is known as the **term structure** or **yield curve**

**US Treasury Yield Curve**


![US Treasury Yield Curve](http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/yield.html)

Interest Rates and Maturity

- A sloping term structure implies we should be using different interest rates for different maturities
  - we should be using \( r_n \) for each maturity \( n \) rather than the same \( r \) for all \( n \)
    - thus the PV formulas should really be
      \[
P V = \frac{C_n}{(1 + r_n)^n}
\]
      so
      \[
P V = \sum_{n=1}^{N} \frac{C_n}{(1 + r_n)^n}
\]
  - alternatively the \( r \) we use needs to be an appropriately weighted average \( r \) for the set of cash flows
  - when “exactly” valuing a set of specific cash flows, one needs to use the rates appropriate for each CF’s timing
    - or an appropriately weighted “average” rate for all flows
      - this is most important for bond pricing and other derivative securities where the cash flows are contracted
      - for investment projects with risky flows it is common to assume a constant “average” rate for all periods
Rates of Return and Taxes

- When taxes are involved, we must distinguish between before- and after-tax interest rates
  - after tax interest rate is what you earn after taxes are paid on the earnings
  - after-tax interest rate = (1-tax rate) x (pre-tax interest rate)
- example
  - before-tax interest rate = 8%  tax rate = 35%
  - after tax interest rate = (1- 0.35) · 8% = 5.2%
  - thus
    - if you invest $100 for 10 years at 8%, compounded annually with a 35% marginal tax rate (after tax interest rate = 5.2%)
    - you would produce a FV of  $100 · (1.052)^{10} = $166.02
      - this is less than compounding at 8% for 10 years and then paying taxes on the interest
      - [$100 · (1.08)^{10} -100]·(1-35%)+100 = $175.33

Real versus Nominal Rates

- Cash values in the future can be misleading as you don’t know their purchasing power
  - suppose you invest $10,000 today for 20 years at 10%
  - in the future we will have a balance of $67,275
    - $10,000 · (1.10)^{20} = $67,275
      - this seems like a lot of money, but what will it buy in 20 years?
        - need to adjust for changes in purchasing power of money balances
  - two ways to deal with this
    1. adjust current prices into future prices using an expected annual inflation rate and compare against future balance of deposit
      - evaluate future investment purchasing power against future prices
    2. adjust the FV balance for the change in purchasing power
      - adjust nominal balance for compound inflation
      - calculate future value using an interest rate net of inflation
        - from macro we know this is rate as the real interest rate
      - using real interest rates is a very useful way to deal with personal financial calculations
FV in Real Terms

- Assume the inflation rate will be 5% per year
  - suppose $10,000 today will cover 10 months of apartment rent
    - apartment rent is currently $1,000 per month
  - thus this investment today is the same as 10 months of rent
- FV of $10,000 in 20 years is $67,275
  - adjust prices for 20 years of expected inflation
    - in 20 years at 5% inflation apartment rent (now $1,000/month) will increase to $2,653 per month
      - price of monthly rent in 20 years = $1,000 \cdot (1.05)^{20} = $2,653
    - the purchasing power of the FV of the investment in 20 years (in terms of apartment rent) is 25.4 months of rent
      - $67,275 / $2,653 = 25.4 months
      - determine real value by dividing future nominal balance of investment by the expected price of goods in the future
    - the $10,000 increases in nominal terms by 6.73 times, but in real terms by only 2.54 times
      - the future value of the investment is worth the equivalent of 25.4 months of apartment rent

FV in Real Terms

- We can also determine purchasing power of the FV of the investment using a real interest rate, \( r_r \)
  \[
  r_r = \left( \frac{1 + r}{1 + \pi_e} \right) - 1 = \frac{r - \pi_e}{1 + \pi_e}
  \]
  - \( \pi_e \) = expected average annual inflation
  - \( r \) = nominal interest rate
  - as an approximation \( r_r = (r - \pi_e) \)
- do FV analysis in real terms (using real interest rate)
  - real interest rate, \( r_r = (1.10)/(1.05) - 1 = 4.7619\%
  - real FV of investment in 20 years at 4.7619% real interest rate is
    \[
    FV_{20} (\text{real}) = $10,000 \cdot (1 + 0.047619)^{20} = $25,355
    \]
    - this is the future value of the investment adjusted to prices of goods today
  - with apartment rent at $1,000/month today, the real value of this investment will be 25.4 months of rent
    - $25,355 / ($1,000/month) = 25.4 months
  - this is identical to adjusting the nominal FV by compound inflation
    - $67.275 / (1.05)^{20} = $25,355 => 25.4 months rent today
Inflation and Annuities

- We can also do annuities in real terms
  - eliminate inflation so real FV can be compared to today's prices
  - again we use real rate of return, $r_r$, instead of nominal
    \[ r_r = \frac{[(1 + r)/(1 + \pi)] - 1}{(1 - \pi)/(1 + \pi)} \]
  - with annuities we must also have a fixed real cash flow
    - requires that cash flow must change with inflation each period

**Example**

What is the real FV of investing $2,500 of today's dollars annually for each of the next 42 years (end of year) for retirement at 10% interest with 3% expected annual inflation?

- $r_r = (1.10)/(1.03)-1 = 6.796\%$, real payment = $2,500/year, $N = 42$ yrs
- real FV = $2,500 \cdot FVAF(N = 42, r_r = 6.796\%)$
- real FV = $2,500 \cdot \frac{(1+.06796)^{42} -1}{0.06796} = $545,312

- this investment produces a FV that would purchase as much in the future as $545,312 would today
  - payments are in real terms so increase with inflation each year

Inflation and Annuities

- Other times we want to know how much to save to purchase something in the future
  - this involves the real PV of an annuity

**Example**

You want to start saving for your new born’s first year of college education. How much in do we need to start saving each year?

- 18 years until tuition payment is due
- future tuition cost (18 years) in real terms (today’s prices) = $30,000
- expect to earn 5% EAR above inflation on investments (i.e. $r_r = 5\%$)

**details:** $N = 18$ yrs, $r_r = 5\%$, FV = $30,000$ (real)

need to solve for annual (real) PMT

\[ PMT \cdot FVAF(N =18, r = 5\%) = $30,000 \]
\[ PMT \cdot [(1.05)^{18} - 1/.05] = $30,000 => PMT = $1,066 \]

- you need to save $1,066 in real terms (today’s $) each year for next 18 years to cover the first year of tuition
  - if inflation is 3%, in year n you must save $1,066 \cdot (1.03)^n$
    - at the end of the first year you have to save $1,066 \cdot 1.03 = $1,098
Inflation and PV of Annuities

- We can also determine the PV of real annuities

**Example**

You are thinking about how much life insurance to buy. If you die, you would like to provide your dependents a real income of $25,000 a year for the next 15 years.

- A safe real rate of interest for the policy proceeds is 3%

**How big a life insurance policy do you need?**

**Details:** \( N = 15, r = 3\% \) \( \text{PMT} = $25,000 \) (real); Determine PV?

\[
\text{PV} = \text{PMT} \cdot \text{PVAF}
\]

- \( \text{PMT} = $25,000 \) in real terms (so must use real interest rate)
- \( \text{PVAF} (N = 15, r = 3\%) = 11.94 \)
- \( \text{PV} = $25,000 \cdot 11.94 = $298,448 \)

- Today you would want an insurance policy of approximately $300,000 to provide your dependents with enough income to purchase equivalent of $25,000 of consumption at today’s prices each year for next 15 years.
- Next year the policy would need to be 3% larger.

### Summary

- **Basic rule of finance**
  - A dollar today is worth more than a dollar tomorrow

- **Time value of money basics**
  - Present value (PV) and future value (FV)
    - Use discount rate \((1/(1+ r))\) or interest rate factor \((1 + r)\)
    - Compound for number of periods

\[
\text{PV} = C_n / (1+r)^n \quad \text{FV}_n = \text{PV} \cdot (1+r)^n
\]

- **Annuities**
  - Fixed payments for period of time

\[
\text{PVAF} = [1 - (1+r)^N] / r \quad \text{FVAF} = [(1+r)^N - 1]/r
\]

- **Perpetuities**
  - Payments that go on forever
    - \( \text{PV} = C / r \)
  - Growing payments that go on forever
    - \( \text{PV} = C_1 / (r - g) \)

- **Growing annuities**
  - Growing payments that last a fixed period of time
    - \( \text{PV} = [C_1/(r-g)] \cdot [1 - ((1+g)/(1+r))^N] \)
Summary

- Other issues
  - net present value as a decision rule
    - \( \text{NPV} = \text{PV(benefits)} - \text{PV(costs)} \)
    - choose project if it has a positive NPV
    - the NPV is the value (wealth) that the project creates
  - time value in different currencies
    - be sure to keep currency of cash flows and interest rates the same
  - interest rates and periods
    - compounding issues and APR versus EFF rates
  - real versus nominal values
    - real interest rates