Pricing Financial Assets

- In normal markets, financial securities are priced so that their NPV = 0
  - this means that what you pay today to acquire the asset must equal the PV of payments you expect to receive from it
- thus to price securities we need to know three things:
  1. what are the cash flows you expect to receive?
  2. when do you expect to receive them?
  3. what is the discount rate for the security’s cash flows?
    - then we use the PV rule so determine the price so NPV = 0
      - we will use this approach to price bonds and stocks today
- one important variable in pricing securities in this way is the security’s Internal Rate of Return (IRR)
  - this is the rate of interest that discounts the security’s cash flows to determine its arbitrage-free price ($P_0 = PV(\text{future CFs})$)
    - lets examine how we determine the IRR for a simple security
Internal Rate of Return (IRR)

To determine a security’s IRR we need to know its cash flows and its current price

Example
Suppose you invest $75 in a savings bond that will be worth $100 in five years. What is the rate of return you are earning?

what interest rate makes $75 today = $100 in 5 years?
terms: $PV = 75$, $FV = 100$, $N = 5$ (period = a year)
for no arbitrage it must be that: $P_0 = \frac{FV_n}{(1 + IRR)^n}$

$75 = \frac{100}{(1 + IRR)^5}$

$IRR = \frac{100}{75}^{(1/5)} - 1 = 5.92\%$

- this is the actual interest rate earned by the investor per period

- for securities with only one future cash flow, $FV_N$, we have
  $IRR = \frac{FV_N}{PV}^{(1/N)} - 1$

- for bonds, the IRR is known as the yield to maturity or sometimes just the yield

Bonds

- Terminology
  - bond certificate – document that describes terms of the bond
  - terms
    - maturity – period at which final payment will be made
      - term (tenor) of bond is its time remaining until maturity
    - coupons – periodic promised interest payments
      - coupon rate – a set rate of interest that determines the coupon payments in conjunction with the bond’s face value
    - face value (principal) – the notional amount (size) of the bond
      - face value usually round figure like $1,000 or $10,000
      - this is amount returned at maturity (along with final coupon)
  - coupon payment (CPN)
    $CPN = \frac{\text{Coupon rate} \times \text{Face Value}}{\text{Number of Coupon Pmts per Year}}$
  - yield to maturity (YTM)
    - this is the discount rate that sets the price of the bond today equal to the PV of its promised future payments
      - this is the bonds’ IRR
Zero Coupon Bond

- Let’s first consider a zero coupon bond (ZCB)
  - it makes zero coupon payments; only pays face value at maturity

**Example**

Consider a 2 year UST $1,000 ZCB

US government bonds have semi-annual compounding periods

this 2 year bond will have 4 semi-annual periods

suppose this bond sells at issuance for $914.84

timeline:

<table>
<thead>
<tr>
<th>periods</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF</td>
<td>-$914.84</td>
<td>...</td>
<td>$1,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

we can determine that the YTM for 4 periods is 2.25% per period

\[
\text{IRR (YTM)} = \left( \frac{\text{FV}_N}{\text{PV}} \right)^{1/N} - 1 = \left( \frac{1000}{914.84} \right)^{1/4} - 1 = 2.25%
\]

with this YTM we can calculate that bond price is $914.84

\[
\text{Price} = \text{PV(CN)} = \frac{1000}{(1.0225)^4} = 914.84
\]

- annualizing the YTM (since the compounding is semiannual)

\[
\text{EAR} = (1 + \text{YTM})^{\text{Compounding periods per year}} - 1 = (1 + 0.0225)^2 - 1 = 4.5506%
\]

\[
\text{APR} = \text{YTM} \times \text{compounding periods per year} = 2.25 \times 2 = 4.5000%
\]

» the annualized YTMs for risk-free zero coupon bonds of maturity n periods define the spot interest rate for n periods

Coupon Bonds

- Coupon bonds make interest payments
  - pay periodic coupon payments as well as face value at maturity

**Example**

Consider 2-year $10,000 bond with coupon rate of 5%

recall US government bonds work on a semiannual period

\[N = 4 \text{ and periodic coupon rate} = (5\% / 2) = 2.5\%
\]

owner will receive 4 coupon payments of $250 ($10,000 x 5%/2) and receive $10,000 at end of period 4

suppose this bond sells today at issuance for $10,041.30

\[
\begin{align*}
\text{N} = 0 & \quad \text{N} = 1 & \quad \text{N} = 2 & \quad \text{N} = 3 & \quad \text{N} = 4 \\
\text{-$10,041.30} & \quad \text{+$250} & \quad \text{+$250} & \quad \text{+$250} & \quad \text{+$250} + \text{+$10,000}
\end{align*}
\]

we can determine the YTM for this coupon bond from

\[
P = (\text{CPN/YTM}) \times [1 - \left( \frac{1}{(1+\text{YTM})^N} \right)] + \text{Face Val} / (1+\text{YTM})^N
\]

but it is easier to use a financial calculator

inputs \( P/Y=1; \ N=4; \ PMT=250, \ PV=-10,041; \ FV=10,000; \ CPT \ I/Y= 2.3905\%

- the YTM for this bond at this price today is 2.3905% per period

» with 2 periods/year this is an APR = \( 2.3905 \times 2 = 4.7810\% \)
Pricing a Coupon Bond

How would we price a newly offered coupon bond?

Example

Consider 3-year UST $10,000 bond with coupon rate of 8% (APR) with 2 compounding periods per year, coupon = $400 per period.

YTM for bond is 10% (APR) so discount rate is 5% per period.

N = 0 N = 1 N = 2 N = 3 N = 4 N = 5 N = 6

<table>
<thead>
<tr>
<th>Period</th>
<th>Required return on investment</th>
<th>Cash flow</th>
<th>End Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1:</td>
<td>$9,492 x 1.05 = $9,967</td>
<td>-$400</td>
<td>$9,567</td>
</tr>
<tr>
<td>P2</td>
<td>$9,567 x 1.05 = $10,046</td>
<td>-$400</td>
<td>$9,646</td>
</tr>
<tr>
<td>P3</td>
<td>$9,646 x 1.05 = $10,128</td>
<td>-$400</td>
<td>$9,728</td>
</tr>
<tr>
<td>P4</td>
<td>$9,728 x 1.05 = $10,214</td>
<td>-$400</td>
<td>$9,814</td>
</tr>
<tr>
<td>P5</td>
<td>$9,814 x 1.05 = $10,305</td>
<td>-$400</td>
<td>$9,905</td>
</tr>
<tr>
<td>P6</td>
<td>$9,905 x 1.05 = $10,400</td>
<td>-$10,400</td>
<td>0</td>
</tr>
</tbody>
</table>

Features of Bond Pricing

Why does the bond sell for a discount?

- because its coupon rate is less than the market's required YTM (the opportunity cost of capital)
  - it has to be worth less than $10,000 (face value) so that given its price it generates the investor a return of 10% APR
- at the price today of $9,492.43, the bond's cash flows provide exactly a 10% (APR) rate of return over the 3 year life.
Bond Pricing

- What if the bond's coupon rate was higher than the market's YTM?
  - the bond would sell for more than face value

**Example**

Consider 3-year UST $10,000 bond with coupon rate of 8% (APR)

YTM for bond is 6% (APR) so discount rate is 3% per period

<table>
<thead>
<tr>
<th>N = 0</th>
<th>N = 1</th>
<th>N = 2</th>
<th>N = 3</th>
<th>N = 4</th>
<th>N = 5</th>
<th>N = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$400</td>
<td>$400/1.03^1</td>
<td>$400/1.03^2</td>
<td>$400/1.03^3</td>
<td>$400/1.03^4</td>
<td>$400/1.03^5</td>
<td>$10,400/1.03^6</td>
</tr>
</tbody>
</table>

PV of payments

Equals $388.35 $377.04 $366.06 $355.39 $345.04 $8709.84

Sum = $10,541.72

- the bond sells for more than $10,000 since its price gets bid up because it offers an attractive coupon rate given market YTM

Features of Bond Prices

- The coupon rate of most bonds is fixed
  - when the market YTM (opportunity cost of capital) moves, the price of the bond changes
  - the relation is an inverse one
    - when the YTM required by the market rises, the price of fixed coupon bonds fall (capital loss)
      - the bond still pays the same coupons but the PV of its cash flows falls
    - when the YTM required by the market falls, the price of fixed coupon bonds rises (capital gain)
      - the bond still pays the same coupons but the PV of its cash flows rises
  - it's all because of present value
Pricing Bonds with Spot Rates

- In reality, newly issued coupon bonds are priced using the actual spot interest rates, not its YTM
  - spot rates come from the YTM of different maturity ZCBs

**Example**

Consider 3-year UST $10,000 bond with coupon rate of 8% (APR)

determine bond price from current spot interest rates (per period)

<table>
<thead>
<tr>
<th>spot rates $r_n$</th>
<th>N = 0</th>
<th>N = 1</th>
<th>N = 2</th>
<th>N = 3</th>
<th>N = 4</th>
<th>N = 5</th>
<th>N = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond's CFs</td>
<td>+$400</td>
<td>+$400</td>
<td>+$400</td>
<td>+$400</td>
<td>+$400</td>
<td>+$10,400</td>
<td></td>
</tr>
<tr>
<td>PV of payments</td>
<td>$400/1.0281</td>
<td>$400/1.0283^2</td>
<td>$400/1.0286^3</td>
<td>$400/1.0292^4</td>
<td>$400/1.0297^5</td>
<td>$10,400/1.03016^6</td>
<td></td>
</tr>
<tr>
<td>Equals</td>
<td>389.07</td>
<td>378.29</td>
<td>367.55</td>
<td>356.50</td>
<td>345.55</td>
<td>8704.76</td>
<td></td>
</tr>
<tr>
<td>Total = $10,541.72</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- this is how the bond prices are actually determined and from the price we determine the bond's YTM

Amortized Bonds

- Some bonds are repaid in equal installments
  - key with amortized bonds (loans) is to determine the payment
    - requires a financial calculator

**Example**

What is the PMT on a 30 yr mortgage of $250,000 at 8% APR?
- mortgage payments are monthly so $N = 30 \times 12 = 360$
- the periodic rate is $8%/12 = 0.667%$
- PV of mortgage is its face value today = $250,000$
- FV of mortgage at end must be zero

calculator: $P/Y=12$; $N = 360$; $I/Y = 8$; $PV = +250,000$; $FV = 0$

CPT => PMT gives us $-1,834.41$

- you pay $1834.41 each month for 30 years to repay the bond

- we can also do this in Excel
  - use PMT function
    - here you must be explicit to use the periodic interest rate, I/P
    - enter in cell =PMT(I/P,N,PV) hit enter and you get $1,834.41
Amortization Tables

- Amortized bonds require amortization tables
  - although every payment is the same size, part of each payment is interest and part of each payment is principal
    - the payment is initially mostly interest and over time becomes increasingly more principal
  - to determine breakdown of payment between interest and principal and remaining balance we build an amortization table
- example: amortization schedule for 5 year bond at 8%
  - P/Y = 1 N = 5, r = 8% PV = $1,000, FV = 0, so PMT is -$250.46
  - 5 payments of $250.46 will totally payoff the loan

### Example Amortization Schedule

<table>
<thead>
<tr>
<th>Year</th>
<th>Beg Balance</th>
<th>Payment</th>
<th>Interest</th>
<th>Principal</th>
<th>End Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1,000.00</td>
<td>$250.46</td>
<td>$80.00</td>
<td>$170.46</td>
<td>$829.54</td>
</tr>
<tr>
<td>2</td>
<td>$829.54</td>
<td>$250.46</td>
<td>$66.36</td>
<td>$184.09</td>
<td>$645.45</td>
</tr>
<tr>
<td>3</td>
<td>$645.45</td>
<td>$250.46</td>
<td>$51.64</td>
<td>$198.82</td>
<td>$446.63</td>
</tr>
<tr>
<td>4</td>
<td>$446.63</td>
<td>$250.46</td>
<td>$35.73</td>
<td>$214.73</td>
<td>$231.90</td>
</tr>
<tr>
<td>5</td>
<td>$231.90</td>
<td>$250.46</td>
<td>$18.55</td>
<td>$231.90</td>
<td>$0.00</td>
</tr>
</tbody>
</table>

- note Interest column is just Beg Balance x 8%, Principal column is just Payment – Interest and End Balance is Beg Balance - Principal

Risk and the YTM

- When bonds are not risk free, the YTM will include a premium for default

  - when lending for projects with uncertain payoffs, the creditor will have to assess the likelihood the borrower will default
    - default occurs when borrower fails to make the promised payments
      » the creditor receives something less than the promised amount
    - default often also triggers change in control of asset
  - to price a risky bond, we have to discount the expected cash flows (E(CF)) we will receive back on the bond
    - to determine the E(CF) we must consider several things
      - probability that default will occur (Prob(D))
      - how much we will collect if default occurs (CF in D)
        » this is also called the recovery rate
      - the promised payment in the bond when no default occurs (PCF)
    - together these are used to estimate E(CF) from bond each period
      \[
      E(CF) = \text{Prob}(D) \times (\text{CF in D}) + (1 - \text{Prob}(D)) \times \text{PCF}
      \]
Certain Default and the Price of Bonds

- The bond’s price is then the PV of these E(CF)s
  - the remaining issue is the interest rate at which to discount it
  - if the probability of default is certain (PR(D) = 1) or completely random, then the discount rate would be the risk free rate, \( r_b = r_f \)
  - if the probability of default is related to other risks, then the discount rate will include a risk premium, \( r_b = r_f + \text{risk premium} \)

**Example**

You are buying a 1 yr $1,000 ZCB from a risky borrower

Suppose that default is certain to occur (\( PR(D) = 1 \)) and in default you know you will only collect $800

\[ E(CF) \text{ in one year} = 800 \ (= 1 \times 800 + 0 \times 1000) \]

Risk free interest rate, \( r_f \), for CF in 1 year = 10%

then price of bond today = $800 / (1 + 10%) = $727.27

YTM for bond = \( (1000 / 727.27)^{(1/1)} - 1 = 37.5% \)

note you only earn a 10% rate of return despite YTM = 37.5%

- but to insure that you earn 10% return, you must price the bond with a 37.5% YTM (27.5% risk premium) because of default

Default and the Price of Bonds

- What if default is not certain or is not random?
  - assume default is related to economic conditions and requires a risk premium of 2% so \( r_b = r_f + \text{risk prem} = 10\% + 2\% = 12\% \)
  - this is the return that I need to earn for the risk

**Example**

You are buying a 1 yr $1,000 ZCB from a risky borrower

You expect default with \( PR(D) = 30\% \) and in default you know you will only recover (CF in D) = $800

\[ E(CF) \text{ in one year} = 0.3 \times 800 + 0.7 \times 1000 = 940 \]

then price of bond today = $940 / (1 + 12\%) = $839.29

YTM for bond = \( (1000 / 839.29)^{(1/1)} - 1 = 19.15\% \)

- note that you do not expect to earn 19.15\%
  - 70\% of the time you earn \( (1000 / 839.29) - 1 = 19.15\% \)
  - 30\% of the time you earn \( (800 / 839.29) - 1 = -4.68\% \)
  - \( E(r_c) = (0.7)(19.15\%) + (0.3)(-4.68\%) = 12.0\% \)
  - to earn an average return of 12% with default you have to price the bond so it has a YTM of 19.15\%
Credit Risk

- The likelihood of default is reflected in the credit spread of the borrower
  - the credit risk of the borrower determines the premium (or return spread) over the risk free rate for similar maturity UST bonds
  - determining the credit risk of a borrower is important for the creditor
    - that’s why banks ask so many questions before making loans
    - for bonds, professional services provide credit ratings
      - Moody’s, Standard & Poors, and Fitch
    - bonds ratings run from AAA, AA, A, BBB, BB, B, CCC, CC, and C
      - BBB and better ratings are investment grade bonds
      - lower than BBB is called high yield or junk bonds
  - the lower the credit rating, the higher the probability of default (and/or lower recovery rate) so the higher the credit spread over the risk free rate
    - AAA rated issues typically price 25 – 50 bp over UST yields
    - B rated issues typically price 250-350 bp over UST treasuries

Using PV to Value Equity

- Equity is a residual claim on the firm’s cash flows
  - since equity is a residual claim, it is riskier than debt
  - however, like other securities, the price of equity is the PV of its future expected cash flows
    - holding a share of equity today entitles you to
      - any dividend the share pays over the next period, DIV
        - dividends are cash payments to equity holders
      - the proceeds of selling the share in the future at P
    - these values are discounted to today at discount rate that reflects the equity cost of capital
  - the arbitrage free price of a share today, P0, is such that
    \[ P_0 = \frac{DIV + P}{1 + r_E} \]
    - where DIV = forecasted dividend in period 1
    - P = forecasted share price at end of period 1
    - r_E = rate of return for equity (equity cost of capital)
Cost of Capital for Equity

- The equity cost of capital comes from the cost of capital for other equity investments of similar risk
  - rearranging the pricing formula we obtain
    \[ r_E = \frac{\text{DIV}_1}{P_0} + \frac{(P_1 - P_0)}{P_0} \]
  - the total return on equity comes from two sources:
    - \( \frac{\text{DIV}_1}{P_0} \) = forward looking dividend yield
    - \( \frac{(P_1 - P_0)}{P_0} \) = capital gain rate

Example
You expect a share of ABC stock to pay a dividend of $2.50 at the end of the year and to trade for $45.
Investments of similar risk produce returns of 12% per year.
Using the formula for \( P_0 \) from above we have:
\[ P_0 = \frac{(2.5 + 45)}{1.12} = 42.41 \]
- given these expectations the price of the equity today, \( P_0 \), should be $42.41 or there is an arbitrage opportunity

Investing for Multiple Periods

- Equity price today \( P_0 = \frac{(\text{DIV}_1 + P_1)}{(1 + r_E)} \)
  - but from our model \( P_1 = \frac{(\text{DIV}_2 + P_2)}{(1 + r_E)} \)
    and \( P_2 = \frac{(\text{DIV}_3 + P_3)}{(1 + r_E)} \) and so on...
  - by substituting in for \( T \) periods we can re-write the pricing equation as
  \[ P_0 = \frac{\text{DIV}_1}{(1 + r_E)^1} + \frac{\text{DIV}_2}{(1 + r_E)^2} + \ldots + \frac{(\text{DIV}_T + P_T)}{(1 + r_E)^T} \]
  - NB: future dividends are discounted at \( r_E \) once for each year
  - since equity has no maturity we usually have \( T \) go to infinity
  \[ P_0 = \sum_{t=1}^{\infty} \frac{\text{DIV}_t}{(1 + r_E)^t} \]
- the stock price is the present value of all future dividends discounted at the opportunity cost of equity
  - this is the Dividend Discount Model (DDM)
    - lots of uncertainty as we have estimates for the future dividends
    - otherwise equity is like a perpetual bond except the expected payments are dividends
Dividends and Equity Valuation

- We can simplify this equation using perpetuities
  - suppose dividends are expected to be constant
    « if \( \text{DIV}_t = \text{DIV} \) for all \( t \) then the DDM becomes a perpetuity
      \[
      P_0 = \frac{\text{DIV}}{r_E}
      \]
    • if dividends that are expected to grow smoothly at rate \( g \)
      « we can write the DDM as a growing perpetuity
        \[
        P_0 = \frac{\text{DIV}_1}{(r_E - g)}
        \]
      • current price is function of just three basic factors
        ∗ current cash flows, represented by \( \text{DIV} \) (positive related)
        ∗ current opportunity cost of equity, \( r_E \) (negatively related)
        ∗ expected growth rate of cash flows, \( g \) (positively related)
      • what is expected \( \text{DIV} \) growth rate?
        ∗ rearrange the formula and compare with previous \( r_E \) formula
          \[
          r_E = \frac{\text{DIV}_1}{P_0} + g
          \]
        • suggests that \( g \) = expected capital gains rate, \( \frac{(P_1 - P_0)}{P_0} \)

Dividend Growth

- Lets think more carefully about growth of dividends
  \[
  \text{DIV}_t = \frac{\text{Earnings}_t}{\text{Shares Outstanding}_t} \times \text{Dividend Payout Rate}_t
  \]
  • the dividends paid to a share are a function of 3 variables
    » note: dividend payout rate = % of earnings paid out as dividend
    • firms can make \( \text{DIV} \) grow by increasing earnings, raising dividend payout rate or reducing shares outstanding
    • but to grow earnings a firm has to invest some current earnings back into the firm and therefore cannot pay them out as a \( \text{DIV} \)
      • to grow earnings fast, the dividend payout rate must be low
    • consider the relations
      \[
      \text{Change in Earnings} = \text{New Investment} \times \text{Return on New Investment}
      \]
      \[
      \text{New Investment} = \text{Earnings} \times \text{Retention Rate} \quad \text{\ensuremath{(\text{Ret Rate} = 1 - \text{Div Payout Rate})}}
      \]
      • substitute and divide by Earnings and we have
      \[
      \text{Earnings Growth Rate} = \text{Retention Rate} \times \text{Return on New Investment}
      \]
      • if a firm keeps payout rate and shares outstanding constant, \( \text{DIV} \) will grow at same rate, \( g \), as earnings grow
        \[
        g = \text{Retention Rate} \times \text{Return on New Investment}
        \]
Valuing Firms when Growth Rates Vary

- What about when DIV growth rate not constant?
  - we can always specify all DIV_t and then manually calculate PV
    \[ P_0 = \frac{DIV_1}{(1 + r_E)^1} + \frac{DIV_2}{(1 + r_E)^2} + \frac{DIV_3}{(1 + r_E)^3} + \frac{DIV_4}{(1 + r_E)^4} + \ldots = \sum_{t=1}^{\infty} \frac{DIV_t}{(1 + r_E)^t} \]
  - sometimes we can model variable growth out to a point N, after which we can assume constant perpetual growth
    - then use growing perpetuity rule for PV of DIV_{N+1} forward
      \[ P_0 = \frac{DIV_1}{(1 + r_E)} + \frac{DIV_2}{(1 + r_E)^2} + \frac{DIV_3}{(1 + r_E)^3} + \ldots + \frac{DIV_N}{(1 + r_E)^N} + \frac{DIV_{N+1} / (r_E - g)}{(1 + r_E)^N} \]
  - in other cases we can assume stages of DIV growth that allow us to partition the stream into parts upon which we can use our special PV cases
    - manual calculation for periods with no assumed growth pattern
    - growing annuity for limited periods of high (constant) growth
      - growth can be higher than \( r_E \), but must last for only a finite time
    - growing perpetuity for final stage into forever
      - growth rate must be lower than \( r_E \), and likely no higher than economy

Various Growth Stages

- Consider a firm with three stages of growth
  1. initial stage: initial period losses and no dividends
  2. high growth stage: positive dividends and high growth
  3. mature stage: stable and slower dividends growth
  - to value such a firm, we simply determine \( PV_0 \) for each piece and sum the three PVs
    - assume that the opportunity cost of equity for the firm is 15%

- Stage 1: initial random stage – standard PV calculation
  - let’s assume that the firm will make the following dividends per share for the next 4 years
    - note negative dividends are like have to invest additional money
    \[
    \begin{array}{c|cccc}
    \text{year} & 0 & 1 & 2 & 3 \\
    \hline
    \text{DIV} \text{ now} & -$3 & -$2 & -$0.75 & $0 \\
    \end{array}
    \]
    \[ PV_0 \text{ of first stage} = -$3/1.15 + -$2/1.15^2 + -$0.75/1.15^3 + 0/1.15^4 \]
    \[ PV_0 = -$4.12 \]
  - the value today of initial random stage is -$4.12
Valuing Firms with Multiple Stages

**Stage 2: limited fast growth – a growing annuity**
- Assume the firm will pay produce dividends in year 5 and DIV will grow at 30% for next 15 years (Y6 – Y20)
  - This stage can be modeled as a 16-year growing annuity
    \[ PV_t = C_{t+1} \times PVAF(N, r, g) \]

**Timeline**

<table>
<thead>
<tr>
<th>Year of Stage</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIV (formula)</td>
<td>$1</td>
<td>$1.30</td>
<td>$1.69</td>
<td>...</td>
<td>$39.37</td>
<td>$51.19</td>
</tr>
</tbody>
</table>

- Growing annuity details (N = 16, r = 15%, g = 30%, C5 = $1)
  - Because DIV begin in year 5, we determine PV as of year 4

\[
PV_4 = \frac{C_5}{r-g} \times \left[ 1 - \frac{(1+g)}{(1+r)}^N \right] = \frac{15}{0.15-0.30} \left[ 1 - \frac{1.3}{1.15} \right]^{16} = 40.74
\]

\[
PV_0 = \frac{PV_4}{1.15^4} = \frac{40.74}{1.15^4} = 23.29
\]
- Thus, PV0 of stage 2 (a growing annuity) = $23.29

**Stage 3: maturity – stable growth into perpetuity**
- Assume that DIV after year 20 grows at 5% forever
  - This stage is a growing perpetuity for DIV for Y20 onward

**Timeline**

<table>
<thead>
<tr>
<th>Year of Stage</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIV (formula)</td>
<td>$53.75</td>
<td>$56.44</td>
<td>...</td>
<td>$51.19 x 1.05^{n-20}</td>
<td></td>
</tr>
</tbody>
</table>

- We can determine PV20 for the perpetuity using CF21, \( r = 15\% \) and g = 5% and then convert PV20 into PV0

\[
C_{21} = CF_{20} \times 1.05 = C_5 \times (1.30)^{15} \times 1.05 = 51.19 \times 1.05 = 53.75
\]
- DIV will continue to grow from this level at 5% forever
- Value perpetuity for year 20 onward (formula gives us PV20)

\[
PV_{20} = \frac{C_{21}}{r-g} = \frac{53.75}{0.15-0.05} = 537.50
\]
\[
PV_0 = \frac{PV_{20}}{1.15^{20}} = \frac{537.50}{1.15^{20}} = 32.84
\]
- To value firm, we sum the PV0 of each stage:

\[
P_0 = \sum PV_0 = -4.12 + 23.29 + 32.84 = 52.01
\]
- Estimate of firm’s share price = $52
Other Methods for Valuing Equity

- DDM is not the only way to value equity
  - it has its problems
    - such as being very sensitive to changes in estimates of $g$
    - dealing with share repurchases (pay out to equity through purchasing shares)
  - other methods
    - Total Payout Model
      - combine dividend and share repurchase amounts
      \[ P_0 = \text{PV(Future Total Dividends and Repurchases)}/\text{Shares Outstanding}_0 \]
    - Discount Free Cash Flow (DCF) Model
      - use DCF to get enterprise value, $V_0$, add cash and subtract debt
        \[ P_0 = (V_0 + \text{Cash}_0 - \text{Debt}_0)/\text{Shares Outstanding}_0 \]
    - Multiples Valuation
      - determining equity value by using average value multiples of others
      - common multiples are P/E ratios or Enterprise Value multiples
      \[ P_0 = E_1 \times (P/E_1)_{AVG} \text{ or } V_0 = \text{EBITDA}_1 \times (V_0 / \text{EBITDA}_1)_{AVG} \]

Final Note on Valuing Securities

- Both bond and equity prices are forward looking
  - based on present value
    - what happened in the past is only important because of how it affects expectations about the future cash flows
  - implications of this forward looking focus
    - historic accounting performance may not relate closely with market pricing behavior
    - the announcement of good news or bad news about the firm’s performance or interest rates might not affect current prices of bonds or stocks if this information was already known by the market

Example

- a firm announces record earnings growth, but slightly lower than analysts had expected
  - stock price will likely fall on the announcement as the record growth was already reflected in current price but the “new information” was that earnings weren’t as good as expected
Investment Decisions and the Net Present Value Rule

- We have learned the NPV rule for securities
  - securities are priced so that NPV = 0
  - this will not be true of real projects (operational assets)
    - real projects can be good or bad
  - we also use NPV to decide on investing in real projects
- decision rule: take projects with positive NPV
  - NPV of project = PV(benefits) - PV (costs)
  - real projects with positive NPV add to firm value, so the general investment decision rule is:
    - take all available positive NPV projects
  - if you must decide among projects, take the projects with the largest NPVs
    - these will add the most to firm value
  - NPV measures that amount of new wealth created today by undertaking the investment activity

Alternative Investment Decision Rules

- Some firms do not use NPV to make investment decisions
- other alternatives
  - Payback Rule
  - Rate of Return Rule (RRR)
    - these are simple rules of thumb for people unwilling or unable to calculate NPV
- let’s look at these alternative rules and compare them to NPV for real investment decisions
  - identify the dangers of using these alternative rules
  - discuss the details of using NPV correctly in unique investment decision situations
Payback Rule

- Decide on project based upon the payback period
  - this is just a benchmark of a number of years for the cumulative expected cash flow to cover the initial investment
    - if the payback period is less (more) than a pre-specified length, take (reject) the project.

**Example**

<table>
<thead>
<tr>
<th>project</th>
<th>$C_0$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>payback</th>
<th>NPV @10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-2</td>
<td>+2</td>
<td>0</td>
<td>0</td>
<td>1 year</td>
<td>-0.2</td>
</tr>
<tr>
<td>B</td>
<td>-2</td>
<td>+1</td>
<td>+1</td>
<td>+5</td>
<td>2 years</td>
<td>+3.5</td>
</tr>
</tbody>
</table>

*project A has shorter payback (1yr vs. 2 yrs for project B) but project A has negative NPV while project B has positive NPV using payback can cause one to accept negative NPV projects*

*problems:*
- does not guarantee that value is created
- does not take into account opportunity cost
  - can do payback with discounted cash flows but problems still remain

Rate of Return Rule

- Compare the rate of return the project earns with a benchmark cost of capital
  - project’s rate of return = project’s IRR
    - we must estimate the project’s internal rate of return (IRR)
      - IRR is the discount rate that makes the project have zero NPV
        \[ C_0 + \frac{C_1}{(1+IRR)} + \frac{C_2}{(1+IRR)^2} + \ldots + \frac{C_T}{(1+IRR)^T} = 0 \]
    - this project rate of return is then compared to a benchmark often referred to as the “hurdle rate”
      - the correct hurdle rate is the opportunity cost of capital for project

**Rate of Return Rule (RRR):**

- take project if its IRR > opportunity cost of capital

  *this is the decision rule most similar to NPV*
  - in most cases when the project’s IRR > opportunity cost of capital for the project, the NPV of project > 0
  - however, the two rules are not exactly equivalent
Rate of Return Rule

**Example**

Consider cash flows of the following 2 projects

<table>
<thead>
<tr>
<th></th>
<th>C₀</th>
<th>C₁</th>
<th>C₂</th>
<th>NPV @10%</th>
<th>IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>project A</td>
<td>-200</td>
<td>+100</td>
<td>+200</td>
<td>56.2</td>
<td>28%</td>
</tr>
<tr>
<td>project B</td>
<td>+200</td>
<td>-100</td>
<td>-200</td>
<td>-56.2</td>
<td>28%</td>
</tr>
</tbody>
</table>

The IRR equation is

\[ \text{NPV} = C₀ + C₁/(1 + IRR) + C₂/(1 + IRR)^2 = 0 \]

Both projects have an IRR = 28%

- Rate of return rule (RRR) says to accept the project if the opportunity cost of capital is less than 28%
  - If project cost of capital is 10%, then using RRR we would accept either or both projects
  - But project B has a negative NPV because RRR cannot distinguish between borrowing or lending projects
    - Borrowing projects with high IRRs will have negative NPV
      - This is an important problem with RRR

Multiple CF Sign Changes and RRR

- Another pitfall of the rate of return rule occurs when cash flows change signs more than once
  - Multiple changes in the signs of the cash flows between negative and positive can result in multiple IRRs

**Example**

Consider a project with 2 changes in sign of the CFs

<table>
<thead>
<tr>
<th></th>
<th>C₀</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>project</td>
<td>-300</td>
<td>+350</td>
<td>+200</td>
<td>+20</td>
<td>-290</td>
</tr>
</tbody>
</table>

This project has 2 IRRs: 9.5% and 24.2%

- Which is the right one to compare to opportunity cost of capital?

[Diagram showing NPV vs. Discount rate for multiple discount rates]
One More Problem with RRR

- More troubling is the problem of using RRR when choosing among mutually exclusive projects

  **Example**
  
  The decision to choose a graduate school is an important financial and largely mutually exclusive decision.
  
  - **public university** - lower cost, good education will get you a good job
  - **private university** - higher costs and excellent education (better contacts) should get better job

  consider rate of return rule versus NPV

  **public graduate school**
  
  - invest $40,000 to increase lifetime income by $25,000/year
  - if $N = 35$, $I/Y = 3\%$, $NPV = 497,180$ and $IRR = 62\%$ ($25k/40k$)

  **private graduate school**
  
  - invest $80,000 to increase lifetime income by $35,000/year
  - if $N = 35$, $I/Y = 3\%$, $NPV = 672,052$ and $IRR = 43\%$ ($35k/80k$)

  - which is more important, the rate of return or wealth created?

Rate of Return Rule and NPV as Decision Rules

- As decision rules, RRR and NPV agree only when there are independent conventional projects
  
  - no project interactions with other projects
  - no change in signs of cash flows

  - problems arise with mutually exclusive projects and projects with non-conventional cash flows
    
    - rate of return can lead to sub-optimal decision

  - NPV is preferable over IRR as only NPV ensures maximization of value
    
    - largest increase in wealth rather than highest return
    
    - high return on small investment not as good as large NPV

  - now let's look at some more complicated decision making situations and see how to apply the NPV rule to make value maximizing choices
Using NPV: Project Interaction

Choosing between short- and long-lived assets

Example

Suppose you have to decide between 2 machines which make the same product.

Machine A costs 17 to buy and will last 3 years and it costs 5 per year to run
Machine B costs 10 to buy but will last 2 years and it costs 6 per year to run
Both machines have the same output per year
all the cost information are in real terms (adjusted for inflation)
machines can be replaced in the future at the same real costs

we must decide which one to buy
- given similar output per period, we can focus on total costs
- but costs run over different periods
- use Equivalent Annuity Cost (EAC) to determine best choice
  - EAC it is the payment for an annual annuity whose PV is the same as the PV of the costs of buying and running the machine
  \[
  EAC \times PVAF = PV(\text{purchase price} + \text{annual maintenance})
  \]

Determining Equivalent Annuity Cost

Costs of buying and running each machine

- assume the real opportunity cost of capital is 6%

<table>
<thead>
<tr>
<th>Machine</th>
<th>( C_0 )</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>PV of costs @6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>17</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>30.4</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
<td>6</td>
<td>6</td>
<td>-</td>
<td>21.0</td>
</tr>
</tbody>
</table>

- machine B has a lower PV of costs; however, it will need to be replaced sooner so it’s not clear which machine is better
  - since the outputs of each machine are the same, the machine with the lower EAC will be the better deal

- determine EAC for each machine independently
  - EAC for machine A
    - find PMT for 3 year annuity whose PV = PV(costs)
    \[
    PMT_A \times PVAF( N = 3, r = 6\%) = 30.4 \implies PMT_A = 30.2/2.67 = 11.4
    \]
  - EAC for machine B
    - find PMT for 2 year annuity whose PV = PV(costs)
    \[
    PMT_B \times PVAF( N = 2, r = 6\%) = 21.0 \implies PMT_B = 21.0/1.83 = 11.5
    \]
Using Equivalent Annuity Cost

Look at the costs of the machines in terms of EAC

<table>
<thead>
<tr>
<th>Machine</th>
<th>$C_0$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>PV of Annuity</th>
<th>PV costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>11.4</td>
<td>11.4</td>
<td>11.4</td>
<td>30.4</td>
<td>30.4</td>
</tr>
<tr>
<td>B</td>
<td>11.5</td>
<td>11.5</td>
<td>-</td>
<td>21.0</td>
<td>21.0</td>
<td>21.0</td>
</tr>
</tbody>
</table>

- the EACs are the payments of an annuity that has the same PV over the life of the asset as the cost of buying and operating the asset
- given these results it is a better decision to purchase Machine A as it has a lower equivalent annuity cost (EAC)
  - Machine A’s equivalent annuity cost of operation is 11.4 per year which is lower than Machine B’s equivalent annuity cost of operating which is 11.5 per year
  - comparison is now valid despite differences in machine life
  - assumption is that when time comes to replace you have similar choice to make

- EAC is also useful for decisions on when it is optimal to replace an existing machine that is wearing out

Deciding When to Replace

Considering to replace an old machine
- old machine will produce a real net cash flow of 4 per year (net of maintenance) this year and next and then fall apart
  - consider a new replacement machine
    - new machine costs 15 but is more efficient and will produce a real net cash flow of 8 per year (net of maintenance) for 4 yrs
  - should we replace the old machine now or wait until it dies?
    - to answer this we need to
      1. calculate NPV of the new machine’s cash flows
      2. calculate EAC of the new machine
      3. compare EAC with annual cash flows of old machine

- calculate the EAC of the new machine
  - all CF are in real terms and real opportunity cost of capital is 6%

<table>
<thead>
<tr>
<th>Cash Flows</th>
<th>$C_0$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>PV @6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>New machine</td>
<td>-15</td>
<td>+8</td>
<td>+8</td>
<td>+8</td>
<td>+8</td>
<td>+12.7</td>
</tr>
</tbody>
</table>

what equivalent annuity cost for 4 years has a PV of 12.7?
Using Equivalent Annuity Cost

- EAC for new machine
  find PMT for 4 year annuity whose PV = +12.7
  \[ PMT \times PVAF(N = 4, r = 6\%) = 12.7 \]
  \[ PMT = \frac{12.7}{3.47} = 3.7 \]
  new machine is equivalent to a 4 yr annuity with CF of 3.7

<table>
<thead>
<tr>
<th>Cash Flows</th>
<th>C₀</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>PV @6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equivalent 4-yr annuity</td>
<td>-</td>
<td>+3.7</td>
<td>+3.7</td>
<td>+3.7</td>
<td>+3.7</td>
<td>+12.7</td>
</tr>
</tbody>
</table>

- we can now use these are the cash flows of the new machine and compare to the existing machine

<table>
<thead>
<tr>
<th>Cash Flows</th>
<th>C₀</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>New Machine</td>
<td>-</td>
<td>+3.7</td>
<td>+3.7</td>
<td>+3.7</td>
<td>+3.7</td>
</tr>
<tr>
<td>Old Machine</td>
<td>-</td>
<td>+4</td>
<td>+4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- don’t replace because the old machine will generate cash flow of 4 for this year and next while the new machine’s equivalent cash flow for these years is only 3.7

Summary of Net Present Value

- Net present value is the most robust rule for making investment decisions
  - superior in almost all situations to alternative rules like payback period or internal rate of return
  - feature of NPV
    - divisible and summable
    - NPV of any project is the NPV of its individual parts
    - be alert to possible project interactions
      - questions can be broken down into smaller pieces and then analyzed
      - often done on an incremental basis
      - figure out annuity equivalents
    - NPV rule and annuity techniques are useful for helping to make value maximizing decisions in these cases
      - this is goal of managerial decision making