Risk and Return

- We have not said much about risk so far
  - in this lecture we will talk about risk and determinants of risk and its relation to return
    - statistics review
    - risk and return overview
    - defining and measuring risk
    - risk and diversification
    - linking risk with the return and the opportunity cost of capital
    - asset pricing models
Statistics Review

- Important statistical terms for finance
  - first, let’s recognize that the return to an investment, \( R_i \), can be a random variable
    - the return differs across different states of the world or observations of time
    - this variability is summarized in the return’s distribution
  - 50 possible outcomes of a return to an investment

- Statistical Measures
  - Probability, \( p_k \)
    - this is the relative likelihood of a particular state of the world or period of time, \( k \)
    - probabilities are always positive; \( p_k > 0 \)
    - from previous histogram the probability of a 4% return \( p_{4\%} = \frac{5}{50} = 0.1 \)
    - sum of all probabilities must be 1; \( \sum_k p_k = 1 \)
  - Expected return, \( E[R] \)
    - a mean or weighted-average of return across all states or periods
    \[
    E[R] = \sum_k p_k \cdot R_k
    \]
    - where \( p_k \) is the probability of state or period \( k \)
    - \( R_k \) is the realized return in state or period \( k \)
  - Variance of return, \( V(R) \) (for a population)
    - a measure of the dispersion of returns around the mean return
    - it is the mean of the squared deviations from the expected return
    \[
    V(R) = \sum_k p_k \cdot [R_k - E(R)]^2
    \]
    - standard deviation = square root of \( V(R) = \sqrt{V(R)} = \text{SD}(R) \)
Examples

2 distributions of returns with 14 observations

- light distribution: \( R_k = \{4,3,5,4,5,4,3,5,3,4,4,5,3,4\} \)

  Expected return: \( \text{E}(R) = \frac{4}{14} \times 3 + \frac{6}{14} \times 4 + \frac{4}{14} \times 5 = 4 \)

  Variance: \( \text{V}(R) = \frac{4}{14} \times (-1)^2 + \frac{6}{14} \times (0)^2 + \frac{4}{14} \times (1)^2 = \frac{5714}{100} = 0.5714 \)

  Standard deviation: \( \text{SD}(R) = \sqrt{0.5714} = 0.76 \)

- dark distribution: \( R_k = \{4,3,4,2,5,4,6,5,3,4,6,5,3,2\} \)

  E(R) = \( \frac{2}{14} \times 2 + \frac{3}{14} \times 3 + \frac{4}{14} \times 4 + \frac{3}{14} \times 5 + \frac{2}{14} \times 6 = 4 \)

  V(R) = \( \frac{2}{14} \times (-2)^2 + \frac{3}{14} \times (-1)^2 + \frac{4}{14} \times (0)^2 + \frac{3}{14} \times 1^2 + \frac{2}{14} \times 2^2 = \frac{15714}{100} = 1.5714 \)

  Standard deviation = \( \text{SD}(R) = 1.25 \)

Covariance

- Another important characteristic of a variable is its covariance with another variable
  - covariance is a measure of the similarity in movement
  - for assets 1 and 2, the covariance of their returns is
    \[ \text{Cov}(R_1, R_2) = \sum_{k=1}^{N} p_k \cdot (R_{1,k} - \text{E}[R_1])(R_{2,k} - \text{E}[R_2]) \]
  - covariance is a measure similar to variance \((\text{return})^2\)
    - in fact if \( R_1 = R_2 \) then \( \text{Cov}(R_1, R_2) = \text{Var}(R_1) = \text{Var}(R_2) \)
  - to make covariance a more interpretable number we normalize it by the standard deviation of each variable
    \[ \text{Corr}(R_1, R_2) = \frac{\text{Cov}(R_1, R_2)}{\text{SD}(R_1) \cdot \text{SD}(R_2)} \]
    - by construction, correlation coefficients range from -1 to +1
Correlations

- series 1 and 2 are perfectly positively correlated, $\rho = 1$
- series 1 and 3 are perfectly negatively correlated, $\rho = -1$
  - any other series that moved perfectly randomly with respect to series 1 would have a correlation of zero, $\rho = 0$

Statistics and Financial Returns

- We often use historical observations as a sample of what the actual distribution looks like

Average return of security $i$ (expected return)

$\bar{R}_i = \mathbb{E}(R) = \frac{1}{T} \times (R_1 + R_2 + R_3 + \ldots + R_T) = \frac{1}{T} \sum_{t=1}^{T} R_i$

Variance (Stnd Deviation) of realized returns for security $i$

$\text{Var}(R_i) = \left( \frac{1}{T-1} \right) \sum_{t=1}^{T} (R_i - \bar{R})^2 \Rightarrow SD(R_i) = \sqrt{\left( \frac{1}{T-1} \right) \sum_{t=1}^{T} (R_i - \bar{R})^2}$

- we use (T-1) because we do not know the “true” average return so we use up a degree of freedom in calculating the expected return
  - covariance would be calculated in a similar fashion to $\text{Var}(R_i)$

- can also determine the standard error of the average return
  - have to assume that the individual observations of returns making up the expected return (mean) are independent
    - independent means that distribution of outcomes are the same and not affected by previous outcomes
  - standard error of Avg Return $= SD(\bar{R}_i) = \frac{SD(R_i)}{\sqrt{T}}$
Returns on US Securities

- US portfolio returns, 1950-2008

<table>
<thead>
<tr>
<th>Return Info</th>
<th>mean</th>
<th>std</th>
<th>excess return</th>
<th>real return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation</td>
<td>3.8%</td>
<td>3.0%</td>
<td></td>
<td>0.0%</td>
</tr>
<tr>
<td>T bills (3 month)</td>
<td>5.1%</td>
<td>3.1%</td>
<td>0.0%</td>
<td>1.3%</td>
</tr>
<tr>
<td>UST Notes</td>
<td>6.4%</td>
<td>6.3%</td>
<td>1.4%</td>
<td>2.6%</td>
</tr>
<tr>
<td>UST Bonds</td>
<td>6.8%</td>
<td>10.6%</td>
<td>1.7%</td>
<td>3.0%</td>
</tr>
<tr>
<td>Corp Bonds</td>
<td>6.6%</td>
<td>9.6%</td>
<td>1.5%</td>
<td>2.8%</td>
</tr>
<tr>
<td>SP500</td>
<td>12.1%</td>
<td>18.1%</td>
<td>7.0%</td>
<td>8.3%</td>
</tr>
<tr>
<td>Small Stocks</td>
<td>16.1%</td>
<td>25.4%</td>
<td>11.1%</td>
<td>12.3%</td>
</tr>
</tbody>
</table>

- these are arithmetic average returns

\[ \bar{R}_t = \frac{1}{T} \sum_{t=1}^{T} \text{Avg Return for year } t \]

- for SP500 Stocks: \( \bar{R}_t = (1/59) \sum_{k=1}^{59} \text{Annual Return } k = 12.1\% \)
  - this is an annual average since our period was a year

- we can also determine a geometric mean return (gmr)
  - let's look at total returns over the same 59 years

US Risk and Return 1950-2008

- Diagram showing the performance of different types of investments from 1950 to 2008, including inflation, corporate bonds, long-term government bonds, short-term government bonds, large stocks, and small stocks.
Geometric Averages

Geometric mean on US stocks (SP500)

\[ \text{gmr} = \left( \prod_{t=1}^{T} (1 + \text{return in year } t) \right)^{\frac{1}{T}} - 1 \]

- this is also the IRR of the investment and can be found using the IRR formula for single beginning and ending values

- $1$ invested in BOY 1950 turns into $374$ by EOY 2008 (59 yrs)

\[ V_0 (1 + \text{IRR})^N = V_N \Rightarrow \text{IRR} = \left( \frac{V_N}{V_0} \right)^{\frac{1}{N}} - 1 \]

implies \( \text{gmr} = \left( \frac{364}{1} \right)^{\frac{1}{59}} - 1 = 1.05\% \)

- this is lower than the arithmetic mean of 12.1%
- geometric mean is always lower than arithmetic mean

Example

Consider three returns: \( R_1 = +25\% \), \( R_2 = -11\% \), \( R_3 = +7\% \)

- the arithmetic mean = \( \frac{1}{3}(+25\% -11\% + 7\%) = 7.0\% \)
- the geometric mean = \( \left( (1+.25)(1-.11)(1+.07) \right)^{\frac{1}{3}} - 1 = 6.0\% \)

- the arithmetic mean is the correct way to evaluate historical data to estimate an expected return for the future
- the geometric mean best describes what was actually earned

Risk Premium

- We have seen that groups of securities with more return variability provide higher average returns
- this is the infamous risk-return trade-off

- this extra return for risk is called the risk premium

- risk premium = return of risky asset – risk free rate
  - risk free rate is from some form of government bond
    - assumed to be default free

- average historical risk premiums are commonly used as a measure of the additional return required for bearing the payoff uncertainty for a particular security
  - using average risk premiums rather than just average return controls for variation in inflation and time preference over the history

- the expected return to asset \( i \) is then the sum of the current risk free rate and the asset’s expected risk premium, \( r_p \)

\[ E[R_i] = r_f + E(\text{risk premium}_i) \]

- to determine the risk premium we first need to better understand risk
Defining Risk

- In finance we distinguish between **types of risk**
  - consider fire and hurricane risk to Florida homeowners
    - both occur to homeowners with 2% probability in a given year and entail large losses per policy
    - you are an insurer and insure both fire and hurricane risk
      - you write $200,000 policies for each risk to 1,000 homeowners
  - **what are the risks to you of having to payout settlements?**
    - hurricane insurance: a 2% chance of a hurricane hitting and all 1,000 policies will require full settlements of $200,000
      - total cost to you would be $200M all at once
      - but there is a 98% chance that no policies will require settlement
        - so risk to you with 1,000 policies is the same as the risk to each homeowner – 2% chance of big loss or 98% chance of no payment
      - your expectation is that you will have to payout on 2% of 1,000 policies but the standard deviation around this payout is large
        
        \[
        E(Claim Pmt) = 2\% \times (200M) + 98\% \times (0) = \$4M
        \]
        
        \[
        SD(Claim Pmt) = \sqrt{0.02 \times (200M - 4M)^2 + 0.98 \times (0 - 4M)^2} = \$28M
        \]

- fire insurance: a 2% chance of a fire but it is a random event
  - you expect to pay settlements of $4M per year but since the risk of fire is independent, your expected payout is less uncertain
    - we can determine the SD of the expected claim payout using the fact that variance of N identical independent events is N·Var(event)
    \[
    SD(Claim Pmt)_{INDEP} = \sqrt{N \cdot Var(Individual event)} = \sqrt{1000 \cdot Var(Individual event)} \]
    \[
    SD(Claim Pmt)_{INDEP} = \sqrt{1000 \cdot [(0.02 \times (200,000 - 4,000)^2 + 0.98 \times (0 - 4,000)^2]} = \$28,000 \]
    \[
    SD(Claim Pmt)_{INDEP} = \sqrt{1000 \times \sqrt{1000}} = \$885,437
    \]
  - since fire risks are independent, the risk of the annual payout on the 1,000 policies is much lower than the hurricane
  - as the number of polices rises, the standard deviation of payout relative to expected payout shrinks with fire, but not hurricane risk
  - while risks to individual are the same, the risk to a group of policies differs because the risks have different features
    - fire risk was largely **random risk** => group risk lower than individual risk
    - hurricane risk was **common risk** => group risk same as individual risk
**Equity Risk**

- A common measure of equity risk is standard deviation of returns
  \[ SD(R_i) = Var(R_i)^{1/2} = \left\{ \frac{1}{(T-1)} \times \Sigma (R_i - E(R_i))^2 \right\}^{1/2} \]
  - standard deviation of return has units of (%)
- this is commonly referred to as the total risk of the equity
- however like above, a firm’s total equity risk will consist of some amount of both random and common risks
  - random risks are things that happen to that firm specifically
    - CEO leaves, hot new product, embezzlement, R&D breakthrough
    - this risk in returns is also referred to as firm-specific, idiosyncratic, unique or diversifiable risk
    - this risk goes away when one holds many of equities
  - common risks are things that affect all firms simultaneously
    - interest rate rise, change in market risk premium, higher GDP growth
    - this risk in returns is also referred to as systematic, undiversifiable or market risk
    - this risk is captured by covariance or correlation

**Portfolio Theory**

- When we combine stocks together into a portfolio, these two forms of risk behave differently
  - systematic risk, since it is common across all stocks, does not disappear and contributes to the volatility of the portfolio
    - this is true regardless of how many stocks are in the portfolio
    - this risk can be measured by the covariance among the stock returns
  - idiosyncratic risk, since it is random, can be diversified away by holding a large enough number of stocks in the portfolio
    - lots of random events end up having little to no impact on the overall volatility of a large portfolio of stocks
- when holding a portfolio of stocks
  - the volatility of a portfolio can be less than the weighted average volatility of its stocks
    - if the stocks' returns are not perfectly correlated
    - however, the expected return of the portfolio will be exactly the weighted average returns of the stocks
- financial theory is based upon the assumption that rational agents hold diversified portfolios of stocks
Portfolio Mathematics

Mathematics of portfolios for 2 stocks

- stock 1 has expected return $E[R_1]$ and variance $V(R_1)$ and standard deviation $SD(R_1)$
- stock 2 has expected return $E[R_2]$ and variance $V(R_2)$ and standard deviation $SD(R_2)$
- covariance $Cov(R_1,R_2) = corr(R_1,R_2) SD(R_1) SD(R_2)$
- we hold $\alpha$% of our wealth in stock 1 and $(1-\alpha)$% in stock 2

Expected return of portfolio:

$$E[R_P] = \alpha E[R_1] + (1-\alpha)E[R_2]$$

- expected return is simple weighted averages of stocks returns

Variance of the portfolio return:

$$V(R_P) = \alpha^2 V(R_1) + (1-\alpha)^2 V(R_2) + 2\alpha(1-\alpha)Cov(R_1,R_2)$$

Standard Deviation (volatility) of portfolio return:

$$SD(R_P) = \sqrt{\alpha^2 SD(R_1)^2 + (1-\alpha)^2 SD(R_2)^2 + 2\alpha(1-\alpha) Cov(R_1,R_2)}$$

- volatilities combine by something less than the weighted average

Portfolio of Two Stocks

**Example**

Assume Stock 1 has $E[R_1] = 10\%$ and $SD(R_1) = 25\%$
Stock 2 has $E[R_2] = 10\%$ and $SD(R_2) = 25\%$
but the stocks move mostly independently on one another so they have a correlation of $Cov(R_1,R_2) = 0.1$
suggests that $Cov(R_1,R_2) = 0.1 \times 0.25 \times 0.25 = 0.00625$
we hold $50\%$ of assets in stock 1 and $50\%$ of assets in stock 2

$$E[R_P] = 50\%(10\%) + 50\%(10\%) = 10\%$$

$$SD(R_P) = \sqrt{(0.5)^2(0.25)^2 + (0.5)^2(0.25)^2 + 2(0.5)(0.5)0.00625} = 18.5\%$$

- because correlation of these stocks is low, they are close to independent so the risk when combining them declines
  - but the expected return remains as the weighted average

- if the correlation had been 1, $Cov = 0.0625$
  - then $SD(R_P) = \sqrt{(0.5)^2(0.25)^2 + (0.5)^2(0.25)^2 + 2(0.5)(0.5)0.0625} = 25.0\%$

- if the correlation had been -1, $Cov = -0.0625$
  - then $SD(R_P) = \sqrt{(0.5)^2(0.25)^2 + (0.5)^2(0.25)^2 - 2(0.5)(0.5)0.0625} = 0.0\%$

- changes in correlation affect portfolio volatility not return
For More than Two Stocks

- Consider a portfolio with N stocks with equal weights
  - expected return of stock = \( E[R_i] \) for \( i = 1, \ldots, N \)
  - variance = \( V(R_i) \) for \( i = 1, \ldots, N \) and covariance = \( \text{Cov}(R_i, R_j) \) (\( i \neq j \))

- Expected portfolio return: \( E[R_p] = \frac{1}{N} \sum \limits_{i=1}^{N} E[R_i] = R \)
  - where \( R \) = the average of expected returns across the N stocks

- Portfolio variance:
  \[
  \text{Var}(R_p) = \frac{1}{N^2} \sum \limits_{i=1}^{N} V(R_i) + \frac{1}{N} \sum \limits_{i=1}^{N} \sum \limits_{j \neq i}^{N} \frac{1}{N} \frac{1}{N} \text{Cov}(R_i, R_j)
  \]
  \[
  = \frac{(1/N) V + (1 - 1/N) \text{Cov}}{N} \text{ as } N \to \infty, \text{ Var}(R_p) \to \text{Cov}
  \]
  - where \( V = \text{average variance} \) and \( \text{Cov} = \text{average covariance} \)

- Implication: the risk of a large portfolio is just the average covariance
  - generally for any large portfolio with weights \( w_i (\sum w_i = 1) \)
  \[
  \text{Var}(R_p) = \sum \limits_{i=1}^{N} w_i \text{Cov}(R_i, R_p) = \sum \limits_{i=1}^{N} w_i \text{SD}(R_i) \text{SD}(R_p) \text{Corr}(R_i, R_p)
  \]
  - dividing both sides by \( \text{SD}(R_p) \) we get (note: \( \text{V}(R_p) / \text{SD}(R_p) = \text{SD}(R_p) \))
  \[
  \text{SD}(R_p) = \sum \limits_{i=1}^{N} w_i (\text{SD}(R_i) \times \text{Corr}(R_i, R_p))
  \]
  - securities contribute to the volatility of a large portfolio as a function of its total risk (\( \text{SD}(R_i) \)) scaled by its correlation with the portfolio

How Do Portfolios Reduce Risk?

- Diversification is the key
  - diversification works by turning the risk into many tiny bets instead of one large one
    - unique or idiosyncratic risk - risk which is specific to the stock
      - this risk essentially disappears as the portfolio gets larger
        - mostly gone with 10 - 20 stocks and completely gone with >100
    - systematic risk is what remains and the portfolios risk is the average level of systematic risk of its stocks

- Idiosyncratic or Unique risk of the portfolio
  - Average Systematic risk of stocks also known as Market Risk = average covariance
  - With a large enough portfolio all of the unique risk disappears
Implication of Diversification for E(R)

- Suppose you had a portfolio that consisted of 100 stocks with unique risk and no systematic risk
  - the covariance (correlation) amongst all these stocks was zero
  - the average variance of a large portfolio is its average covariance
    - since the average covariance of the portfolio is zero, the portfolio would have zero risk
    - even though the stocks could each have significant idiosyncratic risk
  - by arbitrage a portfolio with no risk must be priced in a normal market so that its expected return is the risk free interest rate
    - this would mean that each of the stocks would have an expected return equal to the risk free rate
- Implication: the risk premium for diversifiable risk must be zero
  - investors are not compensated for bearing idiosyncratic risk
- Investors only care about a stock’s systematic risk
  - this is the risk that they cannot eliminate via diversification and must be compensated for via a risk premium for bearing
- Implication: a security’s risk premium is a function of its systematic risk

Expected Return

- The risk that matters to rational (diversified) investors is systematic risk not total risk
  - there should be no relation between total risk and expected return E(R), only between systematic risk and E(R)
  - to determine a security’s expected return we need:
    1. a measure of the security’s systematic risk
    2. a risk premium appropriate for the amount of systematic risk
- The key to measuring the systematic risk of a security is to find a portfolio of securities that is only systematic risk
  - this will have to be a well-diversified portfolio
  - such portfolios are called an efficient portfolio
  - we believe one efficient portfolio is the entire market portfolio
    - all stocks in the market (or at least very large number of them)
    - because there is no unique risk in this portfolio any changes in the value of this portfolio will then represent the systematic shocks to the economy
Measuring Systematic Risk - Beta

- To measure the amount of systematic risk in an investment we measure its beta
  - it is the sensitivity of the security’s return to the efficient portfolio’s return
    - if a security’s beta is 0.5 then when the efficient portfolio moves 1%, the security is expected to move 0.5%
  - we measure beta from covariance (correlation) with an efficient portfolio, P
    - beta for security i = $\beta_i = \frac{\text{Cov}(R_i, R_p)}{\text{Var}(R_p)}$
      - which can also be written as $\beta_i = \frac{\text{Corr}(R_i, R_p) \times \text{SD}(R_i)}{\text{SD}(R_p)}$
  - securities with beta > 0 on average move with the portfolio
    - they have positive portfolio risk
      - this is most stocks and bonds
  - securities with beta < 0 on average move opposite the portfolio
    - this then makes a hedge to portfolio movements and very valuable in reducing the risk of the overall portfolio

Meaning of Beta

- We can also see beta as a security’s index of systematic risk
  - note that in any portfolio the average beta across securities in the portfolio must be 1
    - the average across the securities must be the portfolio’s risk
  - most securities have positive betas
    - securities with beta > 1 tend to amplify portfolio movements
      - they are considered riskier than the portfolio
      - have more systematic risk than average security in the portfolio
      - hence with greater “risk” they must offer investor greater expected return than the market portfolio
    - securities with beta between 0 and 1 tend to dampen the portfolio movement
      - they are considered less risky than the portfolio
      - have less systematic risk than average security in the portfolio
      - hence with lower “risk” they can offer investor a lower expected return than the market portfolio
Beta and Required Return

- By definition the portfolio’s beta with itself is one
  - we also know the portfolio has a risk premium of
    
    \[ Portfolio \ Risk \ premium = E[R_p] - r_f = E(R_p - r_f) \]
    
    - which is also the risk premium one should earn if they hold a security with beta equal to one
  - since a security’s beta is its index portfolio (systematic) risk …
    - a measure of the units of portfolio risk in the security
  - and one unit of portfolio risk has a risk premium of \( E(R_p - r_f) \) …
  - the \( E[R_i] \) for a security \( i \) with a beta, \( \beta_i^{EFF} \), with the efficient portfolio is
    
    \[ E[R_i] = r_f + \beta_i^{EFF} x E(R_p - r_f) \]

**Example**

The efficient portfolio’s risk premium is 8%, the risk-free rate is 5%, and security \( i \) has a beta with the portfolio of 1.5, then

\[ E[R_i] = 5% + 1.5 \times 8% = 17\% \]

- note: any two securities with the same beta will have the same expected return regardless of their level of total risk

Determining the Market Portfolio

- Lets consider how an efficient portfolio is constructed
  - consider a portfolio with 2 stocks with different characteristics
    - stock A: \( \mu_A = 10\% \) SD(\( R_A \)) = 15\%
    - stock B: \( \mu_B = 15\% \) SD(\( R_B \)) = 30\% correlation, Corr(\( A,B \)) = .10
  - can plot of portfolio (SD, E(return)) for changing weights
    - start with 100% Stock A and move in 10% increments to 100% stock B

---

SAS 380.760 Lecture 4 Slide # 26

Investors holding only stock A could have had less risk and more return with 20% invested in stock B

50% - 50% portfolio: 2.5% higher return / 2.5% higher volatility

Rational Investor want to move in this direction
Efficient Frontier

- As more potential stocks are added to the mix, the set of the best risk-return tradeoff portfolios defines the efficient frontier
  - the efficient frontier is best risk-return trade off available for any given level of risk or given level of return

- rational investors will want to hold a portfolio on the efficient frontier
  - this allows them to move further towards the upper-left
  - this improves the risk-return trade-offs that they can choose from
    - the efficient frontiers changes less and less as we add more and more stocks

Diversification and Investing

- Which portfolio on the efficient frontier is best?
  - rational investors choose the tangency portfolio on the efficient frontier given the risk free rate of return
    - the tangency line is called the Capital Market Line (CML)
      - this is the best risk-return tradeoff available to investors
    - we often refer to this tangent efficient portfolio as the "market portfolio"

This tangency line is the Capital Market Line
It's slope is called the Sharpe ratio of the market portfolio
The line starts at $r_f$ and is just tangent to the Efficient Frontier

$$\text{Sharpe ratio} = \frac{E[r_p] - r_f}{\text{SD}(R_p)}$$

Rational investors choose the efficient portfolio so as to maximize their Sharpe ratio (get CML with steepest slope)
Portfolio Theory and Investing

- Rational investors should hold some combination of the efficient portfolio and the risk free asset
  - these two of investments offers the best risk-return trade-off
  - consider the risk / return trade-off of McDonalds stock (MCD)
    - we can achieve the same E[R] as MCD but less total risk by putting 70% of our wealth in the market portfolio and 30% in risk free deposit
    - we achieve the same SD as MCD but higher E[R] by borrowing 55% of our wealth at rf and investing it all in the market portfolio

- thus independent of investor risk preferences, everyone should hold some of the market portfolio and the risk free investment
- holding any set of assets other than this will offer you a lower Sharpe ratio, so a less attractive tradeoff between risk premium and systematic risk

Feature of the Efficient Portfolio

- The efficient portfolio has the highest Sharpe ratio
  - Sharpe ratio (SR) of portfolio P = (E(r_p) - r_f) / SD(R_p)
    - this is the ratio of the risk premium to total risk
    - for a portfolio to be efficient, it must be that adding new securities to it cannot increase its Sharpe ratio
  - consider adding a new security, k, to an efficient portfolio
    - to increase the Sharpe Ratio (SR), security k’s increment to the portfolio’s SR must exceed the portfolio’s SR
      - security k’s increment to the portfolio’s SR will be its risk premium E[R_k] - r_f over it contribution to portfolio risk, (SD(R_k) x Corr(R_k,R_P))
      - Security k’s SR = \( \frac{E[R_k] - r_f}{SD(R_k) x Corr(R_k,R_P)} > \frac{E[R_P] - r_f}{SD(R_P)} = \text{Portfolio’ s SR} \)
      - this can be rewritten as (note: [Corr(R_k,R_P) x SD(R_k) / SD(R_P)] = \( \beta_k^p \))
        - \( E[R_k] > r_f + \beta_k^p (E[R_P] - r_f) = \text{required rate of return} \)
        - if the portfolio is efficient this won’t be; the E[R_k] = required return
  - Implication: for a portfolio to be efficient, the expected return of every security (E[R_k]) must equal its required return = \( r_f + \beta_k^p (E[R_P] - r_f) \)
CAPM

- Portfolio theory becomes an asset pricing model in the Capital Asset Pricing Model (CAPM)
  - three main assumptions
    - investors can buy or sell securities at competitive market prices without transaction costs and can borrow at the risk free rate
    - investors hold only efficient portfolios of traded securities
    - investors have homogeneous expectations on risks and returns
  - under these assumptions investors will identify the same portfolio as the efficient portfolio
    - it will be the portfolio that consists of all securities in the market and we call this portfolio the market portfolio
    - prices of securities will adjust so that \( E[R] = \text{required return} \)
  - then the CAPM is a special case of our earlier pricing rule where the efficient portfolio is the market portfolio
    \[
    E[R_i] = r_f + \beta_i^{MKT} \times (E[R_M] - r_f)
    \]

Security Market Line

- CAPM provides the required return for a security
  - it is a linear function of the security’s market beta
    - in the market if efficient, then required return = \( E[R] \)
      \[
      E[R_i] = r_f + \beta_i^{MKT} \times (E[R_M] - r_f)
      \]
    - a plot of \( (\beta_i^{MKT}, E[R_i]) \) draws out the Security Market Line (SML)
      - if the market portfolio is efficient all securities will be on the SML
      - any deviation from the SML is a measure of the security’s alpha

Beta and Market Risk

- Measuring market betas for the firm
  \[ \beta_i^{MKT} = \beta_i = \text{Corr}(R_i, R_M) \times \frac{SD(R_i)}{SD(R_M)} \]
  \[ = \frac{\text{Cov}(R_i, R_M)}{\text{Var}(R_M)} \]
  - these values can be estimated using past returns
    - we need to obtain a series of historic returns for the firm or other securities with similar types of risk (in similar lines of business)
      - the portfolio of securities of other firms in similar lines of business is preferable as it generally provides a less noisy measure of the beta
    - once we have these returns for the firm or a portfolio of similar firms and the market portfolio we can estimate beta in two ways
      - directly estimate the covariance or correlation and the SD of returns or \( \text{Var}(R_M) \) over the same time period and take the ratio
      - run a regression of the firm’s or portfolio of similar firms’ returns (Y) on the market returns (X)
        » the regression coefficient on the market return variable is the beta
        » from Econometrics: the linear regression coefficient = \( \frac{\text{Cov}(x,y)}{\text{Var}(x)} \)

Estimating Beta

- What if no historical prices are available
  - sometimes even stock prices for comparable investments are not available
  - in these cases one can make an educated guess as to whether the investment would have more, the same, or less sensitivity to changes in economic conditions than the average firm in the market
    - examples of low beta industries (beta < 1)
      - pipelines, agriculture/mining
      - utilities, consumer non-durables
    - examples of high beta industries (beta > 1)
      - financial services, consumer durables
      - electronics, chemicals
Equity Cost of Capital

- Let’s try to determine the $E[R]$ for ABC stock
  - ABC stock’s expected return will be a function of its market beta, the risk-free rate, and the market premium
  - estimate ABC’s market beta directly from its equity returns
    - begin by estimating the beta of ABC
      - $\text{Cov}(R_{ABC}, R_M) = 0.001446$
      - $\text{Var}(R_M) = 0.000964$
      - $\beta_{MKT} = \frac{0.001446}{0.000964} = 1.50$
    - regress ABC’s returns on market returns
      - regression produces $\beta$ estimate
        - $R^2 = \%$ of variance explained by the market return = 33% (67% idiosyncratic)
        - standard error of $\beta$, $SD_\beta = 0.30$
          - confidence interval for $\beta = \pm 2SD_\beta$
        - 95% confidence that actual $\beta$ is between $1.50 - 0.60 = 0.90$ and $2.10 = 1.50 + 0.60$

Equity Cost of Capital

- ABC’s true beta is likely to lie between 0.90 and 2.10
  - lots of uncertainty, can we narrow this range down?
    - we could try estimating beta from an earlier set of data or use longer data sample
    - or we could try estimating beta using returns from a portfolio of similar firms
      - the portfolio will have better ratio of systematic to idiosyncratic risk
        - may get a better measure of systematic risk of this type of assets using a portfolio of returns form this type of assets
    - get returns for a portfolio of companies with similar assets to ABC
      - then estimate the beta of this representative industry portfolio against the market portfolio
      - estimate of $\beta_{IND} = 1.35$, $SD_\beta = 0.15$
        - confidence interval is $\Rightarrow$ estimate $\pm 2SD_\beta$ range: 1.05 to 1.65
          - notice that industry portfolio has more precisely estimated beta
        - portfolio has less idiosyncratic risk so more accurate beta
        - gives us some comfort in that firm’s beta estimate, 1.50, lies within the industry confidence interval
Determining the CoC of ABC Stock

- Use all this information in the CAPM
  - assume $r_f = 5.0\%$ and $(E[R_M] - r_f) = 8\%$
  - CAPM estimate of $E[R]$ for ABC’s stock using $\beta_{\text{ABC}}$
    \[ E[R_{\text{ABC}}] = r_f + \beta_{\text{ABC}}(E[R_M] - r_f) = 5.0\% + 1.5 \times 8\% = 17.0\% \]
  - CAPM estimate of $E[R]$ for ABC using $\beta_{\text{IND}}$
    \[ E[R_{\text{IND}}] = r_f + \beta_{\text{IND}}(E[R_M] - r_f) = 5.0\% + 1.35 \times 8\% = 15.8\% \]

- if we had no prior beliefs as to which data to use, we might surmise that ABC’s cost of equity was something around 16.5\%
- since beta are estimated with noise determining $E[R]$ from CAPM involves some uncertainty
  - easiest way to quantify the uncertainty is to estimate beta using the regression method

How Well Does the CAPM Work?

- The CAPM defines expected returns
  - in comparison with ex-post realized returns it has had some problems
    - ex-post SML line is flatter than predicted
      - fit is not bad on data from 1930-1965 (data on which model was originally based)

![Beta vs. Average Risk Premium 1931-1965](SML)
Problems with the CAPM

- More recently its troubles appear greater
  - actual SML very flat from 1966-1991
    - suggest no real relation between beta and realized returns
  - remember CAPM forecasts expected returns and we only observe actual returns
    - other phenomenon could be causing outcomes to be biased relative to forecasts
      - question is: Are any of these phenomenon identifiable?

Some Pricing Anomalies

- First recognized anomaly is a size anomaly
  - small firms have earned higher returns than large firms
    - look at 10 portfolios sorted by market value
  - there is clearly a pattern that small firms’ returns are higher and large firms’ returns are lower than suggested by the CAPM
    - suggests an additional risk factor for small firms
      - factor is not fully correlated with the market portfolio
        - this could explain errors in CAPM
Another Anomaly

- Most significant anomaly is book-to-market puzzle
  - firms with high book-to-market ratio earned higher returns
    - high book-to-market means low market value of equity relative to book value of equity
      » these are firms that the market has beaten down
      » often referred to as "value" firms

![Graph showing book-to-market ratio vs. average return (%)](image)

New Generation Models

- The problems with the simple CAPM model led to development of more sophisticated models
  - focus was on risk factors rather than correlation with and efficient (market) portfolio
- factor models
  - these models argue that returns are due to sensitivity to multiple risk factors and their risk premiums
- determining return with a factor model
  - identify the set of important risk factors
  - estimate sensitivity of security with risk factors, $\beta_k$'s
    » one factor might be market portfolio
  - estimate risk premium for factors, $(r_{factor} - r_f)$
    » estimate return premium for a unit of factor risk (historic relation)
  - determine expected (required) return on asset from loadings and risk premia
    $$E[R_i] = r_f + \sum_k (\beta_i^{Factor k} \times (E[R_{Factor k}] - r_f))$$
Factor Models

- Versions of factor models
  - formal version: APT model - Arbitrage Pricing Theory
    - in theory identification of factors is done by computer
      » use principle component analysis
    - in practice researchers identify macroeconomic variables as factors
      » GDP, inflation, interest rates, yield spreads (AAA-BBB)
  - modifications to CAPM: three factor model
    - uses firm size (market capitalization) and book-to-market ratio in addition to market portfolio as systematic risk factors
  - problems with multifactor models
    - what factors and how many to use?
      - can one identify a financial portfolio for this factor to measure the risk premium
    - no consensus has emerged on these questions
      - common approach is to use small number of factors

Asset Pricing Models

- Each of these types of models have fans and detractors
  - CAPM, APT, or multiple factor models all used in some degree
    - CAPM is most well known and commonly used
  - however, all three have two important common features
    1. investors require additional return for taking on additional risks
    2. investors appear to be predominantly concerned with risk they cannot eliminate by diversification
      - only non-diversifiable risks matter for required return
      - thus, financial theory always presumes that risks that can be diversified away will be
        » if you choose not to diversify away these risks, you will not be rewarded for bearing them
Lecture 4 Summary

- **Risk and Return**
  - basic statistics of finance
    - means, variances, standard deviations, covariance and correlation
    - difference between arithmetic and geometric averages
  - benefits of portfolios
    - reason investors choose portfolios
    - diversification arguments
  - decomposition of risk
    - unique risk is diversifiable
    - market risk is priced in that it requires higher return
  - asset pricing models
    - CAPM and others
    - be sure to understand common basic idea behind these