1 Proofs

To keep the notation compact, let

\[ g((1-\delta)k+i, z; w') = g(i, k, z; w') = \beta \max \left\{ (1-\delta)k', \int p(k', z'; w')Q(dz'|z) \right\}. \]

This function \(g(.)\) summarizes all the information about the future of an individual firm.

**Proposition 1**

Rewrite the problem of the firm as

\[
p(k, z; w) = \pi(k, z; w) + \max_{i \leq \pi(k, z; w)} \{ -i + g(i, k, z; w') \},
\]

\[
\max_{i > \pi(k, z; w)} \{ -i - \lambda(k; (1-\delta)k + i, z; w) + g(i, k, z; w') \}.
\]

and define the operator

\[
(Tp)(k, z; w) = \pi(k, z; w) + \max_{i \leq \pi(k, z; w)} \{ -i + g(i, k, z; w') \},
\]

\[
\max_{i > \pi(k, z; w)} \{ -i - \lambda(k; (1-\delta)k + i, z; w) + g(i, k, z; w') \}.
\]

Let \(C(K \times Z)\) be the space of all bounded and continuous functions in \(K \times Z\). The proof is then in 2 steps

1. \(T : C(K \times Z) \longrightarrow C(K \times Z)\) (Lemma A1);
2. $T$ is a contraction in $C(K \times Z) \ (\text{Lemma A2})$.

The Contraction Mapping Theorem implies then that there is a unique fixed point to (1).

$T : C(K \times Z) \rightarrow C(K \times Z)$ for any given $w$.

Suppose $p(k, z) \in C(K \times Z)$. Since $Q(dz'|z)$ has the Feller property it follows from Lemma 9.5 in Nancy L. Stokey and Robert E. Lucas (1989) that

$$
\int p(k + i, z'; w) Q(dz'|z) \in C(K \times Z).
$$

hence

$$
g(i, k, z; w') = g((1 - \delta)k + i, z; w') \in C(K \times Z).
$$

- Now consider the problem

$$
i_1(k, z; w) = \arg \max_{i \in \pi(k, z; w)} \{-i + g(i, k, z; w')\}.
$$

Since $i \geq -(1 - \delta)k$, the Maximum Theorem guarantees that the optimal policy correspondence is well defined and upper hemi-continuous. Moreover the function

$$
H_1(k, z; w) = \max_{i \in \pi(k, z; w)} \{-i + g(i, k, z; w')\}.
$$

is also continuous.

- Next consider the problem

$$
i_2(k, z; w) = \arg \max_{i \geq \pi(k, z; w)} \{-i - \lambda(k; (1 - \delta)k + i, z; w) + g(i, k, z; w')\}.
$$

Since maximization over values of $i$ does not take place over a compact set we define the equivalent problem:

$$
i_2(k, z; w) = \arg \max_{\pi(k, z; w) \leq i \leq k} \{-i - \lambda(k; (1 - \delta)k + i, z; w) + g(i, k, z; w')\},
$$

where
\[ \mathcal{E} = \{ k \in K : \pi(k, z; w) - \delta k = 0 \}. \]

Since \( \pi(k, z; w) \) is strictly concave in \( k \) this is a well defined quantity. Since \( k > \mathcal{E} \) is not profitable and can not be sustained. Note that since \( \lambda(k, (1 - \delta)k + i, z; w) > 0 \) we know that \( i_2(k, z; w) < \mathcal{E} \). If follows from the Maximum Theorem that \( i_2(k, z; w) \) is a well defined upper hemi-continuous correspondence and that the function

\[ H_2(k, z; w) = \max_{\pi(k, z; w) \leq i \leq k} \{-i - \lambda(k, (1 - \delta)k + i, z; w) + g(i, k, z; w')\}, \]

is also continuous.

- Combining the two steps above we can write (1) as

\[ (Tp)(k, z; w) = \pi(k, z; w) + \max \{ H_1(k, z; w), H_2(k, z; w) \}. \]

which is a continuous function since \( H_1(k, z; w), H_2(k, z; w) \) and \( \pi(k, z; w) \) are all continuous.

Concavity of \( \pi(k, z; w) \) guarantees that \( i_1(k, z; w) \) and \( i_2(k, z; w) \) are single valued.

\( T \) is a contraction in \( C(K \times Z) \).

The proof uses Blackwell’s sufficient conditions by showing that \( T \) defined above is a contraction. Let us first define the operators

\[ g_j(i, k, z; w') = \beta \max \left\{ (1 - \delta)k', \int p_j(k', z'; w')Q(dz'|z) \right\}, \text{for } j = 1, 2. \]

\[ g(i, k, z; w'; a) = \beta \max \left\{ (1 - \delta)k', \int (p(k', z'; w') + a)Q(dz'|z) \right\} \leq g(i, k, z; w') \]

and

\[ H_{1j}(k, z; w) = \max_{i \leq \pi(k, z; w)} \{-i + g_j(i, k, z; w')\}, \]

\[ H_{2j}(k, z; w) = \max_{i > \pi(k, z; w)} \{-i - \lambda(k, (1 - \delta)k + i, z; w) + g_j(i, k, z; w')\} \]

\( \forall p_1(k, z; w), p_2(k, z; w) \in C(K \times Z) \)
\begin{itemize}
    \item Monotonicity

    Suppose that \( p_1(k, z; w) \geq p_2(k, z; w) \). Then we can use our decomposition in Lemma A1 to obtain

    \[
    H_{11}(k, z; w) \geq H_{12}(k, z; w),
    \]
    \[
    H_{21}(k, z; w) \geq H_{22}(k, z; w).
    \]

    It follows that

    \[
    (Tp_1)(k, z; w) = \pi(k, z; w) + \max \{ H_{11}(k, z; w), H_{21}(k, z; w) \} \geq \pi(k, z; w) + \max \{ H_{12}(k, z; w), H_{22}(k, z; w) \} = (Tp_2)(k, z; w).
    \]

    \item Discounting

    Let \( a \in \mathbb{R} \). Then

    \[
    (Tp + a)(k, z; w) = \pi(k, z; w) + \max \{ \max_{i \leq \pi(k, z; w)} \{ -i + g(i, k, z; w'; a) \}, \max_{i > \pi(k, z; w)} \{ -i - \lambda(k; (1 - \delta)k + i, z; w) + g(i, k, z; w'; a) \} \} \leq (Tp)(k, z; w)
    \]

    \textbf{Proposition 2}

    \item \( p(k; .; .) \) is strictly increasing in \( k \in K \).

    Follows from Lemma 9.5 and Corollary 1 to Theorem 3.2 in Stokey and Lucas (1989).

    \item \( p(., z; .) \) is strictly increasing in \( z \in Z \).

    Take \( \forall z_1 < z_2 \), and define the optimal quantities

    \[
    i_{11}(k, z_1; w) = \arg \max_{i \leq \pi(k, z_1; w)} \{ -i + g(i, k, z_1; w') \},
    \]

    and

    \[
    \]
\[ i_{21}(k, z_1; w) = \arg \max_{\pi(k, z_1; w) \leq i \leq k} \{-i - \lambda(k, (1 - \delta)k + i, z_1; w) + g(i, k, z_1; w')\}. \]

Assumption 2 and the properties of the profit function \( \pi(k, z; w) \) imply that:

\[
Tp(k, z_1; w) = \pi(k, z_1; w) + \max\{-i_{11} + g(i, k, z_1; w'), -i_{21} - \lambda(k, k + i_{21}, z_1; w) + g(i, k, z_1; w')\} \\
\leq \pi(k, z_2; w) + \max\{-i_{11} + g(i, k, z_1; w'), -i_{21} - \lambda(k, (1 - \delta)k + i_{21}, z_2; w) + g(i, k, z_1; w')\} \\
\leq \max_{i \geq \pi(k, z; w)} \{-i - \lambda(k, (1 - \delta)k + i, z_2; w) + g(i, k, z_2; w')\} \\
= (Tp)(k, z_2; w).
\]

• \( p(., .; w) \) is continuous and strictly decreasing in \( w > 0 \).

Continuity (for \( w > 0 \)) follows from a straightforward modification of Lemma A.1 above. Monotonicity follows from Lemma 9.5 and Corollary 1 to Theorem 3.2 in Stokey and Lucas (1989).

**Proposition 3**

We will prove homogeneity for the aggregate output function \( Y(\mu, B; w) \). Proofs for the other cases are straightforward.

Define the measure \( \mu^*(k, z) = B\mu(k, z) \), as the actual measure of firms, and assume the level of entry is \( B \). Since \( \mu^* \) is absolutely continuous with respect to \( \mu \) the Radon-Nikodym theorem implies that

\[ d\mu^* = B d\mu, \]

Then,

\[
Y(\mu^*, B; w) = \int y(k, z; w)x(k, z; w)\mu^*(dk, dz) - Bf = \\
= \int y(k, z; w)x(k, z; w)B\mu(dk, dz) - Bf = BY(\mu, 1; w)
\]
Proposition 4
The assumptions on the utility function guarantee that in a stationary
equilibria the optimal condition for accumulation of shares is

\[ \tilde{p}(k, z) = d(k, z) + \tilde{\beta} \max \left\{ \int \tilde{p}(k', z') Q(\mathrm{d}z' \mid z), (1 - \delta)k' \right\}. \]

Then the balance sheet identity

\[ d(k, z) = \pi(k, z) - i(k, k') - \lambda(k, k', z; w). \]

yields

\[ \tilde{p}(k, z) = \max_{k' \geq 0} \{ \pi(k, z; w) - i(k, k') - \lambda(k, k', z; w) \}
+ \tilde{\beta} \max(\phi(1 - \delta)k', \int \tilde{p}(k', z') Q_z(\mathrm{d}z' \mid z)) \}. \]

with \( \tilde{p}(k, z) = p(k, z) \) and \( \beta = \tilde{\beta} \).

Proposition 5

- A general equilibrium exists.

Let \( w = w(L, \Pi) \) denote the (inverse) labor supply curve for this economy. It is easy to check then that assumptions (a)-(g) in Theorem 3 below are met. It follows that an industry equilibrium exists for this economy. Together with the goods market clearing condition this implies that a general competitive equilibrium exists for this economy.

- There is a unique stationary competitive equilibrium

Since technology is homogeneous in \( k \) and \( l \) we can write the profit function as

\[ \pi(k, z; w) = g(z) h(k, w) \]

The result then follows from Theorem 4 (Hopenhayn (1990)) Define the aggregate demand for final goods as \( \psi(Y) \) and aggregate labor supply as \( w(L) \). An industry equilibrium for this economy exists if the following conditions are met
(a) $Z$ is a compact metric space;
(b) $Q(z'|z)$ is a continuous transition function;
(c) Technology has decreasing returns to scale and the technology set has a closed graph;
(d) $w(L)$ is (weakly) increasing in $L$, $\psi(Y)$ is (weakly) decreasing in $Y$ and both functions are $\mathcal{B}$-measurable;
(e) $w(L)$ and $\psi(Y)$ are uniformly bounded above and $\beta$-integrable;
(f) For any initial measure $\mu^0$ there is a feasible allocation; and
(g) $B \to \infty \lim \|NY\| = \infty$, where $NY = (Y - I - \Lambda, -L)$ is input-output vector.

(Hopenhayn (1992)) Suppose that, in addition to conditions (a)-(g) above, the following condition holds:

(h) The profit function is separable in the following form $\pi(k, z; w) = g(z)h(k, w)$, for some functions $g$ and $h$.

Then the industry equilibrium above is unique and stationary and exhibits positive entry and exit.

## 2 Solution Methods

The behavior of this economy can not be characterized analytically even when one focus on a stationary competitive equilibrium. To study its properties we construct a numerical approximation to the competitive equilibrium described in Definition 1. The computational strategy that we will adopt involves the following steps:

1. Solving the Bellman Equation for the firm and computing the optimal decision rules;
2. Determining the wage rate that satisfies the free entry condition for $B > 0$;
3. Iterating on (12) to compute the stationary measure $\mu$ with $B = 1$, and
4. Using the market clearing conditions (goods or labor) to determine the equilibrium level of entry $B$ and the corresponding stationary measure $\mu$.

Given the properties of our problem the first step can not be accomplished with either policy function iteration since the policy function is not
continuous or methods based on low level polynomial approximations since the value function needs to be approximated well on a wide range of values of $k$, including $k = 0$, the value for new entrants. We implement the less efficient but more robust method of value function iteration on a discrete state space. We specify a grid with a finite number of points for the capital stock together with an upper bound $\bar{K}$, as well as a finite approximation to the normal random variable $z$. The later task was accomplished using in George Tauchen and Robert Hussey’s (1991) method for optimal discrete state space approximations to normal random variables. We use 10 points for this procedure. The space for the capital stock was divided in 301 equally spaced elements. In either case the results were robust to finer grids. The upper bound for capital was chosen to be non-binding at all times.

Homogeneity of the aggregate quantities in the distribution of firms implies that the level of entry can be used only to determine the appropriate scale of the market. In the third step we set $B = 1$ and use the computed optimal decision rules and the law of motion for the aggregate measure $\mu$ to derive the fixed point of (12), assuming $B = 1$. Proposition 3 guarantees that this fixed point is a well defined object.

Finally the level of entry can be determined by using one of the market clearing conditions. Given our preference structure it is easier to use the labor market clearing condition. From Proposition 4 we know that aggregate labor demand is homogeneous. It then follows that

$$L(\mu, B; w) = BL(\mu^*, 1; w) = L^*(w),$$

where $\mu^*$ is the fixed point determined in step 3. With the wage rate $w$ and the measure $\mu^*$ determined above this condition can be used to determine $B$ directly.

### 3 The Problem of a Firm

To derive the optimal value maximizing problem of a firm we explore a simple no-arbitrage condition for the shareholders and show that the value of the firm satisfies the household Euler condition for optimal asset pricing.

The equilibrium (no arbitrage) return for a current shareholder of a firm is given by

$$r = \frac{D^+(k, z; w) + V'(k, z; w) - V(k, z; w) - V^N(k, z; w) - \Lambda}{V(k, z; w)}$$
where we use the state \( (k, z) \) to index the firm. The variables \( V \) and \( D^+ \)
denote, respectively, the beginning of period value and the total dividends
paid by the firm. \( V^N \) denotes the total value of new issues in the period.

Provided the rate of return is strictly positive this condition can be solved
forward to determine the current value of the firm \( V(k, z; w) \). This function
satisfies the following Bellman equation

\[
V(k, z; w) = \max_{k' \geq 0} \{ D^+(k, z; w) - V^N(k, z; w) - \lambda + \beta \int V(k', z'; w)Q(\text{d}z'|z) \}
\]

where \( \beta = 1/(1 + r)^1 \). The obvious symmetry between dividends and new
issues suggests a more convenient way of representing this problem. Define
dividends as \( D(k, z; w) = D^+(k, z; w) - V^N(k, z; w) \), which can now be
negative and normalize the number of shares (defined as \( s(k, z; w) \)) to 1.

Letting \( p(k, z; w) \) denote the price of one such share, \( d(k, z; w) \) the di-
vidend yield and \( \lambda(k, k', z'; w) \) the financing cost, the Bellman equation be-
comes

\[
p(k, z; w) = \max_{k' \geq 0} \{ d(k, z; w) - \lambda(k, k', z; w) + \beta \int p(k', z'; w)Q(\text{d}z'|z) \}.
\] (2)

Then the accounting identity between sources and uses of funds for each
firm

\[
d(k, z; w) = \pi(k, z; w) - i(k, k')
\]

implies the following dynamic program for each firm

\[
p(k, z; w) = \max_{k' \geq 0} \{ \pi(k, z; w) - i(k, k') - \lambda(k, k', z; w) + \beta \int p'(k', z'; w)Q(\text{d}z'|z) \}.
\]

**References**


\(^1\)Since there is no aggregate uncertainty and all risk can be diversified the required rate
of return is constant and identical across firms.