The Price Impact and Survival of Irrational Traders

Leonid Kogan, Stephen Ross, Jiang Wang, and Mark Westerfield*

First Draft: March 25, 2002
This Draft: August 26, 2004

Abstract

Milton Friedman argued that irrational traders will consistently lose money, won't survive and, therefore, cannot influence long run equilibrium asset prices. Since his work, survival and price impact have been assumed to be the same. In this paper, we demonstrate that survival and price impact are two independent concepts. The price impact of irrational traders does not rely on their long-run survival and they can have a significant impact on asset prices even when their wealth becomes negligible. We also show that irrational traders' portfolio policies can deviate from their limits long after the price process approaches its long-run limit. We show, in contrast to a partial equilibrium analysis, these general equilibrium considerations matter for the irrational traders' long-run survival. In sum, we explicitly show that price impact can persist whether or not the irrational traders survive.

*Kogan and Ross are from the Sloan School of Management, MIT and NBER, Wang is from the Sloan School of Management, MIT, CCFR and NBER, and Westerfield is from the Economics Department, MIT. The authors thank Phil Dybvig, David Easley, Ming Huang, Wei Xiong and participants of the 2002 NBER Asset Pricing Summer Conference, the 2003 Econometric Society Meetings, the Liquidity Conference at the London School of Economics, the 2004 American Finance Association Meetings, and seminars at Carnegie-Mellon, Duke, University of Maryland, University of Minnesota, MIT, NYU, University of Oslo, Stockholm School of Economics, and University of Vienna for comments.

Contents

Most neoclassical asset pricing models rely on the assumption that market participants (traders) are rational in the sense that they behave in ways that are consistent with the objective probabilities of the states of the economy (e.g., Radner (1971) and Lucas (1978)). More particularly, they maximize expected utilities using the true probabilities of uncertain economic states. This approach is firmly rooted in the tradition of going from the normative to the positive in economics, yet there is mounting evidence that it is not descriptive of the observed behavior of the average market participant (see, e.g., Alpert and Raiffa (1959), Benartzi and Thaler (2001), Black (1986), Kahneman and Tversky (1979), and Odean 1998)). How the presence of traders with incorrect beliefs may affect the behavior of financial markets remains an open question.

It has long been argued (see, e.g., Friedman (1953)) that irrational traders who use wrong beliefs cannot survive in a competitive market. Trading under the wrong beliefs causes them to lose their wealth. In the long-run, it is the rational traders who control most of the wealth and determine asset prices. Using a partial equilibrium model, De Long, Shleifer, Summers and Waldmann (1991) suggest that traders with wrong beliefs may survive in the long-run because they may hold portfolios with higher growth rates and therefore can eventually outgrow the rational traders.[1] In contrast, in a general equilibrium setting, Sandroni (2000) and Blume and Easley (2001) show that with intermediate consumption, irrational traders do not survive in the long-run.

The efficiency of financial markets is the principal motivation behind the interest in the survival of irrational traders. If irrational traders impact asset prices, then markets will not be efficient, either informationally or allocationally. Implicitly, the discussion on survival is based on the assumption that survival is a necessary condition for long-run price impact. It is thought that irrational traders have to control a significant amount of wealth in order to affect – or 'infect' – prices with their irrational beliefs. In this paper, we show that this assumption is false and that irrational traders can maintain a large price impact even as their relative wealth diminishes towards zero over time.

Our analysis is conducted with a parsimonious general equilibrium model inhabited by both rational traders and irrational traders. Traders only care about their terminal consumption. We are able to derive an explicit solution to the model and obtain conditions under which the irrational traders can survive in the long run in the sense that their share of the total wealth does not go to zero over time. However, we show that even when irrational traders do not survive, with a negligible amount of wealth they can still exert significant

[1]See also Figlewski (1978) for a discussion on the notion of long-run survival.

influence on the asset price over a long period of time.

Underlying this initially counterintuitive result is a solid economic intuition. Under incorrect beliefs, irrational traders express their views by taking positions (bets) on extremely unlikely states of the economy. As a result, the state prices of these extreme states can be significantly affected by the beliefs of the irrational traders, even with negligible wealth. In turn, these states, even though highly unlikely, can have a large contribution to current asset prices. This is especially true for states associated with extremely low levels of aggregate consumption in which the traders' marginal utilities are very high and so too are state prices. The beliefs of the irrational traders on these low probability but high marginal utility states can influence current asset prices and their dynamics. Furthermore, irrational traders need not take extreme positions in order to influence prices. Our formal analysis clearly verifies this conceptual distinction between the long-run price impact and the long-run survival of irrational traders.

The possibility that irrational traders may have a significant price impact with a negligible share of wealth also has important implications for their survival. In the partial equilibrium analysis of De Long, Shleifer, Summers and Waldmann (1991) (DSSW, thereafter), it was assumed that when the irrational traders control only a negligible fraction of the total wealth, they have no impact on asset prices, i.e., asset prices behave as if the irrational traders are absent. Given the rationally determined prices, DSSW then show that the wealth of irrational traders can grow at a faster rate than the wealth of the rational traders, allowing the irrational traders to recover from their losses and survive in the long-run. Although such an argument is illuminating, it is based on unreliable premises. As we have argued, irrational traders may still influence prices with diminishing wealth. Moreover, such a possibility can significantly affect the irrational traders' portfolio policies in ways that make their recovery from losses difficult.

The paper proceeds as follows. In Section 1, we provide a simple example to illustrate the possibility for an agent to affect asset prices with a negligible wealth. Section 2 describes a canonical economy similar to that of Black and Scholes (1973), but in the presence of irrational traders who have persistently wrong beliefs about the economy, and Section 3 describes the general equilibrium of this economy. Section 4 treats the special case of logarithmic preferences and demonstrates that even though irrational traders never survive in this case, they nevertheless can still influence long-run asset prices. Sections 5, 6, and 7 analyze the survival of irrational traders, their price impact and their portfolio policies for the case of risk aversion different from one. Section 8 discusses the importance of equilibrium effects on the survival of irrational traders. Section 9 concludes the paper with a short

summary and some suggestions for future research. All proofs are given in the appendix.

1 An Example

We begin our analysis by considering a simple, static Arrow-Debreu economy and will show that an agent with only a negligible amount of wealth can have a significant impact on asset prices by using certain trading policies.

The economy has two dates, 0 and 1. It is endowed with one unit of a risky asset, which pays a dividend D only at date 1. The realization of D falls in $[0, 1]$ with probability density $p(D) = 2D$, which is plotted in Figure 1(a).

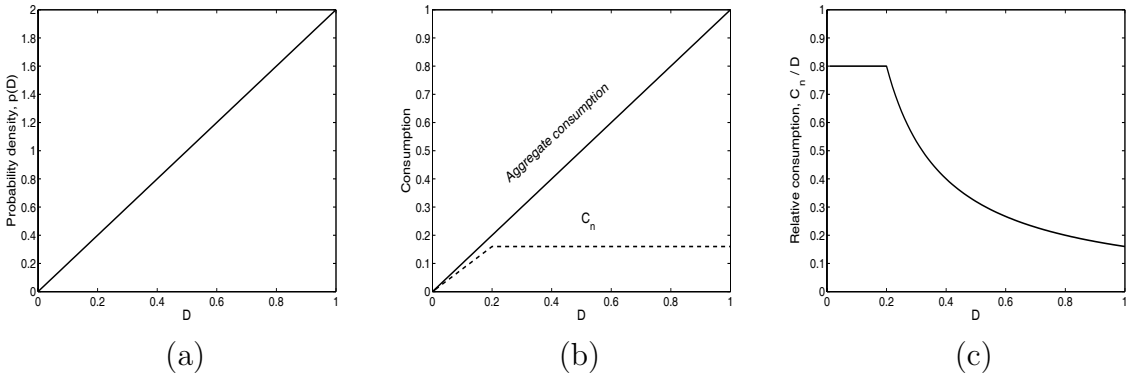

Figure 1: Probability distribution of the stock dividend (the left panel), the aggregate consumption level (D) and the noise trade consumption (C_n) when he is present (the middle panel), and the relative consumption of the noise trader (C_n/D, right panel). Parameter δ is set to 0.2.

There is a complete set of Arrow-Debreu securities traded in a competitive financial market at date 0. Shares of the stock and a risk-free bond with a sure payoff of 1 at date 1, both of which are baskets of the Arrow-Debreu securities, are also traded. We use the bond as the numeraire for the security prices at date 0. Thus, the bond price is always 1.

We first consider the economy when it is populated by a representative agent with a logarithmic utility function over consumption at date 1, $u(C) = \log C$. It immediately follows that $C = D$ and the state price density, denoted by ϕ^*, is

$$\phi^*(D) = a^* u'(D) = \frac{a^*}{D}$$

where a^* is a constant. The price of any payoff X is then given by

$$P = \mathrm{E}\,[\,X \cdot \phi^*\,].$$

In particular, the price of the bond is

$$B = \mathrm{E}\,[\,1 \cdot \phi^*\,] = \int_0^1 \frac{a^*}{D}\, p(D)\, dD = \int_0^1 2a^*\, dD = 2a^* = 1$$

which gives $a^* = \frac{1}{2}$. The price of the stock is then given by

$$S^* = \mathrm{E}\,[\,D \cdot \phi^*] = \int_0^1 D \cdot \frac{1}{2D}\, p(D)\, dD = \int_0^1 D\, dD = \frac{1}{2}.$$

Now we introduce another trader to the economy who has a negligible amount of wealth and desires a particular consumption bundle. We denote this trader with "N" and call him a noise trader. The noise trader demands the following consumption bundle:

$$C_n = (1-\delta)\min(\delta, D), \qquad 0 < \delta < 1$$

which is plotted in Figure 1(b). Figure 1(c) plots C_n as a fraction of the total consumption D. Since $C_n \leq \delta(1-\delta)$, the wealth the noise trader needs to acquire the consumption bundle, is

$$W_n = \mathrm{E}\,[\,C_n \cdot \phi\,] \leq \mathrm{E}\,[\,\delta(1-\delta) \cdot \phi\,] = \delta(1-\delta) < \delta$$

where we have used the fact that the bond price is 1. The consumption for the representative agent (excluding the noise trade) is then $C = D - C_n$, also shown in Figure 1(b). The state price density in this case is

$$\phi = au'(C) = \frac{a}{D - (1-\delta)\min(\delta, D)}$$

Since the price of the bond is one, we have

$$B = \mathrm{E}\,[\,1 \cdot \phi\,] = \int_0^\delta \frac{a}{\delta D}\,(2D)\, dD + \int_\delta^1 \frac{a}{D - \delta(1-\delta)}\,(2D)\, dD = 1$$

which gives

$$a = \frac{1}{4}\left\{1 - \frac{\delta}{2} + \frac{1}{2}\delta(1-\delta)[\ln(1-\delta+\delta^2) - 2\ln(\delta)]\right\}^{-1}$$

As noted above, the wealth needed to acquire the consumption bundle C_n, W_n, is less than δ, so it is small if δ is small. The stock price in the presence of the noise trader is given by

$$\begin{aligned} S &= \mathrm{E}\left[\, D \cdot \phi \,\right] = \int_0^{\delta} D \frac{a}{\delta D}(2D)\, dD + \int_{\delta}^{1} D \frac{a}{(1-\delta)\delta}(2D)\, dD \\ &= a\left\{1 + 3\delta - 5\delta^2 + 2\delta^3 + 2\delta^2(1-\delta)^2\left[\ln(1-\delta+\delta^2) - 2\ln(\delta)\right]\right\} = \frac{1}{4} + O(\delta) \end{aligned}$$

where $O(\delta)$ denotes terms of order δ or higher. Thus, $S/S^* = \frac{1}{2} + O(\delta)$. We can measure the impact of the noise trade on the stock price by

$$1 - \frac{S}{S^*} = \frac{1}{2} + O(\delta)$$

which remains non-negligible even when δ, and therefore the amount of wealth controlled by the noise trader, approaches zero.

This is a stark result: a price-taking trader with negligible wealth can exert finite influence on asset prices. The noise trader spends most of his wealth on consumption in low-dividend states. Given that the marginal utility of the other traders in these states is very high, the state prices for these states are also high and, more importantly, a small change in the consumption level can change the state prices significantly. As we show above, the wealth required for the noise traders to finance their desired consumption profile is small, even though most of their consumption occurs in states with relatively high state prices.

While the above example is rather simple, its intuition holds more generally. In the case of logarithmic preferences, the state price density is proportional to the rational trader's marginal utility $u'(C)$: $\phi = au'(C)$, where a is the proportionality constant. When the irrational trader is introduced into the economy and he purchases ε units of state-contingent claims that pay off only when the aggregate consumption is C, the state-price density will change by $\Delta\phi \approx -au''(C)\varepsilon$. The total cost for the purchase is $w \equiv \phi\varepsilon \approx au'(C)\varepsilon$ when ε is small. Divided by the wealth spent by the irrational trader, we obtain the marginal change in the state-price density:

$$\frac{\Delta\phi}{w} = \frac{u''(C)}{u'(C)} = \frac{1}{C}$$

which is independent of ε. Clearly, in "bad" states, in which C is low (close to zero), irrational traders can have a large impact on the state-price density with little wealth if they decide to bet on these states. Through their impact on the state-price density in bad states, irrational traders can influence asset prices, such as the prices of the stock and the bond. Given that

the bond is used as a numeraire and its price is always one, this influence is captured in the stock price, given by $S = \mathrm{E}[\, D \cdot \phi \,]$, as shown above.

Our example clearly demonstrates the possibility of influencing asset prices with little wealth. The remaining question is whether such a situation can arise in "realistic" settings. In particular, for our purpose in this paper, can the irrational traders with incorrect beliefs maintain a significant price impact even as their relative wealth diminishes from investment losses in the market? In the remainder of the paper, we use a canonical model to address these questions.

2 The Model

We consider a standard setting similar to that of Black and Scholes (1973). For simplicity, we make the model parsimonious.

Information structure

The economy has a finite horizon and evolves in continuous time. Uncertainty is described by a one-dimensional, standard Brownian motion B_t for $0 \leq t \leq T$, defined on a complete probability space (Ω, F, P), where F is the augmented filtration generated by B_t.

The financial market

There is a single share of a risky asset in the economy, the stock, which pays a terminal dividend payment D_T at time T, determined by process:

$$dD_t = D_t \left(\mu dt + \sigma dB_t\right) \tag{1}$$

where $D_0 = 1$ and $\sigma > 0$. There is also a zero coupon bond available in zero net supply. Each unit of the bond makes a sure payment of one at time T. We use the risk-free bond as the numeraire and denote the price of the stock at time t by S_t.

Endowments

There are two competitive traders in the economy, each endowed with a half share of the stock (and none of the bond) at time zero.

Trading strategies

The financial market is frictionless and has no constraints on lending and borrowing. Traders' trading strategies satisfy the standard integrability condition used to avoid pathologies:

$$\int_0^T \theta_t^2 \, d\langle S \rangle_t < \infty$$

where θ_t is the number of stock shares held in the portfolio at time t and $\langle S \rangle_t$ is the quadratic variation process of S_t (see, e.g., Duffie and Huang (1986) and Harrison and Kreps (1979)).

Preferences and beliefs

Both traders have constant relative risk aversion utility over their consumption at time T:

$$\frac{1}{1-\gamma} C_T^{1-\gamma}, \quad \gamma \geq 1.$$

For ease of exposition, we only consider the cases when $\gamma \geq 1$. The cases when $0 < \gamma < 1$ can be analyzed similarly and the results are similar in spirit.

Standard aggregation results imply that each trader in our model can actually represent a collection of traders with the same preferences. This provides a justification for our competitive assumption for each of the traders. The first trader, the rational trader, knows the true probability measure P and maximizes expected utility

$$\mathrm{E}_0^P \left[\frac{1}{1-\gamma} C_{r,T}^{1-\gamma} \right] \tag{2}$$

where the subscript r denotes quantities associated with the rational trader. The second trader, the irrational trader, believes incorrectly that the probability measure is Q, under which

$$dB_t = (\sigma\eta)dt + dB_t^Q \tag{3}$$

and hence

$$dD_t = D_t \left[\left(\mu + \sigma^2 \eta \right) dt + \sigma dB_t^Q \right] \tag{4}$$

where B_t^Q is the standard Brownian motion under the measure Q and η is a constant, parameterizing the degree of irrationality of the irrational trader. When η is positive, the irrational trader is optimistic about the prospects of the economy and overestimates the rate

of growth of the aggregate endowment. Conversely, a negative η corresponds to a pessimistic irrational trader. The irrational trader maximizes expected utility using belief Q:

$$\mathrm{E}_0^Q \left[\frac{1}{1-\gamma} C_{n,T}^{1-\gamma} \right] \tag{5}$$

where the subscript n denotes quantities associated with the irrational trader.

Because η is assumed to be constant, the probability measure of the irrational trader Q is absolutely continuous with respect to the objective measure P, i.e., both traders agree on zero-probability events. Let $\xi_t \equiv (dQ/dP)_t$ denote the density (Radon-Nikodym derivative) of the probability measure Q with respect to P,

$$\xi_t = e^{-\frac{1}{2}\eta^2\sigma^2 t + \eta\sigma B_t}. \tag{6}$$

The irrational trader maximizes

$$\mathrm{E}_0^Q \left[\frac{1}{1-\gamma} C_{n,T}^{1-\gamma} \right] = \mathrm{E}_0^P \left[\xi_T \frac{1}{1-\gamma} C_{n,T}^{1-\gamma} \right]. \tag{7}$$

This permits us to interpret the objective of the irrational trader as the expected value of a state-dependent utility function, $\xi_T \frac{1}{1-\gamma} C_{n,T}^{1-\gamma}$, under the true probability measure P.

The equivalence between incorrect beliefs and state dependent preferences raises a conceptual question about the precise definition of irrationality. It is beyond the scope of this paper to address this question, and our analysis of this form of irrationality is primarily motivated by the fact that it is widely adopted in the recent literature on behavioral models of asset prices.

3 The Equilibrium

The competitive equilibrium of the economy defined above can be solved analytically. Since there is only one source of uncertainty in the economy, the financial market is dynamically complete as long as the volatility of stock returns remains non-zero almost surely. Consequently, the equilibrium allocation is efficient and can be characterized as the solution to a central planner's problem:

$$\max \quad \left[\frac{1}{1-\gamma} C_{r,T}^{1-\gamma} \; + \; b\, \xi_T \frac{1}{1-\gamma} C_{n,T}^{1-\gamma} \right] \tag{8a}$$

$$\text{s.t.} \quad C_{r,T} + C_{n,T} = D_T \tag{8b}$$

where b is the ratio of the utility weights for the two traders. The equilibrium allocation is characterized in the following proposition.

Proposition 1 *For the economy defined in Section 2, the equilibrium allocation between the two traders is*

$$C_{r,T} = \frac{1}{1+(b\,\xi_T)^{1/\gamma}} D_T \tag{9a}$$

$$C_{n,T} = \frac{(b\,\xi_T)^{1/\gamma}}{1+(b\,\xi_T)^{1/\gamma}} D_T \tag{9b}$$

where

$$b = e^{(\gamma-1)\eta\sigma^2 T}. \tag{10}$$

The price of a financial security with the terminal payoff Z_T is given by

$$P_t = \frac{\mathrm{E_t}\left[\left(1+(b\,\xi_T)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma} Z_T\right]}{\mathrm{E_t}\left[\left(1+(b\,\xi_T)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma}\right]}. \tag{11}$$

For the stock, $Z_T = D_T$ and its return volatility is bounded between σ and $\sigma(1+|\eta|)$.

Since the instantaneous volatility of stock returns is bounded below by σ, the stock and the bond dynamically complete the financial market. In the limiting cases when only the rational or the irrational trader is present, the stock prices, denoted by S_t^* and S_t^{**}, respectively, are given by

$$S_t^* = e^{(\mu/\sigma^2-\gamma)\sigma^2 T+\frac{1}{2}(2\gamma-1)\sigma^2 t+\sigma B_t} \tag{12a}$$

$$S_t^{**} = e^{(\mu/\sigma^2-\gamma+\eta)\sigma^2 T+\frac{1}{2}[(2\gamma-1)-2\eta]\sigma^2 t+\sigma B_t} = S_t^* e^{\eta\sigma^2(T-t)}. \tag{12b}$$

We will use this equilibrium model to analyze the survival and extinction of the traders. We employ the following common definition of extinction, and, conversely, of survival.

Definition 1 *The irrational trader is said to experience relative extinction in the long-run if*

$$\lim_{T\to\infty} \frac{C_{n,T}}{C_{r,T}} = 0 \quad a.s. \tag{13}$$

The relative extinction of the rational trader can be defined symmetrically. A trader is said to survive relatively in the long-run if relative extinction does not occur.

In the above definition and throughout the paper, all limits are understood to be almost sure (under the true probability measure P) unless specifically stated otherwise.

In our model, the final wealth of each trader equals their terminal consumption. Thus, the definition of survival and extinction is equivalent to a similar definition in terms of wealth.

4 Logarithmic Preferences

We first consider the case where both the rational and the irrational traders have logarithmic preferences. We have the following result:

Proposition 2 *Suppose $\eta \neq 0$. For $\gamma = 1$, the irrational trader never survives.*

This result is immediate. For $\gamma = 1$, the rational trader holds the portfolio with maximum expected growth (see, e.g., Hakansson (1971)). Any deviation in beliefs from the true probability causes the irrational trader to move away from the maximum growth portfolio, which leads to his long-run relative extinction.

Our interest here, however, is not on the survival of the irrational trader, but on the impact of irrationality on the long-run stock price. Under logarithmic preferences, $b = 1$ and from Proposition 2 the stock price is

$$S_t = \frac{1+\xi_t}{\mathrm{E_t}\left[(1+\xi_T)/D_T\right]} = \frac{1+\xi_t}{1+e^{-\eta\sigma^2(T-t)}\xi_t} S_t^* \tag{14}$$

where S_t^* denotes the stock price in an identical economy populated only by the rational trader, given in (12). We now prove that the irrational trader can maintain a large impact on the stock price despite losing most of his wealth. To state our result formally, we define the relative wealth shares of the rational and irrational traders, respectively,

$$\alpha_{n,t} \equiv \frac{W_{n,t}}{W_{r,t}+W_{n,t}} = \frac{\xi_t}{1+\xi_t}, \qquad \alpha_{r,t} \equiv 1-\alpha_{n,t}$$

The price impact the irrational trader can be measured by $1 - \frac{S_t}{S_t^*}$, the relative deviation in stock price from its limiting value with only the rational trader. We have

Proposition 3 *Consider the case of a pessimistic irrational trader, $\eta < 0$. For any ε as*

small as $e^{-\frac{\sigma^2\eta^2}{12(1+|\eta|)}T}$*, there exists a point in time* $t \geq T/(1+|\eta|)$*, such that*

$$\text{Prob}\left[\alpha_{n,t} \geq \varepsilon\right] \leq \varepsilon \tag{15a}$$

$$\text{Prob}\left[1 - \frac{S_t}{S_t^*} \leq 1 - \varepsilon\right] \leq \varepsilon. \tag{15b}$$

Intuitively, Proposition 3 shows that after a long period of time, which constitutes a nontrivial fraction of the horizon of the economy, the relative wealth of the irrational trader is most likely to be very small (which is consistent with his long-run extinction), but his impact on the stock price is most likely to remain large (the ratio S_t/S_t^* stays far away from one).

Another way to illustrate the persistent nature of the irrational trader's price impact is by examining the long-run behavior of the instantaneous moments of stock returns, which can be derived explicitly. For example, the conditional volatility of stock returns is

$$\sigma_{S,t} = \sigma + \eta\sigma\alpha_{n,t} - \eta\sigma\left[1 - \frac{1}{1 + e^{-\eta\sigma^2(T-t)}\alpha_{n,t}(1-\alpha_{n,t})^{-1}}\right]$$

and the conditional mean is

$$\mu_{S,t} = \sigma_{S,t}^2 - \alpha_{n,t}\eta\,\sigma\sigma_{S,t}.$$

To visualize the behavior of stock return moments, consider the following numerical example. The irrational trader is assumed to be pessimistic ($\eta = -2$). The horizon of the economy is set to $T = 400$, so the relative wealth of the irrational trader becomes relatively small long before the final date. We let the current time t be sufficiently large, so with high probability most of wealth in the economy is controlled by the rational trader. For convenience, we define the following normalized state variable:

$$g_{s,t} \equiv \frac{B_t - B_s}{\sqrt{t-s}} \tag{16}$$

where $s < t$. It is easy to show that $g_{s,t}$ is the unanticipated dividend growth normalized by its standard deviation, which has a standard normal distribution. Figure 2 plots the Sharpe ratio of instantaneous stock returns and the wealth distribution between the two traders at $t = 150$ against the normalized state variable $g_{0,t}$. The probability density for $g_{0,t}$ is illustrated by the shaded area (with the vertical axis on the right). The bottom panel of Figure 2 shows that with almost probability one, the wealth of the economy is all controlled by the rational trader at this time. Yet as the top panel of the figure shows, the conditional Sharpe ratio of stock returns is very different from σ, which is its value in the economy

populated only by the rational trader. In particular, over a large range of values of the dividends, the conditional Sharpe ratio of returns is approximately equal to $\sigma(1-\eta) \neq \sigma$.

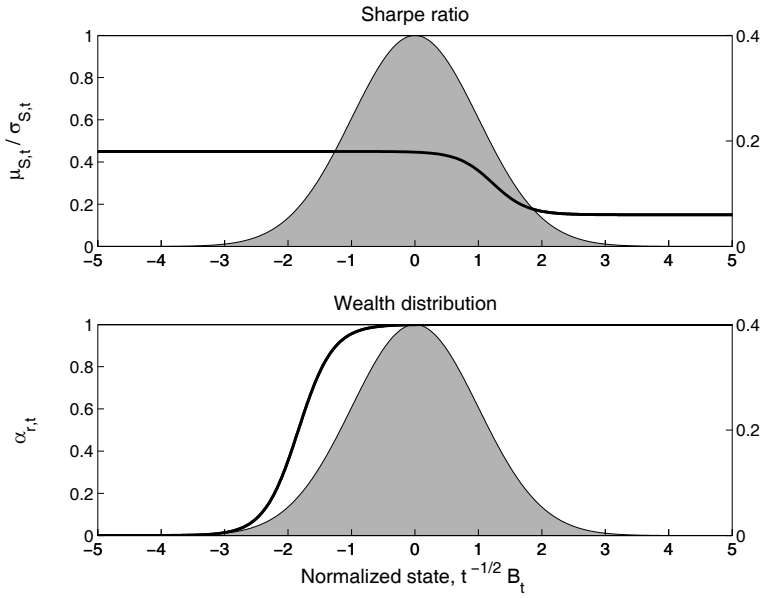

Figure 2: The conditional Sharpe ratio of stock returns, $\mu_{S,t}/\sigma_{S,t}$ and the wealth distribution $\alpha_{r,t} = W_{r,t}/(W_{r,t} + W_{n,t})$ are plotted against the normalized state variable, $g_{0,t} \equiv B_t/\sqrt{t}$. The shaded area is the probability density function of the normalized state variable (vertical axis on the right). The model parameters are set at $\mu = 0.05$, $\sigma = 0.15$, $\eta = -2$, $T = 400$ and $\gamma = 1$. The current time is $t = 150$.

Figure 3 provides a complimentary illustration. It shows the most likely path over time (the path with highest probability) for the irrational trader's wealth share and the Sharpe ratio of stock returns. In fact, the irrational trader's wealth share diminishes to zero exponentially while his price impact diminishes at a much slower rate. The Sharpe ratio stays away from its level in an economy without an irrational trader for an extended period of time before eventually converging to the limiting value.

In order to better understand how the irrational trader can exert influence on the stock price despite having negligible wealth, we examine how his presence affects the state price density (SPD). The left panels of Figure 4 plot the relative consumption shares of the rational and the irrational traders at two different times, $t = 0, 25$, as a function of the normalized state variable, $g_{t,T}$, i.e., the normalized unanticipated dividend growth from t to T defined in (16). At each date, the state of the economy is conditioned on $B_t = 0$, the most likely state. For $t = 0$, the irrational trader owns half of the economy. But at $\eta = -4$, he is very pessimistic and bets on states of low dividends (states toward the left end of the horizontal axis). This is shown in the top left panel of Figure 4. The dashed line plots

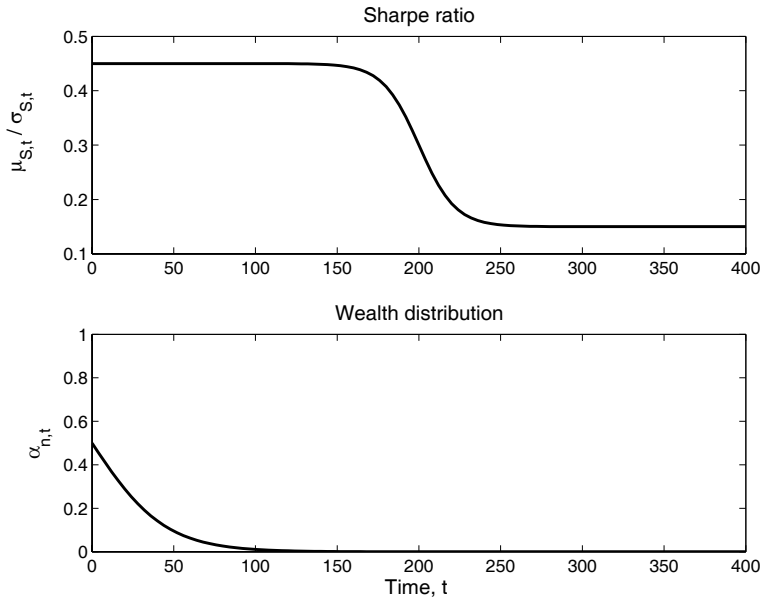

Figure 3: The maximum likelihood path of the irrational trader's wealth share, $\alpha_{n,t} = W_{n,t}/(W_{r,t} + W_{n,t})$, and the Sharpe ratio, $\mu_{S,t}/\sigma_{S,t}$. The model parameters are set at $\mu = 0.05$, $\sigma = 0.15$, $\eta = -2$, $T = 400$ and $\gamma = 1$.

his terminal consumption for different states of the economy. It is worth pointing out that the consumption choice of the irrational trader in this economy is similar to that in the simple one-period economy we considered in Section 1, as shown in Figure 1(c), where the irrational trader consumes a share of $1 - \delta$ of the aggregate endowment in states with low dividends and much smaller share in other states. This explains why in both economies the irrational trader can exert significant influence on prices despite of being left with relatively little wealth.

Over time, the 'bad' states become less likely and the irrational trader's bets become less valuable. Thus, his wealth decreases. At $t = 25$ and $B_t = 0$, these bad states become extremely unlikely and the irrational trader has lost most of his wealth. His wealth as fraction of total wealth has fallen from $1/2$ at $t = 0$ to 0.01. As shown in the bottom left panel of Figure 4, going forward, irrational trader consumes a non-trivial fraction of the total wealth only in the extreme states toward the left end of the horizontal axis. The probability of these states, as shown by the shaded area, becomes very small and so is the irrational trader's wealth.

In the two panels on the right of Figure 4, we plot the state price density against the normalized state variable $g_{t,T}$ at the two times, $t = 0, 25$, conditioned again on $B_t = 0$. With

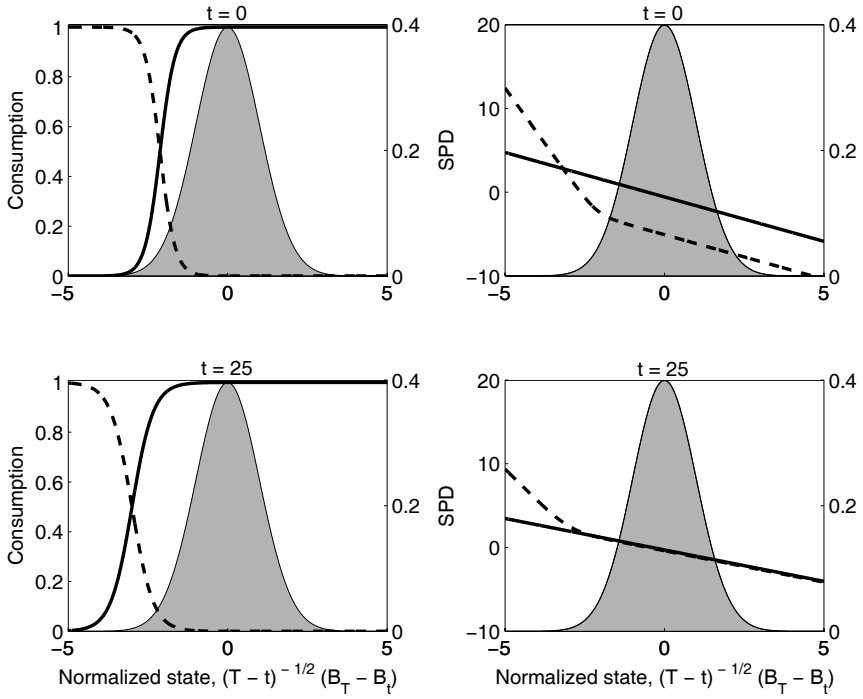

Figure 4: The terminal consumption of the rational and irrational traders as a fraction of the total consumption and the state price density (SPD) in different terminal states of the economy at different times. The model parameters are set to be $\mu = 0.05$, $\sigma = 0.15$, $\eta = -4$, $\gamma = 1$ and $T = 50$. The horizontal axis in all panels is the normalized state variable $g_{t,T} \equiv (B_T - B_t)/\sqrt{T-t}$, which has a standard normal distribution with zero mean and unit variance, which is shown by the shaded area (vertical axis on the right). In the two panels on the left, the terminal consumption for the rational trader (the solid line) and the irrational trader (the dotted line) are plotted against the normalized state variable at times $t = 0, 25$, respectively, when $B_t = 0$. In the two panels on the right, the dashed line plots the logarithm of the state price density at times $t = 0, 25$, respectively, which is $\ln\left\{\left[(1+\xi_T)/D_T\right]/\mathrm{E_t}\left[(1+\xi_T)/D_T\right]\right\}$. The solid line plots the logarithm of the state price density in the economy populated only by the rational traders, which is $\ln\left\{D_T^{-1}/\mathrm{E_t}\left[D_T^{-1}\right]\right\}$.

logarithmic preferences, the equilibrium state price density at time t is given by

$$\phi_t \equiv \frac{(1+\xi_T)D_T^{-1}}{\mathrm{E_t}\left[(1+\xi_T)D_T^{-1}\right]}$$

which is represented by the dashed line in each of the two panels. The solid line plots the state price density when the economy is populated only by the rational traders, which can be obtained by setting $\xi_T = 0$ in the above expression for ϕ_t. The top panel gives the state price density at $t = 0$. At this point, the irrational trader has a half share of the total wealth and his portfolio policy has a significant influence on the state price density over the whole range. In particular, being pessimistic, he is effectively betting on the bad states, which causes the state price density to increase for the bad states and decrease for the good states. This is

shown by the difference between the dashed line, the state price density in the presence of the irrational trader, and the solid line, the state price density without the irrational trader. As time passes, the irrational trader's wealth dwindles and his influence on the state price density diminishes quickly for most of the states, as the bottom panel for $t = 25$ shows. However, for the extremely bad states his influence remains significant because he is still betting heavily on these states.

We can show that the price impact of the irrational trader with negligible wealth does not rely on excessive leverage. The fraction of irrational trader's wealth invested in the stock is given by $\sigma_{S,t} + \eta\sigma(1 - \alpha_{n,t})$, which is bounded in absolute value by $\sigma(1 + 2|\eta|)$. The irrational trader can make bets on states with low aggregate endowment not by taking extreme portfolio positions, but rather by under-weighting the stock in his portfolio over long periods of time.

The simple case of logarithmic preferences developed above clearly shows that survival and price impact are in general not equivalent. In particular, survival is not a necessary condition for the irrational trader to influence long-run prices, and depending on their beliefs, irrational traders can maintain a significant price impact even as their wealth becomes negligible over time.

In the remaining sections, we consider the general case when $\gamma \neq 1$ and analyze the survival of the irrational trader, his price impact, and his portfolio choices.

5 Survival

In the case of logarithmic preferences, the irrational trader does not survive in the long-run simply because his portfolio grows more slowly than the maximum growth rate, the rate achieved by the rational trader. For the coefficient of relative risk aversion different from one, though, the rational trader no longer holds the optimal growth portfolio and under an incorrect belief, the irrational trader may end up holding a portfolio that is closer to the optimal growth portfolio, and so his wealth may grow more rapidly. This was the argument put forward by DSSW using a partial equilibrium setting. In this section, we examine the long-run survival of the irrational trader in our general equilibrium setting.

From the competitive equilibrium derived in Section 3, we have the following result:

Proposition 4 *Suppose $\eta \neq 0$. Let $\eta^\star = 2(\gamma - 1)$. For $\gamma > 1$ and $\eta \neq \eta^\star$, only one of the*

traders survives in the long run. In particular, we have

$$\begin{array}{lcl} \text{Pessimistic irrational trader: } \eta < 0 & \Rightarrow & \text{Rational trader survives} \\ \text{Moderately optimistic irrational trader: } 0 < \eta < \eta^\star & \Rightarrow & \text{Irrational trader survives} \\ \text{Strongly optimistic irrational trader: } \eta > \eta^\star & \Rightarrow & \text{Rational trader survives.} \end{array} \quad (17)$$

For $\eta = \eta^\star$, both rational and irrational traders survive.

For $\gamma > 1$, Proposition 4 identifies three distinct regions in the parameter space as shown in Figure 5. For $\eta < 0$, the irrational trader is pessimistic and does not survive in the long-run. For $0 < \eta < \eta^\star$, the irrational trader is moderately optimistic and survives in the long-run while the rational trader does not. For $\eta > \eta^\star$, the irrational trader is strongly optimistic and does not survive. Clearly, other than the knife-edge case ($\eta = \eta^\star$), only one of the traders can survive.

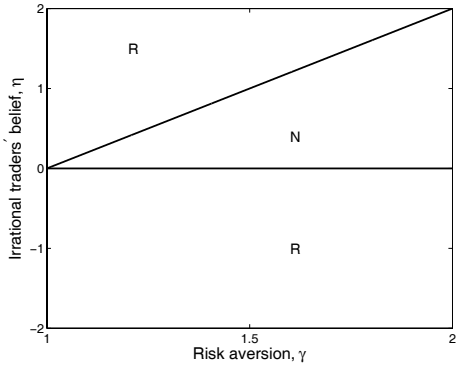

Figure 5: The survival of rational and irrational traders for different values of η and γ. For each region in the parameter space, we document which of the agents survives in the long run. "R" means that survival of the rational trader is guaranteed inside the region, "N" corresponds to the irrational trader.

In order to gain more insight into what determines the survival of each type of traders, we examine their terminal wealth (consumption) profiles. The two panels on the left in Figure 6 show the two traders' terminal wealth profiles for two values of T (10 and 30) when the irrational trader is pessimistic. The solid line shows the terminal wealth share of the rational trader and the dashed line shows that for the irrational trader. As expected, the rational trader ends up with more wealth in good states of the economy (when the dividend is high) while the irrational trader, being pessimistic, ends up with more wealth in the bad states of the economy. As the horizon increases, the irrational trader ends up with non-trivial wealth in more extreme and less likely, low dividend states. When the irrational trader is mildly optimistic, the situation is different. His impact on the prices makes the bad states (i.e.,

the low dividend states) cheaper than the good states. This induces the rational trader to accumulate more wealth in the bad states by giving up wealth in the good states, including those with high probabilities. As a result, the irrational trader is more likely to end up with more wealth. When strongly optimistic, the irrational trader ends up accumulating wealth in very unlikely, good states by giving up wealth in most other states, which leads to his extinction in the long-run.

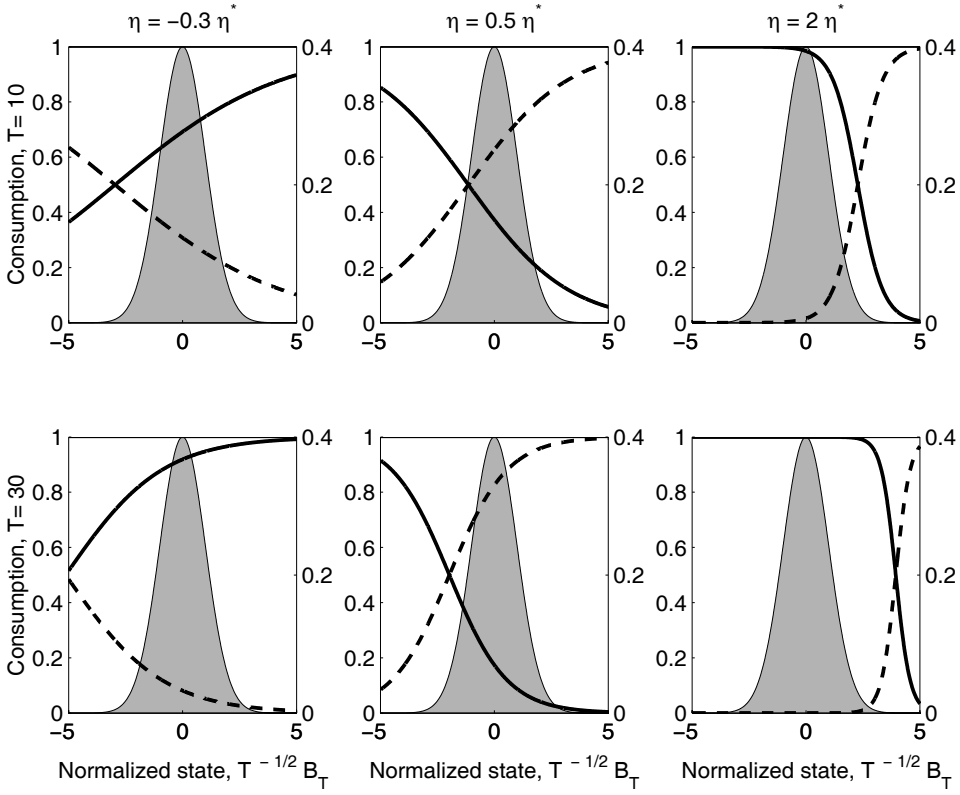

Figure 6: The terminal consumption of rational and irrational traders for different horizon T. We consider two values of T, 10 and 30, respectively. The model parameters are set at $\mu = 0.12$, $\sigma = 0.18$, and $\gamma = 5$. We consider three distinctive cases for the irrational trader's belief: (1) pessimistic, $\eta = -0.3\eta^{\star}$, (2) moderately optimistic, $\eta = 0.5\eta^{\star}$, and (3) strongly optimistic, $\eta = 2\eta^{\star}$. The horizontal axis in all panels is the normalized value of the terminal dividend, i.e., $\left[\ln D_T - \left(\mu - \frac{1}{2}\sigma^2\right) T\right] / (\sigma\sqrt{T})$, which has a standard normal distribution with zero mean and unit variance, shown by the shaded area (vertical axis on the right). The two panels on the left show the terminal consumption, as a fraction of the total consumption, of the rational trader (solid line) and the irrational trader (dashed line) with a pessimistic belief, i.e., $C_{r,T}/D_T$ and $C_{n,T}/D_T$, for the two values of the horizon, $T = 10, 30$, respectively. The two panels in the middle and on the right show the terminal consumption, as a fraction of the total consumption, of the rational trader and the irrational trader with a moderately and strongly optimistic beliefs for the two values of T, respectively.

It is important to recognize that our results on the long-run survival of irrational traders are obtained in absence of intermediate consumption. In other words, these results are primarily driven by the portfolio choices of different traders in the market and their impact

on prices. This allows us to focus on how irrational beliefs influence traders' trading behavior and how it alone affects their wealth evolution. When intermediate consumption is allowed, traders' consumption policies will also be affected by their beliefs, which can significantly affect their wealth accumulation as well. The net impact of irrational belief on a trader's wealth evolution depends on how it affects his portfolio choice *and* his consumption choice. Using an infinite horizon setting with intermediate consumption, Blume and Easley (2001) and Sandroni (2000) have shown that traders with (persistently) irrational beliefs will not survive while traders with rational beliefs will. Their analysis clearly shows that the influence of incorrect beliefs on the irrational traders' consumption policy can reduce their chance of survival. However, their result relies on several conditions imposed on the traders' preferences and the aggregate endowment. For example, they require that aggregate endowment is bounded above and below, away from zero. When these bounds are not imposed, as is the case in this paper, traders with rational beliefs may not always survive while traders with irrational beliefs may.[2] To provide a comprehensive analysis of the survival conditions with intermediate consumption is beyond the scope of this paper and is left for future research. But it suffices to say that even with intermediate consumption, the long-run survival of irrational traders is possible in absence of further restrictions on preferences and/or endowments.

Another difference between our setting and that of Blume and Easley (2001) is that we use a particular and simple form of beliefs of the irrational traders. In our model, such traders maintain a constant belief about the drift of the endowment process and do not update their belief based on realized data. To maintain analytical tractability, we do not allow for a more general form of beliefs, e.g., that resulting from inefficient learning. However, in the setting of Blume and Easley (2001), the specific form of the belief process is less important for the survival results than the aggregate endowment process and agents' preferences. Based on this observation, we would expect the intuition of our model to apply to more general settings as well, in particular to certain types of inefficient learning.

6 The Price Impact of Irrational Traders

We have already seen in the case of logarithmic preferences that the irrational trader's influence on prices does not decay as quickly as his relative wealth share. In this section, we extend our analysis to the general case for γ and characterize the precise combinations of

[2]In a simple case considered by Wang (1996), even among rational traders, survival depends on preferences. In our setting, we did not impose any upper or positive lower bounds on endowments.

model parameters under which such phenomenon is possible.

Our interest is in the behavior of prices in the long run when the horizon of the economy, T, is long. In order to obtain an explicit characterization, we look at the limit when T approaches infinity and derive from the limit an analytical approximation for a large, but finite T. By the definition of the limit, this approximation becomes arbitrarily accurate when T is sufficiently large. Specifically, we call two stochastic processes asymptotically equivalent if for large values of T, their ratio converges to unity with probability one.

Definition 2 *Two stochastic processes, X_t and Y_t, are asymptotically equivalent if*

$$\lim_{T\to\infty} \frac{X_T}{Y_T} = 1 \quad a.s.$$

which we denote $X_T \sim Y_T$.

When studying an economy with a long horizon, T, we need to have a sense about what it means for a particular property of the economy to persist for a significant period of time. Suppose, for example, we claim that the irrational trader's influence on a variable is significant as long as the variable exceeds a fixed level e within a time interval. Such an influence is persistent only if for a larger T, the corresponding time interval of the irrational trader's influence also increases in proportion. Otherwise, the fraction of time the irrational trader does have an influence becomes smaller for a larger T and thus his influence is only transitory and negligible.

To make this more formal, we consider the current time of observation to be $t = \lambda T$, $0 < \lambda \leq 1$. As T grows, the "current" time t increases as well, but it remains at a constant fraction of the horizon of the economy. Moreover, the time remaining until the final date of the economy is also increasing proportionally to T. Since the properties of the equilibrium prices and quantities depend on how much time is remaining until the final date, they depend on λ.

We define three values of λ to help us characterize points of change in the limiting behavior:

$$\lambda_S \equiv \frac{2}{2\gamma - \eta}, \quad \lambda_r \equiv \frac{\eta}{(\gamma-1)(2\gamma-\eta)}, \quad \lambda_n \equiv \frac{\eta}{\eta(\gamma+1) - 2\gamma(\gamma-1)}. \tag{18}$$

It is easy to verify that for $\eta < \eta^\star$, $0 < \lambda_S \leq 1$; for $0 < \eta \leq \eta^\star$, $0 < \lambda_r \leq 1$; and for $\eta < 0$ or $\eta > \eta^\star$, $0 < \lambda_n \leq 1$. The limiting behavior of the stock price process can be characterized as follows.

Proposition 5 *At $t = \lambda T$, the stock price behaves as follows:*

Case 1. Pessimistic Irrational Trader ($\eta < 0$):

$$S_t \sim \begin{cases} S_t^* e^{\eta\left[\sigma^2 T + \frac{1}{2}(\eta - 2\gamma)\sigma^2 t - \sigma B_t\right]}, & 0 < \lambda < \lambda_S \\ S_t^*, & \lambda_S < \lambda \leq 1 \end{cases}$$

Case 2. Moderately Optimistic Irrational Trader ($0 < \eta < \eta^\star$):

$$S_t \sim \begin{cases} S_t^* e^{\eta\left[(\gamma - 1 - \frac{1}{2}\eta)\sigma^2 t + \sigma B_t\right]}, & 0 < \lambda < \lambda_S \\ S_t^{**}, & \lambda_S < \lambda \leq 1 \end{cases}$$

Case 3. Strongly Optimistic Irrational Trader ($\eta^\star < \eta$):

$$S_t \sim S_t^*.$$

The values of the stock price in homogeneous economies, S_t^* and S_t^{**}, are given in Equation (12). The asymptotic values of the instantaneous moments of stock returns are equal to the moments of the corresponding asymptotic expressions for stock prices above.

Observe that in the first two cases, when the irrational trader is pessimistic or moderately optimistic, the stock price process does not converge quickly to its value in the economy populated exclusively by the rational trader who survives in the long-run. Instead, over long periods of time, i.e., for t between 0 and $\lambda_S T$, the stock price process is affected by the presence of both traders. This can occur even when the wealth of the irrational trader becomes negligible way before $\lambda_S T$.[3] We thus have generalized the results obtained in the context of a log-utility economy. A trader can control an asymptotically infinitesimal fraction of the total wealth and yet exert a non-negligible effect on the stock price. In other words, convergence in wealth does not readily imply convergence in prices.

7 Portfolio Policies

Proposition 5 in the previous section established the possibility that a trader whose wealth diminishes over time can have a persistent impact on asset prices. In this section, we study

[3] For brevity, we have omitted the discussion of wealth distribution over time. Interested readers can refer to our working paper, Kogan, Ross, Wang and Westerfield (2003), where we show that for cases 1 and 3, the irrational trader's wealth is asymptotically negligible for any time λT with $\lambda < \lambda_S$.

the traders' portfolio policies. In particular, we show that convergence in the price process does not lead to immediate convergence in policies, which is another and somewhat subtle channel through which traders with asymptotically infinitesimal wealth may affect the long-run behavior of the economy. Moreover, by characterizing the portfolio policy one gains an alternative view on long-run survival in equilibrium, which is complementary to the analysis of state-contingent consumption choices in sections 4 and 5.

Expressions for portfolio policies are not available in closed form. However, using the similar argument as in the proof of the bound on stock price volatility in Proposition 1, we can establish the following result:

Proposition 6 *For both traders, their portfolio weight in the stock, denoted by* w*, is bounded:*

$$|w| \leq 1 + |\eta|(\gamma + 1)/\gamma. \tag{19}$$

The bound on the traders' portfolio holdings is important for our results. It explicitly shows that price impact of the irrational trader with negligible wealth does not rely on excessive leverage. It also implies that our long-run survival results do not rely on the traders' use of high leverage. Our solution for the equilibrium remains valid even if traders are constrained in their portfolio choices, as long as the constraint is sufficiently loose to allow for $w = \pm[1 + |\eta|(\gamma + 1)/\gamma]$.

To analyze the traders' portfolio policies in more detail, we decompose a trader's stock demand into two components, the myopic component and the hedging component. The sum of the two gives the trader's total stock demand. We have the following proposition.

Proposition 7 *At* $t = \lambda T$*, the individual stock holdings behave as follows:*

Case 1. Pessimistic Irrational Trader ($\eta < 0$)*:*[4]

$$w_{r,t} \sim \begin{cases} \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ \frac{\gamma-\eta}{\gamma(1-\eta)} & - & \frac{(\gamma-1)\eta}{\gamma(1-\eta)} & = & 1, & 0 < \lambda < \lambda_S \\ 1 & + & 0 & = & 1, & \lambda_S < \lambda \leq 1 \end{cases}$$

$$w_{n,t} \sim \begin{cases} \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ \frac{1}{1-\eta} & + & 0 & = & \frac{1}{1-\eta}, & 0 < \lambda < \min(\lambda_n, \lambda_S) \\ 1 + \frac{\eta}{\gamma} & + & 0 & = & 1 + \frac{\eta}{\gamma}, & \max(\lambda_n, \lambda_S) < \lambda \leq 1 \end{cases}$$

Case 2. Moderately Optimistic Irrational Trader ($0 < \eta < \eta^\star$)*:*

$$w_{r,t} \sim \begin{cases} \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ \frac{1}{1+\eta} & + & 0 & = & \frac{1}{1+\eta}, & 0 < \lambda < \lambda_r \\ \frac{1}{1+\eta} & + & \frac{\eta(\gamma-1)}{\gamma(1+\eta)} & = & 1 - \frac{\eta}{\gamma(1+\eta)}, & \lambda_r < \lambda < \lambda_S \\ 1 - \frac{\eta}{\gamma} & + & 0 & = & 1 - \frac{\eta}{\gamma}, & \lambda_S < \lambda \leq 1 \end{cases}$$

$$w_{n,t} \sim \begin{cases} \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ \frac{\gamma+\eta}{\gamma(1+\eta)} & + & \frac{\eta(\gamma-1)}{\gamma(1+\eta)} & = & 1, & 0 < \lambda < \lambda_S \\ 1 & + & 0 & = & 1, & \lambda_S < \lambda \leq 1 \end{cases}$$

Case 3. Strongly Optimistic Irrational Trader, ($\eta^\star < \eta$)*:*

$$w_{r,t} \sim 1 + 0 = 1, \quad 0 < \lambda \leq 1$$

$$w_{n,t} \sim \begin{cases} \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ 1 + \frac{\eta}{\gamma} & + & \frac{\eta(\gamma-1)}{\gamma} & = & 1 + \eta, & 0 < \lambda < \lambda_n \\ 1 + \frac{\eta}{\gamma} & + & 0 & = & 1 + \frac{\eta}{\gamma}, & \lambda_n < \lambda \leq 1 \end{cases}$$

Since the moments of stock returns are asymptotically state-independent, it is intuitive to expect that the implied portfolio policies are myopic. Proposition 7 shows, however, that this is not true. In other words, the asymptotic portfolio policy can differ significantly from what

[4]The limit of the portfolio policy for values of $\lambda \in [\min(\lambda_n, \lambda_S), \max(\lambda_n, \lambda_S)]$ can be characterized explicitly as well, but the results depend on the ordering between λ_n and λ_S, which in turn is determined by the values of model parameters. We omit these results to simplify the exposition.

the asymptotic moments of stock returns suggest. Such a surprising behavior can only be due to the hedging component of the traders' portfolio holdings since, by definition, the myopic component of portfolio holdings depends only on the conditional mean and variance of stock returns. Given that the instantaneous moments of stock returns are asymptotically state-independent, it may seem surprising that the hedging component of portfolio holdings remain finite, as Case 3 in Proposition 7 illustrates for the irrational trader. The reason behind this result is that instantaneous moments of stock returns do not fully characterize the investment opportunities traders face. In particular, moments of stock returns do not always stay constant. As we have seen in Figure 2, for example, return volatility can change significantly as the relative wealth distribution changes. After a long time, the likelihood of the reversal of wealth distribution between the rational and irrational traders and a shift in return moments is relatively low. Nonetheless, the possibility of such a change remains important, which gives rise to the significant hedging demand in the traders' portfolio holdings.

Figure 7 illustrates the behavior of the economy when the irrational trader is strongly optimistic ($\eta > \eta^\star$). In this case (Case 3 in Propositions 4, 5 and 7), the irrational trader does not survive and has no price impact in the long-run. For the chosen set of parameter values, $\lambda_n = 0.29$. The time of observation t is set to be $0.15\,T$. Thus $t < \lambda_n T$. As the bottom panel of Figure 7 shows, with almost probability one, the rational trader controls most of the wealth in the economy by this point in time. From Proposition 5, at this point the stock price converge closely to the price in the economy populated by only the rational trader. If we consider the Sharpe ratio of the stock, defined by μ_S/σ_S, which characterizes the instantaneous investment opportunity traders face, it also converges to its value of $\gamma\sigma$ in the limiting economy with the rational trader only. The top panel of Figure 7 plots the value of the Sharpe ratio for different states of the economy at time t. It is obvious that with almost probability one, the value of the Sharpe ratio equals its limit $\gamma\sigma$ (the probability distribution of the state of economy is shown by the shaded area). However, for very large values of D_t (or B_t), the economy will be dominated by the irrational trader (as we see from the bottom panel) and the instantaneous Sharpe ratio of the stock converges to its value in an economy populated by the irrational trader only, which is $(\gamma - \eta)\sigma$. Such a possibility, even though with very low probability under the true probability measure, can be important to the irrational trade because under his belief, its likelihood can be non-trivial. As a result, it can have a significant impact on the irrational trader's portfolio choice.

The importance of these low probability but large changes in the Sharpe ratio is reflected

in the traders' value function, which is given by

$$V(t, W_t, D_t) \equiv \mathrm{E_t}\left[\xi_T \frac{W_T^{1-\gamma}}{1-\gamma}\right] = \frac{1}{1-\gamma} e^{h(t,D_t)} W_t^{1-\gamma} \equiv \mathrm{E_t}\left[\xi_T \frac{1}{1-\gamma} C_{n,T}^{1-\gamma}\right]. \tag{20}$$

State dependence of the indirect utility function, i.e., the effect of possible changes in the Sharpe ratio, is captured by the function $h(t, D_t)$. The second panel of Figure 7 shows that for the irrational trader h is non-constant over a wide range of values of D_t. It exhibits significant state-dependence even when the contemporaneous Sharpe ratio is approximately constant. It is this state-dependence in the indirect utility function that induces hedging demand. The third panel of Figure 7 shows hedging demand of the irrational trader. Over a wide range of values of D_t, his hedging demand is non-zero. In particular, it is close to its asymptotic value $\eta(\gamma - 1)/\gamma$ (see Proposition 7), which equals 12.8 for the chosen values of parameters.

What we conclude from this is that convergence of the stock price to a limiting process does not necessarily imply convergence of the traders' portfolio policies to their policies under the limiting price process. Price paths of small probability under the true probability measure can have a significant impact on the traders' portfolio policies. Thus, an intuitive conjecture that convergence in price gives convergence in portfolio policies does not hold in general. This result has important implications for the analysis of long-run survival as we see in the next section.

8 Heuristic Partial Equilibrium Analysis of Survival

Although general equilibrium analysis is always desirable, its tractability is often limited. Several authors such as DSSW have relied on heuristic partial equilibrium analysis to study the survival of irrational traders. In this section, we want to examine the limitations of partial equilibrium heuristics in our setting.

The essence of the partial equilibrium argument is to examine a limiting situation when one of the two traders controls most of the aggregate wealth. Following DSSW, the argument then assumes that the infinitesimal trader has no impact on market prices and all traders follow portfolio policies close to those under the limiting prices. If the wealth of the infinitesimally small trader has a higher growth rate under the assumed portfolio policies, his share of wealth will grow over time and he will be able to successfully "invade" the economy. Hence, such traders can survive in the long-run, "in the sense that their wealth share does

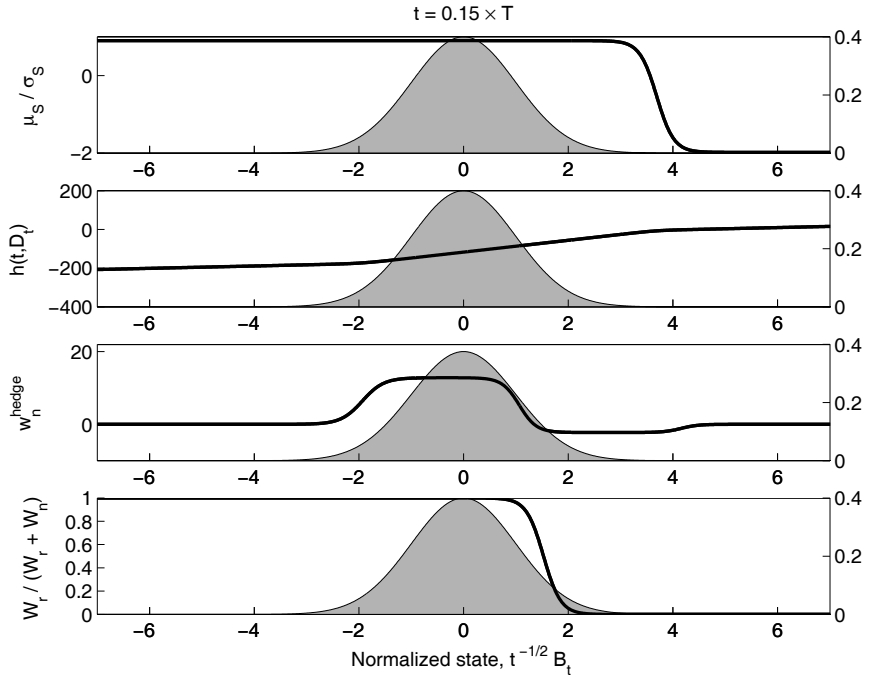

Figure 7: The behavior of the economy for the following parameter values: $\mu = 0.12$, $\sigma = 0.18$, $\gamma = 5$, $T = 30$. Also, $\eta = 2\eta^\star = 16$, i.e., the irrational trader is strongly optimistic. The time of observation is set at $t = 0.15 \times T$. The horizontal axis in all panels is the normalized state variable, $g_{0,T} = B_T/\sqrt{T}$, which has a standard normal distribution with zero mean and unit variance, shown by the shaded area (vertical axis on the right). The four panels from top to bottom show (i) the instantaneous Sharpe ratio of stock returns, μ_S/σ_S; (ii) the state dependence of the indirect value function of the rational trader, as captured by the function $h(t, D_t)$ in (20); (iii) the portion of the portfolio strategy of the irrational trader attributable to hedging demand, defined as $w_n^{\text{hedge}} = w_n - \mu_S + \eta\sigma_S^2/(\gamma\sigma_S^2)$; (iv) the fraction of the aggregate wealth controlled by the rational agent, $W_r/(W_r + W_n)$.

not drop toward zero in the long run with probability one".

In our setting, we can easily derive the survival conditions using this partial equilibrium argument. In the limit when the economy is populated only by either the rational trader or the irrational trader, the stock price follows the geometric Brownian motion:

$$dS_t = S_t \left(\mu_S dt + \sigma_S dB_t\right). \tag{21}$$

If only the rational trader is present, $S_t = S_t^*$ and we have from (12) $\mu_S = \gamma\sigma^2$ and $\sigma_S = \sigma$. He invests only in the stock and rate of his wealth growth is given by $\mu_S - \frac{1}{2}\sigma_S^2 = \frac{1}{2}(2\gamma - 1)\sigma^2$.

Suppose now an irrational trader is injected into the economy. Under his belief (given by the measure Q), the drift of the stock price process is $\widehat{\mu}_S = \mu_S + \sigma^2\eta$ and the volatility remains at σ. He will chose to invest a fraction $w_n = \widehat{\mu}_S / (\gamma\sigma^2) = 1 + \eta/\gamma$ of his wealth in the stock. Thus, the growth rate of the irrational trader's wealth is $\mu_S - \frac{1}{2}\sigma^2 + \frac{1}{2}\frac{\sigma^2}{\gamma^2}\eta\left(\gamma\eta^\star - \eta\right)$ where $\eta^\star = 2(\gamma - 1)$. The growth rate of wealth of the "invading" irrational trader is higher than that of the dominant rational trader if and only if $0 < \eta < \gamma\eta^\star$.

Next, assume that only the irrational trader is dominant. Then, $S_t = S_t^{**}$. Repeating the steps of the previous analysis, the volatility of the limiting stock price remains at σ and the drift becomes $\mu_S = \gamma\sigma^2 - \eta\sigma^2$. The growth rate of the irrational trader's wealth is $\mu_S - \frac{1}{2}\sigma^2$ while for the rational trader it is $\mu_S - \frac{1}{2}\sigma^2 + \frac{1}{2}\frac{\sigma^2}{\gamma^2}(2\gamma - 1)\eta\left(\eta - \frac{\gamma}{2\gamma - 1}\eta^\star\right)$. The rational trader's portfolio grows faster than the irrational trader's portfolio if and only if $\eta < 0$ or $\eta > \frac{\gamma}{2\gamma - 1}\eta^\star$.

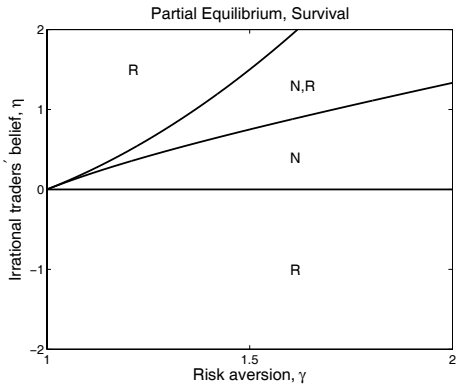

Figure 8: The survival of rational and irrational traders for different values of η and γ in partial equilibrium. For each region in the parameter space, we document which of the agents survives in the long run. "R" means that survival of the rational trader is guaranteed inside the region, "N" corresponds to the irrational trader, "N,R" means that both traders survive.

The partial equilibrium analysis thus appears to provide sufficient conditions for long-run

survival of both types of traders. In particular, for $\gamma > 1$

$$\begin{aligned} 0 < \eta < \tfrac{\gamma}{2\gamma-1}\eta^\star \quad &\Rightarrow \quad \text{Irrational trader survives} \\ \tfrac{\gamma}{2\gamma-1}\eta^\star < \eta < \gamma\eta^\star \quad &\Rightarrow \quad \text{Both traders survive} \\ \eta < 0 \text{ or } \eta > \gamma\eta^\star \quad &\Rightarrow \quad \text{Rational trader survives} \end{aligned} \tag{22}$$

For $\gamma = 1$, only the rational trader survives regardless of the value of η. Figure 8 summarizes these results. Since $\gamma/(2\gamma - 1) \leq 1$ for $\gamma \geq 1$, $\eta^\star$ belongs to the second region in (22).

The survival conditions given in Figure 8 clearly differ from the survival conditions from general equilibrium analysis shown in Figure 5. The difference occurs when $\frac{\gamma}{2\gamma-1}\eta^\star < \eta < \gamma\eta^\star$. In particular, partial equilibrium argument predicts survival of both traders for these parameter values while general equilibrium analysis shows the extinction of the irrational trader when $\eta > \eta^\star$.

The difference in results from the partial equilibrium argument comes from its two assumptions: (1) when the irrational trader becomes small in relative wealth, the stock price behaves as if he is absent, and (2) both traders adopt the portfolio policies that would be optimal under that limiting price process. We know from our analysis in Section 4 that the first assumption is false in general. But the more direct reason for the discrepancy in survival results is because the second assumption is false. For instance, $\eta^\star < \eta < \gamma\eta^\star$ corresponds to Case 3 of Proposition 5, in which the stock price is asymptotically the same as in the economy without the irrational trader. In other words, the irrational trader has no significant impact on the current stock price as his wealth becomes negligible. The moments of stock returns converge to the values implied by the partial equilibrium analysis. However, as we have shown in Section 7, the irrational trader's portfolio policy differs significantly from what the partial equilibrium analysis assumes. In particular, he does not simply hold the portfolio implied by the limiting price process. This explains the deviations in the conclusions about long-run survival from the heuristic partial equilibrium argument and demonstrates the limitations of partial equilibrium arguments and the importance of equilibrium effects on survival.

9 Conclusion

The analysis above has examined the long-run price impact and survival of irrational traders who use persistently wrong beliefs to make their portfolio choices. Using a parsimonious model with no intermediate consumption, we have shown that irrational traders can maintain

a persistent influence on prices even after they have lost most of their wealth. Our analysis of conditions for survival of either type of traders further highlights the importance of taking into account the effect that traders have on asset prices.

For tractability, we have confined our analysis to preferences with constant relative risk aversion. Extensions of our analysis to more general preferences are possible and may yield unexpected results. We have also assumed that the rational and irrational traders differ only in their beliefs but not in their preferences. This allows us to focus on the impact of irrational beliefs on survival and prices. Of course, differences in time and risk preferences can have their own implications for long-run survival. Perhaps more important is the extension of these results to models with intermediate consumption and to alternative preferences. While there is more to be done in this area, it is fair to say that a general message is emerging and is unlikely to be overturned. Namely, survival and price impact are related but distinct concepts and the arguments ignoring such a distinction are unreliable. In our model, irrational traders can survive and even dominate rational traders, but even when they do not survive, they can still have a persistent impact on asset prices.

References

[1] Alpert, M; Raiffa, H. 1959, A progress report on the training of probability assessors. Reprinted in D. Kahneman, P. Slovic, and A. Tversky (eds.) Judgement under Uncertainty: Heuristics and Biases. Cambridge: Cambridge University Press, 1982.

[2] Athey, Susan. 2002, Monotone Comparative Statics Under Uncertainty, *Quarterly Journal of Economics*, 117 (1), 187–223.

[3] Benartzi, Shlomo; Thaler, Richard H. 2001, Naive Diversification Strategies in Defined Contribution Saving Plans, *American Economic Review*, 91(1), 79–98.

[4] Billingsley, Patrick. 1995, Probability and Measure, *Wiley-Interscience*, Third Edition.

[5] Black, Fischer. 1986, Noise, *Journal of Finance*, 41 (3), 529–43.

[6] Black, Fischer; Scholes, Myron S. 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81 (3), 637–54.

[7] Blume, Lawrence; Easley, David. 2001, If You're So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets, Cowles Foundation Discussion Papers, 1319.

[8] Cox, John; Huang, Chi-fu. 1989, Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process, *Journal of Economic Theory*, 49, 33–83.

[9] De Long, J. Bradford; Shleifer, Andrei; Summers, Lawrence H.; Waldman, Robert J. 1990, Noise Trader Risk in Financial Markets, *The Journal of Political Economy*, 98 (4), 703–38.

[10] De Long, J. Bradford; Shleifer, Andrei; Summers, Lawrence H.; Waldman, Robert J. 1991, The Survival of Noise Traders in Financial Markets, *Journal of Business*, 64 (1), 1–19.

[11] Duffie, Darrell; Huang, Chi-fu. 1986, Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities, *Econometrica*, 53 (6), 1337–56.

[12] Figlewski, Stephen. 1978, Market Efficiency in a Market with Heterogeneous Information, *Journal of Political Economy*

[13] Friedman, Milton. 1953, The Case for Flexible Exchange Rates, *Essays in Positive Economics*, University of Chicago Press.

[14] Hakansson, Nils H. 1971, Capital Growth and the Mean-Variance Approach to Portfolio Selection, *Journal of Financial and Quantitative Analysis*, 6 (1), 517–557.

[15] Harrison, J. Michael; Kreps, David M. 1979, Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory*, 2 (3), 381–408.

[16] Kahneman, Daniel; Tversky, Amos. 1979, Prospect Theory: An Analysis of Decision under Risk, *Econometrica*, 47(2), 263–91.

[17] Karatzas, Ioannis; Shreve, Steven E. 1991, Brownian Motion and Stochastic Calculus, *Springer-Verlag New York, Inc.*, Second Edition.

[18] Kogan, Leonid; Ross, Steven; Wang, Jiang; Westerfield, Mark 2003, The Survival and Price Impact of Irrational Traders, NBER working paper 9434.

[19] Lucas, Robert E. 1978, Asset Prices in an Exchange Economy, *Econometrica*, 46 (6), 1429–45.

[20] Odean, Terrance. 1998, Are Investors Reluctant to Realize Their Losses?, *Journal of Finance*, 53(5), 1775–98.

[21] Radner, Roy. 1972, Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets, *Econometrica*, 40 (2), 289–303.

[22] Sandroni, Alvaro. 2000, Do Markets Favor Agents Able to Make Accurate Predictions?, *Econometrica*, 68 (6), 1303–41.

[23] Wang, Jiang. 1996, The Term Structure of Interest Rates in a Pure Exchange Economy with Heterogeneous Investors, *Journal of Financial Economics*, 41 (1), 75–110.

A Proofs

Proof of Proposition 1

The optimality conditions of the maximization problem in (8a) require that

$$C_{r,T} = C_{n,T} \left(b\,\xi_T\right)^{1/\gamma}.$$

Combined with the market clearing condition (8b), this implies (9a) and (9b).

The state price density must be proportional to the traders' marginal utilities. Since we set the interest rate equal to zero, the state price density conditional on the information available at time t is given by

$$\frac{\left(1+\left(b\,\xi_T\right)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma}}{\mathrm{E_t}\left[\left(1+\left(b\,\xi_T\right)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma}\right]}.$$

The price of any payoff Z_T is therefore given by (11).

The individual budget constraint in a dynamically complete market is equivalent to the static constraint that the initial wealth of a trader is equal to the present value of the trader's consumption (e.g., Cox and Huang (1989). Since the two traders in our model have identical endowments at time $t = 0$, their budget constraints imply

$$W_{r,0} = \frac{\mathrm{E}_0\left[D_T^{1-\gamma}\left(1+\left(b\,\xi_T\right)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]}{\mathrm{E}_0\left[D_T^{-\gamma}\left(1+\left(b\,\xi_T\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right]} = \frac{\mathrm{E}_0\left[D_T^{1-\gamma}\left(b\,\xi_T\right)^{\frac{1}{\gamma}}\left(1+\left(b\,\xi_T\right)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]}{\mathrm{E}_0\left[D_T^{-\gamma}\left(1+\left(b\,\xi_T\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right]} = W_{n,0}. \quad \text{(A.1)}$$

We now verify that $b = e^{\eta\sigma^2(\gamma-1)T}$ satisfies (A.1). Note that

$$D_T^{1-\gamma} = e^{[(1-\gamma)(\mu-\frac{\sigma^2}{2})+\frac{1}{2}(1-\gamma)^2\sigma^2]T} e^{-\frac{1}{2}(1-\gamma)^2\sigma^2 T+(1-\gamma)\sigma B_T}$$

Define a new measure Q, such that $\left(\frac{dQ}{dP}\right)_T = e^{-\frac{1}{2}(1-\gamma)^2\sigma^2 T+(1-\gamma)\sigma B_T}$, where P is the original probability measure. Using the translation invariance property of the Gaussian distribution, the random variable $B_T^Q = B_T - (1-\gamma)\sigma T$ is a standard normal random variable under Q. Thus, the equality

$$\mathrm{E}_0\left[D_T^{1-\gamma}\left(b\,\xi_T\right)^{\frac{1}{\gamma}}\left(1+\left(b\,\xi_T\right)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right] = \mathrm{E}_0\left[D_T^{1-\gamma}\left(1+\left(b\,\xi_T\right)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]$$

is equivalent to

$$\mathrm{E}_0^Q\left[(\xi_T^Q)^{\frac{1}{\gamma}}\left(1+(\xi_T^Q)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]=\mathrm{E}_0^Q\left[\left(1+(\xi_T^Q)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]$$

where $\xi_T^Q=\exp(-\frac{1}{2}\sigma^2\eta^2T+\sigma\eta B_T^Q)$. Since the variable B_T^Q is equivalent in distribution to B_T, we can restate the last equality equivalently as

$$\mathrm{E}_0\left[\xi_T^{\frac{1}{\gamma}}\left(1+\xi_T^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]=\mathrm{E}_0\left[\left(1+\xi_T^{\frac{1}{\gamma}}\right)^{\gamma-1}\right].$$

To verify that the above equality holds, consider a function $F(z)$ defined as

$$F(z)=\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}zT}+e^{-\frac{1}{2\gamma}zT}\xi_T^{\frac{1}{\gamma}}\right)^{\gamma}\right].$$

Changing the order of differentiation and expectation operators, (see Billingsley 1995, Th. 16.8),

$$F'(z)\big|_{z=0}=\mathrm{E}\left[\frac{1}{2}\left(1-\xi_T^{\frac{1}{\gamma}}\right)\left(1+\xi_T^{\frac{1}{\gamma}}\right)^{\gamma-1}\right].$$

Thus it suffices to prove that $F'(z)\big|_{z=0}=0$. Since

$$\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}zT}+e^{-\frac{1}{2\gamma}zT}\xi_T^{\frac{1}{\gamma}}\right)^{\gamma}\right]=\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}\left(zT-\frac{1}{2}\eta^2\sigma^2T+\eta\sigma B_T\right)}+e^{-\frac{1}{2\gamma}\left(zT-\frac{1}{2}\eta^2\sigma^2T+\eta\sigma B_T\right)}\right)^{\gamma}\xi_T^{\frac{1}{2}}\right]$$

if we define a new measure Q so that $\left(\frac{dQ}{dP}\right)_T=e^{-\frac{1}{8}\eta^2\sigma^2T+\frac{1}{2}\eta\sigma B_T}$ and use a change of measure similar to its earlier application in this proof, we find that

$$\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}zT}+e^{-\frac{1}{2\gamma}zT}\xi_T^{\frac{1}{\gamma}}\right)^{\gamma}\right]=\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}(zT+\eta\sigma B_T)}+e^{-\frac{1}{2\gamma}(zT+\eta\sigma B_T)}\right)^{\gamma}\right]e^{-\frac{1}{8}\eta^2\sigma^2T}.$$

The symmetry of the distribution of the normal random variable B_T implies that $F(z)=F(-z)$, therefore $F'(z)\big|_{z=0}=0$. This verifies that $b=e^{\eta\sigma^2(\gamma-1)T}$.

We now prove that the conditional volatility of stock returns is bounded between σ and $\sigma(1+|\eta|)$. Define

$$A=e^{\left(\frac{-\eta\sigma^2}{\gamma}\right)(T-t)},\qquad g=e^{-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}T+\sigma^2\eta(\gamma-1)\frac{1}{\gamma}t+\frac{\eta\sigma}{\gamma}B_T}.$$

The stock price can be expressed as

$$S_t = \frac{\mathrm{E_t}\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{1/\gamma}\right)^{\gamma}\right]}{\mathrm{E_t}\left[D_T^{-\gamma}\left(1+(b\xi_T)^{1/\gamma}\right)^{\gamma}\right]} = e^{\left(\mu-\sigma^2\gamma\right)T+\left(-\frac{1}{2}\sigma^2(1-2\gamma)\right)t}e^{\sigma B_t}\frac{\mathrm{E_t}\left[(1+g)^{\gamma}\right]}{\mathrm{E_t}\left[(1+gA)^{\gamma}\right]}.$$

By Ito's lemma, its volatility σ_{St} is given by

$$\sigma_{St} = \frac{\partial \ln S_t}{\partial B_t} = \sigma + \eta\sigma\left(\frac{\mathrm{E_t}\left[(1+gA)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+gA)^{\gamma}\right]} - \frac{\mathrm{E_t}\left[(1+g)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+g)^{\gamma}\right]}\right). \tag{A.2}$$

To establish the bounds on stock return volatility, we prove that

$$\frac{\mathrm{E_t}\left[(1+gA)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+gA)^{\gamma}\right]} - \frac{\mathrm{E_t}\left[(1+g)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+g)^{\gamma}\right]} \geq 0 \tag{A.3}$$

for $A \leq 1$ with the opposite inequality for $A \geq 1$. Note that for any twice-differentiable function $F(A,\gamma)$,

$$\frac{\partial}{\partial\gamma}\frac{\partial}{\partial A}\ln\left(F(A,\gamma)\right) \geq 0 \Rightarrow \frac{\partial}{\partial A}\ln\left(F(A,\gamma-1)\right) - \frac{\partial}{\partial A}\ln\left[F(A,\gamma)\right] \leq 0 \Rightarrow \frac{\partial}{\partial A}\frac{F(A,\gamma-1)}{F(A,\gamma)} \leq 0.$$

Thus, to prove (A.3), it suffices to show that $\partial^2 \ln\left(\mathrm{E_t}\left[(1+gA)^{\gamma}\right]\right)/\partial A\partial\gamma \geq 0$. The function $(1+gA)^{\gamma}$ is log-supermodular in A, g, and γ, since it is positive and it's cross-partial derivatives in all arguments are positive. Thus, according to the additivity property of log-supermodular functions (e.g., Athey (2002)), $\mathrm{E_t}\left[(1+gA)^{\gamma}\right]$ is log-supermodular in A and γ, i.e., $\partial^2 \ln\left(\mathrm{E_t}\left[(1+gA)^{\gamma}\right]\right)/\partial A\partial\gamma \geq 0$.

Because $A > 1$ if and only if $\eta < 0$, we have shown that

$$\eta\left(\frac{\mathrm{E_t}\left[(1+gA)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+gA)^{\gamma}\right]} - \frac{\mathrm{E_t}\left[(1+g)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+g)^{\gamma}\right]}\right) \geq 0$$

and hence $\sigma_{St} \geq \sigma$.

Because $\left(\frac{\mathrm{E_t}\left[(1+gA)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+gA)^{\gamma}\right]} - \frac{\mathrm{E_t}\left[(1+g)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+g)^{\gamma}\right]}\right)$ is bounded between -1 and 0 for $\eta < 0$, and between 0 and 1 for $\eta > 0$, we obtain the upper bound from (A.2): $\sigma_{St} \leq \sigma(1+|\eta|)$. ■

Proof of Proposition 3

We will make use of the following result:

Lemma A.1 *Let $N(x)$ denote the cumulative density function of the standard normal dis-*

tribution: $N(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{z^2}{2}}\, dz$. *For* $x > 0$, $N(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$.

Proof. $N(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{z^2}{2}}\, dz \leq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{x^2}{2} - \frac{(z-x)^2}{2}}\, dz = \frac{1}{2} e^{-\frac{x^2}{2}}$. Note that for convenience, we have defined the cumulative density function as the probability above a given value rather than below. ■

Let $t = T/(1 + |\eta|)$ and define $M = \frac{\sqrt{2}-1}{2} \sigma |\eta| t$. According to Lemma A.1,

$$\text{Prob}\left[|B_t| \geq M\sqrt{t}\right] = 2\, N(M\sqrt{t}) \leq e^{-\frac{M^2}{2} t} = e^{-\frac{(\sqrt{2}-1)^2}{2} \sigma^2 \eta^2 t} \leq e^{-\frac{1}{12}\sigma^2\eta^2 t}.$$

On set $\{|B_t| \leq M\sqrt{t}\}$,

$$\alpha_{n,t} = \frac{\xi_t}{1 + \xi_t} \leq \xi_t \leq e^{-\frac{\sigma^2\eta^2}{2} t + \sigma|\eta| M t} = e^{-\frac{3-2\sqrt{2}}{2} \sigma^2 \eta^2 t} \leq e^{-\frac{1}{12}\sigma^2\eta^2 t} \leq \varepsilon.$$

Therefore,

$$\text{Prob}\left[\alpha_{n,t} \geq \varepsilon\right] \leq \text{Prob}\left[|B_t| \geq M\sqrt{t}\right] \leq e^{-\frac{1}{12}\sigma^2\eta^2 t} \leq \varepsilon$$

which establishes the first result of the proposition. The second result follows from the fact that, on the set $\{|B_t| \leq M\sqrt{t}\}$,

$$\frac{S_t}{S_t^*} \leq e^{-\sigma^2|\eta|(T-t)} \frac{1}{\alpha_{n,t}} \leq e^{-\sigma^2|\eta|(T-t) + \frac{\sigma^2\eta^2}{2} t + \sigma|\eta| M\sqrt{t}} \leq e^{-\frac{\sigma^2\eta^2}{2} t + \sigma|\eta| M t} e^{-\sigma^2|\eta| T + \sigma^2|\eta|(1+|\eta|)t}.$$

Given that on the set $\{|B_t| \leq M\sqrt{t}\}$,

$$e^{-\frac{\sigma^2\eta^2}{2} t + \sigma|\eta| M t} \leq e^{-\frac{1}{12}\sigma^2\eta^2 t}$$

and since $t = T/(1 + |\eta|)$, $e^{-\sigma^2|\eta|T + \sigma^2|\eta|(1+|\eta|)t} \leq 1$ and we conclude that on the set $\{|B_t| \leq M\sqrt{t}\}$, $\frac{S_t}{S_t^*} \leq e^{-\frac{1}{12}\sigma^2\eta^2 t}$ and hence

$$\text{Prob}\left[1 - \frac{S_t}{S_t^*} \leq 1 - \varepsilon\right] \leq e^{-\frac{1}{12}\sigma^2\eta^2 t} \leq \varepsilon$$

which concludes the proof of the proposition. ■

Proof of Proposition 4

According to (9a) and (9b),

$$\frac{C_{n,T}}{C_{r,T}} = (b\,\xi_T)^{1/\gamma} = \exp\left[\frac{1}{\gamma}\left(-\frac{1}{2}\sigma^2\eta^2 + \eta\sigma^2(\gamma-1)\right)T + \frac{1}{\gamma}\eta\sigma B_T\right].$$

Using the strong Law of Large Numbers for Brownian motion (see Karatzas and Shreve (1991, Sec. 2.9.A)), for any value of σ,

$$\lim_{T\to\infty} e^{a\,T+\sigma\,B_T} = \begin{cases} 0, & a<0 \\ \infty, & a>0 \end{cases}$$

where convergence takes place almost surely. The proposition then follows. ■

Proof of Proposition 5

Our analysis will make use of the following technical result.

Lemma A.2 *Consider a stochastic process $X_t = e^{c\,t+v\,B_t}$ and a constant $a \geq 0$. Assume that $ac + \frac{1}{2}v^2a^2(1-\lambda) \neq 0$, $0 \leq \lambda < 1$. Then the limit $\lim_{T\to\infty} \mathrm{E}_t[X_T^a]$ is equal to either zero or infinity almost surely, where we set $t = \lambda T$. The following convergence results hold:*

(i) (Point-wise convergence)

$$\lim_{T\to\infty} \frac{\mathrm{E}_t\left[(1+X_T)^a\right]}{1+\mathrm{E}_t\left[X_T^a\right]} = 1. \tag{A.4}$$

(ii) (Convergence of moments)

$$\lim_{T\to\infty} \frac{\mathrm{mean_t}\ \mathrm{E}_t\left[(1+X_T)^a\right]}{\mathrm{mean_t}\ \left(1+\mathrm{E}_t\left[X_T^a\right]\right)} = 1, \qquad \lim_{T\to\infty} \frac{\mathrm{vol_t}\ \mathrm{E}_t\left[(1+X_T)^a\right]}{\mathrm{vol_t}\ \left(1+\mathrm{E}_t\left[X_T^a\right]\right)} = 1 \tag{A.5}$$

where $\mathrm{mean_t f_t}$ *and* $\mathrm{vol_t f_t}$ *denote the instantaneous mean and standard deviation of the process* $\ln f_t$ *respectively.*

Proof of Lemma A.2

(i) Consider the conditional expectation

$$\mathrm{E_t}\left[X_T^a\right] = \exp\left[ac\,T + \frac{1}{2}v^2a^2(1-\lambda)\,T + av B_t\right]. \tag{A.6}$$

The limit of $\mathrm{E}_t[X_T^a]$ is equal to zero if $ac+\frac{1}{2}v^2a^2(1-\lambda) < 0$ and equal to infinity if the opposite inequality holds (according to the strong Law of Large Numbers for Brownian motion, see Karatzas and Shreve, 1991, Sec. 2.9.A).

Because the function $ac\,T + \frac{1}{2}v^2a^2(1-\lambda)\,T$ is convex in a and equal to zero when $a = 0$, we find that for $a \geq 1$

$$\mathrm{E}_t\left[X_T^a\right] \to \infty \;\Rightarrow\; \frac{\mathrm{E}_t\left[X_T^z\right]}{\mathrm{E}_t\left[X_T^a\right]} \to 0, \quad \forall\; z \in (0,a) \tag{A.7a}$$

$$\mathrm{E}_t\left[X_T^a\right] \to 0 \;\Rightarrow\; \mathrm{E}_t\left[X_T^z\right] \to 0, \quad \forall\; z \in (0,a). \tag{A.7b}$$

We prove the result of the lemma separately for six regions covering the entire parameter space.

Case 1: $0 \leq a \leq 1, \mathrm{E}_t\left[X_T^a\right] \to \infty$.

If $X_T \leq 1$, $(X_T+1)^a \leq 2^a$, while if $X_T \geq 1 \Rightarrow (X_T+1)^a - X_T^a \leq aX_T^{a-1} \leq a$ since $(X_T+1)^a$ is concave and $a-1 \leq 0$. Therefore, $X_T^a \leq (1+X_T)^a \leq X_T^a + 2^a + a$, and hence $\lim_{T\to\infty} \mathrm{E}_t\left[(1+X_T)^a\right]/\mathrm{E}_t[X_T^a] = 1$, which implies $\lim_{T\to\infty} \mathrm{E}_t\left[(1+X_T)^a\right]/(1+\mathrm{E}_t[X_T^a]) = 1$.

Case 2: $1 \leq a \leq 2, \mathrm{E}_t\left[X_T^a\right] \to \infty$.

By the mean value theorem, $(1+X_T)^a = X_T^a + a(w+X_T)^{a-1}$ for some $w \in [0,1]$. Using the analysis of case 1, $(w+X_T)^{a-1} \leq (1+X_T)^{a-1} \leq X_T^{a-1} + 2^{a-1} + a - 1$, which, combined with (A.7a), implies that $\lim_{T\to\infty} \mathrm{E}_\mathrm{t}\left[(1+X_T)^a\right]/\mathrm{E}_\mathrm{t}\left[X_T^a\right] = 1$ and the main result follows.

Case 3: $2 \leq a, \mathrm{E}_\mathrm{t}\left[X_T^a\right] \to \infty$.

By the mean value theorem, $(1+X_T)^a = X_T^a + a(w+X_T)^{a-1}$ for some $w \in [0,1]$. By Jensen's inequality, $[(1+X_T)/2]^{a-1} \leq (1+X_T^{a-1})/2$. Thus,

$$0 \leq (w+X_T)^{a-1} \leq (1+X_T)^{a-1} \leq 2^{a-2} + 2^{a-2}X_T^{a-1}$$

which, combined with (A.7a) implies that $\lim_{T\to\infty} \mathrm{E}_t[(1+X_T)^a]/\mathrm{E}_t\left[X_T^a\right] = 1$ and the main result follows.

Case 4: $0 \leq a \leq 1, \mathrm{E}_\mathrm{t}\left[X_T^a\right] \to 0$:

If $X_T \leq 1$, $(1+X_T)^a \leq 1 + X_T \leq 1 + X_T^a$, while if $X_T \geq 1$, $(1+X_T)^a \leq X_T^a + a \leq 1 + X_T^a$ since $(1+X_T)^a$ is concave. Thus, $1 \leq (1+X_T)^a \leq 1 + X_T^a$ and therefore $\lim_{T\to\infty} \mathrm{E}_\mathrm{t}\left[(1+X_T)^a\right] = 1$, which implies the main result.

Case 5: $1 \leq a \leq 2, \mathrm{E}_\mathrm{t}[X_T^a] \to 0$.

By the mean value theorem, $(1+X_T)^a = 1 + aX_T(1+wX_T)^{a-1}$ for some $w \in [0,1]$. Further, $X_T(1+wX_T)^{a-1} \leq X_T(1+X_T)^{a-1} \leq X_T\left(X_T^{a-1} + 2^{a-1} + a - 1\right)$, using the same argument as in case 1. Since $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = 0$, according to (A.7b), $\lim_{T\to\infty} \mathrm{E}_t[X_T] = 0$ and hence $\lim_{T\to\infty} \mathrm{E}_\mathrm{t}[(1+X_T)^a] = 1$.

Case 6: $2 \leq a, \mathrm{E}_\mathrm{t}[X_T^a] \to 0$.

By the mean value theorem, $(1+X_T)^a = 1 + aX_T(1+wX_T)^{a-1}$ for some $w \in [0,1]$. Further, $X_T(1+wX_T)^{a-1} \leq X_T(1+X_T)^{a-1} \leq 2^{a-2}X_T + 2^{a-2}X_T^a$ by Jensen's inequality. Since $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = 0$, according to (A.7b), and $\lim_{T\to\infty} \mathrm{E}_t[X_T] = 0$ and hence $\lim_{T\to\infty} \mathrm{E}_\mathrm{t}[(1+X_T)^a] = 1$.

(ii) Since the conditional expectations $\mathrm{E}_t[(1+X_T)^a]$ and $\mathrm{E}_t[1+X_T^a]$ are martingales, they have zero drift for all values of T and t. By Ito's lemma, convergence of the first moments of the natural logarithms of the same processes follows from convergence of the second moments.

We now establish convergence of volatility of the process $\mathrm{E}_t[(1+X_T)^a]$. According to Ito's lemma, one must show that

$$\lim_{T\to\infty} \frac{\partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t}{\partial \ln(1+\mathrm{E}_t[X_T^a])/\partial B_t} = 1, \quad \forall\ a \geq 0.$$

Given (A.6), it suffices to prove that $\lim_{T\to\infty} \partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t = 0$ if $\lim_{T\to\infty} \mathrm{E}_t\left[X_T^a\right] = 0$ and $\lim_{T\to\infty} \partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t = av$ if $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \infty$.

First, changing the order of differentiation and expectation operators (see Billingsley 1995, Th. 16.8),

$$\frac{\partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]}{\partial B_t} = av\frac{\mathrm{E}_t\left[X_T(1+X_T)^{a-1}\right]}{\mathrm{E}_t\left[(1+X_T)^a\right]} = av\left(1 - \frac{\mathrm{E}_t\left[(1+X_T)^{a-1}\right]}{\mathrm{E}_t\left[(1+X_T)^a\right]}\right).$$

Furthermore, according to part (i),

$$\frac{\mathrm{E}_t\left[(1+X_T)^{a-1}\right]}{\mathrm{E}_t\left[(1+X_T)^a\right]} \sim \frac{\mathrm{E}_t\left[(1+X_T)^{a-1}\right]}{1+\mathrm{E}_\mathrm{t}\left[X_T^a\right]}. \tag{A.8}$$

Assume $a \geq 1$. As we have shown in case 1 of the proof of part (i), $X_T^{a-1} \leq (1+X_T)^{a-1} \leq X_T^{a-1} + 2^{a-1} + a - 1$. If $\mathrm{E}_\mathrm{t}\left[X_T^a\right] \to \infty$, according to (A.7a), $\mathrm{E}_\mathrm{t}\left[X_T^{a-1}\right]/\mathrm{E}_\mathrm{t}\left[X_T^a\right] \to 0$, which yields $\lim_{T\to\infty} \partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t = av$. Similarly, if $\mathrm{E}_\mathrm{t}\left[X_T^a\right] \to 0$, then, according to (A.7b), $\lim_{T\to\infty} \mathrm{E}_t\left[X_T^{a-1}\right] = 0$, which, according to part (i), implies that $\lim_{T\to\infty} \mathrm{E}_t\left[(1+X_T)^{a-1}\right] = 1$ and $\lim_{T\to\infty} \partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t = 0$.

Next, consider the case of $0 < a < 1$. If $\mathrm{E_t}[X_T^a] \to \infty$, because $\mathrm{E_t}[(1+X_T)^{a-1}] \leq 1$, (A.8) implies $\lim_{T\to\infty} \partial \ln \mathrm{E_t}[(1+X_T)^a]/\partial B_t = a\, v$.

Suppose that $\lim_{T\to\infty} \mathrm{E_t}[X_T^a] = 0$. By Markov's inequality, for any $\epsilon > 0$, $\mathrm{P}_t[X_T > \epsilon] \leq \mathrm{E_t}[X_T^a]/\epsilon^a \to 0$. Similarly, $P_t\left[X_T < \epsilon\right] \leq \mathrm{E_t}[(1+X_T)^{a-1}]/(1+\epsilon)^{a-1}$. Thus, $1 \geq \mathrm{E_t}[(1+X_T)^{a-1}] \geq P_t[X_T < \epsilon](1+\epsilon)^{a-1}$, and $\liminf_{T\to\infty} \mathrm{E_t}[(1+X_T)^{a-1}] \geq (1+\epsilon)^{a-1}$ for any $\epsilon > 0$. This implies that $\lim_{T\to\infty} \mathrm{E_t}[(1+X_T)^{a-1}] = 1$ and $\lim_{T\to\infty} \partial \ln \mathrm{E_t}[(1+X_T)^a]/\partial B_t = 0$. ■

We establish the long-run behavior of S_t for the case when $\gamma > 1$ and $0 < \eta < \eta^\star = 2(\gamma - 1)$. The results for all other regions in the parameter space can be obtained similarly.

The equilibrium stock price and the ratio of the individual wealth processes are given by

$$S_t = \frac{\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right]}{\mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right]}, \qquad \frac{W_{r,t}}{W_{n,t}} = \frac{\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]}{\mathrm{E}_t\left[D_T^{1-\gamma}(b\,\xi_T)^{\frac{1}{\gamma}}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]}.$$

We therefore need to characterize the long-run behavior of the following two quantities:

$$E^{(1)} \equiv \mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right], \qquad E^{(2)} \equiv \mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right].$$

Consider the first expression,

$$E^{(1)} = \mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] = D_t^{1-\gamma}\mathrm{E}_t\left[\left(\frac{D_T}{D_t}\right)^{1-\gamma}\left(1+\left(b\,\xi_t\frac{\xi_T}{\xi_t}\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right].$$

Given the aggregate dividend process,

$$\left(\frac{D_T}{D_t}\right)^{1-\gamma} = e^{(T-t)\left(\mu(1-\gamma)-\frac{1}{2}\sigma^2(1-\gamma)\gamma\right)}e^{-\frac{1}{2}(1-\gamma)^2\sigma^2(T-t)+(1-\gamma)\sigma(B_T-B_t)}.$$

As in the proof of Proposition 1, we introduce a new measure Q with the Radon-Nikodym derivative $\left(\frac{dQ}{dP}\right)_t = e^{-\frac{1}{2}(1-\gamma)^2\sigma^2(T-t)+(1-\gamma)\sigma(B_T-B_t)}$. By Girsanov's theorem, $B_T - B_t = B_T^Q - B_t^Q - (1-\gamma)\sigma(T-t)$, where B_t^Q is a Brownian motion under the measure Q. Using the expression for b from Proposition 1, $b = e^{T(\gamma-1)\sigma^2\eta}$, we find

$$E^{(1)} = e^{T\left(\mu(1-\gamma)-\frac{1}{2}\sigma^2(1-\gamma)\gamma\right)+t\left(-\frac{1}{2}\sigma^2(1-\gamma)^2\right)+B_t(\sigma(1-\gamma))}\mathrm{E}_t^Q\left[\left(1+e^{\left(-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}\right)T+\left(\frac{1}{\gamma}(\gamma-1)\sigma^2\eta\right)t+\frac{\eta\sigma}{\gamma}B_T^Q}\right)^{\gamma}\right].$$

We will omit the superscript Q, since the distribution of B_t^Q under the measure Q is the same as the distribution of B_t under the original measure P.

Using the assumption that $t = \lambda T$, define

$$X_T = e^{\left(-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}+(1-\lambda)\frac{1}{\gamma}(\gamma-1)\sigma^2\eta\right)T+\frac{\eta\sigma}{\gamma}B_T}.$$

We now apply the result of lemma A.2, with

$$c = -\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma} + (1-\lambda)\frac{1}{\gamma}(\gamma-1)\sigma^2\eta, \quad v = \frac{\eta\sigma}{\gamma}, \quad a = \gamma.$$

Since we are assuming $\gamma > 1$ and $0 < \eta < 2(\gamma-1)$, $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \infty$. According to lemma A.2,

$$\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] \sim e^{\left(\mu(1-\gamma)-\frac{1}{2}\sigma^2(1-\gamma)\gamma\right)T+\left(-\frac{1}{2}\sigma^2(\eta+1-\gamma)^2\right)t+\sigma(\eta+1-\gamma)B_t}.$$

We next examine $E^{(2)}$. Using a similar change of measure, we find

$$E^{(2)} = e^{\left(-\mu\gamma+\frac{1}{2}\sigma^2(1+\gamma)\gamma\right)T+\left(-\frac{1}{2}\sigma^2\gamma^2\right)t+(-\sigma\gamma)B_t}\mathrm{E}_t\left[\left(1+e^{\left(-\sigma^2\eta\frac{1}{\gamma}-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}\right)T+\sigma^2\eta t+\frac{\eta\sigma}{\gamma}B_T}\right)^{\gamma}\right].$$

We apply lemma A.2, setting $X_T = e^{c\,T+v\,B_T}$ and

$$c = -\sigma^2\eta\frac{1}{\gamma} - \frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma} + (1-\lambda)\sigma^2\eta, \quad v = \frac{\eta\sigma}{\gamma}, \quad a = \gamma.$$

The value of $\lim_{T\to\infty} \mathrm{E}_t[X_T^a]$ depends on the exact combination of the model parameters. In particular,

$$\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \begin{cases} \infty, & -2\eta+\lambda(2\gamma\eta-\eta^2) > 0, \\ 0, & -2\eta+\lambda(2\gamma\eta-\eta^2) < 0, \end{cases}$$

(see the proof of lemma A.2, part (i)). Define $\lambda_S \equiv \frac{2}{2\gamma-\eta}$. Note that, because $\gamma > 1$ and $0 < \eta < 2(\gamma-1)$, $0 < \lambda_S < 1$. Then, $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \infty$ if $\lambda > \lambda_S$ and $2\gamma\eta - \eta^2 > 0$ or if $\lambda < \lambda_S$ and $2\gamma\eta - \eta^2 < 0$, and the limit is equal to zero otherwise. By lemma A.2, if $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \infty$,

$$\mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] \sim e^{T\left(-\mu\gamma+\frac{1}{2}\sigma^2(1+\gamma)\gamma-\sigma^2\eta\right)T+\left(-\frac{1}{2}\sigma^2(\eta-\gamma)^2\right)t+\sigma(\eta-\gamma)B_t}$$

while if $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = 0$, then

$$\mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] \sim e^{\left(-\mu\gamma+\frac{1}{2}\sigma^2(1+\gamma)\gamma\right)T+\left(-\frac{1}{2}\sigma^2\gamma^2\right)t+(-\sigma\gamma)B_t}.$$

Using our definition of λ_S, we re-state these results as

$$\mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] \;\sim\; \begin{cases} e^{\left(-\mu\gamma+\frac{1}{2}\sigma^2(1+\gamma)\gamma\right)T+\left(-\frac{1}{2}\sigma^2\gamma^2\right)t-\sigma\gamma B_t}, & 0\le\lambda<\lambda_S \\ e^{\left(-\mu\gamma+\frac{1}{2}\sigma^2(1+\gamma)\gamma-\sigma^2\eta\right)T+\left(-\frac{1}{2}\sigma^2(\eta-\gamma)^2\right)t+\sigma(\eta-\gamma)B_t}, & \lambda_S<\lambda\le 1 \end{cases}$$

Having established the behavior of both the numerator and the denominator of the expression for the stock price, we have proven the limiting result for the stock price itself. According to part (ii) of lemma A.2, not only the stock price, but also the mean and volatility of returns behave according to the asymptotic expressions of Proposition 5 in the limit of the economy horizon T approaching infinity. ■

Proof of Proposition 7

When the financial markets are dynamically complete and there is a single source of uncertainty (driven by a Brownian motion), the fraction of the agent's wealth invested in stock can be computed as a ratio of the instantaneous volatility of the agent's wealth to the instantaneous volatility of the cumulative stock return process. Proposition 5 and (Kogan, Ross, Wang, and Westerfield, 2003, Proposition 8) provide expression for the long-run behavior of the volatility of stock returns and individual wealth processes, from which the expression for portfolio holdings follow immediately. To decompose the portfolio holdings of the rational trader into as a sum of the myopic and hedging demands, we compute the hedging demand as $\mu_S/(\gamma\sigma_S^2)$, where μ_S and σ_S are the drift and the diffusion coefficients of the stock return process. The difference between the total portfolio holdings and the myopic component define the agent's hedging demand. For the irrational trader, the calculations are analogous, except the myopic demand is given by $\hat{\mu}_S/(\gamma\sigma_S^2) = (\mu_S+\eta\sigma\sigma_S)/(\gamma\sigma_S^2)$, where $\hat{\mu}_S$ is the expected stock return as perceived by the irrational trader. ■