

Speculation, Overpricing, and Investment - Theory and Empirical Evidence

Stavros Panageas

MIT

<JOB MARKET PAPER>

November 15, 2003*

Abstract

In this paper I investigate whether firms' physical investments react to the speculative over-pricing of their securities. I introduce investment considerations in an infinite horizon continuous time model with short sale constraints and heterogeneous beliefs along the lines of Scheinkman and Xiong (2003). I obtain closed form solutions for all quantities involved. I show that market based q and investment are increased, even though such investment is not warranted on the basis of long run value maximization. Moreover, I show that investment amplifies the effects of speculation on prices through an increase in the value of "growth" options. In the empirical section of the paper, I use a simple episode to test the hypothesis that investment reacts to over-pricing. With publicly available data on short sales during the 1920's, I examine both the price reaction and the investment behavior of a number of companies that were introduced into the "loan crowd" during the first half of 1926. In line with Jones and Lamont (2002), I interpret this as evidence of overpricing due to speculation. I find that investment by these companies follows both the increase and the decline in " q " before and after the introduction, suggesting that companies in this sample reacted to security over-pricing.

JEL Codes: E2, G1, G3, N2

Keywords: Investment, q -theory, speculation, short sales constraints, heterogenous beliefs, bubbles, mis-pricing

*I am grateful to George Marios Angeletos, Olivier Blanchard, Ricardo Caballero, Peter Diamond, Xavier Gabaix, Leonid Kogan, S.P. Kothari, Anna Pavlova, Jim Poterba, Antoinette Schoar, Dimitri Vayanos, Ivan Werning and participants of the Macro, Theory and Finance lunches and seminars at M.I.T. for helpful discussions and comments. I am indebted to Tom Nicholas for providing me with a subset of the company data used in the empirical investigation. All errors are mine.

1 Introduction

Standard neoclassical theory predicts that investment is inherently tied with the stock market through Tobin's "q". The essence of "q" theory is the following argument: if the repurchase cost of capital is less than the net present value of additional profits it will bring at the margin, then the company should invest and vice versa. The only thing preventing the ratio of the two values (known as q) from being always equal to 1 is adjustment costs: it is expensive to install new capital and thus a deviation of q from 1 can exist, but it should diminish over time. The link between investment and the stock market follows: the value of a company is the net present value of its profits and thus¹ whenever one sees the stock market rising, one should simultaneously observe an increase in investment in order to bring the numerator and the denominator of the "q" ratio into line.

However, there is a concern with this line of reasoning. Namely, what happens if the stock market valuation at times does not reflect the net present value of profits but also contains terms that are unrelated to "fundamentals"? Will "q" theory continue to hold or will decision makers in companies be eclectic about the components of stock market valuation to which they will pay attention ?

This is the main question I take up in this paper. I start with an explicit reason for why assets can deviate from fundamentals. Then I introduce investment considerations and study investors' holding horizons, optimal investment, and the resulting equilibrium prices in a unified framework.

To be more specific, I use short sales constraints to derive positive deviations of prices from fundamentals. It is intuitive that the presence of a short sale constraint can cause the price of an asset to deviate from its fundamental value if market participants do not have homogenous beliefs. Agents who believe that the current price is above the net present value of dividends would have to go short in order to take advantage of what they perceive to be mispricing. However, they cannot do this because of the short sale constraint. Accordingly, for pricing purposes, it is as if they do not exist, and the price will only reflect the views of the most optimistic market participants.

This basic intuition was first expressed in a formal intertemporal model by Harrison and Kreps (1978). A number of papers extended the intuition into various directions. A partial listing includes Allen, Morris, and Postlewaite (1993), Detemple and Murthy (1997), Morris (1996) and most recently Scheinkman and Xiong (2003) and Hong and Stein (2002). All of these papers study an exchange setting without a role for investment.

The present paper extends this literature to allow for investment. In particular the model presented here is based on Scheinkman and Xiong (2003) with the difference that I allow firms to adjust their capital stock

¹Under Hayashi's (1982) conditions

by investing. Because the model is set up in continuous time I can derive closed form solutions for prices, investment, trading strategies and investors' horizons. First, I show that traditional "q" theory remains valid if investors have perfect access to financial markets, they are risk neutral and investment is determined in the best interest of current shareholders.² Whether the stock price is high because of fundamentals or resale premia is irrelevant. A shareholder value maximizing company will use the stock market valuation as a guide to how much investors can gain in the stock market by either holding the asset and reaping dividends or by reselling it to more optimistic investors. Second, I show that investment significantly amplifies the effects of speculation on the asset prices by affecting the value of growth options embedded in the company's price.³ In the present framework, young dynamic companies with low adjustment costs and high disagreement associated with their underlying productivity can end up with high levels of q (low levels of book to market) and low expected returns. The closed form solution obtained for the price of the firm allows a quantification of these effects and a comparison with actual data.

It is possible however, to imagine circumstances where investment would not react to market based q and the above logic would fail. For instance, if a major shareholder owns a significant fraction of a firm and values control she would be unlikely to react to resale premia because they are irrelevant for her. Similarly, key investors might be afraid of selling their shares in large amounts because other investors might fear the presence of asymmetric information. In other words, resale premia are only relevant for investors that have short horizons and who can realize the full speculative gains associated with them. If they can't access the markets (or accessing the markets is costly) then the incentive to invest will be attenuated.⁴ I derive optimal investment under this alternative and then discuss a set of observable implications.

Then I address the empirical question: which of the two theories is supported by the data? Answering this question is difficult because one has to identify a shock to resale premia but not to fundamentals. Only then can one study how investment reacts to the former type of shocks. Disentangling fundamental from non-fundamental deviations is a difficult task. The usual approach in the literature has been to try to find proxies for the two components. Such an approach is associated with the usual doubts on how successful one is in creating these proxies. Moreover, certain proxies that are often used are not obviously related to short selling costs and constraints alone, but capture asymmetric information or agency problems.

In this paper I take a direct approach: in the 1920's an entire market, known as the "loan crowd", was active for shorting stock. I use a dataset recently collected by Jones and Lamont (2002) based on daily

²Risk neutrality is not essential if one is willing to make a specific assumption about the valuation of income streams by investors in incomplete markets. See the next section for details.

³Growth options are defined as the difference between the equilibrium price when investment is determined optimally and the equilibrium price when investment is set to 0 throughout.

⁴Similar points were made in Blanchard, Rhee and Summers (1993), Stein (1996) and Morck, Shleifer and Vishny (1990). All three papers emphasize the distinction between short and long horizons.

coverage of this market by the *Wall Street Journal*. The list of companies in this market expanded in several waves. As Jones and Lamont (2002) argue, the introduction of a company into the "loan crowd" reflects a belief by investors that this company is particularly overpriced. I provide some additional evidence to that effect. The behavior of the stock price of the newly introduced companies indeed seems to confirm such an explanation. Stock prices show a marked runup for several quarters before the introduction and decrease dramatically thereafter. Not surprisingly, market-based q presents exactly the same behavior. To complement the dataset of Jones and Lamont (2002) with balance sheet data, I hand-collected financial data on a number of these companies from Moody's manuals and studied the behavior of investment in the years prior to their introduction and thereafter. I find that investment followed exactly the same behavior as market based " q ".

The paper is related to a number of strands in the literature. There is a small number of papers that have addressed the same set of issues, mostly from an empirical angle. These include: Fischer and Merton (1984), Morck, Shleifer, and Vishny (1990), Blanchard, Rhee, and Summers (1993), Stein (1996), Chirinko and Schaller (1996) and more recently Polk and Sapienza (2002), Gilchrist, Himmelberg, and Huberman (2002). A central theme of this literature is the importance of investor's horizons. However, the models developed in these papers do not explicitly characterize the optimal holding horizon (defined as the stopping time at which an investor finds it optimal to resell). Moreover, these models do not allow one to derive intertemporal implications for investment and stock prices jointly. For example this makes it difficult to determine why and when certain Euler relations should hold or fail, and thus is important from an empirical viewpoint.⁵ The present paper models everything explicitly in an infinite horizon continuous time setting and thus one is able to model investor's horizons endogenously and derive testable implications about the relationship between investment and prices in an explicit way. The empirical approach to testing the theory is also more direct. Instead of using proxies to account for mispricing, I use the firms that were perceived to be as most overvalued at the time as evidenced by the fact that they were introduced into the loan crowd.⁶

The paper is also related to a literature in financial economics that uses insights from investment theory

⁵For instance Chirinko and Schaller (1996) derive a test for whether bubbles affect investment or not, by making the interesting assumption that bubbles lead to predictable returns. From that assumption they derive the result that if investment reacts to bubbles, certain Euler relations should fail to hold. However, not every source of predictability can be attributed to bubbles and bubbles will not necessarily lead to predictability. In the explicit framework of this paper, one can determine both the source of predictability and its implications for testing. This issue is explained in detail in the sections that follow.

⁶One direction that is not explored is the behaviour of investment, if decisionmakers are longtermist but the company is financially constrained. It can be conjectured that in this case investment could potentially react to market based q even if managers maximize long run performance. See e.g. Stein (1996), Baker, Stein, and Wurgler (2003). It is interesting to note that in the present paper one does not need to assume anything apart from shareholder value maximization to arrive at the result that investment reacts to market based q even in the absence of constraints. It is also conceivable that constraints could further amplify the result. I discuss this point in further detail in the conclusion of the paper.

to address issues such as the predictability of returns, the role of book to market ratios, etc. A partial listing would include Cochrane (1991,1996), Naik (1994), Berk, Green, and Naik (1999), Lamont (2000). Berk, Green, and Naik (1999) in particular show how a model with investment can account for some apparent irregularities in asset pricing as e.g. the power of the Book to Market ratio to predict returns⁷. In this paper I obtain a closed form solution that decomposes the price into a component related to assets in place and "growth options" or "rents to the adjustment technology". Moreover, I can derive the effects of speculation on both components separately. I find that the "growth options" amplify significantly the effects of speculation. Returns are predictable and predictability of returns becomes strongest when both fundamentals and disagreement about fundamentals are high. This is in contrast to the pure exchange case where predictability only depends on disagreement. It also makes it easier for quantities like B/M or E/P to predict returns since the price of a company captures both fundamental and non-fundamental variations. In a quantitative exercise I show that q can become large even for small degrees of irrationality. This has the potential to explain quantitatively the very low book to market ratios that one observes during speculative episodes. Moreover, the model has the potential to produce reasonable levels of predictability of returns in quantitative terms as is shown by simulating an artificial CRSP dataset and re-running some Fama and MacBeth regressions of simulated monthly returns on Book to Market.

An interesting application of this paper concerns the relationship between return predictability and investment: the reaction of investment to speculative components in prices could potentially help in distinguishing rational and behavioral views of predictability. If investment only reacted to fundamental variations and was powerful at predicting returns, then this would be evidence that the variation in expected returns is due to variation in risk aversion and not to speculation motives and expectational errors. Lamont (2000) indeed documents the ability of investment plans to explain aggregate returns. However, to make the link between investment and variations in risk premia, one would need to establish that investment only reacts to variations in fundamentals and not to potentially irrationally optimistic beliefs. The empirical evidence that I provide in this paper, suggests that investment reacts to both fundamental and speculative terms. Thus, investment does not seem to be able to provide a clear way to distinguish between rational and behavioral theories.

The paper is also complementary to the strand in the macroeconomics literature that models bubbles in the framework of overlapping generations models. Blanchard and Fischer (1989) present a textbook treatment, whereas Caballero and Hammour (2002) and Jacques (2000) are some recent contributions to this literature. This literature assumes short horizons, whereas in the present paper short horizons arise endogenously. Moreover, uncertainty is key in the present paper, whereas uncertainty typically plays a

⁷This fact is documented in the cross section by Fama and French (1992,1995) and in the time series dimension by Kothari and Shanken (1997) among others.

secondary (if any) role in the papers above. However, the simpler setup of overlapping generations allows one to address a richer set of issues (related e.g. to savings and fiscal policy) that would be hard in the present setup. In a sense, the model developed here provides a foundation for models in this literature.

The outline of the paper is as follows: Section 2 presents a simple three-period example that allows an easy presentation of most intuitions of the model. Section 3 presents the model in an infinite horizon continuous time setting with a richer set of dynamics for the beliefs of the agents. In this section I also discuss the properties of the model and its implications for testing. Section 4 presents the empirical evidence. Section 5 concludes. All proofs are given in the appendix.

2 A simple example

In this section I present a simple example that will help fix some ideas. I extend this example in the section that follows to a continuous time setting. I assume that the world lasts for three periods. There are two states of the world h and l and a single productive asset that pays out $f_t K_t$. f_t is 0 in state l and 1 in state h . K_t is the amount of capital available to the economy at time t . For simplicity I also assume that labor is not required to produce output and that the economy is small, i.e. the interest rate is taken as given and normalized to 0. To introduce heterogeneity of beliefs I assume that there are two types of agents, which I label agents A and B . There is a continuum of both types having infinite total wealth and /or an infinite ability to borrow.⁸

Figure 1 depicts the transition probabilities that agents in each group assign to the transition from one state to the other. In particular, agents in group B perceive each state as equiprobable, while agents in group A are originally optimistic (they assign probability 0.9 to the high state occurring). If the high state occurs, then they continue to be optimistic about period 2, otherwise they become pessimistic (in the sense that they assign probability 0.9 to the low state occurring again). The only crucial feature of this setup is that agents do not agree on the transition probabilities. Agents cannot take short positions in the asset. I assume that at time 0 the economy is in state h . The setup is common knowledge to the agents who agree to disagree.

Suppose initially that there is no investment (i.e. I treat $K_t = 1$ as a constant). Moreover there is no depreciation. Equilibrium prices and trading strategies are determined by backwards induction. The joint assumptions of risk neutrality and infinite total wealth allow one to set the price equal to the reservation price of the person who values the asset most. In particular at time 1 and state h the agents who value the asset the most are agents in group A. The reservation price for agents in group A is given as $P_{1h}^A = 0.9 \times 1 + 0.1 \times 0 = 0.9$.

⁸This assumption is made by both Harrison and Kreps (1978) and Scheinkman and Xiong (2003) and is useful in order to drive values towards the reservation price.

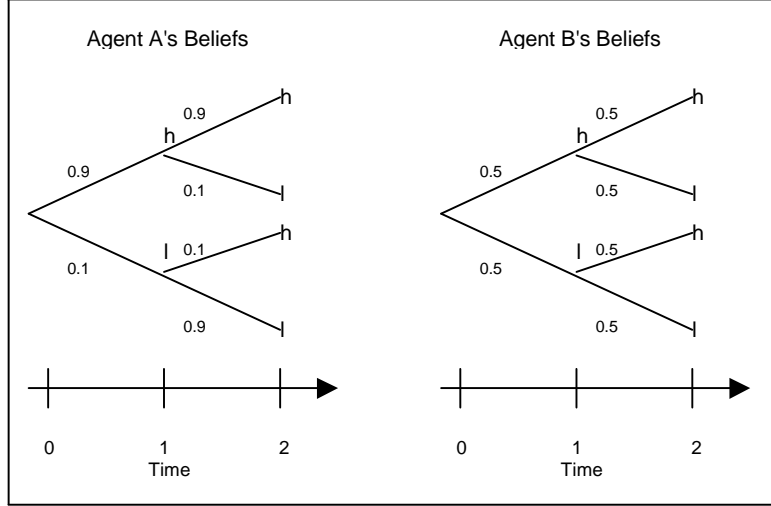


Figure 1: Transition probabilities as perceived by agents in groups A and B

The reservation price for agents in group B is given as $P_{1h}^B = 0.5 \times 1 + 0.5 \times 0 = 0.5$. Even though agents in group B would be happy to short the asset in this state, they can't. Accordingly the price is given as $P_{1h} = \max [P_{1h}^A, P_{1h}^B] = 0.9$. Similarly in state $(1, l)$ the price will be given as $P_{1l} = 0.5 \times 1 + 0.5 \times 0 = 0.5$ since now agents of type A are less optimistic than agents in group B . At time 0 agents in group A value the asset at $P_0^A = 0.9 \times (1 + 0.9) + 0.1 \times (0 + 0.5) = 1.76$. This is the relevant valuation for agents in group A because at the node $(1, l)$ they know that they will resell the asset to agents in group B . For group B the reservation price is $P_0^B = 0.5 \times 1.9 + 0.5 \times 0.5 = 1.2$, so that the equilibrium price of the asset will be given by $P_0 = \max [P_0^A, P_0^B] = 1.76$. A convenient way of summarizing the above discussion is in terms of the Harrison and Kreps (1978) formula:

$$P_t = \max_{o \in \{A, B\}} [P_t^o] = \max_{o \in \{A, B\}} \left[\sup_{\tau} E^o \left(\sum_{s=t+1}^{\tau} D_s + P_{\tau} \right) \right] \quad (1)$$

where D_s are the dividends paid at the state-time pair s and τ is an optimally chosen stopping time at which an agent decides to sell the asset.

Another recursive relation that is true is⁹:

$$\begin{aligned} P_t &= \max_{o \in \{A, B\}} [E^o (D_{t+1} + P_{t+1})] \\ P_T &= 0 \end{aligned}$$

Interestingly, the price at node 0 is strictly higher than what either agent would be willing to pay if she didn't consider the possibility to resale the asset later on. In particular, if one prohibits agents from engaging

⁹See Harrison and Kreps (1978)

in transactions at any point other than at time 0, the price \widetilde{P}_0 of the asset is given by:

$$\widetilde{P}_0 = \max_{o \in \{A, B\}} [\widetilde{P}_0^o] = \max_{o \in \{A, B\}} \left[E^o \left(\sum_{s=1}^T D_s \right) \right]$$

The value has to be lower than the original price, since agents are deprived of the possibility to resell. This possibility is embodied in the optimization over stopping times in formula (1). That is: $P_0 \geq \widetilde{P}_0$. The difference in the two values is the option value ascribed to reselling the asset in the future. An additional implication of this formula concerns predictability of asset returns. If I assume that one of the market participants has the "right" beliefs then the prices are no longer martingales from her perspective. This investor views the asset as having potentially negative expected (excess) returns in certain states of the world. A further interesting interpretation of (1) is in terms of the investor's holding horizon. The interaction of heterogenous beliefs and short sales makes the optimal stopping time problem meaningful. In contrast, if one assumed homogenous beliefs then all stopping times would yield the same payoff and accordingly One could assume that the investor holds the asset until maturity.

Consider now the above example with investment. In particular, assume that there is a technology that allows agents to reduce today's dividends by:

$$i_t + \chi \frac{i_t^2}{2}$$

in order to increase next period capital to

$$K_{t+1} = K_t + i_t$$

I will assume that investment is determined in the best interest of investors who are endowed with the stock at the beginning of the period¹⁰. Once again I start backwards in order to determine equilibrium outcomes. In state $(1, h)$ it is clear that agents of type A will end up holding the stock no matter which investment strategy is chosen and no matter which type of agent is endowed with the stock. This is so because for any investment decision their reservation value will be higher than the reservation value of agents of type B for the stock. Accordingly, the investment decision will be determined according to the beliefs of type A agents *independently of who is endowed with the company stock at the time-state pair* $(1, h)$. In mathematical terms:

$$1 + \chi i_{1,h}^* = E^A (f_2 | s_t = 1, h) \quad (2)$$

where f_2 denotes the productivity in period 2. This first order condition is obvious if agents of type A are endowed with the stock in the state-time pair $(1, h)$. It is also true however, if the the company stock

¹⁰In particular I assume (like Grossman and Hart (1979)) that investors arrive at the beginning of the period with a certain endowment of the stock, they determine the investment policy, dividends are paid and then they trade their shares in a Walrasian market.

belongs to agents of type B . To see this, notice that agents of type B have to balance two effects in making their investment decision. On the one hand they realize that by investing they reduce the dividends that they can obtain. On the other hand they increase the resale value of the asset since agents of type A will be willing to pay more for a company with a larger capital stock. Agents in group B understand that only agents of group A will matter for pricing purposes in state $(1, h)$. Thus they are led to understand that a marginal investment of Δi_t will change the price that they can gain for the asset by $E^A(f_2|s_t = 1, h) \Delta i_t$ while reducing the current dividends by $(1 + \chi i_t) \Delta i_t$. Balancing out these two effects leads to the same first order condition as (2).

For the optimal investment rule derived from (2) one can determine time 1 dividends as:

$$D_1 = f_1 K_1 - i_1^* - \chi \frac{(i_1^*)^2}{2}$$

and period 2 capital as:

$$K_2 = K_1 + i_1^*$$

Working backwards by the same logic one can establish that independently of who controls the company at time 0 the optimal investment strategy is to invest until:

$$1 + \chi i_0^* = E^A(f_1 + q_1)$$

where q_1 is given as $E^A(f_2|s_t = 1, h)$ if the state h realizes and as $E^B(f_2|s_t = 1, l)$ if the state l realizes. It is interesting to note that one can express the stochastic process for investment in terms of the recursive relations:

$$1 + \chi i_0^* = q_t \tag{3}$$

$$q_t = \max_{j \in \{A, B\}} E^j [f_{t+1} + q_{t+1}] \tag{4}$$

$$q_T = 0$$

The second of these equations is exactly the same equation that was obtained for the price in the context of the simple exchange setting. The above discussion motivates the main concept of equilibrium that I will use in this paper. Namely, I will assume that the company is maximizing investor welfare and accordingly I will be searching for investment policies, selling/stopping times, and equilibrium pricing functions that satisfy the relation:

$$P_t(K_t) = \left(\max_{o \in \{A, B\}} \left[\sup_{i_s, \tau} E^o \left(\sum_{s=t+1}^{\tau} D_s(i_s, K_s) + P_{\tau}(K_{\tau}) \right) \right] \right)$$

where $i_{t..T}$ denotes the stochastic process of investment. Clearly, the analysis needs to be modified, if I assume that agents cannot retrade. For instance suppose that markets will only be open at time 0 and never

thereafter. Then (4) should be replaced by the conventional q relationship

$$\begin{aligned} q_t &= E^j [f_{t+1} + q_{t+1}] \\ q_T &= 0 \end{aligned} \tag{5}$$

where j denotes the agent who will bid more for the company at time 0 (in this example agent A). It is evident by comparing (4) and (5) that investment will be necessarily higher in the presence of a resale premium on assets. Also, by the same argument as in the exchange setting marginal " q " no longer satisfies the usual martingale type relationship under the beliefs of any agent. This means that every agent understands that in certain states of the world investment will be undertaken even though its expected (excess) return is negative from her perspective.¹¹¹²

3 A continuous time framework

The primary goal of this section is to derive testable implications of the hypothesis that investment is affected by resale premia and quantify the effects discussed in the last section. Moreover this section focuses on the effects of speculation on the value of a company's growth options and demonstrates how they magnify the effects of speculation on the stock price. I will expand the previous model to an infinite horizon continuous time setup with quadratic adjustment costs independent of the capital stock. Continuous time introduces some tractability into the problem. It allows one to determine closed form solutions for prices, investment

¹¹One might wonder to what extent the conclusions of this section depend on risk neutrality. It can be shown that an extension of the ideas in Grossman and Hart (1979) can be used to address the risk aversion case. In particular, assuming "utility taking" behavior on the most optimistic agents allows one to generalize (4) to a setting with risk aversion by using the marginal utilities of the most optimistic agents at each node of the information tree to construct a pricing kernel. Details on this construction are available upon request.

¹²So far I assumed nothing about financing policies. This was done because the Modigliani Miller Theorem continues to hold in this setup despite the short selling constraint. (If one allowed for debt financing then one would also need to assume unlimited liability in light of the results in Hellwig (1981)). A proof of these claims can be given by arguments identical to DeMarzo (1988) and is available upon request. The intuition for why the MM Theorem holds is straightforward. In this setup the firm cannot do more by trading in its own stock (i.e. by issuing shares) than what the investor can do by selling her shares in the market. This is true because the only direction in which the investor is constrained is the short side. Accordingly, if financial policy could create value then it would have to be by promising to deliver a negative multiple of the company's dividends. This would effectively alleviate the investor's short sale constraint. Of course there is no financial policy that can do that, and accordingly financial policy cannot create value for the investor. This analysis also demonstrates one way to introduce active financial policy in this framework. Suppose for instance that accessing the financial markets directly is costly for existing investors due to e.g. asymmetric information or fears of nonlinear price impact. Then, an easy way for the investor to sell stock and realize speculative gains is by having the firm issue stock and not participating. In reality there seems to be a strong relationship between equity issuance and speculation as documented by Baker and Wurgler (2000), Baker Stein and Wurgler (2003).

and stopping policies. In particular, I will combine a framework proposed by Harrison and Kreps (1978) and Scheinkman and Xiong (2003) to study speculative premia on assets with a standard investment framework with quadratic adjustment costs along the lines of Abel (1983), Abel and Eberly (1994), (1998). I also discuss how one can generalize the basic predictions to a setup with an arbitrary number of groups of agents and arbitrary linear homogenous adjustment technologies.¹³

3.1 Setup

3.1.1 Company Profits and Investment

There is a single company and the goal will be to determine its value as part of the (partial) equilibrium solution of the model. The company's cumulative earnings process is given by:

$$dD_t = K_t f_t dt + K_t \sigma_D dZ_t^D \quad (6)$$

In units of installed capital this expression becomes:

$$\frac{dD_t}{K_t} = f_t dt + \sigma_D dZ_t^D$$

The first component captures a stochastic trend growth rate whereas the second term captures noise in the company earnings that prevents market participants from perfectly inferring the level of productivity f_t . K_t is the amount of physical capital installed in the company which "scales" both the trend growth rate and the "noise" in the cumulative earnings process. σ_D is a constant controlling the "noise", while dZ_t^D is a standard one dimensional Brownian Motion. The variable f_t is not observable and evolves according to an Ornstein Uhlenbeck process as:

$$df_t = -\lambda(f_t - \bar{f})dt + \sigma_f \sqrt{\frac{f_t}{\bar{f}}} dZ_t^f \quad (7)$$

where $\lambda > 0$ is a mean reversion parameter, $\bar{f} > 0$ is a long-run productivity rate, σ_f is the volatility of the Ornstein Uhlenbeck process and dZ_t^f is a second Brownian motion that is independent of dZ_t^D . For simplicity I will also assume that the company is fully financed by equity and there is a finite number of shares of the company whose supply I normalize to 1. The company can invest in physical capital at the rate i_t . The evolution of the capital stock is accordingly given by:

$$dK_t = (-\delta K_t + i_t) dt$$

Investment is subject to quadratic adjustment costs so that the cumulative company earnings net of investment costs are given by:

$$d\Pi_t = dD_t - \left(p i_t + \frac{\chi}{2} (i_t^2) \right) dt$$

¹³Unfortunately, in this case it appears very difficult to obtain closed form solutions for prices, investment etc.

where χ is a constant controlling the significance of adjustment costs and p is the cost of capital. It will be useful to define p as a fraction (\tilde{p}) of $\frac{\bar{f}}{r+\delta}$ so that $p = \tilde{p}\frac{\bar{f}}{r+\delta}$. The assumption of adjustment costs that are independent of K_t has the benefit of allowing reasonably tractable solutions, however it comes at the cost of breaking down the equivalence between average and marginal "q".¹⁴ In the appendix I show how one can generalize (at least qualitatively) the results of this section to a setup with linear homogenous adjustment cost technologies of the sort usually employed in the empirical literature.

3.1.2 Agents and Signals

There are two continuums of risk neutral agents that I will call type A and type B agents. Risk neutrality is convenient both in terms of simplifying the calculations and abstracting from considerations related to spanning etc. In addition to the earnings process (6) both agents observe two signals that I will denote signal s^A and signal s^B . These signals evolve according to:

$$\begin{aligned} ds_t^A &= f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^A \\ ds_t^B &= f_t dt + \sigma_s dZ_t^B \end{aligned}$$

where $(dZ_t^A, dZ_t^B, dZ_t^f, dZ_t^D)$ are standard mutually orthogonal Brownian motions.

Agents have heterogenous perceptions about the informativeness of the various signals. Agents in group A have the correct beliefs, while agents in group B assume that the innovations to the s_t^B process are more and the innovations to the s_t^A process less informative than they actually are. In particular they believe that the signals evolve according to:

$$\begin{aligned} ds_t^A &= f_t dt + \sigma_s dZ_t^A \\ ds_t^B &= f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^B \end{aligned}$$

This setup is meant to capture situations when there has been a regime shift in the economic environment and agents disagree about the informativeness of certain signals because they cannot use past data in order to measure the correlation between various signals with the underlying productivity process. For instance one could interpret the above informational setup as a situation where new signals (e.g. the amount of web site hits of a newly formed dot.com company) arise and analysts are unsure as to how important they are for future profitability.

Finally, as in the previous section, I assume that there is a continuum of agents of each type and the total wealth of each group is infinite.¹⁵

¹⁴ This assumption has been made by several authors in the literature. See e.g. Abel and Eberly (1994) and the references therein (especially footnote 19)

¹⁵ This assumption is made by both Harrison and Kreps (1978) and Scheinkman and Xiong (2003) and is used to drive prices to the reservation value of each group.

In the appendix I establish an approximate filter for this setup.¹⁶ In particular I show that the posterior mean \widehat{f}_t^A of agent A 's beliefs about f , evolves approximately according to:

$$d\widehat{f}_t^A = -\lambda \left(\widehat{f}_t^A - \bar{f} \right) dt + \sqrt{\frac{\widehat{f}_t^A}{\bar{f}}} \sigma_f dB_t^A \quad (8)$$

where dB_t^A is an appropriate linear combination of the innovation processes $\left(ds_t^A - \widehat{f}_t^A dt \right)$, $\left(ds_t^B - \widehat{f}_t^A dt \right)$, $\left(\frac{dD_t}{K_t} - \widehat{f}_t^A dt \right)$ with the property that the volatility of dB_t^A is 1. Similarly for agent B :

$$d\widehat{f}_t^B = -\lambda \left(\widehat{f}_t^B - \bar{f} \right) dt + \sqrt{\frac{\widehat{f}_t^B}{\bar{f}}} \sigma_f dB_t^B \quad (9)$$

where dB_t^B is an appropriate linear combination of the innovation processes $\left(ds_t^A - \widehat{f}_t^B dt \right)$, $\left(ds_t^B - \widehat{f}_t^B dt \right)$, $\left(\frac{dD_t}{K_t} - \widehat{f}_t^B dt \right)$ with the property that the volatility of dB_t^B is 1.¹⁷

A quantity that will be central for what follows is the disagreement process. In the appendix I show that agent A perceives that the process:

$$g_t^A = \widehat{f}_t^B - \widehat{f}_t^A$$

which captures her disagreement with agent B can be approximated by a simple OU process:

$$dg_t^A = -\rho g_t^A dt + \sigma_g dW_g^A$$

with $\langle dB_t^A, dW_g^A \rangle = 0$. The situation for agent B is symmetric. She perceives that the process:

$$g_t^B = -g_t^A$$

evolves approximately as an OU process with increments orthogonal to dB_t^B . Obviously, knowing \widehat{f}_t^A, g_t^A allows one to compute $\widehat{f}_t^B = \widehat{f}_t^A + g_t^A$. Thus, if one is only interested in posterior means, the pair (\widehat{f}_t^A, g_t^A) summarizes the entire belief structure. The appendix presents these approximations in detail and discusses their accuracy. Conditional on these approximate dynamics for the belief processes the rest of the analysis is exact. It is also important to note that one could have chosen any belief structure dynamics as long as it implies disagreement between the agents in some states of the world. The present one was chosen only for tractability reasons.

¹⁶In contrast to Scheinkman and Xiong (2003) I assume a square root process for f_t in (7) instead of a standard OU process in order to guarantee positivity of f_t . This allows one to put a lower bound on f_t which is convenient for some of the proofs in the appendix. The downside of this assumption is that filtering becomes much more involved and I have to settle for an approximate filter, the properties of which seem to be very good.

¹⁷Intuitively agent B will underweight signal A and overweight signal B and thus she will choose a different combination of the innovation processes.

3.2 Equilibrium Investment, Trading and Pricing

3.2.1 Homogenous Beliefs

I start with the simplest possible case¹⁸ where every agent is of type A and accordingly everyone agrees on the interpretation of the signals. One could also think of the discussion in this section as the solution to the investment problem of a long termist risk neutral decision maker who will never resell her shares. The goal will be to maximize

$$P_t = \max_{i_s} E^A \int_t^\infty e^{-r(s-t)} d\Pi_s \quad (10)$$

which can be rewritten as¹⁹

$$P_t = \max_{i_s} E^A \int_t^\infty e^{-r(s-t)} \left(f_s K_s - p i_s - \frac{\chi}{2} (i_s^2) \right) ds$$

One can further rewrite the above objective as²⁰

$$P_t = \max_{i_s} E^A \int_t^\infty e^{-r(s-t)} \left(\widehat{f}_s^A K_s - p i_s - \frac{\chi}{2} (i_s^2) \right) ds$$

This is a problem of exactly the same form as the ones considered in Abel and Eberly (1994),(1997). The solution to this problem (obtained in the appendix) is given by

Proposition 1 *The solution to (10) is given as:*

$$P_t(\widehat{f}_t^A, K_t) = \left(\frac{\bar{f}}{r+\delta} + \frac{\widehat{f}_t^A - \bar{f}}{r+\delta+\lambda} \right) K_t + \left(C_1 (\widehat{f}_t^A - \bar{f})^2 + C_2 (\widehat{f}_t^A - \bar{f}) + C_3 \right) \quad (11)$$

for appropriate constants C_1, C_2, C_3 given in the appendix. Optimal investment is given by:

$$i_t = \frac{1}{\chi} (P_K - p) = \frac{1}{\chi} \left(\frac{\bar{f}(1-\bar{p})}{r+\delta} + \frac{\widehat{f}_t^A - \bar{f}}{r+\delta+\lambda} \right) \quad (12)$$

For $0 \leq \tilde{p} < \bar{p}$ (where \bar{p} is a constant given in the appendix) it can be shown that $P_f > 0$.

¹⁸These results in this section are fairly standard and the reader is referred for details to Abel and Eberly (1997).

¹⁹Throughout I will restrict attention to investment policies that satisfy the requirement:

$$E \left[\int_t^\infty e^{-r(s-t)} K_s dZ_s^D \right] = 0$$

which amounts to a standard square integrability condition on the allowed capital stock processes. Indeed in the present setup the capital stock turns out to be stationary and thus it is easy to verify this condition.

²⁰This is true since the objective is linear in the state and quadratic only in the control i_t . For details on such problems see Bertsekas (1995).

Exactly as in Abel and Eberly (1997), the equilibrium price is comprised of two components. The first is marginal "q" times the capital stock and the other term captures the rents to the adjustment technology or "growth options". The first term captures the expected net present value of profits that can be obtained with the existing capital stock i.e

$$P_K = \frac{\bar{f}}{r + \delta} + \frac{\hat{f}_t^A - \bar{f}}{r + \delta + \lambda} = E \left(\int_t^\infty e^{-(r+\delta)(s-t)} \hat{f}_s^A ds \right) \quad (13)$$

The second term captures the "rents to the adjustment technology" or "growht options", i.e. the value of being able to adjust the capital stock in the future:

$$\left(C_1 \left(\hat{f}_t^A - \bar{f} \right)^2 + C_2 \left(\hat{f}_t^A - \bar{f} \right) + C_3 \right) = \frac{1}{2\chi} E \left(\int_t^\infty e^{-r(s-t)} (P_K(s) - p)^2 ds \right) \quad (14)$$

The first term is clearly increasing in \hat{f}_t^A , while the second term is also increasing in \hat{f}_t^A . This means that not only does a higher belief about current profitability increase the expected profits in the future, it also increases the value of growth options. This is because it becomes more likely that large investments will need to be undertaken in the future and thus the technology to adjust the capital stock becomes more valuable. Small adjustment costs (i.e. low values of χ) will tend to increase the value of the adjustment technology. This is intuitive: the less it costs to adjust the capital stock, the more a company is able to invest (disinvest) and take advantage of temporary increases (decreases) in fundamentals $\left(\hat{f}_t^A \right)$.

3.2.2 Heterogenous beliefs: Optimal investment, trading, and equilibrium prices

This section discusses the recursion:

$$P = \max_{o \in \{A, B\}} \left(\sup_{i_s, \tau} E_t^o \left[\int_t^{t+\tau} e^{-r(s-t)} \left(dD_s - \left(pi_s + \frac{\chi}{2} i_s^2 \right) ds \right) + e^{-r(\tau-t)} P_{t+\tau} \right] \right) \quad (15)$$

As is shown in the appendix the crucial difficulty in dealing with this recursion is that it leads to a multidimensional optimal stopping problem. Fortunately, this problem can be solved explicitly. In the appendix I show the following result:

Proposition 2 *The solution to $P \left(\hat{f}_t^A, g_t^A, K_t \right)$ is given by²¹:*

$$\begin{aligned} P \left(\hat{f}_t^A, g_t^A, K_t \right) &= \left(\frac{\bar{f}}{r + \delta} + \frac{\hat{f}_t^A - \bar{f}}{r + \delta + \lambda} + 1\{g_t^A > 0\} \frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-|g_t^A|) \right) K_t + \\ &+ C_1 \left(\hat{f}_t^A + 1\{g_t^A > 0\} g_t^A - \bar{f} \right)^2 + [C_2 + n(-|g_t^A|)] \left(\hat{f}_t^A + 1\{g_t^A > 0\} g_t^A - \bar{f} \right) \\ &+ d(-|g_t^A|) + C_3 \end{aligned}$$

²¹Under some mild restrictions on the allowed parameters discussed in the appendix.

for functions $y_1(g_t^A), n(g_t^A)$ and $d(g_t^A)$ and a constant β . The functions $y_1(g_t^A), n(g_t^A)$ and $d(g_t^A)$ are integrals and linear combinations of appropriate confluent hypergeometric functions and are given in the appendix. The constants C_1, C_2, C_3 are identical to the ones obtained in Proposition 1. The optimal investment rule is given by:

$$i_t = \frac{1}{\chi} (P_K - p) = \frac{1}{\chi} \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} + \frac{\hat{f}_t^A - \bar{f}}{r + \delta + \lambda} + 1\{g_t^A > 0\} \frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-|g_t^A|) \right)$$

and the optimal stopping time for each investor $o \in \{A, B\}$ is to resell the asset immediately once $\hat{f}_t^o < \hat{f}_t^{\bar{o}}$, where $\bar{o} = A$ if $o = B$ and vice versa.

I organize the discussion of the results in two subsections. I first discuss some properties of the derivative of the equilibrium price w.r.t. K_t (commonly called "marginal q"), i.e:

$$P_K = \frac{\bar{f}}{r + \delta} + \frac{\hat{f}_t^A - \bar{f}}{r + \delta + \lambda} + 1\{g_t^A > 0\} \frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-|g_t^A|)$$

and then I discuss some properties of the rents to the adjustment technology, i.e.:

$$C_1 \left(\hat{f}_t^A + 1\{g_t^A > 0\} g_t^A - \bar{f} \right)^2 + [C_2 + n(-|g_t^A|)] \left(\hat{f}_t^A + 1\{g_t^A > 0\} g_t^A - \bar{f} \right) + d(-|g_t^A|) + C_3 \quad (16)$$

3.2.3 Some observations about marginal "q"

As might be expected from the introductory example discussed in Section 2 investment is unambiguously higher in the presence of speculation. Comparing marginal q in the presence of speculation to the equivalent expression in the presence of homogenous beliefs one observes an extra term, namely:

$$b(g_t^A) = 1\{g_t^A > 0\} \frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-|g_t^A|) \quad (17)$$

The term $\beta y_1(-|g_t^A|)$ is a positive term growing in expectation (instantaneously) at the rate of interest plus the rate of depreciation.²² I.e. it is a pure speculative "bubble" that arises endogenously. In contrast to "rational" bubbles that can grow indefinitely, this term is bounded. Moreover, one can determine its magnitude explicitly and speculative bubbles of this sort can exist even in finite horizon settings.²³

²²Formally, for $g_t^A < 0$ and any $T > t$ this term satisfies

$$y_1(g_t^A) = E(e^{-(r+\delta)(T \wedge \tau - t)} y_1(g_{T \wedge \tau}^A))$$

where:

$$\tau = \inf\{t : g_t^A \geq 0\}$$

and similarly for g_t^B .

²³This term is practically identical to the one obtained in Scheinkman and Xiong (2003) with the sole exception that the effective interest rate in the present setup is increased by the rate of depreciation. The reader is referred to that paper for a detailed discussion on the differences between speculative and rational bubbles.

Of course investment is inflated only if it is determined as part of shareholder value maximization. In this case investors are short termist and invest in order to increase resale value. It is interesting to see what would happen if investment only reacted to "long-run" fundamentals. Such a situation can arise if e.g. the company is run by a set of managers / shareholders who do not have frictionless access to the markets for whatever reason. For instance this group of managers / shareholders might be unwilling to sell its shares because it values control, or because it perceives that its shares might have a large non-linear effect on the price of the stock due to asymmetric information, or simply because there are vesting agreements that preclude sales of stock or finally for reasons related to capital gains taxes. If these managers/shareholders are of type A , then investment will continue to be given by (13). However, the stocks that are traded in the market will still contain speculative components and thus the link between "marginal q " (P_K) and investment will break down. I use this observation to develop tests in section 3.2.5.

A second observation is that marginal "q" (P_K) is now more volatile than the expression obtained in the case of homogenous beliefs. Applying Ito's Lemma to (17) and evaluating this expression at the stationary point ($g_t = 0$) one finds an increase in the volatility of q_t (compared to the homogenous beliefs case) of²⁴:

$$\frac{\sigma_g}{2(r + \delta + \lambda)}$$

In other words the volatility in marginal "q" (P_K) due to the presence of short sale constraints and heterogenous beliefs is increasing in the volatility of the disagreement process and decreasing in the interest rate (r), the rate of depreciation (δ) and the rate of convergence to long run fundamentals (λ).

A third observation concerns predictability. Marginal "q" no longer satisfies the relation:

$$q_t = E^A \left[\int_t^\infty e^{-(r+\delta)(s-t)} \hat{f}_s^A ds | \mathcal{F}_t \right]$$

and more importantly, it will no longer be the case that:

$$q_t = E^A \left[\int_t^{t+\Delta} e^{-(r+\delta)(s-t)} \hat{f}_s^A ds + e^{-(r+\delta)\Delta} q_{t+\Delta} | \mathcal{F}_t \right]$$

In the appendix I show that:

Proposition 3 q_t satisfies the relationship:

$$q_t = E^A \left[\int_t^{t+\Delta} e^{-(r+\delta)(s-t)} \hat{f}_s^A ds + e^{-(r+\delta)\Delta} q_{t+\Delta} | \mathcal{F}_t \right] + E^A \left[\int_t^{t+\Delta} e^{-(r+\delta)(s-t)} \frac{(r + \delta + \rho)}{r + \delta + \lambda} g_s^A 1\{g_s^A > 0\} ds | \mathcal{F}_t \right] \quad (18)$$

²⁴To derive this, apply Ito's Lemma to the expression $1\{g_t^A > 0\} \frac{g_t^A}{r+\delta+\lambda} + \beta y_1(-|g_t^A|)$ keeping terms that multiply the martingale parts. In the appendix I show that $\beta = \frac{1}{2(r+\delta+\lambda)y_1'(0)}$ and so $b(g_t^A)$ is differentiable everywhere, and accordingly there are no terms involving the local time of the process at 0. This in turn is a consequence of smooth pasting.

Defining:

$$Z(g_t^A; \sigma_g) = E^A \left[\int_t^{t+\Delta} e^{-(r+\delta)(s-t)} \frac{(r+\delta+\rho)}{r+\delta+\lambda} g_s^A 1\{g_s^A > 0\} ds | \mathcal{F}_t \right]$$

it can be shown that $Z_g > 0, Z_{\sigma_g} > 0$

These properties of q_t deserve some comment. The first term in (18) is the usual expression one obtains for marginal "q" in the traditional infinite horizon setting. It can easily be derived from formula (13). The second term (Z) is capturing the fact that returns in a setup with heterogenous beliefs and short sale constraints are predictable. The properties $Z_g > 0, Z_{\sigma_g} > 0$ suggest that this predictability will be strongest when the disagreement process is temporarily high and / or when the volatility in the disagreement process increases.

3.2.4 Some observations on growth options, stock prices and returns

The rents to the adjustment technology present a richer set of interactions between speculation, fundamentals and investment. This is to be expected. The ability to adjust the capital stock becomes more valuable when investment is increased due to speculation. This effect becomes magnified, when one takes into account that the differences in beliefs about fundamentals also affect the value of the adjustment technology. As a result investors speculate not only on the ability of the existing capital stock to generate profits in the future, but also on the ability of the company to leverage its value in the future by further increasing its capital stock. As is demonstrated in the quantitative exercises that follow, the effect of these "growth options" on prices can be large.

Applying Ito's Lemma to (16) one can establish the analogs of the results discussed for the case of marginal "q", i.e. excess volatility and predictability. However, it is more interesting to analyze the stock price directly. The following result is proved in the appendix:

Proposition 4 *The equilibrium price satisfies:*

$$\begin{aligned} P_t = & E^A \left[\int_t^{t+\Delta} e^{-r(s-t)} \left(\widehat{f}_s^A K_s - p i_s - \chi \frac{i_s^2}{2} \right) ds + e^{-r\Delta} P_{t+\Delta} | \mathcal{F}_t \right] + \\ & + E^A \left[\int_t^{t+\Delta} e^{-r(s-t)} \frac{(r+\delta+\rho)}{r+\delta+\lambda} g_s^A K_s 1\{g_s^A > 0\} ds | \mathcal{F}_t \right] \\ & + E^A \left[\int_t^{t+\Delta} e^{-r(s-t)} \left(\xi(g_s^A) + C g_s^A \left(\widehat{f}_s^A - \bar{f} \right) \right) 1\{g_s^A > 0\} ds | \mathcal{F}_t \right] \end{aligned} \quad (19)$$

for an appropriate function $\xi(g_s^A)$ and a constant $C > 0$. Denoting

$$\Xi = E^A \left[\int_t^{t+\Delta} e^{-r(s-t)} \left(\xi(g_s^A) + C g_s^A \left(\widehat{f}_s^A - \bar{f} \right) \right) 1\{g_s^A > 0\} ds | \mathcal{F}_t \right]$$

one can show that $\Xi_f, \Xi_{fg}, \Xi_{f\sigma_g} > 0$.

The first term in (19) is the standard recursive relation that connects profits and the price next period to the current price. The second term is the predictability due to marginal "q" that was analyzed in the previous section. The final term is the predictability due to speculation on the value of growth options. Both the second and third terms are positive. The last term is increasing in both g_t^A and \widehat{f}_t^A and moreover the cross partial derivative of the third term with respect to \widehat{f}_t^A and g_t^A is positive. This demonstrates the interaction between beliefs about "fundamentals" i.e. (\widehat{f}_t^A) and the differences in beliefs (g_t^A) which arises in the presence of investment. Predictability can be expected to be strongest in the present setup when *both* fundamentals and the divergence in beliefs are large. By contrast in the absence of investment the extent of predictability is independent of fundamentals. This makes it easier to link predictable variation in returns to variables that react to both fundamentals and speculation like the B/M. These issues are analyzed further in a quantification exercise that follows.

3.2.5 Testing if bubbles affect investment

In this section I will use the theory developed previously in order to derive the properties of some tests concerning bubbles and investment. I will derive the implications of the theory for certain standard statistical tests under alternative hypotheses.

The analysis will be focused mostly on marginal q and its relationship to investment. In the presence of bubbles marginal q is given by:

$$P_K = q_t = \left(\frac{\bar{f}}{r+\delta} + \frac{\widehat{f}_t^A - \bar{f}}{r+\delta+\lambda} + 1\{g_t^A \geq 0\}g_t^A + \beta y_1(-|g_t^A|) \right)$$

Where the two theories differ is to what extent investment reacts to P_K or not. According to H_0 , q theory is valid even in the presence of speculative premia and accordingly:

$$i_t = \frac{1}{\chi}(q_t - p) = \frac{1}{\chi}(P_K - p) \quad (20)$$

According to the alternative

$$i_t = \frac{1}{\chi}(q_t^F - p) \quad (21)$$

where $q_t^F = \frac{\bar{f}}{r+\delta} + \frac{\widehat{f}_t^A - \bar{f}}{r+\delta+\lambda} < q_t$ captures a rational "long run" valuation of marginal profits. One intuitive and straightforward test of the two theories is the following. Suppose one starts with a firm where beliefs are homogenous so that short sale constraints are initially irrelevant. Then, suppose that differences in beliefs arise so that the short sale constraints lead to the creation of speculative premia on the asset. If the company decides to conform with (20) the basic investment- P_K relationship will continue to hold, whereas under (21) investment should be overpredicted.

This idea is effectively behind Blanchard Rhee and Summers (1993). They examine whether an investment-q relationship estimated over roughly 90 years tends to produce negative residuals around periods when the stock market is most likely driven by a bubble. Moreover, they test if positive residuals are observed after these bubbles crash.

This idea is simple and intuitive. The main identifying assumption behind it, is that the researcher is able to identify periods of time or specific stocks where the prices are more likely to be driven by speculative components and not fundamentals. I employ such an empirical strategy in the next section.

An alternative approach is to take advantage of the excess volatility in P_K in the presence of speculation. As demonstrated in section 3.2.3, P_K is excessively volatile compared to long run fundamental marginal "q" as perceived by agent A. Compared to q_t^F , P_K is more volatile by:

$$\sigma_g \beta y_1'(g_t^A)$$

which-evaluated at $g_t^A = 0$ gives:

$$\frac{\sigma_g}{2(r + \delta + \lambda)}$$

This would introduce classical measurement error in a regression of investment on P_K . Accordingly, for companies whose stock contains speculative components, one should expect a biased downward estimate of q compared to companies without speculation. Actually, one can compute the magnitude of this bias. If one decomposes the variance of P_K into a fundamental and a nonfundamental component then the attenuation bias due to classical measurement error would be equal to:

$$\frac{\frac{\sigma_{q_f}^2}{(r+\delta+\lambda)^2}}{\frac{\sigma_{q_f}^2}{(r+\delta+\lambda)^2} + \left(\frac{\sigma_g}{2(r+\delta+\lambda)}\right)^2} = \frac{1}{1 + \frac{1}{4} \left(\frac{\sigma_g}{\sigma_{q_f}}\right)^2}$$

The attenuation bias increases with the ratio of the volatility in the disagreement process relative to the variability in q_t^F . As one approaches homogenous beliefs this volatility goes to 0 and the attenuation bias disappears.²⁵

Another straightforward test of the theory is to create some measure of q_t^F and compare its performance in a "horse" race with P_K . Such a method is devised in Abel and Blanchard (1986) and also used in Blanchard Rhee and Summers (1993). I use such an approach in section 4.3.4 as one of the robustness checks that I perform.

²⁵This basic idea is behind a number of papers that blame the poor performance of q theory on excessively volatile stock prices relative to some notion of long run fundamentals. For instance see Bond and Cummins (2001), or the survey of Chirinko (1993). Of course, this attenuation bias is only present if companies react only to long run fundamentals.

A final methodology is based on Euler equations.²⁶ This approach takes advantage of the predictability introduced into P_K by the relation (18). This methodology is very appealing from a theoretical viewpoint because it does not require a lot of assumptions apart from predictability in the variation of marginal "q" which is true for the model discussed. In particular, following essentially the same ideas as in Chirinko (1993) I show in the appendix that the following relationship holds if investment reacts to fundamentals only (irrespective of whether there are nonfundamental components in the price):

$$E \left[I_t - e^{-(r+\delta)} I_{t+1} - \frac{1}{\tilde{\chi}} \Pi_t + C | \mathcal{F}_{t-1} \right] = 0 \quad (22)$$

I_t denotes the change in the capital stock between $t-1$ and t , Π_t are the observed profits between $t-1$ and t divided by the capital stock, and $\tilde{\chi}, C$ are constants determined in the appendix. If -by contrast- investment reacts to speculation one can use the predictability of returns derived in the previous sections to show that:

$$E \left[I_t - e^{-(r+\delta)} I_{t+1} - \frac{1}{\tilde{\chi}} \Pi_t + C | \mathcal{F}_{t-1} \right] \geq 0$$

I give an explicit derivation in the appendix. An interesting implication of the results in section 3.2.3 is that one can make additional statements about the strength of the predictability as a function of the properties of the disagreement process. Moreover, one can pin down its sign. This gives additional predictions that can be fruitfully used in the cross section.

The appendix to this section also shows how to generalize the findings to arbitrary linear homogenous adjustment technologies and an arbitrary number of investor groups. The main advantage of doing so is that a) marginal and average q become equal, allowing one to obtain a measure of marginal q (P_K) through average q and b) the investment to capital ratio (in contrast to absolute investment) becomes a function of q. Moreover the observations about equation (22) continue to hold with $\frac{I_t}{K_{t-1}}$ replacing I_t . However, it seems difficult to obtain closed form solutions for prices in this case.

3.2.6 A basic quantification exercise

In this section I examine the ability of the model to produce quantitatively plausible magnitudes for q and the extent of predictability. The model has a number of parameters that can be classified in two categories: a) parameters that are mainly related to the underlying productive and adjustment technologies, b) parameters that are related to the beliefs of the rational agent and c) parameters that are related to the beliefs of the

²⁶Similar ideas to the ones discussed here are proposed by Chirinko and Schaller (1996). However in this paper predictability is assumed rather than derived from first principles. This makes it hard to say with confidence whether predictability is due to rational variations in the discount factor. At a theoretical level, without an explanation for why bubbles exist it becomes difficult to connect them to predictability as Chirinko and Schaller (1996) explain in Appendix A of their paper when they study rational bubbles.

irrational agent. The main parameter of interest is the degree of disagreement which is controlled by ϕ . Accordingly, the results are reported as a function of ϕ . The rest of the parameters are used in order to produce sensible first and second time series moments of returns, marginal q, average q, the investment to capital ratio and the dividends to price ratio in the absence of speculation. I chose the values $\delta = 0.1$, $r = 0.05$, $\lambda = 0.1$, $\bar{f} = r + \delta$, $\sigma = 0.25\bar{f}$, $\chi = 2$, $p = 0.6$, $\sigma_D = 0.5\sigma$, $\sigma_s = \sigma$.²⁷

Figure 2 depicts various quantities of interest. The top left panel allows one to "translate" levels of ϕ in terms of the disagreement ratio between the rational and the irrational agent. The disagreement ratio is constructed as the ratio of the standard deviation of the stationary distribution of g_t^A to the average standard deviation of the posterior belief distribution of the rational agent. For instance a ratio of 1 means that the standard deviation of the stationary distribution of the disagreement process is equal in magnitude with one standard deviation of the posterior beliefs of the rational agent.²⁸ The top right panel reports results of the following exercise: fixing the capital stock at its steady state value in the absence of disagreement, I compute average q and marginal q for various level of ϕ . I also report average q in the absence of speculation (i.e. if all agents are of type A) for comparison.²⁹ All quantities are evaluated at $g = 0$, $f = \bar{f}$, so that marginal q is equal to 1. As can be observed, the presence of disagreement increases both marginal "q" and average q or "Market to Book". The increase in marginal q is identical to the effect documented in Scheinkman and Xiong (2003) with the sole exception that r is replaced by $r + \delta$. The second effect is the significant increase

²⁷With these parameters I simulated the model to determine prices, investment and capital if all agents are rational and $\phi = 0$. I simulated 80 years of data dropping 10 years in order to enforce that initial values are drawn from the stationary distribution. The results are given in the following table

	B/M	Marg. Q	D/P	Returns
Mean	0.599	0.995	0.038	0.054
Std. Dev.	0.148	0.264	0.021	0.181

To compare, the study of Kothari and Shanken (1997) reports an average for B/M of 0.69 with a standard deviation of 0.22 whereas the dividend yield is given as 0.036 with a standard deviation of 0.014. The average value of marginal q and the average value of returns are predetermined by construction at 1 and 0.05 by the choices of r and \bar{f} . The simulations unsurprisingly produce values very close to these parameters. Kothari and Shanken (1997) report an equal weighted return of 15.3 and a value weighted return of 9.4 % with standard deviations of 38.8 and 23.9 % respectively. However these returns are not real returns. Adjusting for an average annual inflation of 3.27% from 1926 to 1990 and taking into account the volatility of inflation produces real returns close to the ones reported in the simulation. Moreover the chosen values imply that regressions of the investment to capital ratio on average q return a value of 0.078 which corresponds to the values that are obtained in the empirical section.

²⁸In practical terms this implies that the rational agent will not be able to tell with 95% confidence that the irrational agent is wrong "most" of the time.

²⁹One might be puzzled why average q increases with ϕ in the absence of speculation. This is because ϕ - besides controlling disagreement- also tightens the confidence intervals of the rational agent. However, this effect is of second order. Moreover it can be completely avoided if one were to also modify σ_s with ϕ in order to keep the variance of posterior beliefs of the rational agent constant.

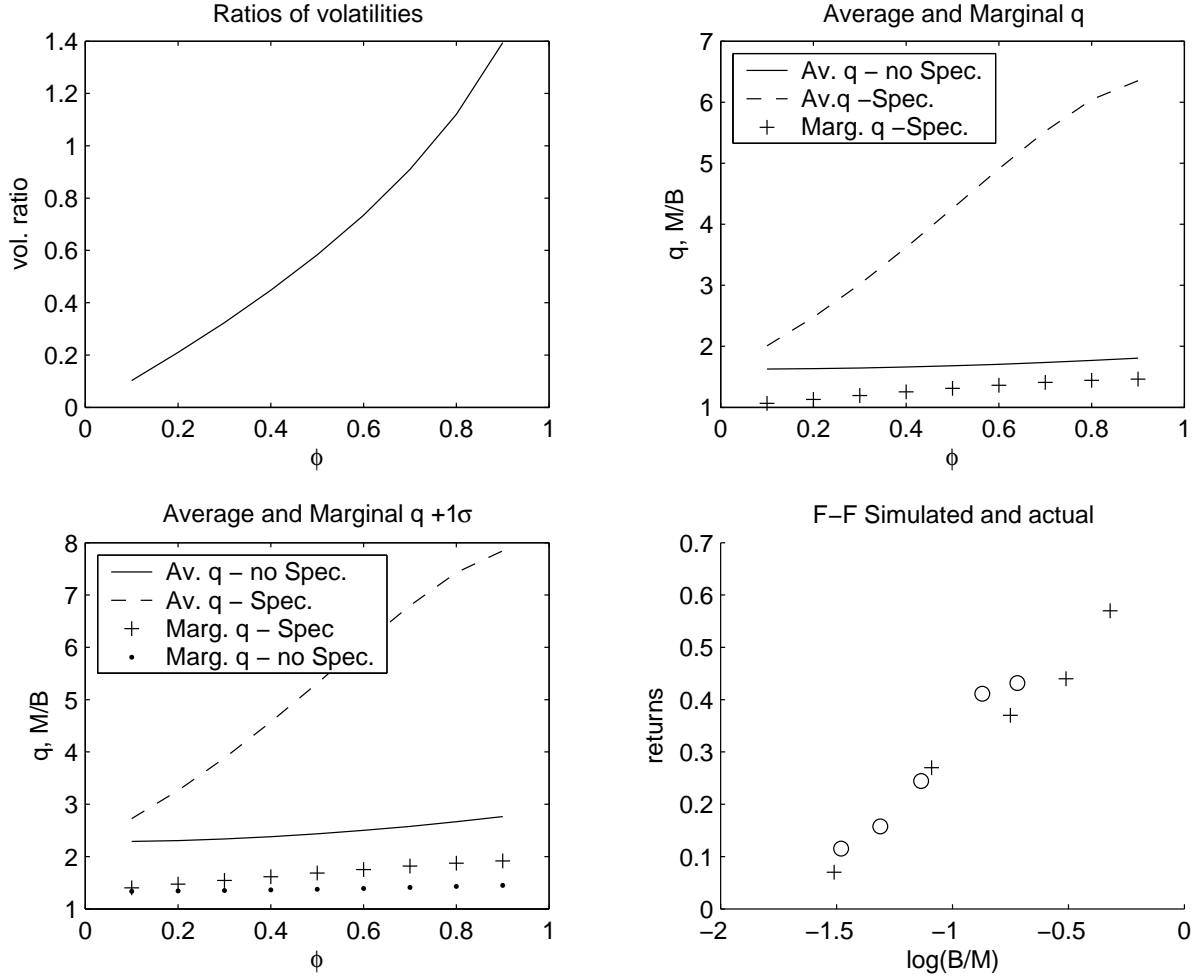


Figure 2: Behaviour of various quantities in the model: The top left panel depicts the "disagreement" ratio as a function of ϕ . The numerator of this ratio is the standard deviation of the stationary distribution of the disagreement process g_t . The denominator is the (average) standard deviation of the beliefs of the rational agent. The top right panel depicts average and marginal q in the presence and absence of speculation. In all cases the capital stock is fixed at its stationary value in the absence of speculation. The fundamentals (f) and the disagreement process (g) are evaluated at the stationary values $f = \bar{f}, g = 0$. The bottom left panel repeats the same exercise as the top right panel with the sole exception that f is evaluated at one positive standard deviation above its mean \bar{f} . The bottom right figure simulates a sample of 2300 companies over 27 years to have a similar setup as Fama and French (1992). The crosses denote the 5 F-F portfolios with the lowest B/M as reported in p.442 of their paper adjusted for an annual inflation rate of 7.2% between 1963 and 1990. The circles indicate simulated values. For 75% of the companies $\phi = 0$ whereas for the rest $\phi = 0.9$. The rest of the parameters are: $\delta = 0.1, r = 0.05, \lambda = 0.1, \bar{f} = r + \delta, \sigma = 0.25\bar{f}, \chi = 2, p = 0.6, \sigma_D = 0.5\sigma, \sigma_s = \sigma$

in average q or market to Book. In this example, if all agents share homogenous rational beliefs, marginal q is 1 and average q is about 1.6. When heterogenous beliefs and speculation enter the picture marginal q is increased mildly (not more than 50 percent) but the rents to the adjustment technology (or growth options) are increased substantially.^{30 31}. The bottom left panel repeats the above exercise when fundamentals are at one positive standard deviation, i.e. $f = \bar{f} + \sigma^{st}$ where σ^{st} is the stationary standard deviation of f . This picture demonstrates that effects are amplified when fundamentals are strong. In summary, growth options form a non-negligible source of valuations in the presence of speculation. The values produced are in line with the relatively large values of market to book found in the data during speculative episodes.

The bottom right panel reports results on the ability of the model to produce both reasonable book to market ratios and predictability. I simulated paths of 2300 companies over 27 years assuming that all companies are identical, except for ϕ ³². The returns of these companies and the Book to Market ratios were simulated under the assumption that for 75% of the companies there is no disagreement ($\phi = 0$) whereas for 25% agents disagree with $\phi = 0.9$. With these assumptions I calculated equal weighted returns for 10 portfolios formed on Book to market as described in Fama and French (1992). The bottom right panel of figure 2 plots the resulting returns and compares them to the results reported in Fama and French (1992)³³. I focused only on the portfolios with the 5 lowest B/M ratios since this paper is concerned with overpricing. The results suggest that the present model can produce degrees of predictability very similar to the ones observed in the data. Fama-MacBeth regressions produce coefficients of roughly 0.38 compared to 0.5 reported in Fama and French (1992). Moreover, a number of alternative parameter values seem to suggest that one needs to assume that only a small number of companies needs to be overpriced in order to explain the data. However the disagreement in these companies needs to be relatively large.

4 Empirical Evidence

4.1 Overview

In this section of the paper I use the theoretical results obtained in order to test the most central predictions of the model. The presence of a short sale constraint should increase valuations for the underlying assets, while the behavior of investment will depend on the shareholders' ability and willingness to sell their shares and take advantage of the speculative components in asset prices. The H_0 hypothesis in this section will be

³⁰It can be shown that this picture is independent of the level of χ , since I normalize by the steady state capital stock.

³¹Of course as time passes average q will fall because the capital stock will start to increase

³²Once again a number of initial years (prior to the 37 that form the simulation study) was dropped to make sure that initial capital stocks, fundamentals and disagreement are drawn from the stationary distribution.

³³To compare the results I subtracted a 7.2% annual (or 0.6% monthly) from the results in Fama and French (1992) in order to compute real returns

that investment reacts to both "long run" fundamental components and short run speculative components. The alternative (H_1) is that it reacts only to the former.

What is difficult, in order to operationalize this notion is to disentangle shocks to fundamental marginal "q" and shocks to the resale premium. If one can identify a negative shock to the resale premium (due for instance to a relaxation of the short sale constraint) then under H_0 the basic "q"-type relation should be able to accurately predict a drop in investment. Under (H_1) there should be no drop in investment and the "q" relationship would falsely predict one. Similarly, during the buildup of speculative components in prices the basic "q" type relationship should overstate the increase in investment under (H_1).

Various studies have used proxies to disentangle fundamental from non-fundamental sources of valuation, such as breadth of ownership, discretionary accruals, equity issuance, etc.³⁴ A problem with this approach is that most of these indications of mispricing could be explained in an alternative way that is not related to the speculative component of prices. They provide indirect ways of controlling for mispricing.

In this paper I adopt a more direct approach to identifying shocks to the speculative component of stock prices. In particular, I test "q" theory on a set of companies for which data on the existence of a market and the costs to market participants of short selling a company's stock is publicly available.

The study focuses on the 1920's, because short selling was done via a public market, and data on short selling was available in the Wall Street Journal. This data set was collected by Jones and Lamont (2002). I describe this data set in more detail in the next section. The interest will be focused on an episode during the beginning of 1926 when 32 industrial companies were added to what was called "the loan crowd"³⁵, i.e. a market for borrowing and lending stock. As is explained in Jones and Lamont (2002) the most likely reason for the introduction into the loan crowd was that market participants considered these companies as particularly overpriced compared to their fundamental value.

This introduction can be interpreted as a relaxation of the short sale constraint. Accordingly, in line with the theory developed, one should expect to observe a drop in the stock price of the companies after their introduction into the "loan crowd", independent of whether H_0 or the alternative holds. Results of this nature were established in Jones and Lamont (2002). I reconfirm their results for the subsample that I consider and provide additional evidence concerning the "q" ratio of these companies.

Then I study investment. The drop in the price that is observed for most of the companies in the subsample can be reasonably interpreted as the effect of a correction to "overpricing". I then proceed to compare the behavior of investment for these companies. There are at least two easily testable implications.

³⁴See e.g. Polk and Sapienza (2002), Gilchirst et. al. (2002)

³⁵Even though there were additions later on to this list most of them came after the August of 1930, a period where the U.S. enters the great depression.

First, I run standard regressions of the form:

$$\frac{I_{i,t}}{K_{i,t-1}} = \alpha_i + \delta_t + \beta q_{i,t-1} + \varepsilon_{i,t} \quad (23)$$

for "control" companies that have been in the loan crowd for some time and the cost of short selling them is low³⁶. I compare the results of these regressions to the equivalent regressions for the companies of the "treatment" group. What one should observe under H_0 is that the coefficients of β are the same up to sampling error. Otherwise, the coefficient β should be biased downward because of the measurement error type problem analyzed in subsection 3.2.3.

Another simple observation is that under (H_1) one should expect the residuals in regressions of the type (23) to be negative on average immediately prior to inclusion and significantly positive thereafter. These intuitive and simple implications of the theory are tested in detail in the sections that follow. Sections (4.3.4) and (4.4) run the tests described in section 3.2.5 to check if the long-termist hypothesis H_1 can be rejected.

4.2 Data

The data for the empirical study come from various sources. Data for the loan crowd market are from Jones and Lamont (2002).³⁷ They collected data from end of the month *Wall Street Journals (WSJ)*. The data collected provide information on rebate rates from 1919-1933. The list of companies that were on the WSJ list was very small in 1919 (less than 20 industrial companies) and expanded in 4 waves described in Jones and Lamont (2002). The first wave occurred in 1926 when 32 industrial companies were added to the list along with a number of railroad companies that I ignore in this study. The other waves came after August 1930, a period during which the U.S. economy was going into a deep recession.

The *Wall Street Journal* reports the names of the companies along with the so-called rebate rates. The difference between a rebate rate and the prevailing interest rate is the cost of short selling. This is illustrated by an example given in Jones and Lamont (2002): suppose A lends shares to B and B sells the stock short. When the sale is made the proceeds go to A and not to B. A is effectively using collateral to borrow and thus must pay interest to B. At the end of the loan A repays cash to B and B returns the shares to A. The rate of interest received by B is called the rebate rate or "loan" rate. Accordingly, stocks with 0 rebate rates are the most expensive to short whereas stocks with positive and high rebate rates are relatively inexpensive to short. In other words, the rebate rate is the price that brings the loan market back to equilibrium.

To form a control group for the study I selected only companies that were in the loan crowd before 1926 and were trading at rebate rates above 2% in February of 1926. This yielded 15 companies that form the

³⁶Companies in the control group are comprised of all companies that were in the loan crowd at least 2 years before 1926 and their rebate rates were at least 2% in February 1926.

³⁷For details of this data set the reader is referred to that paper.

control group. For these companies I assume that short sales were possible and relatively inexpensive since 1919.

The "treatment" group is comprised of companies that enter the loan crowd from January 1926 to June 1926 with the vast majority entering at the end of February. There are 32 industrial companies that meet these criteria³⁸. Virtually all of these companies enter at a rebate rate of 0 which captures either very high shorting demand or limited supply of short selling. Conceivably, it also captures conservatism with the creation of a new market. In either case I adopt the interpretation given in Jones and Lamont (2002) namely, that these are stocks that were considered as particularly overvalued and thus the demand for shorting the stock exceeded the amount the normal broker could accommodate "in house"³⁹.

To produce measurements of "q" I could not rely on standard sources of data like COMPUSTAT, since there is no widely available, electronic source of balance sheet data going back to 1918. Accordingly, data was hand-collected from *Moody's Manuals of Investments* for the years 1918-29.⁴⁰ A particularly difficult problem with balance sheet data from the 20's is that companies did not have to comply with any particular form of data reporting. Especially detailed profit and loss data are typically unavailable. An additional problem is that most companies did not start reporting depreciation and accumulated depreciation reserves until 1926. This introduces measurement error in the investment data which -fortunately- is the left hand side variable. To create the time series for "q" I used the same procedure as Nicholas (2003). This procedure is basically the standard Lindenberg and Ross (1981) procedure adapted to the typical balance sheet data of the 1920's. q is computed as the product of common shares outstanding times the price of common shares plus the market value of preferred stock⁴¹ plus the (book) value of debt. The replacement cost of capital is

³⁸Some of the companies were dropped for one of the following reasons: a) Data could not be found for at least 3 years prior to February 1926 b) the fiscal year ended more than 3 months before or after December 31. c) the company was a pure holding company d) there was an important merger e) Most of the company's balance sheet was undepreciated goodwill. With these selection criteria I tried to address issues related to IPO's, non-synchronous data, issues related to governance and measurement error in q. In contrast to common practice I did not winsorize the data in any way, (i.e. by truncating q) because it is precisely the large variations in q caused by speculation that form the object of this study. The final sample consisted of 25 companies. For 3 of them I was able to construct q but could not find profit data for some of the years 1922-26. To safeguard that the results do not capture IPO related issues, I ran all of the main regressions on the subset of companies that I had data reaching back to at least 1918. The results were unaltered.

³⁹I.e. by using the accounts of one customer who is long the stock to lend it to another who wants to short sell.

⁴⁰I am indebted to Tom Nicholas for providing a data set that contained balance sheet data on some of the companies investigated.

⁴¹I follow Tom Nicholas (2003) here and determine the market value of preferred stock as if it were a perpetuity discounted with Moody's Average yield. This approach is dictated by data availability. To check if this introduces any significant measurement error, I looked at the price of preferred stock for a few companies that I could find data on preferred stock and computed q with actual prices for preferred stock. The estimate of q was practically unaffected.

determined by the usual Lindenber and Ross (1981) type recursion:

$$k_{i,t}^{rc} = k_{i,t-1}^{rc} \left[\frac{1 + \dot{p}_t}{(1 + \rho)(1 + \delta)} \right] + (NAV_{i,t}^{BV} - NAV_{i,t-1}^{BV})$$

where NAV_t^{BV} is the net asset value of physical capital (Plant, Equipment, and Property).⁴² This was the only variable related to physical capital consistently provided for all companies. \dot{p}_t is the inflation rate obtained from the *Historical Statistics of the United States: 1790-1950*. ρ and δ are the rate of technological obsolescence and the depreciation rate respectively and were set to 0. This choice was dictated by the fact that NAV_t^{BV} already includes depreciation. The inventories were computed at book value whereas liquid assets were computed as the difference of total assets and the sum of the book value of plant, equipment and property and inventories.

Investment was hard to compute accurately for many of the firms under consideration. By a basic accounting identity it is the case that:

$$I_{i,t} = D_{i,t} + NAV_{i,t}^{BV} - NAV_{i,t-1}^{BV}$$

where $D_{i,t}$ is the accounting depreciation of the assets during year t and $I_{i,t}$ is gross investment. The above relationship can be rewritten as

$$\frac{I_{i,t}}{NAV_{i,t-1}^{BV}} = \frac{D_{i,t}}{NAV_{i,t-1}^{BV}} + \frac{NAV_{i,t}^{BV} - NAV_{i,t-1}^{BV}}{NAV_{i,t-1}^{BV}}$$

As long as $\frac{D_{i,t}}{NAV_{i,t-1}^{BV}}$ is given as a company specific constant plus some error that is orthogonal to q , i.e.

$$\frac{D_{i,t}}{NAV_{i,t-1}^{BV}} = c + \varepsilon_{i,t}, \quad E(\varepsilon_{i,t} | q_{1...T}) = 0$$

then this induces classical measurement error. However, investment is the left hand side variable so that consistency of the estimated parameters is not affected, only their confidence intervals.⁴³

For some regressions a variable that I label profits is also used. This variable refers to accounting profits after interest and depreciation, $\Pi_{i,t}$, that were reported consistently for most companies. Unfortunately, cash flow variables could not be constructed because depreciation was not reported for most companies. The variable that I call profit rate is defined as $\pi_{i,t} = \frac{\Pi_{i,t}}{K_{i,t-1}}$.⁴⁴

Stock price and capitalization data were obtained from *CRSP* for the months following December of 1925 whereas the *Commercial and Financial Chronicle* was used for stock price data prior to December 1925.

⁴²The algorithm was initialized with $k_{1918}^{rc} = NAV_{1918}$, or setting NAV equal to the first available observation year if data could not be found for 1918.

⁴³To check the influence of measurement error on the results, I ran the investment regressions on a subset of companies where depreciation rates were available and so I could compute investment accurately. The results were practically identical, suggesting that the measurement error is indeed classical, i.e. orthogonal to the regressors.

⁴⁴Unfortunately, separate sales and cost data were not reported for most companies and as a result I cannot address effects of imperfect competition in the usual way that this is done in the literature. Measurement error in the profit rate is partially taken care of in the section on Euler equations by estimating everything with instruments and allowing for fixed effects.

Control Group

	Obs	Mean	S.D.	5%	10%	25%	50%	75%	90%	95%
investment	152	0.076	0.221	-0.077	-0.046	-0.007	0.024	0.078	0.213	0.434
q	167	1.075	1.197	0.377	0.436	0.567	0.683	0.914	2.212	3.875
profits	150	0.207	0.335	-0.030	0.020	0.045	0.092	0.197	0.599	0.858

Treatment Group

	Obs	Mean	S.D.	5%	10%	25%	50%	75%	90%	95%
investment	228	0.080	0.213	-0.078	-0.053	-0.013	0.022	0.116	0.264	0.535
q	254	1.250	0.914	0.382	0.477	0.708	1.012	1.499	2.373	2.839
profits	219	0.392	0.585	-0.010	0.022	0.093	0.195	0.455	0.907	1.811

Table 1: Table of Summary Statistics sorted by treatment and control group. 5%, 25% etc. correspond to the respective quantiles of the distribution.

4.3 Results

4.3.1 Summary Statistics

Table 1 gives some summary statistics of the data. The profit rates of the companies have different distributional properties. Companies in the control group have relatively less dispersed profit rates with a lower mean than the companies in the treatment group. The companies in the treatment group also have a higher and more volatile q compared to the ones in the control group. Both of these observations conform well with the setup of the theoretical model: one would expect a higher variability in the profit rate to leave room for diverging opinions and accordingly cause average q to be more volatile. At first glance there are no obvious differences in the distribution of the investment to capital ratio.

The companies under consideration are relatively large. Companies in the control group belong to the two highest capitalization deciles of CRSP, whereas companies in the treatment group are slightly smaller with the median company in the 7th CRSP capitalization decile.

4.3.2 The behavior of q and excess returns

A central prediction of the theory developed earlier is that the presence of short sale constraints will lead to "overpricing" (irrespective of whether investment reacts to it or not).

Figure 3 gives a visual impression of such an effect. It depicts the average first difference in "q" year

	(1)	(2)	(3)
	V-weight	Eq-weight	$R - R_{size}$
NEWQ(1)	-0.039 (0.014)	-0.023 (0.013)	-0.032 (0.012)
NEWQ(2)	-0.023 (0.006)	-0.015 (0.007)	-0.014 (0.007)
Observations	1147	1147	1147

Table 2: Monthly abnormal returns for companies in the treatment group. NEWQ(1) and NEWQ(2) are dummies that take the value 1 if the return is observed in the first quarter of introduction to the loan crowd and 0 otherwise. NEWQ(2) is defined similarly for the second quarter. A separate beta type model is estimated for each stock with 32 monthly returns. The first column contains results when the index is taken to be the Value weighted CRSP index and the second column contains results for the Equal weighted CRSP index. The third column matches stocks by CRSP capitalization decile and contains results from a regression of this difference on a constant and the two dummies described above. The standard errors are computed with a heteroskedasticity robust covariance matrix that allows for clustering by month.

by year for companies in the treatment and the control group. The average first difference in q is identical for both types of companies until 1923. From that point on, companies in the treatment group start having large positive first differences compared to companies in the control group. 1926 presents a structural break. Companies in the treatment group have a large negative adjustment. To the contrary, companies in the control group have a positive first difference in q in 1926. The difference between groups of the yearly difference in q is -0.314 with a standard error of 0.138 for the year 1926. This drop is large both in economic and statistical terms.

Excess returns can help in testing the overpricing hypothesis statistically. The usual case-study methodology of studying the excess returns of stocks around a particular "event" presents one major difficulty. First and most importantly, CRSP starts in January of 1926, so that one cannot run regressions to determine the "betas" of the stocks on the market before their introduction and I am forced to estimate these "betas" from subsequent observations. Table 2 presents regression results for the model:

$$R_{it} - r_t = a_i + \beta_i(R_{Mt} - r_t) + \gamma 1\{NEWQ1\} + \delta 1\{NEWQ2\} + \varepsilon_{it}$$

where $(R_{Mt} - r_t)$ is the (excess) return on a market wide index, $R_{it} - r_t$ is the excess return of security

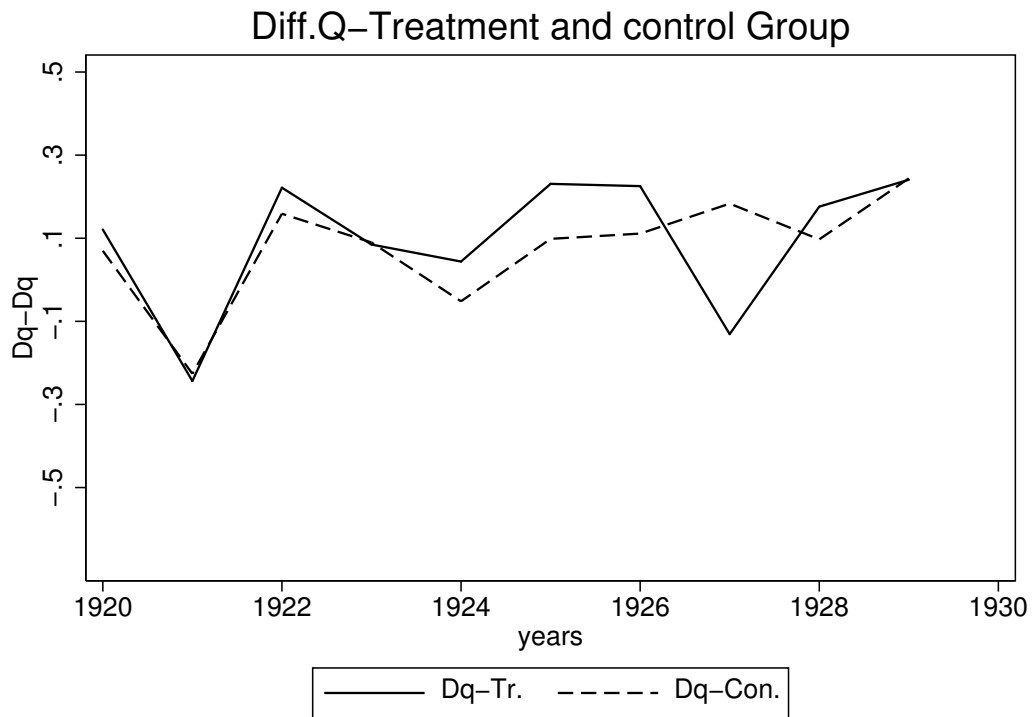


Figure 3: Plot of average first differences in "q" for companies in the treatment and the control group. The solid line denotes companies in the treatment group and the dashed line denotes companies in the control group. q is evaluated at the beginning of each period.

i at time t and the dummy variable $1\{NEQ1\}$ is 1 if the observation belongs to the first quarter in which the stock has been introduced into the market, and $1\{NEWQ2\}$ if the observation belongs to the second quarter. In other words these dummies are capturing the average abnormal return in the months following the introduction of the stocks into the "loan crowd". Columns (1) and (2) show an economically very significant drop in the holding period return in the 2 quarters following the introduction. In column (1) I use the CRSP Value weighted index in order to control for market-wide effects whereas in column (2) I use the equal weighted index. After a stock is introduced into the loan crowd an average -3.9% (monthly) abnormal return can be expected in the first quarter and a -2.3% in the subsequent quarter. To make sure that this is not just a size-related effect column (3) matches the returns of the companies in the sample with the portfolio returns of the CRSP capitalization decile in which they belong. In other words, I construct $R_{it} - R_{Capit}$ and regress this magnitude on a constant and the dummies described above. In all cases the results are very similar varying only in the strength of the effect.

4.3.3 Investment and q

This section studies the relationship between investment and q. Figure 4 depicts the comovement between average first differences in (beginning of period) q and investment for companies in both the treatment and the control group. The only thing that can be said is that the link between investment and q is not apparently different in any way between the two groups. Investment seems to follow both the upturns and downturns of q for companies in the treatment group. Moreover q co-moves with investment even during periods where one would suspect that the stock prices are driven primarily by non-fundamental forces.

Tables 3 and 4 present some formal econometric tests. Table 3 shows results of simple regressions of investment on "q" for various subgroups. Column (1) estimates a regression of investment on beginning of period q allowing for an individual fixed effect and a time fixed effect. The first column runs this regression on all the data in the sample whereas the second column restricts attention to companies in the control group. The third and fourth columns run the same regressions on companies in the treatment group pre and post 1926. The first two rows correspond to different methods of removing individual fixed effects. The first row eliminates individual effects by estimating them out (fixed effects regression) while the second row eliminates fixed effects by first differencing. The third row estimates a fixed effects median regression.

The fixed effects and first differences estimator produce similar results for all the subgroups suggesting that measurement error in q^{45} (due to e.g. mismeasurement of the replacement cost of capital) is not very

⁴⁵See e.g. the results in Grilliches and Hausman (1985) on measurement error in panel data.

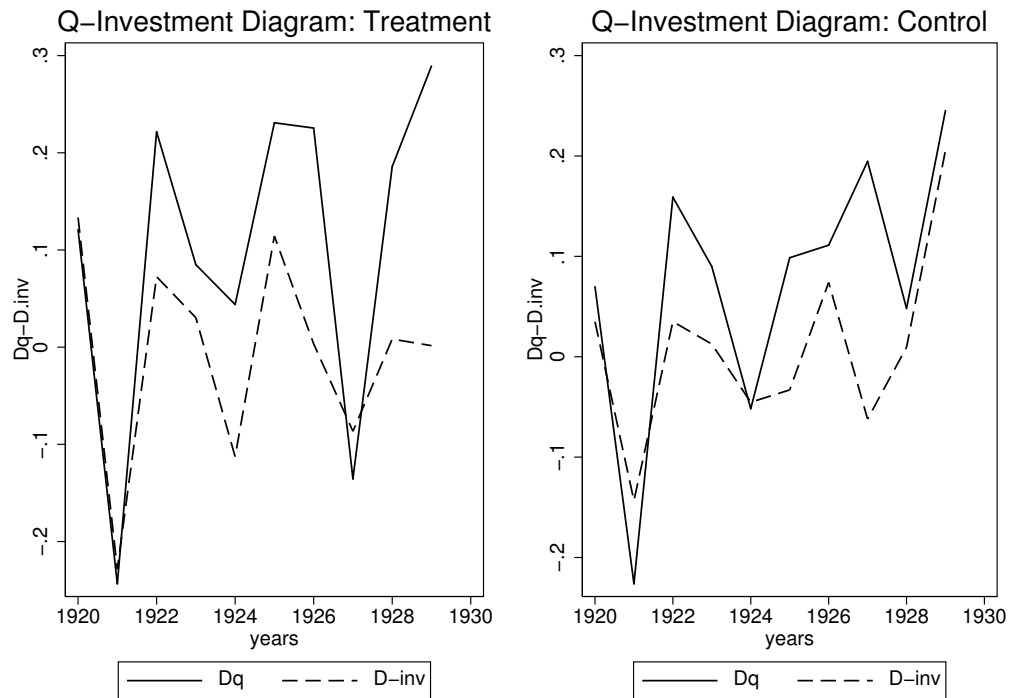


Figure 4: The left panel is a plot of average first differences in investment and q for companies in the treatment group. The solid line denotes average first differences in q and the dashed line average first differences in investment. The right panel depicts the same magnitudes for companies in the control group. q is evaluated at the beginning of the period.

important.⁴⁶ ⁴⁷ The standard errors are wide since the amount of data is very limited. q is significant in the first differences specification if one includes the entire set of companies and is also significant (for both specifications) for the companies in the treatment group prior to 1926. The point estimates are somewhat surprising. They are substantially larger than in the usual Compustat sample.⁴⁸ One potential explanation for this is that most companies are large, industrial stocks so that problems related to financial constraints, intangibles etc. become less prevalent. The point estimates are very large for companies in the treatment group prior to 1926. This suggests that companies do not distinguish between the sources of variation in q . Else, the estimates on q in this regression should be downward biased.

Row (3) in Table 3 and Figure 5 demonstrate some distributional properties of the error term. Row (3) estimates median regressions for subgroups. The estimates are roughly comparable for all subgroups, suggesting that the large estimates of " q " in the fixed effects (or the first differences) specification for companies in the treatment group are driven by a skewed error distribution. This is confirmed by a look at Figure 5 which plots residuals of the fixed effects regression for the two subgroups. This picture reveals two patterns. First the median residual is roughly the same for companies in the treatment and the control group. Second, the distribution of the error term is shifted to the right for companies in the treatment group for the years 1924 and 1925. This suggests that a number of companies adjusted to market based " q " and possibly in a non-linear way, not captured completely by the simple linear q model.

Table 4 contains results on interactions of q with year and treatment effects. Under H_0 one should expect all columns to not be significantly different than 0. Under the alternative the first two columns should be significantly negative. No matter how they are estimated, the interaction effects in the first two columns are positive, suggesting that one cannot reject the base hypothesis that investment reacts to both fundamental and nonfundamental sources of " q ". If instead of interactions of q with treatment and year dummies one uses simple interactions of year and treatment dummies the coefficients prior to 1926 remain positive and become negative thereafter for companies in the treatment group, which again supports H_0 .

I also estimated the model on companies in the treatment group that could be characterized as represen-

⁴⁶Moreover it suggests that the errors satisfy a strict exogeneity condition ($E(\varepsilon_{it}|q_{i1...T}) = 0$) not just a sequential exogeneity assumption ($E(\varepsilon_{it}|q_{i1...t-1}) = 0$). I tested for this directly by including one lead of q in the fixed effects specification. The coefficient was both economically and statistically insignificant. This suggests an interpretation of the errors in the investment regression as adjustment cost shocks. See e.g. Chirinko (1993) and Chirinko and Schaller (1996).

⁴⁷One caveat is in order. If there is correlation in the measurement error then first differences and fixed effects could be producing the same answer even though measurement error is present and as a result coefficients are downward biased. To address this I also estimated the adjustment cost parameter using Euler relations in section 4.4 which produced similar results to the ones reported here.

⁴⁸Abel and Eberly (2003) give estimates of 0.03 and 0.02 for the fixed effects and the first difference regression. The highest estimates for the linear model are produced by using analyst forecasts as instruments. The number they obtain for this specification is 0.11 very close to the numbers reported here.

	(1)	(2)	(3)	(4)
	All	Control	Tr. -pre 26	Tr. - post 26
q-FE	0.084 (0.047)	0.073 (0.084)	0.335 (0.161)	0.094 (0.109)
q-FD	0.112 (0.050)	0.1 (0.133)	0.322 (0.152)	0.077 (0.064)
q-Med.	0.06 (0.015)	0.094 (0.060)	0.076 (0.122)	0.03 (0.119)
Observations	359	142	132	60

Table 3: Results of regressions of investment on beginning of period q . Time and individual fixed effects are included but not reported. The first line contains the results of the fixed effects regression, whereas the second line eliminates fixed effects by first differencing. The last line is a median regression with fixed effects. The columns correspond to the subgroups. The first group includes all companies, the second only companies in the control group. The third and fourth columns report results for the treatment group pre 1926 and post 1926. Standard errors for the fixed effects and the first differences are computed with a robust covariance matrix allowing for clustering by company. For the median regression standard errors, a bootstrap procedure is used.

tative of the "high-tech" companies of the time⁴⁹ (mostly automobile related companies). The motivation behind this estimation is simple: there is increased uncertainty (and hence room for disagreement) about the fundamentals of companies in emerging sectors making short selling constraints more relevant and overvaluation more likely. In addition the automobile sector of the time was characterized as "speculative" by most financial publications⁵⁰. Accordingly, under H_1 this should be the sector in which one would expect to see a heavily downward biased q . Running investment- q regressions in first differences in this subset confirms the previous findings since the estimate on q remains at 0.19, well above the estimate for the control group.⁵¹

⁴⁹I chose American Brake Shoe and Foundry, Simmons Co., Nash Motor Cars, Hudson Motor Cars, Mack Trucks and American Locomotive as a sample of companies that were active in the emerging industries of the time.

⁵⁰(such as the Standard Trade Statistics, a predecessor of S&P)

⁵¹Moreover, to safeguard that the results on companies in the treatment group do not capture phenomena related to IPO's I ran the regressions on the subset of companies in the treatment group for which I could find stock prices in the Commercial and Financial Chronicle at least back to 1919. The coefficients on q were roughly equal to the ones reported for all companies in the treatment group.

	(1)	(2)	(3)
	Pre-1926	24-25	27-28
FE	0.097 (0.047)	0.094 (0.051)	-0.018 (0.034)
FD	0.157 (0.070)	0.06 (0.042)	0.023 (0.045)
Med.	0.004 (0.024)	0 (0.020)	-0.016 (0.022)
Observations	359	359	359

Table 4: Results of regressions of investment on beginning of period q and various interaction terms. Time and individual fixed effects are included but not reported. The first column reports results on an interaction dummy that is equal to q if the company is in the treatment group and the year of observation is prior to 1926. The second and third columns are defined similarly. The first line reports results of the fixed effects regression, whereas the second line eliminates fixed effects by first differencing. The last line is a median regression with fixed effects. Standard errors for the fixed effects and the first differences are computed with a robust covariance matrix allowing for clustering by company. For the median regression standard errors, a bootstrap procedure is used.

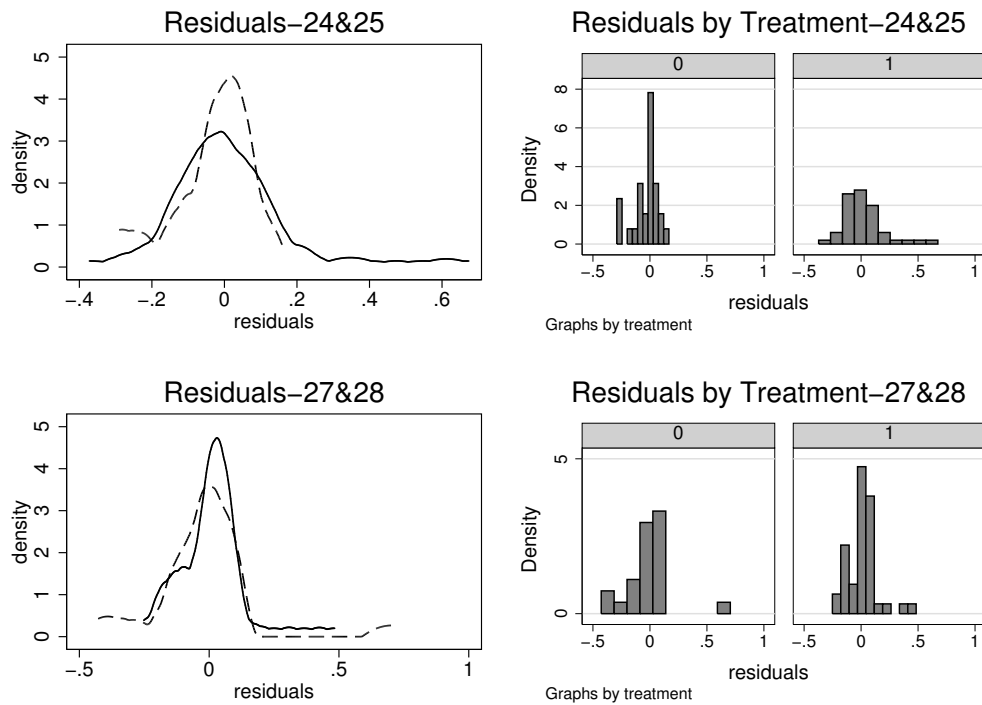


Figure 5: This picture presents the residuals of the (fixed effects) investment on beginning of period q regressions for the years 24 and 25 (top panel) and 27 and 28 (bottom panel). The left figure on each panel is the (kernel-smoothed) density of the residuals whereas the right figure is a histogram (10 bins) of the residuals. The solid line in the left figures corresponds to the residuals for the treatment group whereas the dashed line depicts residuals for the control group. Similarly a 0 in the right figures denotes residuals in the control group and a 1 denotes residuals in the treatment group.

4.3.4 Fundamental q, profits and investment

In this section and the next I run some robustness checks. In particular, I investigate whether the alternative hypothesis (H_1) can be rejected, (namely that the results obtained are attributable to profits and "fundamental" q). By constructing a measure for fundamental q one can also indirectly test the identifying assumption of the previous section, namely that most of the variation in q for companies in the treatment group comes from non-fundamental sources. Ideally, one would like to obtain some measure of fundamental q from analysts' forecasts on company profitability. Bond and Cummins (2001) propose such a method based on I/B/E/S forecasts. Similar data are unfortunately not available for the 1920's. Accordingly, I will use a "brute" force approach to create fundamental q from reported profits.

In particular I use the methodology in Abel and Blanchard (1986) to determine fundamental q for each company. I run a 2x2 first order VAR of company profits and q on lagged company profits and q for the entire sample, assuming that the coefficients are the same for all the companies in the sample.⁵² I then use the estimated coefficient matrix along with a linear approximation to the infinite horizon expression for marginal q to construct a new measure of fundamental q⁵³. I used a separate discount factor for each company. To determine the weighted cost of capital for each company (from the perspective of a long termist investor) I used the CAPM in conjunction with the betas estimated in section 4.3.2 and then created a weighted cost of capital by using an interest rate of 4% for the debt of the company an interest rate of 7% for preferred stock and the remaining share of the capital structure I weighted at the cost of equity implied by the CAPM assuming a market wide expected return for common stock of 10%. Depreciation was taken to be 9%.⁵⁴

Roughly speaking this new measure of q is meant to operationalize the notion that fundamental q is the expected sum of discounted marginal profits which are (roughly) equal to $\pi_s = \frac{\Pi_s}{K_s}$ for linear homogenous technologies. In order to create expectations for the future profit rates, one uses a predictive VAR approach. Then the dynamics of the process are used to create "long run" expectations.

An obvious concern with such a procedure is its accuracy. In particular one could be worried that the estimate of fundamental q obtained in this way would be contaminated by severe measurement error which might make it very difficult to test any hypothesis of interest. A check for this is provided by running a

⁵²I capture the presence of individual heterogeneity by including a fixed effect in each regression of the VAR. This creates a difficult estimation problem, known in the literature as the dynamic panel data problem. The problem arises because the time series dimension is very short in order to invoke standard asymptotic theory. Thus the estimates of the intercepts will be biased. I estimated the coefficients of the VAR with both standard fixed effects and the Arellano and Bond methodology. Even though the coefficients produced by the VAR were somewhat different, in both cases they led to the same conclusions about the role of fundamental "q". In this section I concentrate on the results for the fixed effects regression.

⁵³For details of this procedure see Abel and Blanchard (1986)

⁵⁴I also used a flat discount factor of 0.84 for all companies and varied the required return on the market between 7 and 12%. The results were almost identical to the ones reported here for variable discount factors suggesting that the results are not very sensitive to the specific assumptions one makes about returns etc.

simple regression of first differences in market based q on first differences of fundamental q for the different subgroups of companies. At the very least, one would expect actual q and the constructed measure of fundamental q to co-move closely for companies in the control group. Similarly one would expect the two measures to show disparities for companies in the treatment group prior to 1926. The results of this regression are given in columns (1)-(2) of table 5. Roughly half of the variation in q ($R^2 = 0.43$) is captured by the constructed measure of fundamental q for companies in the control group. The performance of this regression for companies in the treatment group prior to 1926 is -as expected- worse ($R^2 = 0.08$), suggesting that q is driven mostly by non-fundamental sources.

The next 4 columns of table 5 present horse races between fundamental " q " and market based q . Column (3) presents results for the treatment group prior to 1926. Time effects are included, but not reported. Individual fixed effects are eliminated by estimating all equations in first differences. The estimate for market based q is practically the same as that in Table 3, and the estimates on fundamental q are statistically insignificant. Column (4) runs the same regression with lagged profit rates instead of the constructed measure of fundamental q . The motivation for this regression is the following: if one assumes that profit rates follow a first order AR(1), then fundamental q would be just a scalar multiple of the lagged profit rate. Under H_1 this should be the only significant variable. Once again column (4) shows that H_0 cannot be rejected for companies in the treatment group whereas H_1 can. In fact, if one dropped time fixed effects (a Wald test confirms that they are jointly insignificant), then the coefficient on market based q becomes highly significant, whereas the coefficient on fundamental q and lagged profits remain insignificant. These results are reported in columns (5) and (6). Column (7) runs a regression with fundamental q , individual fixed effects and time effects for all observations and includes an interaction between a treatment dummy, the 1925 date effect, and market based q . Under H_1 this coefficient should be insignificant. However, the coefficient on the interaction is significant, suggesting that H_1 can be rejected.

4.4 Euler Equations

I conclude with some Euler tests. This is an alternative robustness check, with the advantage of not requiring an estimate of fundamental q . The test in this section can be motivated by the discussion in section 3.2.5. In particular I will focus on testing the overidentifying restrictions embodied in the Euler relations discussed in 3.2.5. One disadvantage of this test is that its power is likely to be very small. The reason is intuitive. This test can only detect violations of the overidentification restrictions if predictability is strong, adjustment costs are small and the rest of the errors in the investment equation (sometimes called adjustment cost shocks) are relatively unimportant. To increase the power of the test, I will accordingly focus only on investment

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Dq-Tr.-pre 26	Dq-Con.	Inv-Tr.-pre 26		Inv-Tr.-pre 26		Inv.-All
Dq_f	0.39 (0.24)	2.54 (0.57)	0.184 (0.185)		0.13 (0.17)		0.221 (0.14)
Dq			0.302 (0.171)	0.31 (0.170)	0.457 (0.172)	0.467 (0.174)	
Lag Dprofs				0.085 (0.089)		0.060 (0.080)	
Inter25*Treat							-.167 (.128)
Inter25*Treat*Dq							.247 (.117)
Observations	79	112	79	79	79	79	273
R-squared	0.08	0.43	0.29	0.29	0.25	0.25	0.15

Table 5: This table presents results on the relationship between "fundamental" q (q^F), market based q , and investment. The first two columns present results of a regression of first differences in actual "q" on first differences in fundamental "q". Column (1) presents these results for the treatment group prior to 1926 whereas column (2) presents these results for companies belonging to the control group. The next four columns present regressions of investment on actual, fundamental q , and the (lagged) profit rate. Columns (3)-(4) present these results for the treatment group prior to 1926 and columns (5)-(6) present the same results if one drops the (jointly insignificant) time effects. Column (7) introduces interactions between the 1925 time effect and treatment (Inter25*Treat) and interactions between the Treatment, the 1925 dummy, and beginning of period q (Inter25*Treat*Dq). The F-test that these variables are jointly 0 rejects at the 0.022 level. Robust standard errors are reported.

behavior of treatment companies around 1926⁵⁵.

Table 6 presents results on various Euler relationships. Columns (1) and (2) estimate simple Euler relations of the form

$$q_{i,t} = E \left[e^{-(r+\delta)} (\pi_{i,t} + q_{i,t+1}) | \mathcal{F}_t \right] \quad (24)$$

for companies in the treatment and the control group respectively. Instruments include (beginning of period) and lagged q , along with lagged and twice lagged investment to capital ratio and profit rate.⁵⁶ Even though the point estimates are similar and economically plausible, the test of overidentifying restrictions cannot reject for the case of control companies whereas it can reject for companies in the treatment group, suggesting the presence of predictability for companies in the treatment group.⁵⁷ Column (3) presents results from estimating the adjustment cost parameter from the Euler equation

$$E \left[\frac{I_{i,t}}{K_{i,t-1}} - e^{-(r+\delta)} \frac{I_{i,t+1}}{K_{i,t}} - \left(\alpha_i (1 - e^{-(r+\delta)}) + \zeta_t - e^{-(r+\delta)} \zeta_{t+1} + \frac{1}{\chi} \pi_{i,t} \right) | \mathcal{F}_{t-1} \right] = 0 \quad (25)$$

on companies in the control group in two steps. First I estimate $e^{-(r+\delta)}$ from equation (24). Then I substitute into (25) and take first differences to eliminate individual fixed effects. I use twice and three times lagged q , investment/capital ratios and profit rates as instruments and adjust the standard errors for the first step estimation error. This should be viewed as yet another robustness check of the results presented in section 4.3.3. The point estimate obtained is slightly larger than the ones obtained in table 3 and so are the standard errors, reflecting the fact that instrumental variables are used instead of OLS to estimate this parameter. More importantly, the overidentifying restrictions cannot be rejected. These results are to be expected. If companies in the control group are not overpriced then the regressions in 4.3.3 and (25) present just two alternative ways of estimating the adjustment cost parameters as shown in Chirinko (1993). Interestingly, if mispricing exists, does *not* affect investment and the investment decision is made in a rational way, then (25) should continue to hold. In the appendix I show how to construct a test based on this observation. I construct a new variable y as a linear combination of differences in investment and differences in the profit rate for companies in the treatment group as follows:

$$y \doteq \Delta \left(\frac{I_{i,1926}}{K_{i,1925}} \right) - e^{(r+\delta)} \Delta \left(\frac{I_{i,1925}}{K_{i,1924}} \right) + \frac{1}{\chi} e^{(r+\delta)} (\Delta \pi_{i,1925})$$

⁵⁵The increase in the power of the test comes from the fact that investment by a long-termist manager should not have reacted to the large fluctuations in the price during that period. However, if investment is short termist then one should be able to reject H_1 more easily precisely because of the large fluctuations in the price around this period.

⁵⁶To account for risk premia, I also regressed $\frac{\pi_{i,t} + q_{i,t+1}}{q_{i,t}}$ on the a beta estimated separately for each company (on post 1926 CRSP data) and a constant in a Fama-Macbeth fashion. Then I included variables like q that were in the information set of the agents at time t and checked if they are jointly significant in the usual Fama and French (1992) fashion. Variables at time t turned out to be significant for companies in the treatment group even after adjusting for a company specific beta.

⁵⁷It is unlikely that the rejection is driven by other sources of misspecification (e.g. non-linear homogenous technologies, misspecification of π_t etc) because in that case the test would reject for both the control and the treatment group.

	(1)	(2)	(3)	(4)
	q+profits	q+profits	Euler Est.	Euler-Test 1926
Lagged q	1.19 (0.065)	1.162 (0.049)		
Profits			0.192 (0.600)	
q-Lag 2				-0.779 (0.299)
Observations	184	125	82	22
Overid P-val:	0.01	0.43	0.71	
R-squared				0.27

Table 6: Euler Equation Tests and Tests of Overidentifying Restrictions. The first and second columns test the overidentifying restrictions embodied in (24) for the treatment and the control group respectively. The instruments are lagged and twice lagged q, profit rates and investment. The third column estimates the adjustment cost parameter for companies in the control group using (25). An efficient GMM procedure is used with a robust covariance matrix. The fourth column contains estimates for the parameter β in (89). Standard errors for this regression are computed with a robust covariance matrix and are adjusted for first step estimation error as described in the appendix.

This linear combination depends on parameters that can be consistently estimated from control group observations using (24) and from simple regressions of investment on q like the ones performed in section 4.3.3. If (24) holds for companies in the treatment group, β should be 0 in the following regression of y on q_{1924} and a constant under H_1 :

$$y \stackrel{d}{=} \Delta \left(\frac{I_{i,1926}}{K_{i,1925}} \right) - e^{(r+\delta)} \left(\frac{I_{i,1925}}{K_{i,1924}} \right) + \frac{1}{\chi} e^{(r+\delta)} (\Delta \pi_{i,1925}) = \beta q_{i,1924} + \zeta + \varepsilon_{i,1926} - \varepsilon_{i,1925} \quad (26)$$

Column (4) presents results on the parameter β in (26) estimated on companies in the treatment group. Standard errors are adjusted for two step estimation. The test rejects H_1 since β is significant.

In conclusion, no matter how one runs the test of H_0 vs. H_1 there seems to be evidence that the companies in the treatment group did react to market based q. This seems to be true despite the fact that a significant fraction of the variation in market based q seems to have been driven by non-fundamental sources.

5 Conclusion

This paper addressed the question of whether investment should be expected to react to "speculative" components in stock prices. The answer obtained in the theoretical section of the paper is affirmative. In the presence of short sales restrictions and heterogenous beliefs, investors can gain by either holding the asset and reaping its dividends, or by reselling it. From an individual point of view, both are sources of value. If one further assumes that the purpose of a company is to maximize shareholder value, then the conclusion that investment will react to both fundamental and speculative sources of value follows. One can however think of situations where this short-term reasoning is no longer optimal. Indeed, investors with holding horizons that are sufficiently long might choose to disregard speculation altogether.

This raises the empirical question: Which theory is supported by the data? The theoretical framework developed allowed a discussion of empirical tests in a unified framework. More importantly, it provided more concrete predictions about sources of predictability, excess volatility, and their strength depending on the dispersion of beliefs. These implications were tested in the framework of an episode in the 1920's. At that time, a number of companies were introduced into a market for lending stock (the so called "loan crowd"). The main finding of this paper is that the buildup of speculative components was followed by company investment as well.

One could argue that the incident studied is isolated. However, in many respects one can find many parallels between the '90's and the '20's. Large technological progress was followed by widely varying views on the growth potential of various sectors. The information superhighway was to the '90's what the automobile was to the '20's. The radio and the new advertising and distribution channels (shopping through catalogues - the birth of large retail stores) were in many respect analogous to on-line shopping in the '90's. These technological innovations in production and distribution fueled speculation in the stock market and reduced the hurdle rates for investment in both historical periods.

One direction that was left unexplored in this paper concerns active financial policy. In the model of this paper I did not introduce any frictions or financing constraints, that would lead to a role for active financial policy. However, as discussed in Stein (1996) the presence of financing constraints can provide a further argument why investment and q would be tightly linked even if decisionmakers are long-termist. It would be interesting to study the behaviour of long-termist decisionmakers in this intertemporal model under the assumption that they have to rely on equity to finance investment. It is likely that in such a setup one would be able to derive additional relationships between equity issuance, investment and returns that would allow one to disentangle whether investment reacts to q because of short termism or because of an active financing channel. I pursue this line in current research.

References

- [1] Andrew B. Abel. Optimal investment under uncertainty. *American Economic Review*, 73(1):228–33, 1983.
- [2] Andrew B. Abel and Olivier J. Blanchard. The present value of profits and cyclical movements in investment. *Econometrica*, 54(2):249–73, 1986.
- [3] Andrew B. Abel and Janice C. Eberly. A unified model of investment under uncertainty. *American Economic Review*, 84(5):1369–84, 1994.
- [4] Andrew B. Abel and Janice C. Eberly. An exact solution for the investment and value of a firm facing uncertainty, adjustment costs, and irreversibility. *Journal of Economic Dynamics and Control*, 21(4-5):831–52, 1997.
- [5] Andrew B. Abel and Janice C. Eberly. Investment and q with fixed costs: An empirical analysis. Working Paper, 2002.
- [6] Milton Abramowitz and Irene Stegun. *Handbook of Mathematical Functions*. Dover, New York, 1965.
- [7] Franklin Allen, Stephen Morris, and Andrew Postlewaite. Finite bubbles with short sale constraints and asymmetric information. *Journal of Economic Theory*, 61(2):206–29, 1993.
- [8] Malcolm Baker, Jeremy C. Stein, and Jeffrey Wurgler. When does the stock market matter? stock prices and the investment of equity dependent firms. *Quarterly Journal of Economics*, (3):203–218, 2003.
- [9] Malcolm Baker and Jeffrey Wurgler. The equity share in new issues and aggregate stock returns. *Journal of Finance*, 55(5):2219–57, 2000.
- [10] Steven A. Barnett and Plutarchos Sakellaris. A new look at firm market value, investment, and adjustment costs. *Review of Economics and Statistics*, 81(2):250–60, 1999.
- [11] Jonathan B. Berk, Richard C. Green, and Vasant Naik. Optimal investment, growth options, and security returns. *Journal of Finance*, 54(5):1553–1607, 1999.
- [12] Dimitri P. Bertsekas. *Dynamic Programming and Optimal Control*. Athena Scientific, 1995.
- [13] Olivier Blanchard, Changyong Rhee, and Lawrence Summers. The stock market, profit, and investment. *Quarterly Journal of Economics*, 108(1):115–36, 1993.
- [14] Olivier Jean Blanchard. Bubbles, liquidity traps, and monetary policy. comments on jinushi et al, and on bernanke. Working Paper, M.I.T., 2000.

- [15] Olivier Jean Blanchard and Stanley Fischer. *Lectures on macroeconomics*. MIT Press, Cambridge, Mass. and London, 1989.
- [16] Stephen Bond and Jason Cummins. Noisy share prices and the q model of investment. IFS Working Papers W01/22, Institute for Fiscal Studies, 2001.
- [17] Stephen R. Bond and Jason G. Cummins. The stock market and investment in the new economy: Some tangible facts and intangible fictions. *Brookings Papers on Economic Activity*, 0(1):61–108, 2000.
- [18] Ricardo J. Caballero and Mohamad L. Hamour. Speculative growth. Working Paper, M.I.T., 2002.
- [19] Robert S. Chirinko. Business fixed investment spending: Modeling strategies, empirical results, and policy implications. *Journal of Economic Literature*, 31(4):1875–1911, 1993.
- [20] Robert S. Chirinko and Huntley Schaller. Bubbles, fundamentals, and investment: A multiple equation testing strategy. *Journal of Monetary Economics*, 38(1):47–76, 1996.
- [21] Robert S. Chirinko and Huntley Schaller. Business fixed investment and "bubbles": The japanese case. *American Economic Review*, 91(3):663–80, 2001.
- [22] John H. Cochrane. Production-based asset pricing and the link between stock returns and economic fluctuations. *Journal of Finance*, 46(1):209–37, 1991.
- [23] John H. Cochrane. A cross-sectional test of an investment-based asset pricing model. *Journal of Political Economy*, 104(3):572–621, 1996.
- [24] Peter M DeMarzo. An extension of the modigliani-miller theorem to stochastic economies with incomplete markets and interdependent securities. *Journal of Economic Theory*, 45(2):353–69, 1988.
- [25] Jerome Detemple and Shashidhar Murthy. Equilibrium asset prices and no-arbitrage with portfolio constraints. *Review of Financial Studies*, 10(4):1133–74, 1997.
- [26] Eugene F. Fama and Kenneth R. French. The cross-section of expected stock returns. *Journal of Finance*, 47(2):427–65, 1992.
- [27] Eugene F. Fama and Kenneth R. French. Size and book-to-market factors in earnings and returns. *Journal of Finance*, 50(1):131–55, 1995.
- [28] Stanley Fischer and Robert C. Merton. Macroeconomics and finance: The role of the stock market. *Carnegie-Rochester Conference Series on Public Policy*, 21:57–108, 1984.
- [29] Wendell H. Fleming and H. Mete Soner. *Controlled Markov processes and viscosity solutions*, volume 25. Springer-Verlag, New York, 1993.

- [30] Eric Fournie, Jean-Michel Lasry, Jérôme Lebuchoux, Pierre-Louis Lions, and Nizar Touzi. Applications of malliavin calculus to monte carlo methods in finance. *Finance and Stochastics*, 3(4):391–412, 1999.
- [31] Simon Gilchrist, Charles P. Himmelberg, and Gur Huberman. Do stock price bubbles influence corporate investment. Working Paper, 2002.
- [32] Zvi Griliches and Jerry A. Hausman. Errors in variables in panel data. *Journal of Econometrics*, 31(1):93–118, 1986.
- [33] Sanford J. Grossman and Oliver D. Hart. A theory of competitive equilibrium in stock market economies. *Econometrica*, 47(2):293–329, 1979.
- [34] Sanford J. Grossman and Joseph E. Stiglitz. On value maximization and alternative objectives of the firm. *Journal of Finance*, 32(2):389–402, 1977.
- [35] J. Michael Harrison and David M. Kreps. Speculative investor behavior in a stock market with heterogeneous expectations. *Quarterly Journal of Economics*, 92(2):323–36, 1978.
- [36] Oliver D. Hart. On the optimality of equilibrium when the market structure is incomplete. *Journal of Economic Theory*, 11(3):418–43, 1975.
- [37] Fumio Hayashi. Tobin’s marginal q and average q: A neoclassical interpretation. *Econometrica*, 50(1):213–24, 1982.
- [38] Martin F. Hellwig. Bankruptcy, limited liability, and the modigliani-miller theorem. *American Economic Review*, 71(1):155–70, 1981.
- [39] Harrison Hong and Jeremy Stein. Differences of opinion, short sales constraints and market crashes. *Review of Financial Studies (forthcoming)*, 2002.
- [40] Andrew H. Jazwinski. *Stochastic Processes And Filtering Theory*. Academic Press, New York and London, 1970.
- [41] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer-Verlag, New York, 2 edition, 1991.
- [42] Ioannis Karatzas and Steven E. Shreve. *Methods of mathematical finance*, volume 39. Springer-Verlag, New York, 1998.
- [43] Ioannis Karatzas and Hui Wang. Utility maximization with discretionary stopping. *SIAM J. Control Optim.*, 39(1):306–329, 2000.

- [44] S.P. Kothari and Jay Shanken. Book-to-market, dividend yield, and expected market returns: A time-series analysis. *Journal of Financial Economics*, pages 169–203, 1997.
- [45] Owen A. Lamont. Investment plans and stock returns. *Journal of Finance*, 55(6):2719–45, 2000.
- [46] Owen A. Lamont and Charles M. Jones. Short sales constraints and stock returns. *Journal of Financial Economics (forthcoming)*, 2002.
- [47] Eric B. Lindenberg and Stephen A. Ross. Tobin’s q ratio and industrial organization. *Journal of Business*, 54(1):1–32, 1981.
- [48] Michael Magill and Martine Quinzii. *Theory of incomplete markets Volume 1*. MIT Press, Cambridge and London, 1996.
- [49] Edward J. Meeker. *Short Selling*. Harper and Brothers Publishers, New York and London, 1932.
- [50] Randall Morck, Andrei Shleifer, and Robert W. Vishny. The stock market and investment: Is the market a sideshow? *Brookings Papers on Economic Activity*, 0(2):157–202, 1990.
- [51] Stephen Morris. Speculative investor behavior and learning. *Quarterly Journal of Economics*, 111(4):1111–33, 1996.
- [52] Vasanttilak Naik. Asset prices in dynamic production economies with time-varying risk. *Review of Financial Studies*, 7(4):781–801, 1994.
- [53] Whitney K. Newey and Daniel McFadden. Large sample estimation and hypothesis testing. *Handbook of Econometrics, North Holland*, 4.
- [54] Tom Nicholas. Why schumpeter was right: Innovation, market power, and creative destruction in 1920s america. *The Journal of Economic History*, 63(4), 2003.
- [55] Bernt Øksendal. *Stochastic differential equations*. Springer-Verlag, Berlin, 5 edition, 1998. An introduction with applications.
- [56] Jacques Olivier. Growth-enhancing bubbles. *International Economic Review*, 41(1):133–51, 2000.
- [57] Christopher Polk and Paola Sapienza. The real effects of investor sentiment. Working Paper, Kellogg School of Management, 2002.
- [58] Earl D. Rainville, Phillip E. Bedient, and Richard E. Bedient. *Elementary Differential Equations*. Prentice Hall, Eighth Edition, 1997.

- [59] Jose Scheinkman and Wei Xiong. Overconfidence and speculative bubbles. *Journal of Political Economy* (*forthcoming*), 2003.
- [60] Andrei Shleifer. *Inefficient markets: An introduction to behavioral finance*. Oxford University Press, Oxford and New York, 2000.
- [61] Jeremy C. Stein. Rational capital budgeting in an irrational world. *Journal of Business*, 69(4):429–55, 1996.

6 Appendix: Proofs

6.1 Proofs for section 3.1.2

The essential difficulty in solving the filtering problem in section 3.1.2 consists in dealing with the non-linearity introduced by (7). If one were to replace (7) by:

$$df_t = -\lambda(f_t - \bar{f})dt + \sigma dZ_t^f$$

then one could replicate the arguments in Scheinkman and Xiong (2003) to show that the posterior mean process is given by:

$$\begin{aligned} d\hat{f}_t^A &= -\lambda(\hat{f}_t^A - \bar{f})dt + \frac{\phi\sigma_s\sigma + \gamma}{\sigma_s^2} (ds^A - \hat{f}_t^A dt) \\ &\quad + \frac{\gamma}{\sigma_s^2} (ds^B - \hat{f}_t^A dt) + \frac{\gamma}{\sigma_D^2} (dD - \hat{f}_t^A dt) \end{aligned} \quad (27)$$

for agent A and similarly

$$\begin{aligned} d\hat{f}_t^B &= -\lambda(\hat{f}_t^B - \bar{f})dt + \frac{\phi\sigma_s\sigma + \gamma}{\sigma_s^2} (ds^B - \hat{f}_t^B dt) \\ &\quad + \frac{\gamma}{\sigma_s^2} (ds^A - \hat{f}_t^B dt) + \frac{\gamma}{\sigma_D^2} (dD - \hat{f}_t^B dt) \end{aligned} \quad (28)$$

for agent B where γ is given as:

$$\bar{\gamma} = \frac{\sqrt{\left(\lambda + \phi\frac{\sigma}{\sigma_s}\right)^2 + (1 - \phi^2)\left(2\frac{\sigma^2}{\sigma_s^2} + \frac{\sigma^2}{\sigma_D^2}\right)} - \left(\lambda + \phi\frac{\sigma}{\sigma_s}\right)}{\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}} \quad (29)$$

Then the arguments in Scheinkman and Xiong (2003) can be used to arrive at the dynamics of the disagreement process g_t^A (respectively g_t^B):

$$dg_t^A = -\rho g_t^A dt + \sigma_g dW_g^A \quad (30)$$

where ρ, σ_g are given by:

$$\begin{aligned} \rho &= \sqrt{\left(\lambda + \phi\frac{\sigma}{\sigma_s}\right)^2 + (1 - \phi^2)\sigma^2\left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)} \\ \sigma_g &= \sqrt{2}\phi\sigma \end{aligned}$$

Moreover it is easy to show that dW_g^A is orthogonal to the innovations in $d\widehat{f}_t^A$ and dW_g^B is orthogonal to the innovations in $d\widehat{f}_t^B$. The reader is referred to Scheinkman and Xiong (2003) for details.

If one wants to account for the fact that the volatility in (7) is non-constant, an approximate way to proceed is by means of the extended Kalman filter, which is proposed in Jazwinski (1970)⁵⁸. This filter can be constructed by using a time varying γ (i.e. depending on the path of \widehat{f}_t^i) instead of the constant γ in formula (29):

$$\frac{d\gamma_t^i}{dt} = -2 \left(\lambda + \phi \frac{\sigma}{\sigma_s} \frac{\widehat{f}_t^i}{f} \right) \gamma_t^i + (1 - \phi^2) \sigma^2 \frac{\widehat{f}_t^i}{f} - (\gamma_t^i)^2 \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right), \quad i \in \{A, B\} \quad (31)$$

It is easy to verify that substituing $\widehat{f}_t^i = \bar{f}$ and requiring $\frac{d\gamma_t^i}{dt} = 0$ one can recover equation (29). In principle one could solve γ_t^i explicitly for a given path of \widehat{f}_t^i . Agent A 's beliefs about the mean of f would then be characterized (approximately) by the two-dimensional system (31), (27). For small λ , small σ and large $\frac{\sigma}{\sigma_s}, \frac{\sigma}{\sigma_D}$, γ_t^i will be given approximately by:

$$\gamma_t^i = \sqrt{\frac{\widehat{f}_t^i}{f} \frac{\left(\phi \frac{\sigma}{\sigma_s} \right)^2 + (1 - \phi^2) \left(2 \frac{\sigma^2}{\sigma_s^2} + \frac{\sigma^2}{\sigma_D^2} \right) - \phi \frac{\sigma}{\sigma_s}}{\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}}} \quad (32)$$

To see why, rewrite equation (29) to get:

$$\begin{aligned} d\gamma_t^i &= \left(-2 \left(\lambda + \phi \frac{\sigma}{\sigma_s} \frac{\widehat{f}_t^i}{f} \right) \gamma_t^i + (1 - \phi^2) \sigma^2 \frac{\widehat{f}_t^i}{f} - \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right) [\gamma_t^i - \bar{\gamma} + \bar{\gamma}]^2 \right) dt = \\ &= \left(-2 \left(\lambda + \phi \frac{\sigma}{\sigma_s} \frac{\widehat{f}_t^i}{f} \right) \gamma_t^i + (1 - \phi^2) \left(\frac{\sigma^2}{f} \right) \widehat{f}_t^i - \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right) \left((\gamma_t^i - \bar{\gamma})^2 + \bar{\gamma}^2 + 2\bar{\gamma}(\gamma_t^i - \bar{\gamma}) \right) \right) dt = \\ &= \left(-2 \left(\lambda + \phi \frac{\sigma}{\sigma_s} \frac{\widehat{f}_t^i}{f} + \bar{\gamma} \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right) \right) \gamma_t^i + (1 - \phi^2) \left(\frac{\sigma^2}{f} \right) \widehat{f}_t^i - \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right) \left((\gamma_t^i - \bar{\gamma})^2 - \bar{\gamma}^2 \right) \right) dt \end{aligned}$$

If one approximates $\frac{\widehat{f}_t^i}{f} \sim 1$ then the "solution" to this ODE is given by:

$$\gamma_t = \gamma_0 e^{-2wt} + \int_0^t e^{2w(\xi-t)} \left[(1 - \phi^2) \left(\frac{\sigma^2}{f} \right) \widehat{f}_\xi - \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right) \left((\gamma_\xi - \bar{\gamma})^2 - \bar{\gamma}^2 \right) \right] d\xi \quad (33)$$

where:

$$w = \lambda + \phi \frac{\sigma}{\sigma_s} + \bar{\gamma} \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right)$$

w is the factor by which past γ_t are weighted. For large t one can ignore the first term in (33). Moreover, if w is large, then one can basically ignore the effect of past γ and approximate the above integral (as $t \rightarrow \infty$) by

$$\gamma_t = \frac{\left(\left(\frac{\sigma^2}{f} \right) (1 - \phi^2) \widehat{f}_t + \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right) \bar{\gamma}^2 \right) - \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right) (\gamma_t - \bar{\gamma})^2}{2w}$$

⁵⁸Unfortunately, this filter does not make a claim to approximate the optimal non-linear filter, even though in applications it seems to have quite reasonable properties. Various sources discuss the properties and the efficiency of this filter for "small" noise.

Solving this quadratic equation and setting $\lambda = 0$ one gets (32). With this simplification the dimensionality of the problem can be reduced since now γ^i depends only on \hat{f}_t . Replacing (32) into (27) in the place of γ leads to the approximate belief processes:

$$d\hat{f}_t^A = -\lambda(\hat{f}_t^A - \bar{f}) dt + \sqrt{\frac{\hat{f}_t^A}{f}} \left[\frac{\phi\sigma_s\sigma + \tilde{\gamma}}{\sigma_s^2} (ds^A - \hat{f}_t^A dt) + \frac{\tilde{\gamma}}{\sigma_s^2} (ds^B - \hat{f}_t^A dt) + \frac{\tilde{\gamma}}{\sigma_D^2} (dD - \hat{f}_t^A dt) \right] \quad (34)$$

$$d\hat{f}_t^B = -\lambda(\hat{f}_t^B - \bar{f}) dt + \sqrt{\frac{\hat{f}_t^B}{f}} \left[\frac{\phi\sigma_s\sigma + \tilde{\gamma}}{\sigma_s^2} (ds^A - \hat{f}_t^B dt) + \frac{\tilde{\gamma}}{\sigma_s^2} (ds^B - \hat{f}_t^B dt) + \frac{\tilde{\gamma}}{\sigma_D^2} (dD - \hat{f}_t^B dt) \right] \quad (35)$$

where

$$\tilde{\gamma} = \frac{\sqrt{\left(\phi\frac{\sigma}{\sigma_s}\right)^2 + (1-\phi^2)\left(2\frac{\sigma_s^2}{\sigma_s^2} + \frac{\sigma_D^2}{\sigma_D^2}\right)} - \phi\frac{\sigma}{\sigma_s}}{\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}}$$

Since $\frac{ds^A - \hat{f}_t^A dt}{\sigma_s}$, $\frac{ds^B - \hat{f}_t^B dt}{\sigma_s}$, $\frac{dD - \hat{f}_t^A dt}{\sigma_D}$ are (standard) Brownian motions in the mind of agents of type A , it will be convenient to define the "total volatility" of \hat{f}_t^A (or \hat{f}_t^B) by

$$\sigma_f = \sqrt{\left(\frac{\phi\sigma_s\sigma + \tilde{\gamma}}{\sigma_s}\right)^2 + \left(\frac{\tilde{\gamma}}{\sigma_s}\right)^2 + \left(\frac{\tilde{\gamma}}{\sigma_D}\right)^2}$$

which leads to formulas (9) and (8). Moreover, as long as $\sqrt{\frac{\hat{f}_t^A}{f}}$, $\sqrt{\frac{\hat{f}_t^B}{f}}$ do not differ significantly from 1 (i.e. the volatility in \hat{f}_t^A, \hat{f}_t^B is relatively small) then (30) will continue to be a reasonable approximation to the disagreement process.⁵⁹ Figure (6) demonstrates the performance of these approximations for the quantitative calibration in section 3.2.6. The top left figure compares the solution to (31) (obtained by an Euler Scheme) to (32). There are two observations about the figures. First the two volatilities comove quite closely and second the posterior standard deviation (captured by γ_t^i) does not vary too much. These two observations help understand the next three panels. The top right panel is depicting the exact solution to the extended Kalman filter obtained by solving the two dimensional system (27) and (31) and the approximate filter obtained by using (34) instead. The two processes basically cannot be disentangled from each other, since they practically coincide. The bottom left panel depicts the performance of the extended Kalman Filter against the actual process f_t . It is easy to see that the extended Kalman Filter performs well in "recovering" the path of f_t . Finally, the bottom left panel depicts the difference in beliefs between agents A and B obtained from the approximate equation (30). Once again, the approximation is sufficiently good that one cannot disentangle the two processes, since they are practically identical. From these simulations it can be reasonably claimed that the approximation used is sufficiently accurate for all practical purposes.

⁵⁹One could derive an alternative approximation to this disagreement process by subtracting df_t^A from df_t^B and then approximating all terms to the first order. Such an approximation would yield something close to the OU process used here for reasonably small ϕ . For simplicity I chose the approximate OU process described in the beginning of this section to be able to compare the results to Scheinkman and Xiong (2003).

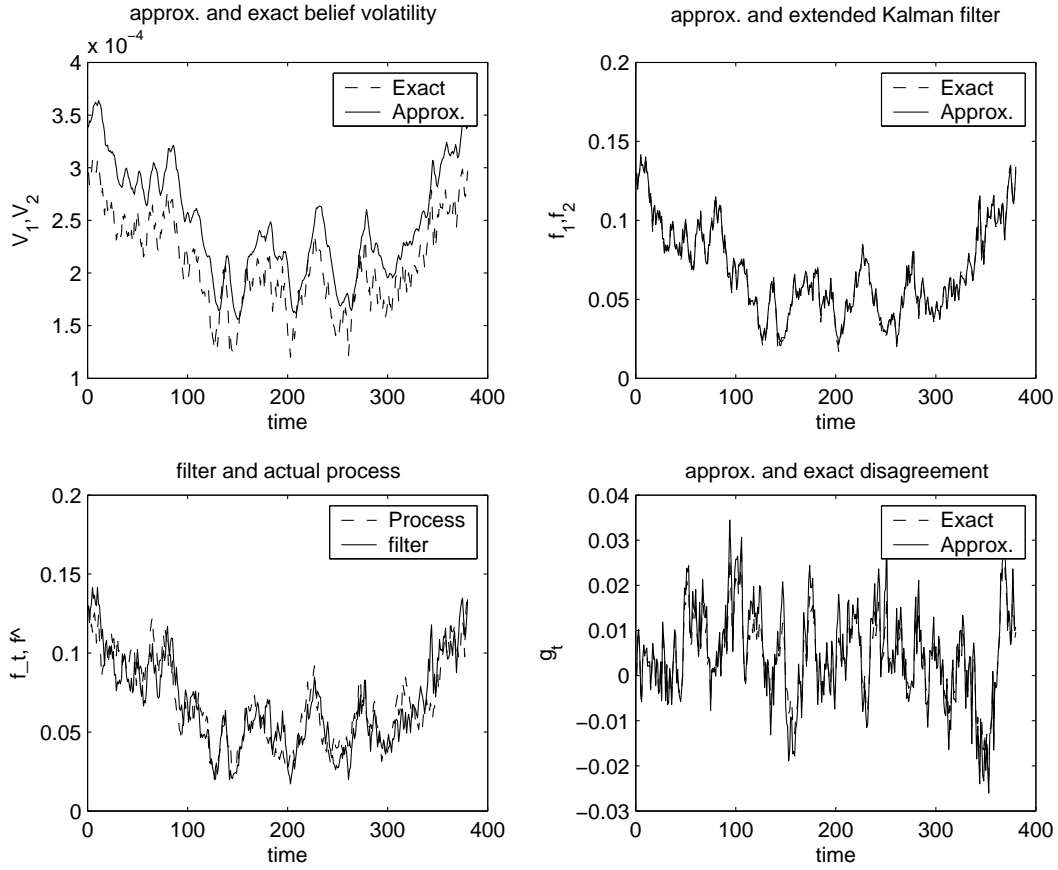


Figure 6: Simulation of a typical path of the model. The top left panel depicts the behaviour of the exact (under the extended Kalman Filter) and the approximate volatility of the posterior beliefs. The top right panel depicts the exact conditional mean process (under the extended Kalman Filter) and the approximation to the exact solution. The bottom left picture depicts the true f and the posterior mean \hat{f} as obtained by the extended Kalman Filter. Finally, the bottom right panel depicts the exact disagreement process (assuming both agents use the extended Kalman Filter) and the approximation proposed. The parameters for this example are the same as the ones used in Figure 2, namely: $\delta = 0.1$, $r = 0.05$, $\lambda = 0.1$, $\bar{f} = r + \delta$, $\sigma = 0.25\bar{f}$, $\sigma_D = 0.5\sigma$, $\sigma_s = \sigma$

6.2 Proofs for section 3.2.1

Proof. Proposition (1) I use a standard verification argument to verify that (11) provides the solution to (10). One can start by conjecturing a solution of the form:

$$P(\widehat{f}_t^A, K_t) = q^F(\widehat{f}_t^A) K_t + u^F(\widehat{f}_t^A) \quad (36)$$

Substituting this conjecture into the Hamilton Jacobi Bellman equation:

$$\max_{i_t} \left[\frac{1}{2} \sigma_f^2 \frac{\widehat{f}_t^A}{\bar{f}} P_{ff} - \lambda(\widehat{f}_t^A - \bar{f}) P_f + P_K (-\delta K + i_t) - rP + \widehat{f}_t^A K_t - p i_t - \frac{\chi}{2} (i_t^2) \right] = 0 \quad (37)$$

one arrives at the conclusion that (36) satisfies (37) if and only if the functions $q^F(\widehat{f}_t^A)$ and $u^F(\widehat{f}_t^A)$ solve the ordinary differential equations:

$$\frac{1}{2} \sigma_f^2 \frac{\widehat{f}_t^A}{\bar{f}} q_{ff}^F - \lambda(\widehat{f}_t^A - \bar{f}) q_f^F - (r + \delta) q^F + \widehat{f}_t^A = 0 \quad (38)$$

$$\frac{1}{2} \sigma_f^2 \frac{\widehat{f}_t^A}{\bar{f}} u_{ff}^F - \lambda(\widehat{f}_t^A - \bar{f}) u_f^F - r u^F + \frac{(q^F - p)^2}{2\chi} = 0 \quad (39)$$

The solution to equation (38) can easily be determined as⁶⁰:

$$q^F(\widehat{f}_t^A) = \frac{\bar{f}}{r + \delta} + \frac{\widehat{f}_t^A - \bar{f}}{r + \delta + \lambda} \quad (40)$$

whereas the solution to $u_1(\widehat{f}_t^A)$ is given as

$$z_1(\widehat{f}_t^A) = C_1 (\widehat{f}_t^A - \bar{f})^2 + C_2 (\widehat{f}_t^A - \bar{f}) + C_3$$

with:

$$C_1 = \frac{1}{\chi} \frac{1}{r + 2\lambda} \frac{1}{2} \left(\frac{1}{r + \delta + \lambda} \right)^2 \quad (41)$$

$$C_2 = \frac{1}{\chi} \frac{1}{(r + \lambda)(r + \delta + \lambda)} \left[\left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} \right) + \frac{\sigma_f^2}{2} \frac{1}{\bar{f}} \frac{1}{r + 2\lambda} \frac{1}{r + \delta + \lambda} \right] \quad (42)$$

$$C_3 = \frac{1}{r} \left[\frac{1}{2\chi} \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} \right)^2 + \sigma_f^2 C_1 \right] \quad (43)$$

where $\tilde{p} = p \frac{\bar{f}}{r + \delta}$. The derivative of P_t w.r.t \widehat{f}_t^A is given by:

$$\frac{1}{r + \delta + \lambda} K_t + 2C_1 (\widehat{f}_t^A - \bar{f}) + C_2$$

As long as $2C_1 (\widehat{f}_t^A - \bar{f}) + C_2 > 0$, $P_f > 0$. Since \widehat{f}_t^A will always be positive it remains to check that:

$$-2C_1 \bar{f} + C_2 > 0$$

or

$$\tilde{p} < 1 - \frac{r + \delta}{r + \delta + \lambda} \frac{1}{r + 2\lambda} \left(r + \lambda - \frac{\sigma_f^2}{2\bar{f}^2} \right) \doteq \bar{p}$$

As might be expected, for the special case where $\tilde{p} = 0$ the above equation is always satisfied ■

⁶⁰In this paper only particular solutions of ODE's will be considered. Economically this means that "rational bubbles" will not be allowed, i.e. terms that grow unboundedly in expectation at the riskless rate. See Abel and Eberly (1997) on this point. In contrast only "resale premia" will be analyzed, that are determined in the next section.

6.3 Proofs for section 3.2.2

Two preliminary results will help in the construction of an explicit solution.

Lemma 1 Consider the linear second order ordinary differential equation (ODE):

$$\frac{\sigma_g^2}{2}y'' - \rho xy' - (r + \delta)y = 0 \quad (44)$$

Then there are two linearly independent solutions to this ODE and are given by

$$\begin{aligned} y_1(x) &= \begin{cases} U\left(\frac{r+\delta}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) & \text{if } x \leq 0 \\ \frac{2\pi}{\Gamma\left(\frac{1}{2} + \frac{r+\delta}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}M\left(\frac{r+\delta}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) - U\left(\frac{r+\delta}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) & \text{if } x > 0 \end{cases} \\ y_2(x) &= y_1(-x) \end{aligned}$$

where $U()$ and $M()$ are Kummer's M and U functions⁶¹. $y_1(x)$ is positive, increasing and satisfies $\lim_{x \rightarrow -\infty} y_1(x) = 0$, $\lim_{x \rightarrow +\infty} y_1(x) = \infty$. Accordingly, $y_2(x)$ is positive, decreasing and satisfies: $\lim_{x \rightarrow -\infty} y_1(x) = \infty$, $\lim_{x \rightarrow +\infty} y_1(x) = 0$. Moreover any positive solution that satisfies equation (44) and $\lim_{x \rightarrow -\infty} y(x) = 0$ is given by: $\beta y_1(x)$ where $\beta > 0$ an arbitrary constant. Similarly any solution to (44) that is positive and satisfies: $\lim_{x \rightarrow \infty} y(x) = 0$ is given by $\beta y_2(x)$ where $\beta > 0$ is an arbitrary constant.

Proof. Lemma (1) The proof is essentially the same as the proof of proposition 2 in Scheinkman and Xiong (2003) and therefore large portions are omitted. If $v(z)$ is a solution to:

$$zv''(z) + \left(\frac{1}{2} - z\right)v'(z) - \frac{r + \delta}{2\rho}v(z) = 0 \quad (45)$$

then $y(x) = v\left(\frac{\rho}{\sigma_g^2}x^2\right)$ satisfies (44). The general solution to equation (45) is given by⁶²:

$$v(z) = \alpha M\left(\frac{r + \delta}{2\rho}, \frac{1}{2}, z\right) + \beta U\left(\frac{r + \delta}{2\rho}, \frac{1}{2}, z\right)$$

where the functions $M()$ and $U()$ are given in terms of their power series expansion in 13.1.2. and 13.1.3. of Abramowitz and Stegun (1964). The properties $y_1 > 0$, $y_{1x} > 0$, $\lim_{x \rightarrow -\infty} y_1(x) = 0$, $\lim_{x \rightarrow +\infty} y_1(x) = \infty$ can be established as in Scheinkman and Xiong (2003). It remains to show that the Wronskian of the two solutions ($y_1 y_2' - y_1' y_2$) is different from 0 everywhere. This is immediate since $y_1(x), y_2(x) > 0$ and $y_1'(x) > 0, y_2'(x) < 0$. ■

⁶¹These functions are described in Abramowitz and Stegun (1965) p.504.

⁶²See. Abramowitz and Stegun (1965) p. 504

Lemma 2 Consider the linear (inhomogenous) second order ODE:

$$\frac{\sigma_g^2}{2}y'' - \rho xy' - (r + \delta)y = -f(x) \quad (46)$$

Then the general solution to (46) is given as:

$$y(x) = \left[\int_x^{+\infty} \left(\frac{\frac{2}{\sigma^2} f(z)y_2(z)}{y_1'(z)y_2(z) - y_1(z)y_2'(z)} \right) dz + C_1 \right] y_1(x) + \left[\int_{-\infty}^x \left(\frac{\frac{2}{\sigma^2} f(z)y_1(z)}{y_1'(z)y_2(z) - y_1(z)y_2'(z)} \right) dz + C_2 \right] y_2(x)$$

provided that the above integrals exist. Moreover the derivative $y'(x)$ is given as:

$$y'(x) = \left[\int_x^{+\infty} \left(\frac{\frac{2}{\sigma^2} f(z)y_2(z)}{y_1'(z)y_2(z) - y_1(z)y_2'(z)} \right) dz + C_1 \right] y_1'(x) + \left[\int_{-\infty}^x \left(\frac{\frac{2}{\sigma^2} f(z)y_1(z)}{y_1'(z)y_2(z) - y_1(z)y_2'(z)} \right) dz + C_2 \right] y_2'(x)$$

Proof. Lemma (2) The proof is a basic variations of parameters argument and is omitted. For details see e.g. Section 9.3. in Rainville, Bedient and Bedient (1997). ■

By setting $C_1 = C_2 = 0$ in the above Lemma one gets the so called particular solution, which will depend on σ, ρ, r and the specific functional form of $f(x)$. I will denote the solution $y(x)$ to this equation as:

$$G(f(x); \sigma_g, \rho, r + \delta)$$

and proceed in steps to provide a proof to proposition 2. The first step is to make a guess on the form of optimal investment that is verified later. In particular suppose that the firm's investment policy is given by:

Conjecture 5 The optimal investment policy in equilibrium is given as:

$$i_t = \frac{1}{\chi} \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} + \frac{\hat{f}_t^A - \bar{f}}{r + \delta + \lambda} + 1_{\{g_t^A > 0\}} \frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-|g_t^A|) \right) \quad (47)$$

where β is a constant that can be determined as:

$$\beta = \frac{1}{2(r + \delta + \lambda)y_1'(0)} \quad (48)$$

and y_1 is the function described in the Lemma 1.

The next step will be to determine the equilibrium prices, stopping times for agents etc. conditional on the investment policy described. To do this it is easiest to compute the "infinite" horizon value of the company to an investor of type A conditional on the policy (47). One can focus without loss of generality on the determination of the reservation price for agent A since the problem for agent B is symmetric. Formally, the goal will be to determine the functional:

$$V(K_t, \hat{f}_t^A, g_t^A) = E_t^A \left[\int_t^\infty e^{-r(s-t)} \left(\hat{f}_s^A K_s - p i_s - \frac{\chi}{2} (i_s^2) \right) ds \right] \quad (49)$$

This function captures the value of the asset to an "infinite" horizon investor of type A who takes the conjectured investment policy (47) as given.

Proposition 6 *The solution to (49) is given by:*

$$\begin{aligned} V(K_t, \widehat{f}_t^A, g_t^A) &= \left[\frac{\bar{f}}{r + \delta} + \frac{\widehat{f}_t^A - \bar{f}}{r + \delta + \lambda} \right] K_t + \\ &+ \left(C_1 (\widehat{f}_t^A - \bar{f})^2 + C_2 (\widehat{f}_t^A - \bar{f}) + C_3 \right) \\ &+ u(g_t^A) \end{aligned}$$

where $u(g_t^A) < 0$ and C_1, C_2, C_3 are the same constants as in Proposition 1. Moreover $u_g < 0$.

Proof. *Proposition (6)* According to the Feynman Kac Theorem the solution $V(K_t, \widehat{f}_t^A, g_t^A)$ to (49) must satisfy the partial differential equation:

$$\mathcal{A}V + \widehat{f}_t^A K_t - i_t \left(p + \frac{\chi}{2} i_t \right) = 0 \quad (50)$$

where \mathcal{A} is the infinitesimal operator given by:

$$\mathcal{A}V = \frac{\sigma_f^2 \widehat{f}_t^A}{2} V_{ff} + \frac{\sigma_g^2}{2} V_{gg} - \lambda(\widehat{f}_t^A - \bar{f})V_f - \rho g_t^A V_g + V_k(-\delta K_t + i_t) - rV$$

Conjecturing a solution of the form:

$$V = h(\widehat{f}_t^A) K_t + z(\widehat{f}_t^A, g_t^A)$$

and substituting this conjecture back into (50) one can determine conditions that $h()$ and $z()$ have to satisfy in order to satisfy (50). $h()$ has to satisfy:

$$\frac{\sigma_f^2 \widehat{f}_t^A}{2} h_{ff} + \frac{\sigma_g^2}{2} h_{gg} - \lambda(f - \bar{f})h_f - \rho g_t^A h_g - (r + \delta)h + \widehat{f}_t^A = 0$$

A particular solution is given by⁶³:

$$h(\widehat{f}_t^A) = \frac{\bar{f}}{r + \delta} + \frac{\widehat{f}_t^A - \bar{f}}{r + \delta + \lambda}$$

while $z(\widehat{f}_t^A, g_t^A)$ solves the partial differential equation:

$$\frac{\sigma_f^2 \widehat{f}_t^A}{2} z_{ff} + \frac{\sigma_g^2}{2} z_{gg} - \lambda(f - \bar{f})z_f - \rho g_t^A z_g - rz + h(f, g)i_t - i_t \left(p + \frac{\chi}{2} i_t \right) = 0 \quad (51)$$

It is easy to show that:

$$h(f, g)i_t - i_t \left(p + \frac{\chi}{2} i_t \right) = \frac{1}{2\chi} \left(\frac{\bar{f}}{r + \delta} + \frac{\widehat{f}_t^A - \bar{f}}{r + \delta + \lambda} - p \right)^2 - \frac{1}{2\chi} (\tilde{b}(g_t^A))^2 \quad (52)$$

⁶³Obviously there are other solutions that "explode" at the rate r but we will only be interested in bounded solutions in this paper.

and

$$\tilde{b}(g_t^A) = \beta y_1(-|g_t^A|) + 1\{g_t^A > 0\} \frac{g_t^A}{r + \delta + \lambda}$$

(52) allows one to derive an explicit solution for (51) by solving two ordinary differential equations $z_1(\hat{f}_t^A), u(g_t^A)$ that satisfy:

$$\frac{\sigma_f^2}{2} \frac{\hat{f}_t^A}{\bar{f}} z_{1ff} - \lambda(\hat{f}_t^A - \bar{f}) z_{1f} - r z_1 + \frac{1}{2\chi} \left(\frac{\bar{f}}{r + \delta} + \frac{\hat{f}_t^A - \bar{f}}{r + \delta + \lambda} - p \right)^2 = 0 \quad (53)$$

$$\frac{\sigma_g^2}{2} u_{gg} - \rho g u_g - r u - \frac{1}{2\chi} (\tilde{b}(g_t^A))^2 = 0 \quad (54)$$

$z_1(\hat{f}_t^A)$ solves the exact same ODE as $u^F(\hat{f}_t^A)$ in Proposition (1) and thus it will be the case that:

$$z_1(\hat{f}_t^A) = C_1 (\hat{f}_t^A - \bar{f})^2 + C_2 (\hat{f}_t^A - \bar{f}) + C_3$$

for the same constants as in Proposition (1). Finally, one can use the results in Lemma 2 to construct the solution to (54). It is given by:

$$u(g) = G \left(-\frac{1}{2\chi} (\tilde{b}(g_t^A))^2; \sigma_g, \rho, r \right) < 0$$

To show that $u_g < 0$, observe that $\tilde{b}(g^A)$ is a strictly increasing, positive and continuously differentiable function of g_t^A , so that:

$$\left[-\frac{1}{2\chi} (\tilde{b}(g_t^A))^2 \right]' = -\frac{1}{\chi} \tilde{b}(g_t^A) \tilde{b}_g < 0$$

Differentiating (54) w.r.t. g_t^A gives:

$$\frac{\sigma_g^2}{2} u_{ggg} - \rho g u_{gg} - (r + \rho) u_g - \frac{1}{\chi} \tilde{b}(g_t^A) \tilde{b}_g = 0$$

Defining $u_g = z^d$ one can rewrite this equation as:

$$\frac{\sigma_g^2}{2} z_{gg}^d - \rho g z_g^d - (r + \rho) z^d - \frac{1}{\chi} \tilde{b}(g_t^A) \tilde{b}_g = 0$$

which has the particular solution:

$$G \left(-\frac{1}{\chi} \tilde{b}(g_t^A) \tilde{b}_g; \sigma_g, \rho, r + \rho \right)$$

which is unambiguously negative. This formal analysis can be made rigorous by invoking a set of results known as Malliavin Calculus (see e.g. Fournie et. al. (1999)).⁶⁴ ■

With an expression for the value of the asset to an agent who does not intend to resell it ever in the future, one can proceed to guess an equilibrium pricing function and an optimal stopping policy. An informed "guess" is that

⁶⁴Details are available upon request

the optimal stopping policy is of a particularly simple form: Agent A should sell once $\widehat{f}_t^A < \widehat{f}_t^B$ and agent B should sell once $\widehat{f}_t^B < \widehat{f}_t^A$. This is the case because there are no transactions costs in this model. Accordingly, each agent sells the asset once she stops having the most optimistic beliefs in the market. In particular one can re-express the reservation price for agent A as long as $g_t^A < 0$ (so that it is agent A who is the highest bidder) as

$$P(K_t, \widehat{f}_t^A, g_t^A) = V\left(K_t, \widehat{f}_t^A, g_t^A\right) + s(K_t, \widehat{f}_t^A, g_t^A) \quad (55)$$

Similarly, the reservation price for agent B (as long as $g_t^A > 0$ or equivalently $g_t^B < 0$) is given by symmetry as:

$$P(K_t, \widehat{f}_t^B, g_t^B) = V\left(K_t, \widehat{f}_t^B, g_t^B\right) + s(K_t, \widehat{f}_t^B, g_t^B) \quad (56)$$

A further conjecture that will be verified shortly is that for $g_t^A < 0$, s is given as:

$$s(K, f, g) = \beta y_1(g)K + n(g)\left(\widehat{f}_t^A - \bar{f}\right) + v(g) \quad (57)$$

for some functions n and v . In other words the reservation price for each agent is just the infinite horizon valuation of the dividends plus a speculative component. Using (55) one can get the following result:

Lemma 3 *If the reservation price function for agent B is given by (56) then the reservation price for agent A is given by:*

$$P(K_t, \widehat{f}_t^A, g_t^A) = V\left(K_t, \widehat{f}_t^A, g_t^A\right) + \sup_{\tau} Ee^{-r\tau} \left[V\left(K_{\tau}, \widehat{f}_{\tau}^B, g_{\tau}^B\right) - V\left(K_{\tau}, \widehat{f}_{\tau}^A, g_{\tau}^A\right) + s(K_{\tau}, \widehat{f}_{\tau}^B, g_{\tau}^B) \right] \quad (58)$$

$$= V\left(K_t, \widehat{f}_t^A, g_t^A\right) + \sup_{\tau} Ee^{-r\tau} \left[\left(\frac{g_{\tau}^A}{r + \delta + \lambda} + \beta y_1(-g_{\tau}^A) \right) K_{\tau} + w(\widehat{f}_{\tau}^A, g_{\tau}^A) \right] \quad (59)$$

for $w(\widehat{f}_{\tau}^A, g_{\tau}^A)$ given by:

$$w(\widehat{f}_{\tau}^A, g_{\tau}^A) = \left[C_2 + n(-g_{\tau}^A) \right] g_{\tau}^A + C_1 \left(g_{\tau}^A \right)^2 + u(-g_{\tau}^A) - u(g_{\tau}^A) + \left[n(-g_{\tau}^A) + g_{\tau}^A 2C_1 \right] \left(\widehat{f}_{\tau}^A - \bar{f} \right) + v(-g_{\tau}^A)$$

Proof. Lemma (3). The argument is essentially identical to the one given in Scheinkman and Xiong (2003). I give it here for completeness. Using (55) one can get:

$$P(K_t, \widehat{f}_t^A, g_t^A) =$$

$$\begin{aligned}
&= \sup_{\tau} E \left[\int_0^{\tau} e^{-rt} \left(\widehat{f}_t^A K_t - p i_t - \frac{\chi}{2} i_t^2 \right) dt + e^{-r\tau} P(K_{\tau}, \widehat{f}_{\tau}^B, g_{\tau}^B) \right] = \\
&= \sup_{\tau} E \left[\int_0^{\tau} e^{-rt} \left(\widehat{f}_t^A K_t - p i_t - \frac{\chi}{2} i_t^2 \right) dt + e^{-r\tau} \left[V \left(K_{\tau}, \widehat{f}_{\tau}^B, g_{\tau}^B \right) + \beta y_1(g_{\tau}^B) K_{\tau} + n(g_{\tau}^B) \left(\widehat{f}_{\tau}^B - \bar{f} \right) + v(g_{\tau}^B) \right] \right] = \\
&= \sup_{\tau} E \left[\int_0^{\infty} e^{-rt} \left(f K_t - p i_t - \frac{\chi}{2} i_t^2 \right) dt + e^{-r\tau} \left[V \left(K_{\tau}, \widehat{f}_{\tau}^B, g_{\tau}^B \right) - V \left(K_{\tau}, \widehat{f}_{\tau}^A, g_{\tau}^A \right) + \beta y_1(g_{\tau}^B) K_{\tau} + n(g_{\tau}^B) \left(\widehat{f}_{\tau}^B - \bar{f} \right) + v(g_{\tau}^B) \right] \right] = \\
&= V \left(K_t, \widehat{f}_t^A, g_t^A \right) + \sup_{\tau} E e^{-r\tau} \left[\left(\frac{g_{\tau}^A}{r+\delta+\lambda} + \beta y_1(g_{\tau}^B) \right) K_{\tau} + C_2 g_{\tau}^A + u(g_{\tau}^B) - u(g_{\tau}^A) + \right. \\
&\quad \left. C_1 \left[\left(\widehat{f}_{\tau}^B - \bar{f} \right)^2 - \left(\widehat{f}_{\tau}^A - \bar{f} \right)^2 \right] + n(g_{\tau}^B) \left(\widehat{f}_{\tau}^B - \bar{f} \right) + v(g_{\tau}^B) \right] = \\
&= V^A + \sup_{\tau} E e^{-r\tau} \left[\left(\frac{g_{\tau}^A}{r+\delta+\lambda} + \beta y_1(-g_{\tau}^A) \right) K_{\tau} + [C_2 + n(-g_{\tau}^A)] g_{\tau}^A + C_1 (g_{\tau}^A)^2 + \right. \\
&\quad \left. + u(-g_{\tau}^A) - u(g_{\tau}^A) + [n(-g_{\tau}^A) + g_{\tau}^A 2C_1] \left(\widehat{f}_{\tau}^A - \bar{f} \right) + v(-g_{\tau}^A) \right]
\end{aligned}$$

where the last line follows from the identities:

$$\begin{aligned}
\widehat{f}_{\tau}^B &= \widehat{f}_{\tau}^A + g_{\tau}^A \\
g_{\tau}^B &= -g_{\tau}^A
\end{aligned}$$

Defining the function $w(\widehat{f}_{\tau}^A, g_{\tau})$ as:

$$\begin{aligned}
w(\widehat{f}_{\tau}^A, g_{\tau}) &= [C_2 + n(-g_{\tau}^A)] g_{\tau}^A + C_1 (g_{\tau}^A)^2 + u(-g_{\tau}^A) - u(g_{\tau}^A) + \\
&\quad + [n(-g_{\tau}^A) + g_{\tau}^A 2C_1] \left(\widehat{f}_{\tau}^A - \bar{f} \right) + v(-g_{\tau}^A)
\end{aligned}$$

concludes the proof. ■

In light of Lemma (3) it remains to establish that (57) is right, i.e. that there exist appropriate functions $n(\cdot)$, $v(\cdot)$, and an appropriate constant β such that:

$$\beta y_1(g_t^A) K_t + n(g_t^A) \left(\widehat{f}_t^A - \bar{f} \right) + v(g_t^A) = \sup_{\tau} E e^{-r\tau} \left[\left(\frac{g_{\tau}^A}{r+\delta+\lambda} + \beta y_1(-g_{\tau}^A) \right) K_{\tau} + w(\widehat{f}_{\tau}^A, g_{\tau}^A) \right] \quad (60)$$

In other words it remains to establish the existence of functions $n(g_t^A), v(g_t^A)$ and a constant β so that the Value function of the optimal stopping problem on the right hand side has the form on the left hand side inside the continuation region, i.e. inside the region where agent A finds it optimal to hold the asset. The right hand side problem is a three dimensional optimal stopping problem (in $K_t, \widehat{f}_t^A, g_t^A$) and in general there is no method to solve such problems analytically. This is in contrast to one dimensional optimal stopping problems where continuity along with smooth pasting is enough to determine the stopping region and the associated value function in most cases. Fortunately, the simple form of the conjectured continuation region allows one to solve this problem as is demonstrated in the next proposition:

Proposition 7 *There exist functions $n(g_t^A), v(g_t^A)$ and a constant β such that the function:*

$$s(K_t, \widehat{f}_t^A, g_t^A) = \begin{cases} \beta y_1(g_t^A) K_t + n(g_t^A) (\widehat{f}_t^A - \bar{f}) + v(g_t^A) & \text{if } g_t^A < 0 \\ \left(\frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-g_t^A) \right) K_t + w(\widehat{f}_t^A, g_t^A) & \text{if } g_t^A \geq 0 \end{cases} \quad (61)$$

satisfies

$$\mathcal{A}V = 0 \text{ if } g_t^A < 0$$

is twice continuously differentiable in the region $g_t^A > 0$ and in the region $g_t^A < 0$ and once cont. differentiable everywhere. The constant β is given by:

$$\beta = \frac{1}{2(r + \delta + \lambda)} \frac{1}{y_1'(0)}$$

and the functions $n(\cdot)$ and $v(\cdot)$ are given in the proof.

Proof. Proposition (7) The first step is to construct the Value function under the assumption that both the conjecture for the optimal stopping region and the equilibrium investment strategy is correct. Since for $g_t^A < 0$ the conjectured optimal strategy is to hold the asset, one can formulate a necessary condition for the value function s of the optimal stopping problem on the right hand side of (60). Namely, it has to be the case that inside this region:

$$\mathcal{A}s = 0$$

or:

$$\frac{\sigma_f^2}{2} \frac{\widehat{f}_t^A}{\bar{f}} s_{ff} + \frac{\sigma_g^2}{2} s_{gg} - \lambda(\widehat{f}_t^A - \bar{f}) s_f - \rho g_t^A s_g + s_K \left[-\delta K + \frac{1}{\chi} \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} + \frac{\widehat{f}_t^A - \bar{f}}{r + \delta + \lambda} + \beta y_1(g_t^A) \right) \right] - rs = 0 \quad (62)$$

An informed guess is that this PDE has a solution of the form:

$$\beta y_1(g_t^A) K_t + \zeta(\widehat{f}_t^A, g_t^A)$$

Plugging this into (62) one gets the set of equations:

$$\frac{\sigma_g^2}{2} y_{1gg} - \rho g_t^A y_{1g} - (r + \delta) y_1 = 0 \quad (63)$$

$$\frac{\sigma_f^2}{2} \frac{\widehat{f}_t^A}{\bar{f}} \zeta_{ff} + \frac{\sigma_g^2}{2} \zeta_{gg} - \lambda(\widehat{f}_t^A - \bar{f}) \zeta_f - \rho g_t^A \zeta_g + \beta y_1(g_t^A) \frac{1}{\chi} \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} + \frac{\widehat{f}_t^A - \bar{f}}{r + \delta + \lambda} + \beta y_1(g_t^A) \right) - r\zeta = 0 \quad (64)$$

It is immediate that the function $y_1(g_t^A)$ constructed in Lemma 1 satisfies (63) by construction⁶⁵. One can determine a solution to equation (64) by postulating that the solution u is given by:

$$\zeta(\widehat{f}_t^A, g_t^A) = v(g_t^A) + n(g_t^A) (\widehat{f}_t^A - \bar{f})$$

⁶⁵Moreover it is the only solution that vanishes as $g_t^A \rightarrow -\infty$

and upon substituting this conjecture into (64) it is easy to see that $v(g_t^A)$ and $n(g_t^A)$ have to satisfy the two ordinary differential equations:

$$\frac{\sigma_g^2}{2} v_{gg} - \rho g_t^A v_g - r v + \frac{1}{\chi} \left(\beta y_1(g_t^A) \left(\frac{\bar{f}}{r + \delta} + \beta y_1(g_t^A) \right) \right) = 0 \quad (65)$$

$$\frac{\sigma_g^2}{2} n_{gg} - \rho g_t^A n_g - (r + \lambda) n(g_t^A) + \frac{1}{\chi} \left(\frac{\beta}{r + \delta + \lambda} \right) y_1(g_t^A) = 0 \quad (66)$$

In the $g_t^A < 0$ region, the general solution to (65) and (66) is given by:

$$\begin{aligned} v(g_t^A) &= c_1 y_1^{(r)}(g_t^A) + c_2 y_2^{(r)}(g_t^A) + v_P(g_t^A) \\ n(g_t^A) &= \tilde{c}_1 y_1^{(r+\lambda)}(g_t^A) + \tilde{c}_2 y_2^{(r+\lambda)}(g_t^A) + n_P(g_t^A) \end{aligned}$$

where $v_P(g_t^A), n_P(g_t^A)$ are the particular solutions to the above equations obtained by Lemma 2:

$$\begin{aligned} v_P(g_t^A) &= G \left[\frac{1}{\chi} \left(\beta y_1(-|g_t^A|) \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} + \beta y_1(-|g_t^A|) \right) \right); \sigma_g, \rho, r \right] \\ n_P(g_t^A) &= G \left[\frac{1}{\chi} \left(\frac{\beta}{r + \delta + \lambda} \right) y_1(-|g_t^A|); \sigma_g, \rho, r + \lambda \right] \end{aligned}$$

and $y_1^{(x)}(g_t^A), y_2^{(x)}(g_t^A)$ are defined in an identical way to $y_1(g_t^A)$ and $y_2(g_t^A)$ of Lemma 1 with the only exception that $r + \delta$ is replaced by x . It is also clear that since $y_1(-|g_t^A|)$ is a bounded function, the above integrals are finite. Moreover, it is easy to check that the particular solutions to the above equations satisfy $v'_P(0) = 0$ and $n'_P(0) = 0$.⁶⁶ Finally, to keep only solutions that do not explode as $g_t^A \rightarrow -\infty$ one can set $c_2 = \tilde{c}_2 = 0$.

Observe that the conjectured Value function is of the form posited in the left hand side of equation (60). To conclude with the construction of a candidate pricing function, it remains to determine the constants in such a way that the resulting value function for the optimal stopping problem (60) is both continuous and cont. differentiable everywhere. For $g_t^A > 0$ the conjecture is that agent A resells to agent B , so that the value function for this case is given by the value of "immediate exercise" i.e.

$$s(K_t, \hat{f}_t^A, g_t^A) = \left(\frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-g_t^A) \right) K_t + w(\hat{f}_t^A, g_t^A) \text{ if } g_t^A > 0$$

In each of the two regions ($g_t^A < 0, g_t^A > 0$) the function V is twice cont. differentiable, accordingly continuity and differentiability only needs to be enforced at $g_t^A = 0$. The left limit of V at $g_t^A = 0$ is given by:

$$\beta y_1(0) K_t + v(0) + n(0) \left(\hat{f}_t^A - \bar{f} \right)$$

⁶⁶Since they are symmetric around 0 and continuously differentiable.

whereas the right limit is obtained by evaluating $\left(\frac{g_t^A}{r+\delta+\lambda} + \beta y_1(-g_t^A)\right) K_t + w(\widehat{f}_t^A, g_t^A)$ around $g_t^A = 0$. This yields after obvious simplifications:

$$\beta y_1(0)K_t + v(0) + n(0) \left(\widehat{f}_t^A - \bar{f}\right)$$

so that continuity is immediately satisfied. Differentiability requires that:

$$\begin{aligned} \beta y_1'(0) &= \frac{1}{r + \delta + \lambda} - \beta y_1'(0) \\ \tilde{c}_1 y_1^{(r+\lambda)'}(0) &= -\tilde{c}_1 y_1^{(r+\lambda)'}(0) + 2C_1 \\ 2c_1 y_1^{(r)'}(0) &= -2u'(0) + C_2 + n(0) \end{aligned}$$

which implies that:

$$\begin{aligned} \beta &= \frac{1}{2(r + \delta + \lambda)} \frac{1}{y_1'(0)} \\ \tilde{c}_1 &= \frac{C_1}{y_1^{(r+\lambda)'}(0)} \\ c_1 &= \frac{-2u'(0) + C_2 + n(0)}{2y_1^{(r)'}(0)} \end{aligned}$$

■

In order to be able to invoke a verification Theorem for optimal stopping one needs additionally the following two results:

Proposition 8 *The function s constructed in Proposition (7) satisfies:*

$$\begin{aligned} s(\widehat{f}_t^A, g_t^A, K) &= \\ \left(\frac{g_t^A}{r + \delta + \lambda} K_t + 2C_1 g_t^A (\widehat{f}_t^A - \bar{f}) + C_1 (g_t^A)^2 + C_2 g_t^A + u(-g_t^A) - u(g_t^A)\right) + s(\widehat{f}_t^A + g_t^A, -g_t^A, K_t) &= \\ = V(K_t, \widehat{f}_t^B, g_t^B) - V(K_t, \widehat{f}_t^A, g_t^A) + s(K_t, \widehat{f}_t^B, g_t^B) \end{aligned}$$

Proof. Proposition (8) The definition of s in Proposition (7) allows one to compute $s(\widehat{f}_t^A + g_t^A, -g_t^A, K_t)$ as:

$$s(\widehat{f}_t^A + g_t^A, -g_t^A, K_t) = \begin{cases} \beta y_1(-g_t^A)K_t + n(-g_t^A) \left(\widehat{f}_t^A + g_t^A - \bar{f}\right) + v(-g_t^A) & \text{if } g_t^A > 0 \\ \left(\frac{-g_t^A}{r+\delta+\lambda} + \beta y_1(g_t^A)\right) K_t + w(\widehat{f}_t^A + g_t^A, -g_t^A) & \text{if } g_t^A \leq 0 \end{cases}$$

and since

$$\begin{aligned} w(\widehat{f}_t^A + g_t^A, -g_t^A) &= -\left[C_2 + n(g_t^A)\right] g_t^A + C_1 (g_t^A)^2 + u(g_t^A) - u(-g_t^A) + \\ &+ \left[n(g_t^A) - g_t^A 2C_1\right] \left(\widehat{f}_t^A + g_t^A - \bar{f}\right) + v(g_t^A) \end{aligned}$$

one gets after a number of simplifications irrespective of whether $(g_t^A > 0)$ or $(g_t^A \leq 0)$

$$s(\widehat{f}_t^A, g_t^A, K_t) = \left(\frac{g_t^A}{r + \delta + \lambda} K_t + 2C_1 g_t^A (\widehat{f}_t^A - \bar{f}) + C_1 (g_t^A)^2 + C_2 g_t^A + u(-g_t^A) - u(g_t^A) \right) + s(\widehat{f}_t^A + g_t^A, -g_t^A, K_t)$$

■

The single most important step towards verifying the results is to verify that:

$$\mathcal{A}s \leq 0$$

in the region $(g_t \geq 0)$. In particular, the following result is true:

Proposition 9 *Suppose*

$$(A) \tilde{p} < \frac{\lambda}{r + \delta + \lambda}^{67}$$

$$(B) \rho - 3\lambda - 2r > 0$$

Then

$$\mathcal{A}s \leq 0$$

for $g_t^A \geq 0$.

Proof. Proposition (9) For $g_t^A \geq 0$ the operator A is given for any function \tilde{V} as:

$$\mathcal{A}\tilde{V} = \frac{\sigma_f^2 \widehat{f}_t^A}{2 \bar{f}} \tilde{V}_{ff} + \frac{\sigma_g^2}{2} \tilde{V}_{gg} - \lambda (\widehat{f}_t^A - \bar{f}) \tilde{V}_f - \rho g_t^A \tilde{V}_g + \tilde{V}_k \left(-\delta K_t + \frac{1}{\chi} \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} + \frac{\widehat{f}_t^A + g_t^A - \bar{f}}{r + \delta + \lambda} + \beta y_1(-g_t^A) \right) \right) - r \tilde{V}$$

The conjectured s is given as:

$$s(K_t, \widehat{f}_t^A, g_t^A) = \left(\frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-g_t^A) \right) K_t + w(\widehat{f}_t^A, g_t^A)$$

where (from Lemma 3) $w(\widehat{f}_t^A, g_t^A)$ is given as:

$$w(\widehat{f}_t^A, g_t) = \left[C_2 + n(-g_t^A) \right] g_t^A + C_1 (g_t^A)^2 + u(-g_t^A) - u(g_t^A) + \left[n(-g_t^A) + g_t^A 2C_1 \right] (\widehat{f}_t^A - \bar{f}) + v(-g_t^A)$$

⁶⁷It is trivial to show that under this condition $K_t \geq 0$ and moreover $\tilde{p} < \bar{p}$

It will be easiest to apply the operator on each term separately:

$$\begin{aligned}
\mathcal{A} \left[\left(\frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-g_t^A) \right) K_t \right] &= \left(-\frac{(\rho + r + \delta)}{r + \delta + \lambda} g_t^A \right) K_t \\
&\quad + \frac{1}{\chi} \left(\frac{g_t^A}{r + \delta + \lambda} + \beta y_1(-g_t^A) \right) \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} + \frac{\hat{f}_t^A + g_t^A - \bar{f}}{r + \delta + \lambda} + \beta y_1(-g_t^A) \right) \\
\mathcal{A} \left[\left[C_2 + n(-g_t^A) \right] g_t^A + C_1 (g_t^A)^2 \right] &= -(\rho + r) g_t^A C_2 + C_1 \sigma_g^2 - C_1 (2\rho + r) (g_t^A)^2 + \\
&\quad + g \left(\frac{\sigma_g^2}{2} n_{gg}(-g_t^A) - \rho(-g_t^A) n_g(-g_t^A) - r n(-g_t^A) \right) \\
&\quad - \rho g_t^A n(-g_t^A) - \sigma_g^2 n_g(-g_t^A) \tag{67}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A} \left[\left[n(-g_t^A) + g_t^A 2C_1 \right] (\hat{f}_t^A - \bar{f}) \right] &= \\
&= \left(\frac{\sigma_g^2}{2} n_{gg}(-g_t^A) - \rho g_t^A (-n_g(-g_t^A) + 2C_1) - (r + \lambda) [n(-g_t^A) + g_t^A 2C_1] \right) \cdot \\
&\quad \cdot (\hat{f}_t^A - \bar{f}) \\
&= \left(\frac{\sigma_g^2}{2} n_{gg}(-g_t^A) - \rho(-g_t^A) (n_g(-g_t^A)) - (r + \lambda) n(-g_t^A) \right) (\hat{f}_t^A - \bar{f}) \\
&\quad - (\rho + r + \lambda) 2C_1 g_t^A (\hat{f}_t^A - \bar{f}) \tag{68}
\end{aligned}$$

Using the definitions of $u()$ and $v()$ from Propositions 7 and 49 the above expressions become:

$$\begin{aligned}
\mathcal{A} \left[u(-g_t^A) - u(g_t^A) \right] &= \left(\frac{\sigma_g^2}{2} u_{gg}(-g_t^A) - \rho(-g_t^A) u_g(-g_t^A) - r u(-g_t^A) \right) - \left(\frac{\sigma_g^2}{2} u_{gg} - \rho g_t^A u_g - r u \right) = \\
&= -\frac{1}{2\chi} \left(\frac{2\beta y_1(-g_t^A) g_t^A}{r + \delta + \lambda} + \frac{(g_t^A)^2}{(r + \delta + \lambda)^2} \right) \\
\mathcal{A} \left[v(-g_t^A) \right] &= \frac{\sigma_g^2}{2} v_{gg}(-g_t^A) - \rho((-g_t^A) v_g(-g_t^A)) - r v(-g_t^A) \\
&= -\frac{1}{\chi} \left(\beta y_1(-g_t^A) \left(\frac{\bar{f}(1 - \tilde{p})}{r + \delta} + \beta y_1(-g_t^A) \right) \right)
\end{aligned}$$

Collecting terms and using from Proposition 7 the fact that:

$$\frac{\sigma_g^2}{2} n_{gg} - \rho g_t^A n_g - (r + \lambda) n(g_t^A) = -\frac{1}{\chi} \left(\frac{\beta}{r + \delta + \lambda} \right) y_1(g_t^A)$$

and substituting into (67) and (68) it follows that:

$$\begin{aligned}
\mathcal{A} \left[\left[C_2 + n(-g_t^A) \right] g_t^A + C_1 \left(g_t^A \right)^2 \right] &= -(\rho + r)g_t^A C_2 + C_1 \sigma_g^2 - C_1(2\rho + r) \left(g_t^A \right)^2 + \\
&+ g \left(-\frac{1}{\chi} \left(\frac{\beta}{r + \delta + \lambda} \right) y_1(-g_t^A) + \lambda n(-g_t^A) \right) \\
&- \rho g_t^A n(-g_t^A) - \sigma_g^2 n_g(-g_t^A) \\
\mathcal{A} \left[\left[n(-g_t^A) + g_t^A 2C_1 \right] \left(\widehat{f}_t^A - \bar{f} \right) \right] &= \\
&= \left(-\frac{1}{\chi} \left(\frac{\beta}{r + \delta + \lambda} \right) y_1(-g_t^A) \right) \left(\widehat{f}_t^A - \bar{f} \right) \\
&- (\rho + r + \lambda) 2C_1 g_t^A \left(\widehat{f}_t^A - \bar{f} \right)
\end{aligned}$$

Collecting terms, simplifying and using the definitions of C_1, C_2, C_3 in Proposition 49 one can conclude that:

$$\begin{aligned}
AV &= \left(-\frac{(\rho + r + \delta)}{r + \delta + \lambda} g_t^A \right) K_t + \\
&- \frac{1}{\chi} (\rho - \lambda) g_t^A \left(2C_1 \left(\widehat{f}_t^A - \bar{f} \right) + \left[C_2 + \frac{1}{\rho - \lambda} \frac{1}{\chi} \frac{1}{2} \frac{\sigma_f^2}{\bar{f}} \frac{1}{r + 2\lambda} \left(\frac{1}{r + \delta + \lambda} \right)^2 \right] \right) \\
&- \frac{1}{2\chi} (\rho - \lambda) \frac{1}{r + 2\lambda} \left(\frac{(g_t^A)^2}{(r + \delta + \lambda)^2} \right) \\
&- (\rho - \lambda) g_t^A n(-g_t^A) + \sigma_g^2 \left[C_1 - n_g(-g_t^A) \right]
\end{aligned} \tag{69}$$

The first term is unambiguously negative if $K_t > 0$, the second term is negative since $(\rho - \lambda)$ is positive⁶⁸ and we have assumed $p < \bar{p}$ so that $2C_1 \left(\widehat{f}_t^A - \bar{f} \right) + C_2 > 0$. The third term will be negative since $\rho - \lambda > 0$. The terms in the fourth line requires some further analysis. In particular, notice that

$$\begin{aligned}
\left(C_1 - n_g(-g_t^A) \right) \sigma_g^2 &= \\
&= \sigma_g^2 \left(C_1 - \frac{C_1 y_1^{(r+\lambda)' }(-g_t^A)}{y_1^{(r+\lambda)' }(0)} - n_{Pg}(-g_t^A) \right) = \\
&= \sigma_g^2 C_1 \left(1 - \frac{\widetilde{y}_1'(-g_t^A)}{\widetilde{y}_1'(0)} \right) - \sigma_g^2 n_{Pg}(-g_t^A)
\end{aligned}$$

Thus one can rewrite (69) as:

$$-(\rho - \lambda) g_t^A n(-g_t^A) + \sigma_g^2 C_1 \left(1 - \frac{\widetilde{y}_1'(-g_t^A)}{\widetilde{y}_1'(0)} \right) - \sigma_g^2 n_{Pg}(-g_t^A) \tag{70}$$

⁶⁸ by the definition of ρ in 3.1.2

As expected, (70) at $g_t^A = 0$ is 0 by smooth pasting. Thus it is sufficient to show that (70) is declining for $g_t^A > 0$. To do this, differentiate once w.r.t to g_t^A , to get:

$$-(\rho - \lambda)n(-g_t^A) + (\rho - \lambda)g_t^A n_g(-g_t^A) + \sigma_g^2 n_{gg}(-g_t^A) \quad (71)$$

Now one can use the definition of $n(g)$ to get:

$$\sigma_g^2 n_{gg}(-g_t^A) = -\frac{2}{\chi} \left(\frac{\beta}{r + \delta + \lambda} \right) y_1(-g_t^A) + 2\rho(-g_t^A)n_g(-g_t^A) + 2(r + \lambda)n(-g_t^A)$$

and substitute into (71) to arrive at:

$$\begin{aligned} & -(\rho + \lambda)g_t^A n_g(-g_t^A) \\ & -\frac{2}{\chi} \left(\frac{\beta}{r + \delta + \lambda} \right) y_1(-g_t^A) \\ & -[\rho - 3\lambda - 2r]n(-g_t^A) \end{aligned}$$

which will be smaller than 0 under assumption (B), since both $n(\cdot)$ and $n_g(\cdot)$ are greater than 0. ■

Combining the properties established above one arrives at the equilibrium pricing function of Proposition 2. By Proposition 6 and the results in Proposition 9 it is straightforward to verify that $P(\hat{f}_t^A, g_t^A, K_t)$ satisfies the following properties:

$$\begin{aligned} \mathcal{L}P &= 0 \text{ if } g_t^A < 0 \\ \mathcal{L}P &\leq 0 \text{ if } g_t^A \geq 0 \end{aligned}$$

where:

$$LP = \max_i \left(AP + fK - i \left(p + \frac{\chi}{2} i \right) \right) \leq 0$$

and

$$AP = \frac{1}{2} \frac{\sigma_f^2}{\bar{f}} \hat{f}_t^A P_{ff} - \lambda(\hat{f}_t^A - \bar{f})P_f + \frac{1}{2} \sigma_g^2 P_{gg} - \rho g P_g + P_K(-\delta K_t + i_t) - rP$$

Moreover $P(\hat{f}_t^A, g_t^A, K_t)$ is C^1 everywhere and C^2 except at $g_t^A = 0$. Consider now any policy i_t and a stopping time τ . Then Ito's Lemma implies:

$$e^{-r\tau} P_\tau = P_0 + \int_0^\tau e^{-rt} AP dt + \int_0^\tau dM_t$$

where dM_t is a (local) martingale. Since $P_t \geq 0$, one can conclude that $E \left(\int_0^\tau dM_t \right) \leq 0$ and thus:

$$E(e^{-r\tau} P_\tau) \leq P_0 + E \left[\int_0^\tau AP + \hat{f}_t^A K_t - i_t \left(p + \frac{\chi}{2} i_t \right) dt \right] - E \left[\int_0^\tau \hat{f}_t^A K_t - i_t \left(p + \frac{\chi}{2} i_t \right) dt \right]$$

Then the following set of inequalities follows

$$\begin{aligned} P_0 &\geq P_0 + E \left[\int_0^\tau \mathcal{L}P dt \right] \geq P_0 + E \left[\int_0^\tau \mathcal{A}P + \widehat{f}_t^A K_t - i_t \left(p + \frac{\chi}{2} i_t \right) dt \right] \\ &\geq E \left(e^{-r\tau} P_\tau \right) + E \left[\int_0^\tau \widehat{f}_t^A K_t - i_t \left(p + \frac{\chi}{2} i_t \right) dt \right] \end{aligned}$$

Thus there is no set of investment policy / stopping policies that can yield more than P_0 . Moreover, the conjectured investment and stopping policies turn the above inequalities into equalities. Thus, the conjectured equilibrium prices and policies form an equilibrium.

6.4 Proofs for section 3.2.3

Proof. (*Proposition 3*) Ito's Lemma for continuously differentiable functions implies that $b(g_t^A)$ satisfies:

$$e^{-(r+\delta)\Delta} b_{t+\Delta} = b_t + \int_t^{t+\Delta} e^{-(r+\delta)(s-t)} (\mathcal{A}b_s) ds + M_{t+\Delta} \quad (72)$$

where M_{t+T} is a martingale difference satisfying:

$$E [M_{t+\Delta} | \mathcal{F}_t] = 0$$

and $\mathcal{A}b_s$ is given as:

$$\frac{\sigma_g^2}{2} b_g - \rho g b_g - (r + \delta) b$$

It is easy to show by using the definition of $y_1(\cdot)$ that:

$$\mathcal{A}b_s = -\frac{(r + \delta + \rho)}{r + \delta + \lambda} g_t^A \mathbf{1}\{g_t^A > 0\} < 0$$

Combining this with the fact that:

$$q_t = \frac{\bar{f}}{r + \delta} + \frac{\widehat{f}_t^A - \bar{f}}{r + \delta + \lambda} + b \left(g_t^A \right)$$

leads to the first assertion. The last assertions can be proved by using the results in Fournie et. al (1999). In particular:

$$\frac{\partial Z}{\partial g} = E \left[\int_t^{t+\Delta} e^{-(r+\delta+\rho)(s-t)} \mathbf{1}\{g_s^A > 0\} ds \right]$$

which is clearly positive. A similar method can be used to show that $\frac{\partial Z}{\partial \sigma_g} > 0$. ■

6.5 Proofs for section 3.2.4

Proof. The proof is a straightforward application of Ito's Lemma to

$$P_t = V_t + s_t$$

taking into account equation (69). To establish the rest of the result one can focus only on:

$$\tilde{\Xi} = CE^A \left[\int_t^{t+\Delta} e^{-r(s-t)} g_s^A \left(\hat{f}_s^A - \bar{f} \right) 1\{g_s^A > 0\} ds | \mathcal{F}_t \right]$$

and pass the expectation inside the integral and use the independence of \hat{f}_s^A and g_s^A to get that:

$$\tilde{\Xi} = C \left[\int_t^{t+\Delta} e^{-r(s-t)} \left(E^A \left(\hat{f}_s^A - \bar{f} \right) | \mathcal{F}_t \right) \left(E^A g_s^A 1\{g_s^A > 0\} | \mathcal{F}_t \right) ds \right]$$

and since

$$E^A \left(\hat{f}_s^A - \bar{f} | \mathcal{F}_t \right) = \left(\hat{f}_t^A - \bar{f} \right) e^{-\lambda s}$$

one gets:

$$\tilde{\Xi} = \left(\hat{f}_t^A - \bar{f} \right) C \left[\int_t^{t+\Delta} e^{-(r+\lambda)(s-t)} \left(E^A g_s^A 1\{g_s^A > 0\} | \mathcal{F}_t \right) ds \right]$$

so that:

$$\tilde{\Xi}_{\hat{f}_t^A} = C \left[\int_t^{t+\Delta} e^{-(r+\lambda)(s-t)} \left(E^A g_s^A 1\{g_s^A > 0\} | \mathcal{F}_t \right) ds \right] > 0$$

This term is of the same form as the term obtained in Proposition 3 and the rest of the results can be proved in an identical manner. ■

6.6 Proofs for section 3.2.5

In order to give a proof of (22) it is useful to start by modelling the evolution of the capital stock for discrete time intervals:

$$K_T = K_t e^{-\delta(T-t)} + \int_t^T e^{-\delta(T-s)} i_s ds + \varepsilon_{it}$$

where ε_{it} captures adjustment cost shocks⁶⁹ and satisfies a strict exogeneity condition:

$$E(\varepsilon_{it} | q_{t=0..T}) = 0$$

⁶⁹I haven't modelled adjustment costs and time variation in capital prices explicitly. Such a modification is easy to do. One just assumes that the adjustment cost technology is given by:

$$\frac{\chi}{2} (i_t + n_t)^2$$

where n_t is some stochastic process. Similarly one can introduce variability in prices by modifying the dividend stream to:

$$dD_s - \frac{\chi}{2} (i_t + n_t)^2 - p_t i_t$$

so that:

$$E\left(K_T - K_t e^{-\delta(T-t)} | \mathcal{F}_t\right) = E\left(\int_t^T e^{-\delta(T-s)} i_s ds | \mathcal{F}_t\right) = \quad (73)$$

$$= \frac{1}{\chi} E\left(\int_t^T e^{-\delta(T-s)} q_s ds | \mathcal{F}_t\right) + \tilde{C} \quad (74)$$

where $\tilde{C} = -\frac{p \int_t^T e^{-\delta(T-s)} ds}{\chi}$. Now if $q_t = \frac{\bar{f}}{r+\delta} + \frac{\widehat{f_t^A} - \bar{f}}{r+\delta+\lambda}$ (i.e. marginal q is equal to the long run fundamental notion of q) it is easy to demonstrate that

$$E\left(\int_0^T e^{-\delta(T-s)} q_s ds | \mathcal{F}_t\right) = B_1 + B_2 q_t \quad (75)$$

where $B_2 \rightarrow \int_t^T e^{-\delta(T-s)} ds$ as $\lambda \rightarrow 0$ and $B_2 \rightarrow 0$ as $\lambda \rightarrow \infty$. This is in essence the Barnett and Sakellaris (1999) critique. Normalizing $T - t = 1$, it will be the case that $B_2 < 1$ and thus the estimate that will be obtained in a regression of i_t on *beginning of period* marginal "q" will produce a downwards biased estimate of $\frac{1}{\chi}$ in the sense that OLS will consistently estimate $B_2 \frac{1}{\chi}$. If depreciation (δ) is small and fundamentals persist for a while, (i.e λ is small) then B_2 will be not much different than 1.⁷⁰

Under the assumption that investment only reacts to long run fundamental q one can rewrite (74) as:

$$E(I_t | \mathcal{F}_{t-1}) = \frac{1}{\chi} Q_{t-1}^F + \tilde{C} \quad (76)$$

where -in order to simplify notation- I have defined:

$$I_t = K_t - K_{t-1} e^{-\delta}$$

If adjustment costs are independent of the capital stock all of these modifications affect the rents to the adjustment technology only.

⁷⁰In the presence of predictability this issue becomes even more involved because equation (18) has to be augmented by terms involving g_t^A which is correlated with beginning of period q_t . One can base a test on this fact by testing the orthogonality between beginning of period q_t and the error in the regression of investment on beginning of period q_t as proposed by Chirinko and Schaller (1996) equation (15). However, this is a test of whether bubbles exist, not whether they influence investment. It also appears to be less powerful than a test based on equation (18). The reason is that predictability in the Chirinko and Schaller (1996) test is multiplied by $\frac{1}{\chi}$ and the error term consists of the (possibly biased expectations error) and the adjustment cost shock. These two facts might make it difficult to observe predictability even if it exists. A practical way to obtain consistent estimates of χ is to approximate $\int_t^T e^{-\delta(T-s)} q_s ds$ by a weighted average of beginning and end of period q_t , project this quantity on beginning of period quantities and then use the predicted values in the regression. In other words to estimate two stage least squares. I used such an approach too in the empirical section of the paper and the results were unaltered.

and

$$Q_{t-1}^F = E \left(\int_{t-1}^t e^{-\delta(t-s)} q_s^F ds | \mathcal{F}_{t-1} \right)$$

As demonstrated before, Q_{t-1}^F can be expressed as $B_1 + B_2 q_{t-1}$ under the assumptions of the model. Therefore one can rewrite (76) as

$$E(I_t | \mathcal{F}_{t-1}) = \frac{1}{\chi} B_1 q_{t-1}^F + \tilde{C} \quad (77)$$

Leading this once, one gets:

$$E(I_{t+1} | \mathcal{F}_t) = \frac{1}{\chi} B_1 q_t^F + \tilde{C} \quad (78)$$

However notice that q_t^F and q_{t-1}^F are related by:

$$q_{t-1}^F = E \left[\int_{t-1}^t e^{-(r+\delta)(s-(t-1))} \hat{f}_s ds + e^{-(r+\delta)} q_t^F | \mathcal{F}_t \right] \quad (79)$$

so that (77),(78), and (79) are related by:

$$E \left[I_t - e^{-(r+\delta)} I_{t+1} - \frac{1}{\chi} B_1 \int_{t-1}^t e^{-(r+\delta)(s-(t-1))} \hat{f}_s ds + \tilde{C}(1 - e^{-(r+\delta)}) | \mathcal{F}_{t-1} \right] = 0$$

For small r, δ this Euler Equation can be well approximated in terms of the observed profit rate:

$$E \left[I_t - e^{-(r+\delta)} I_{t+1} - \frac{1}{\chi} B_1 \pi_t + \tilde{C}(1 - e^{-(r+\delta)}) | \mathcal{F}_{t-1} \right] = 0 \quad (80)$$

where the profit rate is given as:

$$\pi_t = \int_{t-1}^t \frac{dD_s}{K_s} = \int_{t-1}^t f_s ds + \sigma_D \int_{t-1}^t dZ_s^D$$

The results in the text follow once one defines : $C = \tilde{C}(1 - e^{-(r+\delta)})$ and $\frac{1}{\chi} B_1 = \frac{1}{\chi}$

Generalizing to arbitrary linear homogenous adjustment cost technologies

To generalize the results to arbitrary linear homogenous adjustment cost technologies and an arbitrary number of investor groups it will be most useful for expositional reasons to consider a discrete time setup and focus on a quadratic adjustment cost function for simplicity. Moreover I will assume 0 depreciation and a price of investment of 1. Once again the equilibrium price, investment and selling times will have to satisfy:

$$V_t \stackrel{d}{=} D_t + P_t = \max_{j \in J} \sup_{\tau, i_s} E^j \left(\sum_{s=0}^{\tau} d^s D_s + d^\tau V_\tau \right) \quad (81)$$

where $j \in J$ is indexing the various groups of investors who have heterogenous beliefs, $d = \frac{1}{1+r}$, D_t is defined as

$$D_t = \left[f_t - \frac{i_{t+1}}{K_t} - \frac{\chi}{2} \left(\frac{i_{t+1}}{K_t} \right)^2 \right] K_t$$

and:

$$K_{t+1} = K_t + i_{t+1}$$

I will also assume that investment is determined at the beginning of the period. The basic idea is to show the following:

Lemma 4 *A set of prices, investment policies and stopping policies satisfies (81) if and only if it satisfies:*

$$V_t = D_t + P_t = \max_j \sup_{i_{t+1}} E^j \left(\left[f_t - \frac{i_{t+1}}{K_t} - \frac{\chi}{2} \left(\frac{i_{t+1}}{K_t} \right)^2 \right] K_t + dV_{t+1} \right) \quad (82)$$

Proof. Lemma (4). The proof is a generalization of the result shown in Harrison and Kreps (1978) and is available upon request. ■

With Lemma (4) the rest of the steps follow essentially standard arguments. One can show that marginal q is equal to average q , where q is now given by the recursion:

$$q_t = E^{j_t^*} d \left(f_{t+1} + \frac{\chi}{2} \left(\frac{i_{t+2}}{K_{t+1}} \right)^2 + q_{t+1} \right)$$

, j_t^* is given as

$$j_t^* = \arg \max_j \sup_{i_t} E^j \left(\left[f_t - \frac{i_{t+1}}{K_t} - \frac{\chi}{2} \left(\frac{i_{t+1}}{K_t} \right)^2 \right] K_t + dV_{t+1} \right)$$

and optimal investment is:

$$\frac{i_{t+1}}{K_t} = \frac{q_t - 1}{\chi}$$

From here it is not difficult to derive that marginal and average q are equal by standard arguments (see e.g Chirinko (1993)). Suppose that now one were to define:

$$\Pi_t = f_t + \frac{\chi}{2} \left(\frac{i_{t+1}}{K_t} \right)^2 = \frac{\partial D_t}{\partial K_t}$$

If one assumes the presence of a rational agent A in the model it is immediate that under her beliefs it will no longer be the case that:

$$E^A [q_t - d[\Pi_{t+1} + q_{t+1}] | \mathcal{F}_t] = 0 \quad (83)$$

nor that:

$$E^A \left[\frac{i_{t+1}}{K_t} - d \left(\frac{i_{t+2}}{K_{t+1}} + \frac{1}{\chi} \Pi_t \right) + C | \mathcal{F}_t \right] = 0 \quad (84)$$

However, if investment is determined by a long termist rational investor (84) will hold even if (83) fails for the reasons explained in the text.

6.7 Proofs for section 4.4

For this section I use the standard assumption in the literature that adjustment costs are linear homogenous in capital, time is discrete and the adjustment cost technology contains both time and individual fixed effects. Then by steps similar to 3.2.5 one can derive that:

$$E \left[\frac{I_{i,t}}{K_{i,t-1}} - e^{-(r+\delta)} \frac{I_{i,t+1}}{K_{i,t}} - \left(\alpha_i (1 - e^{-(r+\delta)}) + \zeta_t - e^{-(r+\delta)} \zeta_{t+1} + \frac{1}{\chi} \pi_{i,t} \right) | \mathcal{F}_{t-1} \right] = 0 \quad (85)$$

which can be rewritten as:

$$\frac{I_{i,t+1}}{K_{i,t}} = e^{(r+\delta)} \frac{I_{i,t}}{K_{i,t-1}} - \tilde{\alpha}_i (e^{(r+\delta)} - 1) - \zeta_t e^{(r+\delta)} + \zeta_{t+1} - \frac{1}{\chi} e^{(r+\delta)} \pi_{i,t} + \varepsilon_{it}$$

where

$$E [\varepsilon_{it} | \mathcal{F}_{t-1}] = 0 \quad (86)$$

In particular:

$$E [\varepsilon_{it} | q_{1..t-1}] = 0$$

I am going to formulate the test in first differences in order to eliminate fixed effects, so that:

$$\Delta \left(\frac{I_{i,t+1}}{K_{i,t}} \right) = e^{(r+\delta)} \Delta \left(\frac{I_{i,t}}{K_{i,t-1}} \right) - \zeta - e^{(r+\delta)} \kappa (\Delta \pi_{i,t}) + \varepsilon_{it+1} - \varepsilon_{it} \quad (87)$$

where I have defined for convenience:

$$\begin{aligned} \kappa &= \frac{1}{\chi} \\ \zeta &= \Delta \zeta_{t+1} - \Delta \zeta_t e^{(r+\delta)} \end{aligned}$$

The unknown parameters are $e^{(r+\delta)}$, ζ and κ . If one knew these parameters then one could determine $\varepsilon_{it+1} - \varepsilon_{it}$. To utilize the entire sample I estimate both κ and $e^{-(r+\delta)}$ from the system of Euler relations:

$$\begin{aligned} q_{i,t} &= e^{-(r+\delta)} E[\pi_{i,t} + q_{i,t+1} | \mathcal{F}_t] \\ \frac{I_{i,t}}{K_{i,t-1}} &= \alpha_i + \zeta_t + \kappa q_{i,t-1} + \varepsilon_{it} \end{aligned} \quad (88)$$

estimated on all data in the control group.⁷¹ Then one can substitute the estimated parameters into (87) to get:

$$\Delta \left(\frac{I_{i,1926}}{K_{i,1925}} \right) = e^{\widehat{(r+\delta)}} \Delta \left(\frac{I_{i,1925}}{K_{i,1924}} \right) - \zeta - \widehat{\kappa} e^{\widehat{(r+\delta)}} (\Delta \pi_{i,1925}) + \varepsilon_{i1926} - \varepsilon_{i1925}$$

To test the hypothesis of interest one can modify this equation to:

$$y \stackrel{d}{=} \Delta \left(\frac{I_{i,1926}}{K_{i,1925}} \right) - e^{\widehat{(r+\delta)}} \Delta \left(\frac{I_{i,1925}}{K_{i,1924}} \right) + \widehat{\kappa} e^{\widehat{(r+\delta)}} (\Delta \pi_{i,1925}) = \beta q_{i,1924} + \zeta + \varepsilon_{i1926} - \varepsilon_{i1925} \quad (89)$$

According to H_1 , β should be 0. Moreover, q_{1924} should be orthogonal to the errors, so that an OLS regression of y on $q_{i,1924}$ will produce a consistent estimate of β . Of course standard errors need to be adjusted for the first step error. I undertake this adjustment by using the results in Newey and McFadden (2000) section 6.

⁷¹I use only the control group to simplify the computation of standard errors for the two step estimator of β .