Asset Prices with Heterogeneity in Preferences and Beliefs*

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Abstract

In this paper, we study asset prices in a dynamic, general-equilibrium Lucas endowment economy where agents have power utility and differ with respect to both beliefs and their preference parameters for time preference and risk aversion. Our main contribution is to solve in closed form for the following quantities: optimal consumption and portfolio policies of individual agents; the riskless interest rate and market price of risk; the stock price, equity risk premium, and volatility of stock returns; and, the term structure of interest rates. Our solution allows us to identify the strengths and limitations of the model with heterogeneity in both preferences and beliefs. We find that beliefs about the growth rate of aggregate endowment that are pessimistic on average (across investors) lead to a significant increase in the market price of risk; and, heterogeneity in risk aversion increases stock-return volatility. Consequently, the equity risk premium, which is the product of the market price of risk and stock return volatility, is considerably higher in the model where average beliefs are pessimistic and risk aversions are heterogeneous, and this is accompanied neither by an increase in the level nor the volatility of the short-term riskless rate. The main limitation of the model is that it is stationary only for a restricted set of parameter values, and for these parameter values one can obtain a high market price of risk and equity risk premium but not excess stock return volatility.

1 Introduction and Motivation

Two key characteristics of economic agents are their beliefs and preferences. Our objective in this paper is to study the effect of heterogeneity in *both* of these characteristics on the choices of individual agents and the resulting asset prices. The main contribution of our work is to solve *in closed form* for consumption policies, portfolio policies, and the stock and bond prices in a general equilibrium stochastic dynamic exchange economy with heterogeneous agents who have power utility.[1] This allows us to identify the strengths and limitations of the model with heterogeneity in both preferences and beliefs. We find that, compared to the standard representative agent model, both heterogeneous preferences and heterogeneous beliefs (that are pessimistic on average) play a significant role in improving the ability of the model to match properties of asset returns. In particular, heterogeneity in risk aversion increases stock return volatility relative to the volatility of aggregate dividends, but heterogeneity in beliefs has a negligible effect. Neither heterogeneity in risk aversion nor heterogeneity in beliefs can generate a large enough market price of risk; to get close to the empirically observed market price of risk, we need average beliefs to be pessimistic. Thus, heterogeneous risk aversion together with average beliefs that are pessimistic, generate a significantly larger equity risk premium compared to the standard representative-agent model. Moreover, the average risk aversion in the model is countercyclical, just like the Campbell and Cochrane (1999) model, and so asset returns have the appropriate cyclical properties. The main limitation of the model is that for parameter values for which the model is stationary, one can obtain a high market price of risk and equity risk premium but not excess stock-return volatility.

The importance of studying models with heterogeneous agents rather than a representative agent has been recognized recently by both policymakers and academics. For instance, the April 15, 2010 issue of the *Economist* describing the Soros-sponsored conference on "The Economic Crisis and the Crisis in Economics" says that, "The conference rehearsed many familiar complaints, bashing ... the use of representative agents (a kind of economic Everyman, whose behavior mimics the macroeconomy in microcosm)." Hansen (2010) in his talk at this conference lists one of the challenges for macroeconomic models to be "Building in explicit heterogeneity in beliefs, preferences ...." Stiglitz (2010) in his presentation at the same conference also criticizes the representative agent model and highlights the importance of heterogeneous agents as a key modeling challenge. Sargent (2008), in his presidential address to the American Economic Association, discusses extensively the implications of the common beliefs assumption for policy, and Hansen (2007, p. 27) in his Ely lecture says: "While introducing heterogeneity among investors will complicate model solution, it has intriguing possibilities. ... There is much more to be done." Empirical work by

[1] In particular, we obtain the following quantities in closed form: the equilibrium consumption allocation across agents and its dynamics over time; the optimal portfolios of individual investors; the state price density and its dynamics, which are characterized in terms of the riskless interest rate and the market price of risk; the stock price, the equity risk premium, and the volatility of stock returns; and, the term structure of interest rates and the term premium.

Beber, Buraschi, and Breedon (2009), Berrada and Hugonnier (2010), Buraschi and Jiltsov (2006), Buraschi, Trojani, and Vedolin (2009, 2010), and Ziegler (2007) also suggests the importance of allowing for heterogeneous beliefs and preferences in asset-pricing models.

A key contribution of our paper is to demonstrate how one can obtain a closed-form solution to the consumption sharing rule for agents with heterogeneous beliefs and preferences without restricting the risk aversion of the two agents to special values.[2] In the case of two agents, the consumption-sharing rule is a non-linear algebraic equation, which reduces to a polynomial of degree η if the ratio of the risk aversion of one agent to that of the other is a natural number. If η equals two, three or four, then this polynomial equation can of course be solved in closed-form. We show how to construct a closed-form solution for all real values of η. Central to our approach is a theorem due to Lagrange. Given the ubiquity of nonlinear sharing rules in solutions to problems in economics and finance (see Peluso and Trannoy (2007) for examples of such problems), the approach we develop can be applied also to other problems, which previously would have called for numerical methods.

The paper that is closest to our work is Cvitanić, Jouini, Malamud, and Napp (2009), which also studies asset prices in an economy where agents have expected utility and differ with respect to both beliefs and their preference parameters. Their paper provides *bounds* on asset prices and characterizes prices *in the limit* when only one agent survives. However, it does not provide closed-form solutions for these quantities. In fact, Cvitanić and Malamud (2009b, p. 3) write that:

> "when risk aversion is heterogeneous, SDF [stochastic discount factor] is the solution to highly non-linear equation (1) [in their paper], and no explicit solution is possible, except for some very special values of risk aversion; see, for example, Wang (1996)."

In contrast to Cvitanić, Jouini, Malamud, and Napp (2009), we provide a closed-form solution for the stochastic discount factor without restricting the risk aversion of the two agents to special values. In particular, we show how the stochastic discount factor can be expressed as a weighted average of stochastic discount factors from a set of underlying *single-agent* economies, each with a *constant* market price of risk and risk-free rate. We should point out that, in contrast to our analysis that is for the case of two agents, the limit analysis of Cvitanić, Jouini, Malamud, and Napp (2009) studies an economy with more than two agents.

Most of the other papers in the existing literature with heterogeneous agents allow for *either* differences in beliefs *or* differences in preferences. We first discuss the literature that considers heterogeneity in beliefs and then the literature that considers differences in preferences. Essentially, there are two ways to generate heterogeneity in beliefs. In the first approach, agents receive different

[2] Our work can be viewed as complementary to that of Calin, Chen, Cosimano, and Himonas (2005), who provide an analytic representation (that is, a convergent power series) for the price-dividend function of one state variable in an economy with a *single* representative agent whose utility function displays habit formation.

information. This is the classical approach, adopted in the early noisy-rational-expectations literature (see, for instance, Grossman and Stiglitz (1980), Hellwig (1980), Wang (1993), and Shefrin and Statman (1994)). In this class of models, one group of (informed) agents receives private signals and then there is a second group of agents (noise-traders), which trades for exogenous reasons and thereby prevents the price from fully revealing the private information of the informed agents. The second approach for generating heterogeneity, which is the one we adopt, is to have agents who "agree to disagree" about some aspect of the underlying economy, and in this class of models it is assumed that agents do not learn from each other's behavior. Morris (1995) provides a good philosophical discussion of this modeling approach. Examples of papers using this approach include Basak (2000), Beber, Buraschi, and Breedon (2009), Berrada (2006), Borovička (2009), Buraschi and Jiltsov (2006), Buraschi, Trojani, and Vedolin (2009, 2010), Cecchetti, Lam, and Mark (2000), David (2008), David and Veronesi (2002), Duffie, Garleânu, and Pedersen (2002) Dumas, Kurshev, and Uppal (2009), Gallmeyer (2000), Gallmeyer and Hollifield (2008), Kogan, Ross, Wang, and Westerfield (2006), Scheinkman and Xiong (2003), Veronesi (1999), Xiong and Yan (2009), Yan (2008), and Zapatero (1998).[3] Excellent reviews of this literature are provided in Basak (2005) and Jouini and Napp (2007).

We now discuss the literature on the effect of heterogeneous preferences on asset prices. The effect of different time-discount factors on efficient allocation of consumption is studied in Gollier and Zeckhauser (2005). The effect of heterogeneity in risk aversion on asset prices is examined in several papers, most of which assume that investors have expected utility. For example, Dumas (1989) studies the riskfree rate and the risk premium in a production economy; Wang (1996) examines the term structure in an exchange economy; Basak and Cuoco (1998) and Kogan, Makarov, and Uppal (2007) analyze the effect of constraints on borrowing and short-sales on the equity risk premium in an exchange economy; Bhamra and Uppal (2009) and Tran (2009) examine the volatility of stock market returns; Benninga and Mayshar (2000) and Weinbaum (2001) study option prices; Longstaff and Wang (2009) investigate the relation between open interest in the bond market and stock market returns; Cvitanić and Malamud (2009a,b,c) consider equilibrium with multiple heterogeneous traders who maximize utility of only terminal wealth; and, Garleânu and Panageas (2008) study the effect of heterogeneous preferences in an overlapping-generations model that leads to a stationary equilibrium. In contrast to these papers that assume investors have expected utility, Chan and Kogan (2002) and Xiouros and Zapatero (2010) study asset prices in an economy where agents have "catching-up-with-the-Joneses" preferences, where habit formation ensures that the model is stationary. And, finally there are papers that work with Epstein and Zin (1989) recursive preferences that allow for a distinction between risk aversion and the elasticity of intertemporal substitution. For example, Guvenen (2005), studies asset pricing in a model with

[3] Yan (2008) also studies a model where agents have both heterogeneous beliefs and preferences, but he solves for asset prices in terms of exogenous variables only for the case where both agents have the same risk aversion, which is a natural number (see his Proposition 3).

heterogeneity in elasticity of intertemporal substitution, Isaenko (2008) studies the term structure in a model where agents differ in both their risk aversion and elasticity of intertemporal substitution, and Gomes and Michaelides (2008) study portfolio decisions of households and asset prices in a model where agents are heterogeneous not just in terms of preferences but are also exposed to uninsurable income shocks in the presence of borrowing constraints.

When there are multiple agents who differ in their risk aversion, there is no paper in the literature that provides a complete characterization of equilibrium that is exact and entirely analytical. For example, for the case of expected utility, Wang (1996) provides closed form expressions for only particular parameter values; Kogan and Uppal (2001) characterize the equilibrium in production and exchange economies approximately using perturbation analysis in the neighborhood of log utility; Bhamra and Uppal (2009) and Tran (2009) study stock-market-return volatility, but without solving explicitly for volatility; Dumas (1989) solves numerically for the interest rate in a production economy; for the case of "catching-up-with-the-Joneses" preferences, Chan and Kogan (2002) rely on numerical solutions, and the working-paper version of Chan and Kogan (2002) provides approximate analytic results in the neighborhood of log utility using perturbation analysis. Xiouros and Zapatero (2010) provide an expression for the value function of the central planner assuming a Gamma distribution for the risk tolerances of the investors, but asset prices are obtained using numerical methods. The models in Guvenen (2005), Isaenko (2008), and Gomes and Michaelides (2008) are also solved using numerical methods.

To summarize, the main contribution of our paper is that, in contrast to the existing literature on general equilibrium models of asset pricing that considers either heterogeneous preferences *or* heterogeneous beliefs, we allow for heterogeneity in preferences *and* beliefs, we do *not* restrict the preference parameters of the agents to particular values, and we solve in *closed form* not just for the interest rate and market price of risk, but also for the stock price, equity premium, volatility of stock market returns, and the term structure of interest rates. We show explicitly that our results nest the results in the models that consider an exchange economy with agents who have expected utility with different degrees of risk aversion, such as Wang (1996) and Bhamra and Uppal (2009), and that they nest also the results in models where agents have expected utility with heterogeneous beliefs, for instance, Basak (2005) and Yan (2008). A major advantage of our characterization of equilibrium is that it allows us to identify which empirical features of asset returns can be explained by heterogeneity in preferences and/or beliefs, as discussed in the opening paragraph of the introduction.

The rest of the paper is arranged as follows. In Section 2, we describe our model of an exchange economy with heterogenous agents. The equilibrium consumption allocation, derived by solving the problem of a "central planner", is given in Section 3, which also includes a discussion of survival of the agents and stationarity of the equilibrium. We derive the state price density and its dynamics in Section 4. A full characterization of asset prices and the properties of asset returns is provided in Section 5. We conclude in Section 6. Our main results are highlighted in propositions, results for special cases are given in corollaries,[4] and detailed proofs for all the results are provided in Appendix A, while Appendix B contains two lemmas that we use to derive these results, Appendix C contains a statement of Lagrange's Theorem, and Appendix D gives the wealth of individual investors along with their optimal portfolios, which correspond to the solution of the central planner's problem.[5]

2 The model

In this section, we describe the features of the economy we are considering. Below, we explain our assumptions about the information structure and the endowment process, the financial assets in the economy, the beliefs and preferences of agents, the definition of equilibrium, and how this equilibrium can be identified by solving the problem of a central planner, whose utility is a weighted average of the utilities of the individual agents, where the weights are stochastic.

We consider a continuous-time, pure-exchange economy with an infinite time horizon. There is a single consumption good that serves as the numeraire. It is modeled as an exogenously specified endowment process. There are two types of investors, $k \in \{1, 2\}$. We adopt the convention of subscripting by k the quantities related to Agent k, where $k \in \{1, 2\}$. Each investor has constant relative risk averse utility (CRRA). The two types of agents are allowed to differ in their rates of time preference and relative risk aversions. Furthermore, the two types of agents have different beliefs about the expected growth rate of the endowment, which they do not update.[6] In summary, our model differs from the standard Lucas (1978) model along two dimensions: one, preferences are heterogeneous; two, agents may not have the correct beliefs, and the beliefs of one agent may differ from those of the other.

2.1 The information structure and endowment process

The uncertainty in the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ on which is defined a one-dimensional Brownian motion Z. The economy is modeled as being endowed with a single non-storable consumption good. The true evolution of the aggregate endowment, Y, which in our model is equivalent to both aggregate dividends and aggregate consumption, is:

$$\frac{dY_t}{Y_t} = \mu_Y \, dt + \sigma_Y \, dZ_t, \; Y_0 > 0,$$

[4]There are three special cases that we consider: one, where investors have identical beliefs but different preferences (risk aversion and rate of time preference); two, where investors differ in beliefs but have identical risk aversion; and three, where the investors differ in beliefs but have identical risk aversion, which is a natural number.

[5]We recognize that these appendices increase the length of the manuscript, but we have opted to provide details of our derivations so that they can be verified with ease; at a later stage, one could shorten/remove the appendices.

[6]We model beliefs as in Basak (2005); see Section 2.1 and Remark 1 of his paper about the generality of this specification and how it can be extended further.

in which μ_Y and σ_Y are constants.

2.2 Financial assets

There are two financial assets in the economy: a risky asset (stock) with one share outstanding and a locally riskfree bond in zero net supply. The stock is a claim on the aggregate endowment. The price of the stock, which can be interpreted as the market portfolio, is denoted S_t, and its *cumulative* return, R_t, which consists of capital gains plus dividends, is described by the process:

$$\frac{dS_t + Y_t dt}{S_t} = dR_t = \mu_{R,t}\, dt + \sigma_{R,t}\, dZ_t.$$

The price of the locally riskfree bond is B_t, and it's riskfree return r_t is described by the process

$$\frac{dB_t}{B_t} = r_t\, dt.$$

The expected return on the stock, $\mu_{R,t}$, the volatility of stock returns, $\sigma_{R,t}$, and the locally riskfree rate, r_t, will be determined endogenously in equilibrium.

2.3 Beliefs of the two agents

Agent k believes that the expected growth rate of the endowment process takes the constant value, $\mu_{Y,k}$. Therefore, Agent k's beliefs can be represented by an exponential martingale $\xi_{k,t}$, given by $\xi_{k,t} = e^{-\frac{1}{2}\sigma^2_{\xi,k}t+\sigma_{\xi,k}Z_t}$, where $\sigma_{\xi,k} \equiv \frac{\mu_{Y,k}-\mu_Y}{\sigma_Y}$.[7] Hence, by Girsanov's Theorem, Agent k believes that the process for aggregate endowments is

$$\frac{dY_t}{Y_t} = \mu_{Y,k}\, dt + \sigma_Y\, dZ_{k,t},$$

where $Z_{k,t} = Z_t - \sigma_{\xi,k}t$ is a standard Brownian motion under $\mathbb{P}^k$. Hence, we see that under $\mathbb{P}^k$, which represents Agent k's beliefs, the expected growth rate of aggregate endowment is $\mu_{Y,k}$.[8]

We quantify the *level of disagreement* between the two agents via the process, ξ_t, where $\xi_t \equiv \frac{\xi_{2,t}}{\xi_{1,t}} = e^{-\frac{1}{2}(\sigma^2_{\xi,2}-\sigma^2_{\xi,1})t+(\sigma_{\xi,2}-\sigma_{\xi,1})Z_t}$, and its dynamics are

$$\frac{d\xi_t}{\xi_t} = \mu_\xi dt + \sigma_\xi dZ_t,$$

where

$$\begin{aligned}\mu_\xi &\equiv -\sigma_{\xi,1}(\sigma_{\xi,2} - \sigma_{\xi,1}),\\ \sigma_\xi &\equiv (\sigma_{\xi,2} - \sigma_{\xi,1}).\end{aligned}$$

[7] The exponential martingale, $\xi_{k,t}$, defines the probability measure $\mathbb{P}^k$ on $(\Omega, \mathcal{F})$, via $\mathbb{P}^k(e_T) = E_t[1_{e_T}\xi_{k,T}]$, $\forall t, T \in [0,\infty)$, $t \leq T$, where e_T is an event which occurs at time T and $\mathbb{P}^k(e_T)$ is the probability of its occurrence based on information known at time t.

[8] Note that the measures $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}$ are all equivalent; that is, they agree on which events are impossible.

2.4 Preferences of the two agents

The consumption of Agent k at instant u is denoted by $C_{k,u}$ and the instantaneous utility from consumption is assumed to be time additive and given by a power function:

$$U_k(C_{k,u}) \equiv e^{-\beta_k u} \frac{C_{k,u}^{1-\gamma_k}}{1-\gamma_k},$$

where β_k is the constant subjective discount rate (that is, the rate of time preference) and γ_k is the degree of relative risk aversion. Without loss of generality, we assume that Agent 1's relative risk aversion is less than that of Agent 2: $\gamma_1 < \gamma_2$.

Given her beliefs, represented by the measure $\mathbb{P}^k$, the expected lifetime utility of Agent k at time t from consuming $C_{k,u}$ is given by

$$V_{k,t} = E_t^k \left[\int_t^\infty e^{-\beta_k(u-t)} \frac{C_{k,u}^{1-\gamma_k}}{1-\gamma_k} du \right], \tag{1}$$

where E_t^k denotes the time-t conditional expectation operator with respect to the measure $\mathbb{P}^k$. Existence of a solution requires that the integral in (1) is well defined, for which the condition is:

$$\beta_k > (1-\gamma_k)\mu_Y - \frac{1}{2}\gamma_k(1-\gamma_k)\sigma_Y^2.$$

2.5 The optimization problem of each agent

Each agent k is assumed to have an initial allocation of a_k shares of the stock, with $a_1 + a_2 = 1$. Thus, the initial wealth of agent k is $a_k S_0$. The problem of agent k is to maximize lifetime utility, given by $V_{k,0}$ in (1), subject to a static budget constraint, which restricts the present value of all future consumption to be no more than the initial wealth of each agent:[9]

$$E_0^k \left[\int_0^\infty \frac{\pi_{k,u}}{\pi_{k,0}} C_{k,u} du \right] \leq a_k S_0, \tag{2}$$

in which $\pi_{k,u}$ is the marginal utility of investor k at date u:

$$\pi_{k,u} \equiv \frac{\partial U(C_{k,u})}{\partial C_{k,u}} = e^{-\beta_k u} C_{k,u}^{-\gamma_k}. \tag{3}$$

[9]The budget constraint for Agent k in (2) is written in terms of the state prices perceived by this agent; one could write an equivalent expression in terms of the state prices (and expectation) of the central planner.

2.6 The equilibrium

The notion of equilibrium that we use is an extension of the equilibrium in the single-agent model of Lucas (1978). Both agents optimize their expected lifetime utility and all markets must clear. So, in equilibrium, the two individuals consume all of the aggregate endowment, and in the financial market the two investors together hold all the shares that are a claim on aggregate endowment, while their aggregate holding of the zero-supply riskfree bond must net to zero.

2.7 The central planner

Given our assumption that investors can trade in a stock and a locally riskfree asset, financial markets are dynamically complete relative to the filtrations of the two agents. When markets are dynamically complete, one can solve for equilibrium consumption policies using a "central-planner," whose social welfare function is a weighted average of the value functions of individual agents, as shown in Basak (2005). In contrast to the case of identical beliefs, if agents have heterogeneous beliefs, Basak (2005) shows that the weights used to construct the central planner's utility function are stochastic. The central planner's utility function is given by

$$\sup_{C_1+C_2\leq Y} \sum_{k=1}^{2} \lambda_{k,t}\, U_k(C_{k,t}), \quad \text{where } \lambda_{k,t} = \lambda_{k,0}\xi_{k,t}. \tag{4}$$

Note that even though the model described above does not have any exogenous source of time variation, the equilibrium in this model will not be static; because the distribution of wealth between the two agents is stochastic, there will be endogenously generated dynamics arising as a consequence of the stochastic wealth distribution.

3 Equilibrium Consumption Allocations and Stationarity

In the first part of this section, we derive exact closed-form expressions for equilibrium consumption allocations and also characterize the dynamics of the equilibrium consumption-sharing rule. In the second part of this section, we identify the conditions under which the equilibrium is stationary, that is, both agents survive in the long run.

3.1 The consumption-sharing rule and its dynamics

The first-order condition for optimal consumption, from the central planner's problem in (4), gives the consumption sharing rule, which shows how aggregate consumption is allocated between the two agents in equilibrium:

$$
\begin{aligned}
\lambda_{1,t}\, e^{-\beta_1 t} C_{1,t}^{-\gamma_1} &= \lambda_{2,t}\, e^{-\beta_2 t} C_{2,t}^{-\gamma_2}, \qquad (5)\\
(\lambda_{1,0}\, \xi_{1,t})\, e^{-\beta_1 t} C_{1,t}^{-\gamma_1} &= (\lambda_{2,0}\, \xi_{2,t})\, e^{-\beta_2 t} C_{2,t}^{-\gamma_2}.
\end{aligned}
$$

In order to solve explicitly for the equilibrium allocations, we write Agent k's consumption share as $\nu_{k,t} = \frac{C_{k,t}}{Y_t}$, where $0 \le \nu_k \le 1$, and $\nu_1 + \nu_2 = 1$. Then the consumption sharing rule is

$$
\lambda_{1,0}\xi_{1,t}\, e^{-\beta_1 t}\nu_{1,t}^{-\gamma_1} Y_t^{-\gamma_1} = \lambda_{2,0}\xi_{2,t}\, e^{-\beta_2 t}\nu_{2,t}^{-\gamma_2} Y_t^{-\gamma_2}, \qquad (6)
$$

which can be rewritten as

$$
\hat{\pi}_{1,t}\nu_{1,t}^{-\gamma_1} = \hat{\pi}_{2,t}\nu_{2,t}^{-\gamma_2}. \qquad (7)
$$

where[10]

$$
\begin{aligned}
\hat{\pi}_{k,t} &= \lambda_{k,0}\xi_{k,t}\, e^{-\beta_k t} Y_t^{-\gamma_k} \qquad (8)\\
&= \lambda_{k,0} e^{-\hat{r}_k t} e^{-\frac{1}{2}\hat{\theta}_k^2 t - \hat{\theta}_k Z_t}. \qquad (9)
\end{aligned}
$$

In the expression above, $\hat{\pi}_{k,t}$ is the state-price density when Agent k is the sole agent in the economy, and $\hat{r}_k$ and $\hat{\theta}_k$ are the risk-free rate and market price of risk in this single-agent economy:

$$
\begin{aligned}
\hat{r}_k &= \beta_k + \gamma_k \mu_{Y,k} - \frac{1}{2}\gamma_k(1+\gamma_k)\sigma_Y^2, \qquad (10)\\
\hat{\theta}_k &= \gamma_k \sigma_Y + \frac{\mu_Y - \mu_{Y,k}}{\sigma_Y}. \qquad (11)
\end{aligned}
$$

Thus, the consumption sharing rule in (7) can be expressed as

$$
\nu_{2,t}^{\eta} A_t = \nu_{1,t}, \qquad (12)
$$

where

$$
\begin{aligned}
\eta &= \gamma_2/\gamma_1,\\
A_t &= \left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{\frac{1}{\gamma_1}}, \qquad (13)\\
\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} &= e^{(\beta_2-\beta_1)t}\, Y_t^{\gamma_2-\gamma_1}\, \frac{\lambda_{1,0}}{\lambda_{2,0}} \xi_t^{-1}. \qquad (14)
\end{aligned}
$$

When $\eta \in \{1, 2, 3, 4\}$, the above equation can be written as a polynomial of degree 4 or less, thus allowing us to solve for the equilibrium consumption allocation in closed-form in terms of

[10] Equations (9)–(11) are obtained by applying Ito's Lemma to $\hat{\pi}_{k,t}$ in (8) and using the standard asset-pricing result (see, for instance, Duffie (2001)) that $\frac{d\hat{\pi}_{k,t}}{\hat{\pi}_{k,t}} = -\hat{r}_t dt - \hat{\theta}_{k,t} dZ_{k,t}$.

radicals, using standard results from polynomial theory, as pointed out in Wang (1996). Because polynomials of order 5 and above do not admit closed-form solutions in terms of radicals, it would be appear that going beyond the results in Wang (1996) by solving for the consumption-sharing rule in closed-form when η is a natural number greater than or equal to 5 is not possible. However, when η is a natural number greater than or equal to 5, the consumption shares can be obtained in closed-form by using hypergeometric functions. We go further still by showing that when η is *any real number*, it is possible to derive closed-form, convergent, series solutions for the sharing rule.[11] The series solutions are derived using a theorem of Lagrange (see Appendix C), which to the best of our knowledge has not been used before in finance or economics. However, Lagrange's Theorem does not provide the radius of convergence for the series, which is essential if we want to use these series to study the long-run behavior of the consumption shares. We show, in the proof of Proposition 1, how to identify the radius of convergence.

Proposition 1 *Agent 2's equilibrium share of the aggregate endowment, $\nu_{2,t} = \frac{C_{2,t}}{Y_t}$, is given by*

$$\nu_{2,t} = \begin{cases} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \left(\frac{\hat{\pi}_{2,t}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_2}} \binom{n\frac{\gamma_1}{\gamma_2}}{n-1} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R, \\ 1 - \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}} \binom{n\frac{\gamma_2}{\gamma_1}}{n-1} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R, \end{cases} \tag{15}$$

where

$$R = \frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}} \left(\frac{\gamma_2}{\gamma_1} - 1\right)^{\gamma_2 - \gamma_1} = \left(\frac{(\eta-1)^{\eta-1}}{\eta^\eta}\right)^{\gamma_1},$$

and, for $z \in \mathbb{C}$ and $k \in \mathbb{N}$, $\binom{z}{k} = \Pi_{j=1}^{k} \frac{z-k+j}{j}$ is the generalized binomial coefficient.

The proof of the proposition shows that, depending on whether $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R$ or $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R$, we get a different convergent series solution for the sharing rule; the solutions corresponding to these two regions are given in (15).[12] We also see from (15) that the consumption shares of the two agents will depend on the ratio of the single-agent economy state-price densities, $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}$, which from (13) depends on the difference in the subjective discount rates, β_1 and β_2, the difference in risk aversions, γ_1 and γ_2, and the difference in beliefs, $\xi_t^{-1} = \xi_{1,t}/\xi_{2,t}$.

From (14), we also see that the ratio $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}$ will evolve over time, and that its evolution will have a deterministic component and a stochastic component, where the stochastic component depends on the stochastic behavior of aggregate endowment and the differences in beliefs. Below, we first define aggregate risk aversion, and then describe the dynamics of the consumption-sharing rule.

[11]Because the derivation of the sharing rule for the general case where η is any real number is given in Appendix A, the derivation showing how the sharing rule can be expressed in terms of hypergeometric functions when η is an natural number greater than or equal to 5 is not included but is available upon request; see also Dumas, Kurshev, and Uppal (2009) and Chabakauri (2010) and the papers cited therein.

[12]Observe that Equation (15) does not depend on the assumption that Y is a geometric Brownian motion, and is valid for any stochastic process (including discontinuous processes) as along as the optimization problems of each agent are well defined and financial markets are complete.

Definition 1 *The aggregate relative risk aversion,* $\mathbf{R}_t$*, in the economy is defined as the consumption-share weighted harmonic average of individual agents' relative risk aversions:*

$$\mathbf{R}_t = \left(\nu_{1,t}\frac{1}{\gamma_1} + \nu_{2,t}\frac{1}{\gamma_2}\right)^{-1}. \tag{16}$$

Equivalently, the aggregate risk tolerance *in the economy,* $1/\mathbf{R}_t$*, is the consumption-share weighted average of individual agents' risk tolerances,* $1/\gamma_k$*.*

Proposition 2 *The true evolution of the sharing rule is given by:*[13]

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_{1,t}}dt + \sigma_{\nu_{1,t}}dZ_t,$$

where

$$\sigma_{\nu_{1,t}} = \nu_{2,t}\frac{1}{\gamma_1}\frac{1}{\gamma_2}\mathbf{R}_t\left[(\gamma_2-\gamma_1)\,\sigma_Y - \sigma_\xi\right], \tag{17}$$

$$\mu_{\nu_{1,t}} = \nu_{2,t}\frac{1}{\gamma_1}\frac{1}{\gamma_2}\mathbf{R}_t\Bigg\{(\beta_2-\beta_1) + (\gamma_2-\gamma_1)\,\mu_Y \tag{18}$$

$$+\left[\frac{\mu_Y - \frac{1}{2}(\mu_{Y,1}+\mu_{Y,2})}{\sigma_Y^2} - \left(\frac{\nu_{2,t}^2}{\gamma_2} - \frac{\nu_{1,t}^2}{\gamma_1}\right)\mathbf{R}_t^2\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\right](\mu_{Y,2}-\mu_{Y,1}) \tag{19}$$

$$+\frac{1}{2}(\gamma_2-\gamma_1)\left(\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2} - 2\right)\sigma_Y^2 + \frac{1}{2}\left(\frac{\nu_{2,t}^2}{\gamma_2} - \frac{\nu_{1,t}^2}{\gamma_1}\right)\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2}\sigma_\xi^2\Bigg\}. \tag{20}$$

From (17), we see that the volatility of the sharing rule, $\sigma_{\nu_{1,t}}$, is driven by differences in risk aversion and differences in beliefs, but not differences in subjective discount rates, which have only a deterministic effect and so appear only in the expression for $\mu_{\nu_{1,t}}$. The expression for $\sigma_{\nu_{1,t}}$ in (17) shows that, if agents have identical beliefs ($\sigma_\xi = 0$), then an increase in heterogeneity in risk aversion leads to an increase in the volatility of the consumption share of Agent 1 because of an increase in consumption risk sharing. Similarly, if agents have identical risk aversions ($\gamma_1 = \gamma_2$), then an increase in heterogeneity in beliefs leads to an increase in the volatility of the consumption share of Agent 1.

However, when both risk aversion and beliefs are heterogeneous, then the effect of an increase in the heterogeneity in either one of these factors on the volatility of the consumption share depends on whether it reinforces or counteracts the effect of the other factor. From (17) we observe that $\sigma_{\nu_{1,t}} > 0$ if and only if

$$\gamma_2 - \gamma_1 > \frac{\mu_{Y,2}-\mu_{Y,1}}{\sigma_Y^2}; \tag{21}$$

[13]The expressions for what each agent *believes* to be the evolution of the sharing rule are given in the appendix; see Equations (A14) and (A15).

that is, if the more risk averse agent is not too optimistic relative to the less risk averse agent.[14] If this condition is satisfied, then we see from the definition of aggregate risk aversion in (16) that $\mathbf{R}_t$ will be countercyclical, because when the aggregate endowment has a positive shock, the weight on the risk aversion of Agent 1 increases, and so the aggregate risk aversion in the economy decreases. Therefore, the heterogeneity in risk aversion and beliefs can generate countercyclical aggregate risk aversion endogenously. Moreover, if Agent 2, who has the higher risk aversion, is also the more pessimistic agent, then the heterogeneity in beliefs reinforces the effect arising from heterogeneity in risk aversions. This countercyclical behavior of aggregate risk aversion has previously been recognized in the multiagent models of Chan and Kogan (2002) and Xiouros and Zapatero (2010), where agents have heterogeneous risk aversions but homogeneous beliefs, and this feature appears in the single-agent model of Campbell and Cochrane (1999) as a consequence of the assumption of habit-formation.

Equation (18) shows how $\mu_{\nu_{1,t}}$ depends on differences in subjective discount rates and risk aversions (or, more accurately, the inverse of the elasticities of intertemporal substitution). The impact of differences in beliefs is given in (19), where we see that disagreement impacts the drift of the sharing rule only if the equally weighted arithmetic average belief does not equal the true growth rate, $\frac{1}{2}(\mu_{Y,1} + \mu_{Y,2}) \neq \mu_Y$, or there is heterogeneity in risk aversion, $\gamma_1 \neq \gamma_2$. We also see how $\mu_{\nu_{1,t}}$ is affected by the volatility of aggregate endowment growth, σ_Y, and the volatility of the disagreement process, σ_ξ, both of which appear in (20).

The discussion above illustrates the benefit of having the closed-form results in Propositions 1 and 2. Because we have explicit expressions for the sharing rule and its dynamics, we can understand exactly how these are affected by the parameters for preferences, beliefs, and the endowment process. In the absence of closed-form results, numerical analysis could be used to characterize the sharing rule, but it would be difficult to understand precisely the relation between the different forces driving the results.

3.2 Survival of agents and stationarity in the economy

In this section, we derive the conditions under which both agents survive in the long run. We say that the economy is stationary if both agents survive. To formalize the concept of survival, we introduce two complementary concepts of survival: *almost-sure (a.s.) survival* with respect to a particular measure, and *mean survival* with respect to a particular measure. The definition of almost-sure survival is the same as in Kogan, Ross, Wang, and Westerfield (2006). The concept of mean survival is novel to this paper.

We define *almost sure survival* as follows.

[14] In the case where agents have different risk aversion but the same beliefs, $\sigma_{\nu_{1,t}}$ is always positive.

Definition 2 *Agent k survives $\mathbb{P}$-a.s. if*

$$\lim_{t\to\infty} \nu_{k,t} > 0,\ \mathbb{P}\text{-}a.s.$$

Similarly, Agent k survives $\mathbb{P}^j$-a.s. if

$$\lim_{t\to\infty} \nu_{k,t} > 0,\ \mathbb{P}^j\text{-}a.s.$$

To understand the above concept of survival, note that if an agent's consumption share is strictly above zero with a probability of less than one, under $\mathbb{P}$ say, then she does not survive $\mathbb{P}$–almost surely. Furthermore, the probability measure is important, because an agent may *believe* she survives almost surely (with respect to the measure representing her beliefs), when in fact, she almost surely does not survive under the true measure $\mathbb{P}$.

We define *mean survival* with respect to a particular measure as follows.

Definition 3 *Agent k survives in the mean with respect to $\mathbb{P}$ if*

$$\lim_{u\to\infty} E_t \nu_{k,t+u} > 0.$$

Similarly, Agent k survives in the mean with respect to $\mathbb{P}^j$ if

$$\lim_{u\to\infty} E^j_t \nu_{k,t+u} > 0.$$

The economy is stationary if both agents survive. Each concept of survival leads to a corresponding concept of stationary: *almost sure stationarity* under a particular measure, and *mean stationarity* under a particular measure. We now determine the conditions for these two concepts of stationarity. Start by recalling the standard result that if $a > 0$, then $\lim_{t\to\infty} e^{at+bZ_t} = \infty$, $\mathbb{P}$-a.s., while if $a < 0$ then this limit is 0. Moreover, when $a = 0$, then $\limsup_{t\to\infty} e^{bZ_t} = \infty$, while $\liminf_{t\to\infty} e^{bZ_t} = 0$. From the above results it follows that to ensure that $\lim_{t\to\infty} e^{at+bZ_t}$ is strictly between zero and infinity, we need to have both a and b equal to zero. Now, substituting for Y_t and ξ_t in (12), we get

$$\nu^{\eta}_{2,t}\left(Y_0^{(\gamma_2-\gamma_1)}\frac{\lambda_{1,0}}{\lambda_{2,0}}e^{(\beta_2-\beta_1)t}e^{\frac{1}{2}(\sigma^2_{\xi,2}-\sigma^2_{\xi,1})t+(\sigma_{\xi,1}-\sigma_{\xi,2})Z_t}e^{(\gamma_2-\gamma_1)[(\mu_Y-\frac{1}{2}\sigma^2_Y)t+\sigma_Y Z_t]}\right)^{1/\gamma_1} = \nu_{1,t}.$$

Thus, both agents survive almost surely under the true measure $\mathbb{P}$, and the economy is almost surely stationary under $\mathbb{P}$, if the exponential decay rates of the deterministic and stochastic components in the expression above equal zero. We can also show that these two conditions are not only sufficient, but are also necessary. Formally, we have the following result.

Proposition 3 *The economy is almost surely stationary under* $\mathbb{P}$ *if and only if*

$$\mu_{Y,1} - \gamma_1 \sigma_Y^2 = \mu_{Y,2} - \gamma_2 \sigma_Y^2. \tag{22}$$

and

$$\beta_1 + \gamma_1 \left(\mu_Y - \frac{1}{2}\sigma_Y^2 \right) + \left(\frac{\mu_{Y,1} - \mu_Y}{\sigma_Y} \right)^2 = \beta_2 + \gamma_2 \left(\mu_Y - \frac{1}{2}\sigma_Y^2 \right) + \left(\frac{\mu_{Y,2} - \mu_Y}{\sigma_Y} \right)^2, \tag{23}$$

Equation (22) can be interpreted as stating that the two agents have the same beliefs about the risk-adjusted growth rate of the economy. Equation (23) equates the marginal utility of savings for the two agents: β_k is the subjective time preference for each agent, which determines the agent's patience; the second term, $\gamma_k \left(\mu_Y - \frac{1}{2}\sigma_Y^2 \right)$ determines the response of an agent to growth in the economy, with the growth rate of the economy being $(\mu_Y - \frac{1}{2}\sigma_Y^2)$ and the γ_k being the inverse of the elasticity of substitution parameter that drives savings; and, the final term determines the magnitude of the error in an agent's beliefs. Equation (23) is the same as the "survival index" derived in Yan (2008) and also discussed in Cvitanić, Jouini, Malamud, and Napp (2009).

For mean stationarity we need to find the conditions under which $\lim_{u \to \infty} E_t \nu_{k,t+u} > 0$ for both agents. As we show in the proof of this proposition, the only condition required for this is that the exponential decay rate of the *deterministic* component of (12) be equal to zero.

Proposition 4 *The economy is mean stationary under* $\mathbb{P}$ *if and only if the condition in* (23) *is satisfied.*

The consumption sharing rule, $\nu_{1,t}$, is a constant in the $\mathbb{P}$-a.s. stationary economy, and so risk premia and return volatilities will be the same as in a homogeneous-agent economy. In contrast, in the $\mathbb{P}$-mean stationary economy, $\nu_{1,t}$, is a function purely of the Brownian motion, Z_t. Consequently, $\nu_{1,t}$ is stochastic, and so risk premia and return volatilities will *not* be the same as in a homogeneous-agent economy.

The following corollary shows that when preferences of the two agents are identical, but there are differences in beliefs, the economy can still be mean stationary.

Corollary 1 *Suppose agents have identical preferences, but different beliefs. Then the economy is mean stationary under* $\mathbb{P}$ *if and only if*

$$\frac{\mu_{Y,1} + \mu_{Y,2}}{2} = \mu_Y. \tag{24}$$

The above corollary tells us that if agents have identical preferences but different beliefs, then the economy is mean stationary if and only if the equally weighted arithmetic mean belief equals the true expected growth rate of the economy. For example, if both agents have incorrect beliefs about the expected growth rate of the economy, which are on average correct, then both agents will survive in the mean. Equivalently, the disadvantage of having incorrect beliefs that are optimistic about the growth rate of aggregate endowment relative to the true growth rate is the same as that for beliefs that are pessimistic.

4 The Equilibrium State-Price Density

In this section, we first define the aggregate rate of time preference, the aggregate beliefs, and the aggregate prudence in this economy, all of which will appear in the characterization of the state-price density. Then, we determine the *dynamics* of the state-price density, and hence, the equilibrium riskfree rate and market price of risk. Finally, we derive an expression for the *level* of the state-price density, which is expressed as an average of state-price densities of *single-agent* economies.

Definition 4 *The* aggregate rate of time preference *in the economy, $\boldsymbol{\beta}_t$, is given by the weighted arithmetic mean of individual agents' rates of time preference, where the weights are the consumption-share weighted relative risk tolerances of the two investors:*

$$\begin{aligned} \boldsymbol{\beta}_t &= w_{1,t}\,\beta_1 + w_{2,t}\,\beta_2, \\ w_k &= \frac{\frac{1}{\gamma_k}\nu_{k,t}}{\frac{1}{\gamma_1}\nu_{1,t} + \frac{1}{\gamma_2}\nu_{2,t}}, \quad \textit{and} \;\; w_1 + w_2 = 1. \end{aligned} \tag{25}$$

Definition 5 *The* aggregate belief, *$\boldsymbol{\mu}_{Y,t}$, is given by the weighted arithmetic mean of the beliefs of individual agents, where the weights are the consumption-share weighted relative risk tolerances of the two investors as defined in* (25)*:*

$$\boldsymbol{\mu}_{Y,t} = w_{1,t}\,\mu_{Y,1} + w_{2,t}\,\mu_{Y,2}.$$

The prudence of an individual investor who has power utility is given by $(1+\gamma_k)$. Below, we define aggregate prudence.

Definition 6 *The quantity $\mathbf{P}_t$ is the* aggregate prudence *in the economy:*[15]

$$\mathbf{P}_t = (1+\gamma_1)\left(\frac{\mathbf{R}_t}{\gamma_1}\right)^2 \nu_{1,t} + (1+\gamma_2)\left(\frac{\mathbf{R}_t}{\gamma_2}\right)^2 \nu_{2,t}.$$

[15] Note that aggregate prudence may be larger than the prudence of either agent; that is, aggregate prudence is not necessarily bounded between the prudence of the individual agents. Consequently, the interest rate in the two-agent economy, which depends on aggregate prudence as shown in Equation (29), may not be bounded between the interest rates in the economies with only one of the two agents, as observed in Wang (1996). For a further discussion of this result, see Tran (2009, Proposition 2).

4.1 The riskless interest rate and its volatility

The central planner's state-price density, π_t, is given by[16]

$$\pi_t = \lambda_{k,t} e^{-\beta_k t} \nu_{k,t}^{-\gamma_k} Y_t^{-\gamma_k}. \tag{26}$$

From standard results in asset pricing (see Duffie (2001, Section 6.D, p. 106)), the evolution of the central planner's state-price density, π_t, is:

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \theta_t dZ_t, \tag{27}$$

and the evolution of Agent k's state-price density, $\pi_{k,t}$, is:

$$\frac{d\pi_{k,t}}{\pi_{k,t}} = -r_t dt - \theta_{k,t} dZ_{k,t}. \tag{28}$$

Note that each agent has her own market price of risk; however, because the instantaneously riskfree bond is a traded security, the two agents must agree on its price, and hence, on the interest rate. The following proposition gives the closed-form expression for the riskfree rate.

Proposition 5 *The locally riskfree rate is given by:*

$$\begin{aligned} r_t \quad = \quad & \boldsymbol{\beta}_t + \mathbf{R}_t \boldsymbol{\mu}_{Y,t} - \frac{1}{2}\mathbf{R}_t \mathbf{P}_t \sigma_Y^2 \\ & + \frac{1}{2} w_{1,t} w_{2,t} \mathbf{R}_t^2 \left(1 - \frac{\mathbf{R}_t}{\gamma_1 \gamma_2}\right) \sigma_\xi^2 - w_{1,t} w_{2,t} \mathbf{R}_t^3 \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)(\mu_{Y,1} - \mu_{Y,2}), \end{aligned} \tag{29}$$

where the weights w_k are defined in (25).

The corollary below gives the riskfree rate for the special cases where agents differ only with respect to their risk aversions or their beliefs.

Corollary 2 *If agents have identical and correct beliefs, then the locally riskfree rate is given by*

$$r_t = \boldsymbol{\beta}_t + \mathbf{R}_t \mu_Y - \frac{1}{2}\mathbf{R}_t \mathbf{P}_t \sigma_Y^2. \tag{30}$$

On the other hand, if agents have identical relative risk aversion, $\gamma_1 = \gamma_2 = \gamma$, but different beliefs and rates of time preference, then the locally riskfree rate is given by

$$r_t = \sum_{k=1}^{2} \nu_{k,t}\beta_k + \gamma \sum_{k=1}^{2} \nu_{k,t}\mu_{Y,k} - \frac{1}{2}\gamma(1+\gamma)\sigma_Y^2 + \frac{1}{2}\nu_{1,t}\nu_{2,t}\left(1 - \frac{1}{\gamma}\right)\sigma_\xi^2. \tag{31}$$

[16]Because financial markets are effectively complete, marginal utilities of consumption are equal across agents for each state, and therefore the first order condition for consumption in (5) ensures that the expression in (26) is the same for $k \in \{1, 2\}$.

To interpret the expression for the interest rate, recall that in an economy where all agents have correct and identical beliefs, and identical preferences, the expression for the interest rate is

$$r = \beta + \gamma \mu_Y - \frac{1}{2}\gamma(1+\gamma)\sigma_Y^2.$$

From the expression above, we see that the interest is positively related to the rate of impatience, β; positively related to the growth rate of aggregate endowment, μ_Y, scaled by risk aversion γ (that is, the inverse of the elasticity of intertemporal substitution); and the third term arises because of precautionary savings in the face of aggregate endowment risk, which leads to a drop in the interest rate, where the magnitude of the drop depends on $(1+\gamma)$, the prudence of agents.

Equation (30) of the corollary shows that if only risk aversions are heterogeneous but beliefs are homogeneous and correct, then the riskfree rate has the same form as that for a single-agent economy, but with aggregate quantities $\boldsymbol{\beta}_t$, $\mathbf{R}_t$, and aggregate prudence, $\mathbf{P}_t$, replacing their single-agent counterparts. On the other hand, if only beliefs are heterogeneous but preferences are homogeneous, then we see from the last term in (31) that if $\gamma < 1$ the differences in beliefs will decrease the interest rate, or equivalently, increase the price of the instantaneously riskless bond. This effect is similar to the premium ("bubble") in asset prices that has been studied in Harrison and Kreps (1978) and Scheinkman and Xiong (2003) for the case of risk neutrality ($\gamma = 0$) in the presence of shortsale constraints; over here, we get a similar effect for agents who are risk averse without needing to constrain shortsales. However, if $\gamma > 1$ then the price of the bond *decreases* with heterogeneity in beliefs, an observation made also in Dumas, Kurshev, and Uppal (2009).

When agents have both heterogeneous beliefs and preferences, the risk-free rate in given by (29). The terms in the first line of (29) correspond to the three terms in (30); note, however, that because the weights used to construct these aggregate measures vary over time, the aggregate measures will be time-varying rather than constant. The first term in the second line of (29) arises because of volatility of the differences in beliefs, σ_ξ, and corresponds to the last term in (31). This term increases the risk-free rate when the aggregate risk aversion is less than the square of the geometric mean of risk aversion; that is, $\mathbf{R}_t < \gamma_1\gamma_2$, which is true if and only if $\gamma_1 > 1$.[17] It follows that if $\gamma_1 > 1$ ($\gamma_1 < 1$), then heterogeneity in beliefs increases (decreases) the risk-free rate. The second term in the second line of (29) arises because of differences in both risk aversion *and* in beliefs; that is, $\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)(\mu_{Y,1} - \mu_{Y,2})$. When the less risk averse agent is also the more optimistic agent, that is, $\mu_{Y,1} > \mu_{Y,2}$, this term decreases the risk-free rate.

One of the limitations of the representative-agent general-equilibrium model of asset pricing is that, when risk aversion is increased in order to improve the match of the equity risk premium in the model to that in the data, the riskfree interest rate in the model becomes too high relative to the data; this is the "riskfree rate puzzle" identified in Weil (1989). From the discussion above,

[17] Note that since $\mathbf{R}_t \leq \gamma_2$, $\mathbf{R}_t < \gamma_1\gamma_2$ if and only if $\gamma_1 > 1$.

we see that both heterogeneity in beliefs and preferences have the potential to reduce the interest rate relative to a homogeneous agent economy. Using the parameter values listed in Table 1, the riskfree interest rate is plotted as a function of the consumption share of Agent 1 in Figure 1. From this figure, we see that the riskfree rate in the homogeneous agent economy is more than 6% p.a. However, in the data it is about 1% p.a. (see Campbell (2003)). The figure shows that heterogeneity in both risk aversion and in beliefs reduce the interest rate, and when both sources of heterogeneity are present, the interest rate is about 2% p.a. For the case where the parameters are restricted so that the model is stationary, the level of the interest rate is about 3%.

We can also derive an explicit expression for the volatility of the instantaneously riskless interest rate. This is an important quantity because often models that can generate a sufficiently high equity risk premium run into the problem of having a volatility for the real riskfree rate that is too high relative to its empirical value of about 1.7% p.a. The gap between the low volatility of the real interest rate and the relatively higher volatility of real stock returns (about 16% p.a. in the data) is called the "equity volatility puzzle" in Campbell (2003).

Proposition 6 *The volatility of the instantaneously riskless interest rate is:*

$$\begin{aligned}\sigma_{r,t} \quad = \quad & \Bigg\{(\gamma_2-\gamma_1)(\beta_2-\beta_1)+(\gamma_2-\gamma_1)(1+\gamma_1)(1+\gamma_2)\left(\frac{\mu_{Y,1}}{1+\gamma_1}-\frac{\mu_{Y,2}}{1+\gamma_2}\right) \\ & +\left(\frac{3\mathbf{R}_t^2}{2\gamma_1\gamma_2}-\mathbf{R}_t\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+1\right)\right)\left[2(\gamma_2-\gamma_1)(\mu_{Y,1}-\mu_{Y,2})+(\gamma_2-\gamma_1)^2\sigma_Y+\sigma_\xi^2\right] \\ & +\frac{1}{2}(1+\gamma_1+\gamma_2)\sigma_\xi^2\Bigg\}\frac{\mathbf{R}_t(\mathbf{R}_t-\gamma_1)(\mathbf{R}_t-\gamma_2)}{\gamma_1\gamma_2(\gamma_2-\gamma_1)^3}[(\gamma_2-\gamma_1)\sigma_Y-\sigma_\xi].\end{aligned} \tag{32}$$

For the special cases where either risk aversions are the same, or beliefs are the same and are also correct, the expression for the volatility of the riskless interest rate simplifies to the following.

Corollary 3 *If the two agents have identical risk aversion, $\gamma_1=\gamma_2=\gamma$, then the volatility of the interest rate in* (32) *reduces to*

$$\sigma_{r,t}=\frac{\nu_{1,t}\nu_{2,t}}{\gamma}\frac{\mu_{Y,1}-\mu_{Y,2}}{\sigma_Y}\left[(\beta_1-\beta_2)+\gamma(\mu_{Y,1}-\mu_{Y,2})-(\nu_{1,t}-\nu_{2,t})\left(1-\frac{1}{\gamma}\right)\frac{1}{2}\left(\frac{\mu_{Y,1}-\mu_{Y,2}}{\sigma_Y}\right)^2\right]. \tag{33}$$

On the other hand, if the two agents have identical beliefs, $\mu_{Y,1}=\mu_{Y,2}=\mu_Y$, then the volatility of the interest rate in (32) *reduces to*

$$\sigma_{r,t}=\left(\frac{\beta_2-\beta_1}{\gamma_2-\gamma_1}+\mu_Y+\mathbf{R}_t\left[\frac{3\mathbf{R}_t}{2\gamma_1\gamma_2}-\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+1\right)\right]\sigma_Y^2\right)\frac{\mathbf{R}_t\left(\mathbf{R}_t-\gamma_1\right)\left(\mathbf{R}_t-\gamma_2\right)}{\gamma_1\gamma_2}\sigma_Y. \tag{34}$$

From the expressions for the volatility of the riskfree interest rate for the two special cases in (33) and (34), or the general case in (32), we see that heterogeneity in beliefs and heterogeneity in preferences (both rates of time preference and risk aversions) contribute to the volatility of the riskfree rate. The low volatility of the real riskfree rate in the data, about 1.7% p.a., imposes discipline on our model, by limiting the differences across agents in our choice of the parameters for preferences and beliefs. In Figure 2, we show the volatility of the riskfree rate of interest as a function of the consumption share of Agent 1 (using the same parameter values listed in Table 1). We see from the figure that heterogeneity in beliefs has only a small effect on the volatility of the riskfree rate, but heterogeneity in risk aversions increases the interest rate. However, for the parameter values we consider, the maximum volatility of the riskfree rate is less than 0.4% p.a. For the case where the parameters are restricted so that the model is stationary, the volatility of the interest rate is even lower.

4.2 The market price of risk

From (27), we see that the volatility of the central planner's state price density (also know as the stochastic discount factor) is given by the market price of risk, θ_t, while from (28) we see that the volatility of the state price density for each individual agent is given by the perceived market price of risk, $\theta_{k,t}$. The following proposition gives the closed-form expressions for these market prices of risk.

Proposition 7 *The market price of risk of the central planner, θ_t, is:*

$$\theta_t = \mathbf{R}_t \, \sigma_Y + \left[\frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y} \right], \tag{35}$$

and the market prices of risk perceived by the two agents are:

$$\theta_{1,t} = \mathbf{R}_t \left(\sigma_Y + \frac{\nu_{2,t}}{\gamma_2} \left[\frac{\mu_{Y,1} - \mu_{Y,2}}{\sigma_Y} \right] \right), \tag{36}$$

$$\theta_{2,t} = \mathbf{R}_t \left(\sigma_Y + \frac{\nu_{1,t}}{\gamma_1} \left[\frac{\mu_{Y,2} - \mu_{Y,1}}{\sigma_Y} \right] \right). \tag{37}$$

The corollary below gives the market prices of risk for the central planner and the two agents for the special cases where agents have identical preferences or identical beliefs.

Corollary 4 *If agents have identical and correct beliefs, then the central planner's market price of risk, θ_t, and the market price of risk perceived by the two agents, $\theta_{k,t}$ are given by:*

$$\theta_t = \theta_{k,t} = \mathbf{R}_t \sigma_Y. \tag{38}$$

On the other hand, if agents have identical relative risk aversion, $\gamma_1 = \gamma_2 = \gamma$, but different beliefs and rates of time preference, then the central planner's equilibrium market price of risk is

$$\theta_t = \gamma\,\sigma_Y + \left[\frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}\right],$$

and the market prices of risk perceived by Agents 1 and 2 are given by

$$\theta_{1,t} = \gamma\sigma_Y + \nu_{2,t}\left[\frac{\mu_{Y,1} - \mu_{Y,2}}{\sigma_Y}\right], \tag{39}$$

$$\theta_{2,t} = \gamma\sigma_Y + \nu_{1,t}\left[\frac{\mu_{Y,2} - \mu_{Y,1}}{\sigma_Y}\right]. \tag{40}$$

To understand the expressions for the market price of risk in the above corollary and proposition, note that in an economy where all agents have correct and identical beliefs, and identical risk aversion, $\gamma_1 = \gamma_2 = \gamma$, the market price of risk is $\theta = \gamma\,\sigma_Y$. When only preferences are different across agents, then γ is replaced by the average risk aversion in the economy, $\mathbf{R}_t$, and the market price of risk is given by (38), with both agents agreeing with this market price of risk. On the other hand, if preferences are identical but beliefs are heterogeneous, then we see from (39) and (40) that agents do not agree on the market price of risk. From (39) we see that if Agent 2 is pessimistic relative to Agent 1, $\mu_{Y,1} > \mu_{Y,2}$, then the market price of risk perceived by Agent 1 will be increased. The magnitude of this increase depends on the consumption-share of Agent 2, $\nu_{2,t}$, because this determines Agent 2's influence on equilibrium stock market returns. For the general case in (36) where both beliefs and risk aversions are different, we see that the increase in the market price of risk perceived by Agent 1 will depend on the consumption share of Agent 2, $\nu_{2,t}$, and the agent's risk tolerance, $1/\gamma_2$, relative to aggregate risk tolerance in the economy, $1/\mathbf{R}_t$, because these are the two factors that determine the size of the position Agent 2 takes in the stock market. Finally, from the expression in (35) for the general case where there is heterogeneity in both preferences and beliefs, we see that the market price of risk for the central planner will increase if average beliefs are pessimistic; that is, $\mu_Y > \boldsymbol{\mu}_{Y,t}$. The intuition for this is that, if agents are pessimistic on average, then the compensation for bearing risk must be relatively higher than what it needs to be in an economy where agents have the correct average beliefs.

We now discuss the implications of heterogeneity in preferences and beliefs for the market price of risk in the data. From Corollary 4, we see that in a model *without* heterogeneity of beliefs, the market price of risk is given by $\mathbf{R}_t\,\sigma_Y$, the product of aggregate risk aversion and the volatility of aggregate endowment. But, in the data the volatility of aggregate endowment is about 3% p.a., which means that to obtain the empirically observed market price of risk of about 30%–50%, we need aggregate risk aversion to be about 10–17, which is much higher than what many people view as reasonable. More importantly, increasing risk aversion leads to a riskfree rate that is high (because investors wish to borrow in order to consume today rather than in the future), but in the data the riskfree rate is only about 1% p.a., and thus choosing a high value for relative risk aversion would lead to the "riskfree rate puzzle" of Weil (1989).

On the other hand, in a model in which average beliefs do not coincide with the true beliefs, the expression for the market price of risk in (35) has a second term, $(\mu_Y - \boldsymbol{\mu}_{Y,t})/\sigma_Y$, which contributes to the magnitude of the market price of risk. In the second term, the volatility of aggregate endowment *divides* the difference between the true growth rate of aggregate endowment and the average belief about this in the economy. Thus, if investors are pessimistic on average, $\mu_Y > \boldsymbol{\mu}_{Y,t}$, even small differences between the true expected growth rate and the aggregate belief about the expected growth rate will have a large impact on the magnitude of the market price of risk implied by the model.[18] Figure 3 plots the market price of risk against the consumption share of Agent 1. The figure shows that while heterogeneity in risk aversion (or beliefs) does not increase the market price of risk relative to the homogeneous-agent benchmark, beliefs that are pessimistic on average lead to a significant increase in the market price of risk. This is true even for the the case where the parameters are restricted so that the model is stationary.

Note also that the market price of risk is countercyclical in the data and in the model of Campbell and Cochrane (1999). This will be true also in our model if $\mathbf{R}_t$ is countercyclical, which requires that the more risk averse agent not be too optimistic relative to the less risk averse agent—the exact condition is given in Equation (21). Therefore, to obtain a market price of risk which is close to the data in both its level and cyclical behavior, we need both heterogeneity in risk aversion *and* average beliefs that are pessimistic. Ziegler (2007) also finds that one needs beliefs that are pessimistic on average if one wishes to obtain the "smile" that is observed in prices of options.

4.3 The state price density

In the section above, we have characterized the dynamics of the state price density for the central planner and also for each individual agent. We now give the *level* of the equilibrium state-price density using convergent series, where the individual terms depend solely on exogenous variables and are written in terms of the state-price densities of *single-agent* economies, that is, $\hat{\pi}_{k,t}$, $k \in \{1, 2\}$, defined in (9).

Proposition 8 *The equilibrium state-price density is given by*

$$\pi_t = \begin{cases} \sum_{n=0}^{\infty} a_{n,1}^{\pi} \hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R, \\ \sum_{n=0}^{\infty} a_{n,2}^{\pi} \hat{\pi}_{1,t}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R, \end{cases} \tag{41}$$

[18] For example, if the difference between the true growth rate of aggregate endowment and the average belief about this in the economy is 1%, then dividing this by the volatility of the growth rate of endowment of 3% will contribute an additional 33% to the market price of risk. So, for instance, if the average risk aversion in the economy is 3, then the first term in (35) is 9%, and the second term is 33%, for a total market price of risk that is 42% p.a.

where $a^{\pi}_{n,1} = a^{\pi}_{n,2} = 1$ *for* $n = 0$, *and*

$$a^{\pi}_{n,1} = \gamma_1 \frac{(-1)^{n+1}}{n} \binom{n\frac{\gamma_1}{\gamma_2} - \gamma_1 - 1}{n-1}, n \in \mathbb{N}, \tag{42}$$

$$a^{\pi}_{n,2} = \gamma_2 \frac{(-1)^{n+1}}{n} \binom{n\frac{\gamma_2}{\gamma_1} - \gamma_2 - 1}{n-1}, n \in \mathbb{N}. \tag{43}$$

To interpret the expression for the state price density, observe that in (41) the term on the first line can be written as[19]

$$\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}} = \lambda_{1,0}^{1-\frac{n}{\gamma_2}} \lambda_{2,0}^{\frac{n}{\gamma_2}} e^{-r^{n,1}t} e^{-\frac{1}{2}\left(\theta^{n,1}\right)^2 t - \theta^{n,1} Z_t},$$

where

$$\theta^{n,1} = \left(1 - \frac{n}{\gamma_2}\right)\hat{\theta}_1 + \frac{n}{\gamma_2}\hat{\theta}_2, \tag{44}$$

$$r^{n,1} = \left(1 - \frac{n}{\gamma_2}\right)\hat{r}_1 + \frac{n}{\gamma_2}\hat{r}_2 + \frac{1}{2}\left(1 - \frac{n}{\gamma_2}\right)\frac{n}{\gamma_2}(\hat{\theta}_1 - \hat{\theta}_2)^2. \tag{45}$$

Thus, the above proposition shows that the equilibrium state-density in (41) can be expressed as a *linear* combination of state-price densities from a set of *underlying* economies with a *constant* market price of risk and risk-free rate. Note that the market price of risk, $\theta^{n,1}$, is itself a weighted arithmetic mean of the market prices of risk in the economies where Agents 1 and 2, respectively, are the sole agents, and the risk-free rate, $r^{n,1}$, is the weighted arithmetic mean of the individual agent economy risk-free rates *but* with an additional term, $(\hat{\theta}_1 - \hat{\theta}_2)^2$. This term appears because both heterogeneity in beliefs and risk aversion give risk to an additional demand for precautionary savings. When $n < \gamma_2$, the additional term is positive, leading to a premium ("bubble") in asset prices relative to the representative-agent setting, and negative when $n > \gamma_2$, implying a discount in asset prices.

The expression for the equilibrium state-density in (41) can be simplified if agents have the same risk aversion, $\gamma_1 = \gamma_2 = \gamma$, and a further simplification is possible if γ is a natural number. These simpler expressions are given in the corollary below.

Corollary 5 *Suppose agents have identical risk aversion, that is,* $\gamma_1 = \gamma_2 = \gamma$, *but different beliefs. Then the equilibrium state-price density is given by*

$$\pi_t = \begin{cases} \sum_{n=0}^{\infty} a^{\pi}_n \hat{\pi}_{2,t}^{\frac{n}{\gamma}} \hat{\pi}_{1,t}^{1-\frac{n}{\gamma}}, & \hat{\pi}_{2,t} < \hat{\pi}_{1,t}, \\ \sum_{n=0}^{\infty} a^{\pi}_n \hat{\pi}_{1,t}^{\frac{n}{\gamma}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma}}, & \hat{\pi}_{2,t} > \hat{\pi}_{1,t}, \end{cases} \tag{46}$$

[19]The interpretation for the second line of (41) is analogous, and the expressions corresponding to (44) and (45) are given in equations (A23) and (A24) of the appendix.

where, denoting by $\mathbb{N}_0$ *the set of natural numbers that includes* 0*,*

$$a_n^\pi = \binom{\gamma}{n}, n \in \mathbb{N}_0. \tag{47}$$

If relative risk aversion, γ*, is a natural number, then the equilibrium state-price density can be further simplified to a finite sum:*

$$\pi_t = \sum_{n=0}^{\gamma} a_n^\pi \hat{\pi}_{1,t}^{1-\frac{n}{\gamma}} \hat{\pi}_{2,t}^{\frac{n}{\gamma}}. \tag{48}$$

Observe that using Newton's Binomial Theorem (for non-integral powers), we can rewrite the series expansion in (46) as[20]

$$\pi_t = \left(\hat{\pi}_{1,t}^{\frac{1}{\gamma}} + \hat{\pi}_{2,t}^{\frac{1}{\gamma}}\right)^\gamma. \tag{49}$$

The expression for the equilibrium state-price density in (49) is a power mean (with exponent $\frac{1}{\gamma}$) of the individual agent state-price densities.[21]

The special case considered in Corollary 5 where $\gamma_1 = \gamma_2 = \gamma$, with γ being a natural number, is similar to the model studied in Dumas, Kurshev, and Uppal (2009, Equation (35)), where they obtain a similar expression for the state price density. Because γ needs to be a natural number, this special case does not allow one to study the case of risk aversion smaller than one. Our Proposition 8, in contrast, allows for different risk aversion parameters for the two agents and does not restrict their values to be natural numbers.

5 Prices and Risk Premia of Stocks and Bonds

In this section, we derive the stock price, the equity risk premium, the volatility of stock market returns, and the term structure of interest rates. We then use these results to analyze how heterogeneity in beliefs, rates of time preference, and risk aversion impact the equity risk premium, the volatility of stock market returns, the price-dividend ratio, and the term premium.

5.1 The equity risk premium and volatility of stock market returns

The price of the stock, which pays out the cash flow Y_t in perpetuity, is given by

$$P_t^Y = Y_t p_t^Y,$$

[20] When γ is a natural number, the expression in (49) also follows from (48) by using the Binomial Theorem for integral powers, and one could also obtain (49) directly from the first-order condition for consumption in (7).

[21] It follows from well known properties of the power mean, that the state-price density in Equation (49) is increasing in relative risk aversion, γ. The intuition for this is that more risk averse agents will be more willing to pay for a unit of consumption in a given state. If $\gamma = 1$, the power mean reduces to the arithmetic mean; if $\gamma \to \infty$ it reduces to the geometric mean; and, if $\gamma \to 0$, it reduces to the maximum of the individual-agent state-price densities.

where the price-dividend ratio p_t^Y is:

$$p_t^Y = E_t \int_t^\infty \frac{\pi_u}{\pi_t}\frac{Y_u}{Y_t} du. \tag{50}$$

The risk premium on equity, which pays Y_t in perpetuity, is given by the standard asset pricing equation:

$$E_t\left[\frac{dP_t^Y + Y_t dt}{P_t^Y} - r_t dt\right] = -E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t^Y}{P_t^Y}\right]. \tag{51}$$

Applying Ito's Lemma to $P_t^Y = Y_t p_t^Y$ and using Equation (51) leads to the following proposition.

Proposition 9 *The volatility of stock market returns, $\sigma_{R,t}^Y$, is*

$$\sigma_{R,t}^Y = \sigma_Y + \sigma_{\nu_{1,t}} \frac{\nu_{1,t}}{p_t^Y}\frac{\partial p_t^Y}{\partial \nu_{1,t}}, \tag{52}$$

and the risk premium on equity is

$$\mu_{R,t}^Y - r_t = \theta_t\, \sigma_{R,t}^Y = \left(\mathbf{R}_t\, \sigma_Y + \left[\frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}\right]\right)\sigma_{R,t}^Y; \tag{53}$$

Agent 1's perception of the risk premium is given by

$$\mu_{R,1,t}^Y - r_t = \mathbf{R}_t\left(\sigma_Y + \frac{\nu_{2,t}}{\gamma_2}\left[\frac{\mu_{Y,1} - \mu_{Y,2}}{\sigma_Y}\right]\right)\sigma_{R,t}^Y;$$

and, Agent 2's perception of the risk premium is given by

$$\mu_{R,2,t}^Y - r_t = \mathbf{R}_t\left(\sigma_Y + \frac{\nu_{1,t}}{\gamma_1}\left[\frac{\mu_{Y,2} - \mu_{Y,1}}{\sigma_Y}\right]\right)\sigma_{R,t}^Y.$$

In a model with a single representative investor, stock return volatility, $\sigma_{R,t}$ is equal to fundamental volatility, σ_Y. From (52) we see that in a model with heterogeneous investors, stock market return volatility is the sum of fundamental volatility, σ_Y, and *excess volatility*, $\sigma_{\nu_1,t}\,\frac{\nu_{1,t}}{p_t^Y}\,\frac{\partial p_{\nu_{1,t}}^Y}{\partial \nu_{1,t}}$, which depends on fluctuations in the price-dividend ratio. When demand for precautionary savings is not too large, the price-dividend ratio is monotonic and countercyclical, and so excess volatility is positive, as in the data. Figure 4 shows the stock market return volatility σ_R as a function of the consumption share of Agent 1. We see from this figure that the excess volatility generated by heterogeneity in beliefs is not significant,[22] but the excess volatility arising from heterogeneous risk aversions is substantial. Overall, in the model with heterogeneous investors, stock return volatility is 2–4 times higher than volatility in a model with identical investors. However, for the case where

[22]This is partly because the value of risk aversions for the two agents in the base case is specified to be 3; if risk aversion was less than 1, belief heterogeneity would have a larger effect on stock market return volatility.

the parameters are restricted so that the model is stationary, the model does not generate excess volatility.

We now discuss the equity risk premium. From Proposition 9, we see that while agents agree on conditional stock return volatility, they may disagree on the conditional risk premium. The central planner's view of the conditional risk premium is given in (53), which is the product of the market price of risk, θ_t, and the volatility of stock market returns, $\sigma^Y_{R,t}$. The risk premium will be high when: (i) in aggregate, agents are pessimistic, $\boldsymbol{\mu}_{Y,t} < \mu_Y$; (ii) the aggregate risk aversion in the economy, $\mathbf{R}_t$, is high; and (iii) stock return volatility, $\sigma^Y_{R,t}$, is high. Quantitatively, the first and third channels are the most important for generating a risk premium that is high relative to the risk premium in an economy where agents are homogeneous.[23] This can be seen in Figure 5, where the equity risk premium is substantially higher than what it would be in a homogeneous-agent economy.[24] Even for the case where the parameters are restricted so that the model is stationary, the equity risk premium is substantially higher than one would get in a representative-agent model, though it is less than what is observed empirically.

5.2 Price-dividend ratio for equity

We derive an exact closed-form solution for the price-dividend ratio, p^Y_t, by using the series expression for the state-price density in Proposition 8 to directly evaluate the expectation of the integral in the right-hand side of (50). Because the state-price density is one of two linear combinations of state-price densities from a set of underlying economies with constant risk-free rates and market prices of risk, depending on whether $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} \gtrless R$, the price-dividend ratio p^Y_t in (56) is a sum of two weighted averages. The first is a weighted average of price-dividend ratios from a set of underlying economies with constant risk-free rates and market prices of risk conditional on $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R$, and the second is a weighted average of price-dividend ratios from a set of underlying economies conditional on $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R$.

To identify the price-dividend ratio for equity, we first identify the price-dividend ratio $\zeta^Y_{n,1,t}$ ($\zeta^Y_{n,2,t}$) for a claim that pays Y_t in perpetuity *if* $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R$ $\left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R\right)$; that is,

$$\zeta^Y_{n,1,t} \quad = \quad E_t\left[\int_t^\infty \frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,u}^{\frac{n}{\gamma_2}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}}\frac{Y_u}{Y_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}}du\right],\ n \in \mathbb{N}_0, \tag{54}$$

[23]Note that if stock return volatility, $\sigma^Y_{R,t}$, is higher than fundamental volatility, σ_Y, the risk premium can be higher than in either of the two homogeneous agent economies.

[24]Above, we have seen that the market price of risk is about 40% p.a., while stock market return volatility is 2–4 times fundamental volatility, so 6%–12% p.a., and therefore, the product of these gives an equity risk premium that is as much as 2.4% to 4.8% p.a. In contrast, in the homogeneous agent economy the equity risk premium would be the product of fundamental volatility, 3% p.a., stock market return volatility which in the homogeneous agent economy is also 3% p.a., and average risk aversion, which we have assumed to be 3, for an equity risk premium that is only 0.27% p.a.

$$\zeta^Y_{n,2,t} = E_t\left[\int_t^\infty \frac{\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}\frac{Y_u}{Y_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<R\right\}}du\right], n \in \mathbb{N}_0. \tag{55}$$

Closed-form expressions for $\zeta^Y_{n,1,t}$ and $\zeta^Y_{n,2,t}$ are given in (A35) and (A36) in the appendix.[25] We now express the price of equity in terms of the prices of the two claims described above.[26]

Proposition 10 *The time-t price of equity, which pays the cash flow stream, Y_t in perpetuity, is given by $P^Y_t = p^Y_t Y_t$, where*

$$p^Y_t = \sum_{n=0}^\infty \omega_{n,1,t}\zeta^Y_{n,1,t} + \sum_{n=0}^\infty \omega_{n,2,t}\zeta^Y_{n,2,t},$$

where the weights $\omega_{n,1,t}$, $n \in \mathbb{N}_0$, and $\omega_{n,2,t}$, $n \in \mathbb{N}_0$, are given by

$$\omega_{n,1,t} = a^\pi_{n,1}(\nu^{\gamma_1}_{1,t})^{1-\frac{n}{\gamma_2}}(\nu^{\gamma_2}_{2,t})^{\frac{n}{\gamma_2}}, n \in \mathbb{N}_0 \tag{56}$$

$$\omega_{n,2,t} = a^\pi_{n,2}(\nu^{\gamma_1}_{1,t})^{\frac{n}{\gamma_1}}(\nu^{\gamma_2}_{2,t})^{1-\frac{n}{\gamma_1}}, n \in \mathbb{N}_0, \tag{57}$$

and each set of weights sums to one:

$$\sum_{n=0}^\infty \omega_{n,1,t} = \sum_{n=0}^\infty \omega_{n,2,t} = 1.$$

Note that the price-dividend ratio can be non-monotonic. This is possible, because the expressions for the risk-free rates in the underlying economies, given in (45) and (A24), are weighted arithmetic means of the individual agent economy risk-free rates plus an additional term, arising from demand for precautionary savings. When demand for precautionary savings is high, the price-dividend ratio will be non-monotonic. From (45) and (A24), we can see this will occur when the individual-agent economy market prices of risk, given in (11), are more heterogeneous. We can see from (11) that heterogeneity in the market prices of risk will be higher when the more risk averse agent, Agent 2, is also more pessimistic relative to Agent 1.

Finally, we consider two special cases: the first where the two agents have the same risk aversion, $\gamma_1 = \gamma_2 = \gamma$, and the second, where the two agents have the same risk aversion *and* γ is a natural

[25] Recall that in each of the n economies above, the riskfree rate is given by $r^{n,1}$ or $r^{n,2}$, and the market price of risk is given by $\theta^{n,1}$ or $\theta^{n,2}$, which are defined in Equations (44), (45), (A23) and (A24).

[26] Observe that this result is valid when Y is any stochastic process, such that the optimization problems of individual agents are well defined and markets are complete. When Y is Markovian, we can derive a differential equation that the price of the claim must satisfy. The price-dividend ratio, p^Y_t, depends on the distribution of consumption across the two agents in the economy, and hence, is a function of the consumption share, that is, $p^Y_t = p^Y(\nu_{1,t})$. The differential equation has natural boundary conditions: $p^Y(0) = \frac{1}{\hat{r}_2+\gamma_2\sigma^2_Y-\mu_{Y,2}}$ and $p^Y(1) = \frac{1}{\hat{r}_1+\gamma_1\sigma^2_Y-\mu_{Y,1}}$, which are a consequence of the equation's limiting behavior at $\nu_{k,t} = 0$, $k \in \{1,2\}$. Using the further assumption that Y is a geometric Brownian motion, this differential equation can be transformed into an inhomogeneous second order linear differential equation with constant coefficients, which can be solved exactly in closed-form in terms of the Gaussian hypergeometric function. The latter function can be defined as an infinite series, so we can verify that this result is a special case of (56), together with (54) and (55).

number. For these two special cases, the price-dividend ratio for equity is expressed in terms of $\zeta^Y_{n,1,t}$ ($\zeta^Y_{n,2,t}$), which is the price-dividend ratio of the claim which pays the cashflow stream, Y_t, in perpetuity, provided $\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > 1$ $\left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < 1\right)$:

$$\begin{aligned}
\zeta^Y_{n,1,t} &= E_t\left[\int_t^\infty \frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma}}\hat{\pi}_{2,u}^{\frac{n}{\gamma}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma}}\hat{\pi}_{2,t}^{\frac{n}{\gamma}}}\frac{Y_u}{Y_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>1\right\}}du\right],\ n \in \mathbb{N}_0, && (58)\\
\zeta^Y_{n,2,t} &= E_t\left[\int_t^\infty \frac{\hat{\pi}_{1,u}^{\frac{n}{\gamma}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma}}}\frac{Y_u}{Y_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<1\right\}}du\right],\ n \in \mathbb{N}_0. && (59)
\end{aligned}$$

Closed-form expressions for $\zeta^Y_{n,1,t}$ and $\zeta^Y_{n,2,t}$ are given by (A37) and (A38) in the appendix. We now give the price-dividend ratio for equity.

Corollary 6 *When risk aversions are identical, $\gamma_1 = \gamma_2 = \gamma$, then*

$$p^Y_t = \sum_{n=0}^{\infty}\omega_{n,1,t}\zeta^Y_{n,1,t} + \sum_{n=0}^{\infty}\omega_{n,2,t}\zeta^Y_{n,2,t},$$

where

$$\begin{aligned}
\omega_{n,1,t} &= \binom{\gamma}{n}(\nu^\gamma_{1,t})^{1-\frac{n}{\gamma}}(\nu^\gamma_{2,t})^{\frac{n}{\gamma}},\ n \in \mathbb{N}_0 && (60)\\
\omega_{n,2,t} &= \binom{\gamma}{n}(\nu^\gamma_{1,t})^{\frac{n}{\gamma}}(\nu^\gamma_{2,t})^{1-\frac{n}{\gamma}},\ n \in \mathbb{N}_0. && (61)
\end{aligned}$$

If in addition to risk aversions being identical, $\gamma_1 = \gamma_2 = \gamma$, we also have that $\gamma \in \mathbb{N}$, then the above expressions simplify further to:

$$p^Y_t = \sum_{n=0}^{\gamma}\omega_{n,t}\, p^Y_n, \qquad (62)$$

where

$$\begin{aligned}
p^Y_n &= (r^n + \gamma\sigma^{sys}_Y\sigma_Y - \mu^n_Y)^{-1},\\
r^n &= \beta_n + \gamma\mu^n_Y - \frac{1}{2}\gamma(1+\gamma)\sigma^2_Y + \frac{1}{2}\left(\frac{n}{\gamma}\right)\left(1-\frac{n}{\gamma}\right)\sigma^2_\xi,\\
\beta^n &= \left(1-\frac{n}{\gamma}\right)\beta_1 + \left(\frac{n}{\gamma}\right)\beta_2,\\
\mu^n_Y &= \left(1-\frac{n}{\gamma}\right)\mu_{Y,1} + \left(\frac{n}{\gamma}\right)\mu_{Y,2},\\
\omega_{n,t} &= \begin{pmatrix}\gamma\\ n\end{pmatrix}\left(\nu_{1,t}^{1-\frac{n}{\gamma}}\nu_{2,t}^{\frac{n}{\gamma}}\right)^{\gamma}.
\end{aligned}$$

From (62), we see that the price-dividend ratio in the economy with heterogenous beliefs is a weighted sum of the price-dividend ratios in $1+\gamma$ homogeneous agent economies, where in the n'th such economy, the agent has a rate of time preference given by β_n, and her beliefs about the expected growth rate of the endowment are a weighted average of the beliefs in the heterogenous agent economy, where the weights are $1-\frac{n}{\gamma}$ and $\frac{n}{\gamma}$, respectively.[27] The special case considered in Corollary 6 is similar to the model studied by Yan (2008, Proposition 3), where he obtains closed-form results for only the case in which the risk aversion parameter γ is identical across agent *and* γ is a natural number, which then excludes the case of risk aversion smaller than one. Our Proposition 10, in contrast, allows for different risk aversion parameters for the two agents and does not restrict their values to be natural numbers.

5.3 Valuation of risky and riskless zero-coupon claims

In the previous section, we studied the price of equity, which is an asset that pays a stream of cashflows in perpetuity. We now explore the term structure of zero-coupon risky and riskfree claims in the presence of heterogeneity in beliefs and preferences. We start by defining the yield on a risky zero-coupon claim, y^Y_{T-t}:

$$y^Y_{T-t} = -\frac{1}{T-t}\ln\frac{V^Y_{T-t}}{Y_t}.$$

The following proposition describes this yield when the maturity of the claim is infinite, that is, the "long-term" yield.

Proposition 11 *The long-term yield on the* risky *zero-coupon claim, y^Y_{T-t}, which pays the cash flow Y_T at time T is given by*

$$\lim_{T\to\infty} y^Y_{T-t} = \min\left(\hat{r}_1 + \gamma_1\,\sigma_Y^2 - \mu_{Y,1},\ \hat{r}_2 + \gamma_2\,\sigma_Y^2 - \mu_{Y,2}\right).$$

The long-term yield on the riskfree *zero-coupon discount bond, y^1_{T-t}, as $T\to\infty$ is*

$$\lim_{T\to\infty} y^1_{T-t} = \min\left(\hat{r}_1, \hat{r}_2\right),$$

and the limit of the term premium, *the difference between y^1_{T-t} and the short rate, r_t, is:*

$$\lim_{T\to\infty} y^1_{T-t} - r_t = \min\left(\hat{r}_1, \hat{r}_2\right) - r_t.$$

[27] The n'th weight in the sum is given by the expression in (63); observe that the weights sum to one, because

$$\sum_{n=0}^{\gamma}\left(\begin{array}{c}\gamma\\ n\end{array}\right)\left(\nu_{1,t}^{1-\frac{n}{\gamma}}\nu_{2,t}^{\frac{n}{\gamma}}\right)^{\gamma} = (\nu_{1,t}+\nu_{2,t})^{\gamma} = 1.$$

Observe that each term inside the min operator in (63) has the following interpretation: $\hat{r}_k$ is the riskless interest rate in a homogeneous-agent economy where the agent is of type k; the term $\gamma_k\, \sigma_Y^2$ is the adjustment to the riskless return for bearing risk in this economy, so the sum of the first two terms gives the expected return adjusted for risk; and, the last term is the growth rate expected by Agent k. Together, the three terms give the "discount rate" used by Agent k for valuing risky cashflows.

Proposition 11 implies that the long-term yield will be set by whichever agent has the *lower* discount rate, and *not* necessarily the agent who survives $\mathbb{P}$-almost surely in the long run. The intuition is that even though an agent may not survive in the long-run in the almost-surely sense, she may still be the dominant agent in rare states of the world, which are also high marginal utility states for this investor, and thus important for asset prices, as explained in Kogan, Ross, Wang, and Westerfield (2006).

Corollary 7 *Suppose agents have identical preferences, and Agent 1 has correct beliefs, whereas Agent 2 has incorrect beliefs about the expected growth rate of the economy. Then the economy is $\mathbb{P}$-a.s. non-stationary, since Agent 2 (with incorrect beliefs) does not survive $\mathbb{P}$-a.s. The long-term yield, y^Y_{T-t}, is set by Agent 2 if and only if (i) $\mu_{Y,2} < \mu_Y$ and $\gamma > 1$, or (ii) $\mu_{Y,2} > \mu_Y$ and $\gamma < 1$.*

Empirically, the nominal term premium (for riskless bonds) is smaller than the equity risk premium (see Campbell (2003)), while there is little empirical evidence on the magnitude of the real term premium. In our model, we also find that the term premium is smaller than the equity risk premium, though the difference is not as substantial as in the data. From Figure 6, we see that the term premium is around 1.5% in magnitude over most of the state space, even when we restrict the parameters so that the model is stationary. The figure also shows that heterogeneity in risk aversion alone would generate a very large term premium, but heterogeneity in beliefs plays an important role in reducing the magnitude of the term premium.

6 Conclusion

In this paper, we study an endowment economy where there are two types of agents, each with expected (power) utility. The two agents are heterogeneous with respect to their preference parameters for the subjective rate of time preference and relative risk aversion, and also with respect to their beliefs. The two agents can invest in a stock, which is a claim on endowment, and a instantaneously risk free asset, which is in zero net supply. Our main contribution is to solve in closed form for the equilibrium in this economy and to identify the optimal consumption-sharing rule, without restricting the risk aversions of the two agents to particular values. We use this closed-form solution to identify the market price of risk, the locally risk free interest rate and its volatility, the stock price, the equity market risk premium, the volatility of stock returns , and the term structure of interest rates. We then analyze how heterogeneity in preferences and beliefs affects the properties of asset returns.

We find that beliefs about the growth rate of aggregate endowment that are pessimistic on average lead to a significant increase in the market price of risk. Moreover, heterogeneity in preferences and beliefs increase stock-return volatility by as much as two to three times the fundamental volatility of aggregate endowment. Consequently, the equity risk premium, which is the product of the market price of risk and stock return volatility, is considerably higher in a model where both beliefs and preferences are heterogeneous, and this is accompanied neither by an increase in the level of the short-term riskless rate, nor an increase in its volatility. When the parameters values are restricted so that the model is stationary, one can still obtain a high market price of risk and equity risk premium, but not stock return volatility that is in excess of fundamental volatility.

A Appendix: Proofs for Propositions and Corollaries

Proof of Proposition 1: Consumption-sharing rule

Equation (12) is equivalent to

$$A_t(1-\nu_{1,t})^{\eta} = \nu_{1,t},$$

which implicitly defines $\nu_{1,t}$ in terms of A_t. To solve explicitly for $\nu_{1,t}$, we apply Theorem C2, expanding around the point $\nu_{1,t} = 0$, with

$$\begin{aligned} f(z) &= z(1-z)^{-\eta}, & \text{(A1)} \\ \varphi(z) &= (1-z)^{\eta}, & \text{(A2)} \\ g(z) &= z, \end{aligned}$$

after showing that f is complex analytic in some neighborhood of 0. We know from the binomial series expansion that for $z \in \mathbb{C}$, such that $|z| < 1$,

$$(1-z)^{-\eta} = \sum_{n=0}^{\infty} \binom{-\eta}{k} (-)^n z^n,$$

where $\binom{-\eta}{k} = \Pi_{j=1}^{k} \frac{-\eta-k+j}{j}$ is the generalized binomial coefficient. Therefore, $(1-z)^{-\eta}$ is complex analytic in the open ball $\{z \in \mathbb{C} : |z| < 1\}$. Since z is complex analytic for all $z \in \mathbb{C}$, it follows that f as defined in (A1) is complex analytic in the open ball $\{z \in \mathbb{C} : |z| < 1\}$. It therefore follows from Theorem C2 that

$$\nu_{1,t} = \sum_{n=1}^{\infty} \frac{A_t^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[(1-x)^{\eta n}\right]_{x=0}.$$

Since

$$\frac{d^{n-1}}{dx^{n-1}} \left[(1-x)^{n\eta}\right] = (-)^{n-1} \eta n(\eta n - 1)(\eta n - 2)\ldots(\eta n - (n-2))(1-x)^{\eta n-(n-1)},$$

it follows that

$$\begin{aligned} \nu_{1,t} &= -\sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{\eta n}{n-1}, & \text{(A3)} \\ \nu_{2,t} &= 1 + \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{\eta n}{n-1}. & \text{(A4)} \end{aligned}$$

We shall now determine the radius of convergence of the above series. From d'Alembert's ratio test, it follows that the above series converge absolutely for all $A \in \mathbb{C}$ s.t. $|A| < \overline{R}$, where

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \left| \frac{\binom{\eta n}{n-1}}{\binom{\eta(n+1)}{n}} \right|.$$

We wish to evaluate the above limit for all $\eta \in \mathbb{R}$ such that $\eta > 1$. Hence, $\binom{\eta n}{n-1}$ and $\binom{\eta(n+1)}{n}$ are positive and real, and so

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\binom{\eta n}{n-1}}{\binom{\eta(n+1)}{n}}.$$

We note that the generalized binomial coefficient, $\binom{z}{k} = \Pi_{j=1}^{k} \frac{z-k+j}{j}$, can be written as

$$\binom{z}{k} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)\Gamma(k+1)}, \tag{A5}$$

where $\Gamma(z)$ is the Gamma function, which for $\Re(z) > 0$ (where $\Re(z)$ denotes the real part of z), has the integral representation,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The Euler Beta function, $B(x, y)$, defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

can be written in terms of the Gamma function as follows,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{A6}$$

Together with (A5), the above expression implies that the generalized binomial coefficient is given by

$$\binom{z}{k} = \frac{1}{(z+1)B(z-k+1, k+1)}. \tag{A7}$$

Hence,

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{B((\eta-1)(n+1), n+1)}{B((\eta-1)n, n)}.$$

To evaluate the above limit, we start by recalling Stirling's series for the Gamma function

$$\Gamma(z) = \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \left(1 + O\left(\frac{1}{z}\right)\right), \tag{A8}$$

which together with (A6) implies that

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{\frac{((\eta-1)(n+1))^{(\eta-1)(n+1)-\frac{1}{2}}(n+1)^{(n+1)-\frac{1}{2}}}{((\eta-1)(n+1)+(n+1))^{((\eta-1)(n+1)+(n+1))-\frac{1}{2}}}}{\frac{((\eta-1)n)^{((\eta-1)n)-\frac{1}{2}} n^{n-\frac{1}{2}}}{(((\eta-1)n)+n)^{(((\eta-1)n)+n)-\frac{1}{2}}}}.$$

Simplifying the above expression gives

$$\begin{aligned} \overline{R} &= \lim_{n\to\infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{\frac{(\eta-1)^{(\eta-1)(n+1)-1/2}(n+1)^{\eta(n+1)-1}}{[\eta(n+1)]^{\eta(n+1)-1/2}}}{\frac{(\eta-1)^{(\eta-1)n-1/2} n^{\eta n-1}}{(\eta n)^{\eta n-1/2}}} \\ &= \frac{(\eta-1)^{\eta-1}}{\eta^\eta}. \end{aligned} \tag{A9}$$

Since A_t is a geometric Brownian motion, it is positive and real. Hence, the right-hand side of (A4) is absolutely convergent for $A_t < \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$.

We now derive a series expansion for $\nu_{2,t}$ in terms of A_t, which is absolutely convergent for $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$. We start by rearranging (12) to obtain

$$\nu_{2,t} = A_t^{-1/\eta}(1-\nu_{2,t})^{1/\eta}.$$

To find $\nu_{2,t}$, we apply Theorem C2, expanding around the point $\nu_{2,t} = 0$, with f, φ and g, defined as below

$$\begin{aligned} f(z) &= z(1-z)^{-1/\eta} && \text{(A10)} \\ \varphi(z) &= (1-z)^{1/\eta} && \text{(A11)} \\ g(z) &= z. \end{aligned}$$

We can show that our newly defined f is complex analytic in the open ball, $\{z \in \mathbb{C} : |z| < 1\}$, in the same way as for (A1). Hence, Theorem C2 implies that

$$\nu_{2,t} = \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[(1-x)^{n/\eta}\right]_{x=0}.$$

Because

$$\frac{d^{n-1}}{dx^{n-1}} \left[(1-x)^{n/\eta}\right] = (-)^{n-1}\frac{n}{\eta}\left(\frac{n}{\eta}-1\right)\left(\frac{n}{\eta}-2\right)\ldots\left(\frac{n}{\eta}-(n-2)\right)(1-x)^{\frac{n}{\eta}-(n-1)},$$

it follows that

$$\nu_{2,t} = -\sum_{n=1}^{\infty} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \binom{\frac{n}{\eta}}{n-1} = \sum_{n=1}^{\infty} \frac{(-)^{n-1}(A_t^{-\frac{1}{\eta}})^n}{n} \binom{\frac{n}{\eta}}{n-1}. \tag{A12}$$

By comparing the above expression with (A3), we can see that (A12) is absolutely convergent if $A_t^{-1/\eta} < \frac{(\frac{1}{\eta}-1)^{\frac{1}{\eta}-1}}{\frac{1}{\eta}^{\frac{1}{\eta}}}$, that is, if $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$. To summarize, we have

$$\nu_{2,t} = \begin{cases} -\sum_{n=1}^{\infty} \frac{\left(-A_t^{-\frac{1}{\eta}}\right)^n}{n} \binom{\frac{n}{\eta}}{n-1} & , A_t > \overline{R}, \\ 1 + \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta}{n-1} & , A_t < \overline{R}, \end{cases}$$

where $\overline{R}$ is given in (A9). Using (13) we can write the expressions for the sharing rule as (15).

Proof of Proposition 2: Dynamics of the consumption-sharing rule

We first derive a stochastic differential equation satisfied by $\nu_{1,t}$ by treating $\nu_{1,t}$ as a function of t, Y and ξ. Differentiating (6) implicitly with respect to t gives

$$\beta_1 + \gamma_1 \frac{1}{\nu_{1,t}} \frac{\partial \nu_{1,t}}{\partial t} = \beta_2 - \gamma_2 \frac{1}{\nu_{2,t}} \frac{\partial \nu_{1,t}}{\partial t}.$$

Solving for $\partial \nu_{1,t}/\partial t$, we obtain

$$\frac{\partial \nu_{1,t}}{\partial t} = \frac{1}{\gamma_1}\frac{1}{\gamma_2} \nu_{1,t}\nu_{2,t}\left(\beta_2 - \beta_1\right)\mathbf{R}_t,$$

where $\mathbf{R}_t$ is the average relative risk aversion in the economy, defined in (16). Differentiating (6) implicitly with respect Y_t and solving for $\partial\nu_{1,t}/\partial Y_t$ gives

$$Y_t \frac{\partial \nu_{1,t}}{\partial Y_t} = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) \nu_{1,t}\nu_{2,t}\mathbf{R}_t. \tag{A13}$$

Partial differentiation of each side of (A13) with respect to Y_t and solving for $\partial^2\nu_{1,t}/\partial Y_t^2$ gives

$$Y_t^2 \frac{\partial^2 \nu_{1,t}}{\partial Y_t^2} = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) \nu_{1,t}\nu_{2,t}\mathbf{R}_t \left(\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2} - 2\right).$$

Differentiating (6) implicitly with respect to ξ gives

$$\begin{aligned} a_1 e^{-\beta_1 t} Y_t^{-\gamma_1} \nu_{1,t}^{-\gamma_1} \left(-\frac{\gamma_1}{\nu_{1,t}} \frac{\partial \nu_{1,t}}{\partial \xi_t}\right) &= a_2 e^{-\beta_2 t} Y_t^{-\gamma_2} \nu_{2,t}^{-\gamma_2} \xi_t \frac{1}{\xi_t} + a_2 e^{-\beta_2 t} Y_t^{-\gamma_1} \nu_{2,t}^{-\gamma_2} \xi_t \left(-\frac{\gamma_2}{\nu_{2,t}} \frac{\partial \nu_{2,t}}{\partial \xi_t}\right) \\ \frac{\partial \nu_{1,t}}{\partial \xi_t} &= -\frac{\xi_t^{-1}\nu_{1,t}\nu_{2,t}}{\gamma_1\gamma_2}\mathbf{R}_t. \end{aligned}$$

Therefore,

$$\frac{\partial^2 \nu_{1,t}}{\partial^2 \xi_t} = -\frac{1}{\gamma_1\gamma_2}\frac{\partial}{\partial \xi_t}\left[\xi_t^{-1}\nu_{1,t}\nu_{2,t}\mathbf{R}_t\right] = -\frac{1}{\gamma_1\gamma_2}\left[-\xi_t^{-2}\nu_{1,t}\nu_{2,t}\mathbf{R}_t + \xi_t^{-1}\frac{\partial(\nu_{1,t}\nu_{2,t}\mathbf{R}_t)}{\partial \xi_t}\right].$$

Now note that

$$\begin{aligned} \frac{\partial(\nu_{1,t}\nu_{2,t}\mathbf{R}_t)}{\partial \xi_t} &= \nu_{1,t}\nu_{2,t}\frac{\partial \mathbf{R}_t}{\partial \xi_t} + \mathbf{R}_t\left(\nu_{1,t}\frac{\partial \nu_{2,t}}{\partial \xi_t} + \nu_{2,t}\frac{\partial \nu_{1,t}}{\partial \xi_t}\right) \\ &= \nu_{1,t}\nu_{2,t}\frac{\partial \mathbf{R}_t}{\partial \xi_t} + \mathbf{R}_t\frac{\partial \nu_{1,t}}{\partial \xi_t}\left(\nu_{2,t} - \nu_{1,t}\right). \end{aligned}$$

We now compute $\frac{\partial \mathbf{R}_t}{\partial \xi_t}$:

$$\frac{\partial \mathbf{R}_t}{\partial \xi_t} = -\mathbf{R}_t^2\left(\frac{1}{\gamma_1}\frac{\partial \nu_{1,t}}{\partial \xi_t} + \frac{1}{\gamma_2}\frac{\partial \nu_{2,t}}{\partial \xi_t}\right) = -\mathbf{R}_t^2\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\frac{\partial \nu_{1,t}}{\partial \xi_t}.$$

Therefore,

$$\begin{aligned} \frac{\partial(\nu_{1,t}\nu_{2,t}\mathbf{R}_t)}{\partial \xi_t} &= -\nu_{1,t}\nu_{2,t}\mathbf{R}_t^2\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\frac{\partial \nu_{1,t}}{\partial \xi_t} + \mathbf{R}_t\frac{\partial \nu_{1,t}}{\partial \xi_t}\left(\nu_{2,t} - \nu_{1,t}\right) \\ &= -\xi_t^{-1}\nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2}\left(-\nu_{1,t}\nu_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) + \nu_{2,t} - \nu_{1,t}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial^2 \nu_{1,t}}{\partial \xi_t^2} &= -\frac{1}{\gamma_1\gamma_2}\left[-\xi_t^{-2}\nu_{1,t}\nu_{2,t}\mathbf{R}_t - \xi_t^{-2}\nu_{1,t}\nu_{2,t}\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2}\left(-\nu_{1,t}\nu_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) + \nu_{2,t} - \nu_{1,t}\right)\right] \\ &= \frac{1}{\gamma_1\gamma_2}\xi_t^{-2}\nu_{1,t}\nu_{2,t}\mathbf{R}_t\left[1 + \frac{\mathbf{R}_t}{\gamma_1\gamma_2}\left(-\nu_{1,t}\nu_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) + \nu_{2,t} - \nu_{1,t}\right)\right]. \end{aligned}$$

The mixed partial derivative, $\frac{\partial^2 \nu_{1,t}}{\partial Y \partial \xi_t}$, is given by

$$\frac{\partial^2 \nu_{1,t}}{\partial Y \partial \xi_t} = -\frac{1}{\gamma_1\gamma_2}\frac{\partial}{\partial Y_t}\left[\xi_t^{-1}\nu_{1,t}\nu_{2,t}\mathbf{R}_t\right] = -\frac{1}{\gamma_1\gamma_2}\xi_t^{-1}\left\{\mathbf{R}_t\frac{\partial}{\partial Y_t}\left[\nu_{1,t}\nu_{2,t}\right] + \nu_{1,t}\nu_{2,t}\frac{\partial \mathbf{R}_t}{\partial Y_t}\right\}.$$

Hence, we compute

$$\frac{\partial}{\partial Y_t}\left[\nu_{1,t}\nu_{2,t}\right] = \frac{\partial \nu_{1,t}}{\partial Y_t}\nu_{2,t} + \frac{\partial \nu_{2,t}}{\partial Y_t}\nu_{1,t} = \frac{\partial \nu_{1,t}}{\partial Y_t}\nu_{2,t} - \frac{\partial \nu_{1,t}}{\partial Y_t}\nu_{1,t} = \frac{\partial \nu_{1,t}}{\partial Y_t}(\nu_{2,t} - \nu_{1,t}),$$

and

$$\frac{\partial \mathbf{R}_t}{\partial Y_t} = -\mathbf{R}_t^2\left(\frac{1}{\gamma_1}\frac{\partial \nu_{1,t}}{\partial Y_t} + \frac{1}{\gamma_2}\frac{\partial \nu_{2,t}}{\partial Y_t}\right) = -\mathbf{R}_t^2\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\frac{\partial \nu_{1,t}}{\partial Y_t}.$$

Thus, we obtain

$$\begin{aligned}\frac{\partial^2 \nu_{1,t}}{\partial Y \partial \xi_t} &= -\frac{1}{\gamma_1\gamma_2}\xi_t^{-1}\left\{\mathbf{R}_t\frac{\partial}{\partial Y_t}\left[\nu_{1,t}\nu_{2,t}\right] + \nu_{1,t}\nu_{2,t}\frac{\partial \mathbf{R}_t}{\partial Y_t}\right\} \\ &= \frac{-1}{\gamma_1\gamma_2}Y_t^{-1}\xi_t^{-1}\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\nu_{1,t}\nu_{2,t}\mathbf{R}_t^2\left\{(\nu_{2,t} - \nu_{1,t}) - \nu_{1,t}\nu_{2,t}\mathbf{R}_t\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\right\}.\end{aligned}$$

From Ito's Lemma

$$\begin{aligned}d\nu_{1,t} &= \left(\frac{\partial \nu_{1,t}}{\partial t} + Y_t\frac{\partial \nu_{1,t}}{\partial Y_t}\mu_Y + \xi_t\frac{\partial \nu_{1,t}}{\partial \xi_t}\mu_\xi + \frac{1}{2}Y_t^2\frac{\partial^2 \nu_{1,t}}{\partial Y_t^2}\sigma_Y^2 + \frac{1}{2}\xi_t^2\frac{\partial^2 \nu_{1,t}}{\partial \xi_t^2}\sigma_\xi^2 + \xi_t Y_t\frac{\partial \nu_{1,t}^2}{\partial \xi_t \partial Y_t}\sigma_Y\sigma_\xi\right)dt \\ &\quad + \left(Y_t\frac{\partial \nu_{1,t}}{\partial Y_t}\sigma_Y + \xi_t\frac{\partial \nu_{1,t}}{\partial \xi_t}\sigma_\xi\right)dZ_t,\end{aligned}$$

which under measure $\mathbb{P}^1$ becomes

$$\begin{aligned}d\nu_{1,t} &= \left(\frac{\partial \nu_{1,t}}{\partial t} + Y_t\frac{\partial \nu_{1,t}}{\partial Y_t}\mu_{Y,1} + \frac{1}{2}Y_t^2\frac{\partial^2 \nu_{1,t}}{\partial Y_t^2}\sigma_Y^2 + \frac{1}{2}\xi_t^2\frac{\partial^2 \nu_{1,t}}{\partial \xi_t^2}\sigma_\xi^2 + \xi_t Y_t\frac{\partial \nu_{1,t}^2}{\partial \xi_t \partial Y_t}\sigma_Y\sigma_\xi\right)dt \\ &\quad + \left(Y_t\frac{\partial \nu_{1,t}}{\partial Y_t}\sigma_Y + \xi_t\frac{\partial \nu_{1,t}}{\partial \xi_t}\sigma_\xi\right)dZ_{1,t},\end{aligned}$$

and under measure $\mathbb{P}^2$ is

$$\begin{aligned}d\nu_{1,t} &= \left(\frac{\partial \nu_{1,t}}{\partial t} + Y_t\frac{\partial \nu_{1,t}}{\partial Y_t}\mu_{Y,2} + \xi_t\frac{\partial \nu_{1,t}}{\partial \xi_t}\sigma_\xi^2 + \frac{1}{2}Y_t^2\frac{\partial^2 \nu_{1,t}}{\partial Y_t^2}\sigma_Y^2 + \frac{1}{2}\xi_t^2\frac{\partial^2 \nu_{1,t}}{\partial \xi_t^2}\sigma_\xi^2 + \xi_t Y_t\frac{\partial \nu_{1,t}^2}{\partial \xi_t \partial Y_t}\sigma_Y\sigma_\xi\right)dt \\ &\quad + \left(Y_t\frac{\partial \nu_{1,t}}{\partial Y_t}\sigma_Y + \xi_t\frac{\partial \nu_{1,t}}{\partial \xi_t}\sigma_\xi\right)dZ_{2,t}.\end{aligned}$$

Substituting the partial derivatives into these equations, followed by some straightforward algebra, leads to the results stated in the proposition.

Similarly, one can show that Agent k believes the evolution of the sharing rule is given by

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_{1,t}}^{\mathbb{P}^k}dt + \sigma_{\nu_{1,t}}dZ_{k,t},$$

where

$$\begin{aligned}\mu_{\nu_{1,t}}^{\mathbb{P}^1} &= \mu_{\nu_{1,t}} + \sigma_{\xi,1}\sigma_{\nu_{1,t}} \\ &= \nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\left\{(\beta_2 - \beta_1) + (\gamma_2 - \gamma_1)\mu_{Y,1} - \left(\frac{\nu_{2,t}^2}{\gamma_2} - \frac{\nu_{1,t}^2}{\gamma_1}\right)\mathbf{R}_t^2\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)(\mu_{Y,2} - \mu_{Y,1})\right. \\ &\quad \left. + \frac{1}{2}(\gamma_2 - \gamma_1)\left(\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2} - 2\right)\sigma_Y^2 + \frac{1}{2}\sigma_\xi^2\left(\left(\frac{\nu_{2,t}^2}{\gamma_2} - \frac{\nu_{1,t}^2}{\gamma_1}\right)\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2} + 1\right)\right\}, \qquad \text{(A14)}\end{aligned}$$

and

$$
\begin{aligned}
\mu^{\mathbb{P}^2}_{\nu_{1,t}} &= \mu_{\nu_{1,t}} + \sigma_{\xi,2}\sigma_{\nu_{1,t}} \\
&= \nu_{2,t}\frac{\mathbf{R}_t}{\gamma_1\gamma_2}\left\{(\beta_2-\beta_1)+(\gamma_2-\gamma_1)\mu_{Y,2}-\left(\frac{\nu_{2,t}^2}{\gamma_2}-\frac{\nu_{1,t}^2}{\gamma_1}\right)\mathbf{R}_t^2\left(\frac{1}{\gamma_1}-\frac{1}{\gamma_2}\right)(\mu_{Y,2}-\mu_{Y,1})\right. \\
&\quad \left. +\frac{1}{2}(\gamma_2-\gamma_1)\left(\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2}-2\right)\sigma_Y^2+\frac{1}{2}\sigma_\xi^2\left(\left(\frac{\nu_{2,t}^2}{\gamma_2}-\frac{\nu_{1,t}^2}{\gamma_1}\right)\frac{\mathbf{R}_t^2}{\gamma_1\gamma_2}-1\right)\right\}.
\end{aligned}
$$

Proof of Proposition 3: Almost-sure survival

Equation (12) can be rewritten as

$$
\nu_{2,t}^{\gamma_2} = Y_0^{-(\gamma_2-\gamma_1)}\frac{\lambda_{2,0}}{\lambda_{1,0}}e^{-(\beta_2-\beta_1)t}e^{-\frac{1}{2}(\sigma_{\xi,2}^2-\sigma_{\xi,2}^2)t+(\sigma_{\xi,2}-\sigma_{\xi,1})Z_t}e^{-(\gamma_2-\gamma_1)[(\mu_Y-\frac{1}{2}\sigma_Y^2)t+\sigma_Y Z_t]}\nu_{1,t}^{\gamma_1}.
$$

Thus,

$$
\nu_{2,t}^{\eta} = \left(Y_0^{-(\gamma_2-\gamma_1)}\frac{\lambda_{2,0}}{\lambda_{1,0}}e^{-(\beta_2-\beta_1)t}e^{-\frac{1}{2}(\sigma_{\xi,2}^2-\sigma_{\xi,1}^2)t+(\sigma_{\xi,2}-\sigma_{\xi,1})Z_t}e^{-(\gamma_2-\gamma_1)[(\mu_Y-\frac{1}{2}\sigma_Y^2)t+\sigma_Y Z_t]}\right)^{1/\gamma_1}\nu_{1,t},
$$

which implies that

$$
\nu_{2,t}^{\eta}\left(Y_0^{(\gamma_2-\gamma_1)}\frac{\lambda_{1,0}}{\lambda_{2,0}}e^{\left[\beta_2-\beta_1-\frac{\left(\mu_{Y,2}-\mu_{Y,1}\right)\left(\mu_Y-\frac{1}{2}(\mu_{Y,1}+\mu_{Y,2})\right)}{\sigma_Y^2}+(\gamma_2-\gamma_1)\left(\mu_Y-\frac{1}{2}\sigma_Y^2\right)\right]t}e^{\left[\frac{\mu_{Y,1}-\mu_{Y,2}}{\sigma_Y}+(\gamma_2-\gamma_1)\sigma_Y\right]Z_t}\right) = \nu_{1,t}. \tag{A15}
$$

Now, recall the standard results that

$$
\lim_{t\to\infty} e^{at+bZ_t} = \begin{cases} \infty, \ \mathbb{P}-a.s., & a>0, \\ 0, \ \ \mathbb{P}-a.s., & a<0, \end{cases}
$$

and

$$
\begin{aligned}
\limsup_{t\to\infty} e^{bZ_t} &= \infty, \\
\liminf_{t\to\infty} e^{bZ_t} &= 0.
\end{aligned}
$$

From the above results it follows that to ensure that $\lim_{t\to\infty} e^{at+bZ_t}$ is strictly between zero and infinity, we need to have both a and b equal to zero. It then follows from the expression in (A15) that both agents will survive $\mathbb{P}$-a.s., that is, the economy will be stationary almost surely under $\mathbb{P}$, if and only if (22) and (23) hold.

Under $\mathbb{P}^1$, (A15) becomes

$$
\nu_{2,t}^{\eta}\left(Y_0^{(\gamma_2-\gamma_1)}\frac{\lambda_1}{\lambda_2}e^{(\beta_2-\beta_1)t}e^{\frac{1}{2}\sigma_\xi^2 t+(\sigma_{\xi,1}-\sigma_{\xi,2})Z_t}e^{(\gamma_2-\gamma_1)[(\mu_{Y,1}-\frac{1}{2}\sigma_Y^2)t+\sigma_Y Z_{1,t}]}\right)^{1/\gamma_1} = \nu_{1,t}.
$$

It follows that Agent 1 believes the economy is almost surely stationary if and only if (22) and (A16) hold:

$$(\beta_2-\beta_1)+\frac{1}{2}\sigma_\xi^2+(\gamma_2-\gamma_1)(\mu_{Y,1}-\frac{1}{2}\sigma_Y^2)=0. \tag{A16}$$

Under $\mathbb{P}^2$, (A15) becomes

$$\nu_{2,t}^{\eta}\left(Y_0^{(\gamma_2-\gamma_1)}\frac{\lambda_{1,0}}{\lambda_{2,0}}e^{(\beta_2-\beta_1)t}e^{-\frac{1}{2}\sigma_\xi^2 t+(\sigma_{\xi,1}-\sigma_{\xi,2})Z_t}e^{(\gamma_2-\gamma_1)[(\mu_{Y,2}-\frac{1}{2}\sigma_Y^2)t+\sigma_Y Z_{2,t}]}\right)^{1/\gamma_1}=\nu_{1,t}.$$

So, Agent 2 believes the economy is almost surely stationary if and only if (22) and (A17) hold:

$$(\beta_2-\beta_1)-\frac{1}{2}\sigma_\xi^2+(\gamma_2-\gamma_1)(\mu_{Y,2}-\frac{1}{2}\sigma_Y^2)=0. \tag{A17}$$

Proof of Proposition 4: Survival in the mean

First we compute $E_t\nu_{2,t+u}$, $E_t^1\nu_{2,t+u}$, and $E_t^2\nu_{2,t+u}$. Then we take limits as $u\to\infty$. Thus,

$$\begin{aligned}E_t[\nu_{2,t+u}] &= E_t\left[\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\left(\frac{\hat{\pi}_{2,t+u}}{\hat{\pi}_{1,t+u}}\right)^{\frac{n}{\gamma_2}}\binom{n\frac{\gamma_1}{\gamma_2}}{n-1}1_{\left\{\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}>R\right\}}\right]\\ &\quad+E_t\left[1-\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\left(\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}\right)^{\frac{n}{\gamma_1}}\binom{n\frac{\gamma_2}{\gamma_1}}{n-1}1_{\left\{\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}<R\right\}}\right].\end{aligned}$$

The infinite series in the expressions above are complex analytic functions of $\left(\frac{\hat{\pi}_2}{\hat{\pi}_1}\right)^{\frac{1}{\gamma_2}}$ for $\left\{\frac{\hat{\pi}_1}{\hat{\pi}_2}\in\mathbb{C}:|\frac{\hat{\pi}_1}{\hat{\pi}_2}|>R\right\}$, and $\left(\frac{\hat{\pi}_1}{\hat{\pi}_2}\right)^{\frac{1}{\gamma_1}}$ for $\left\{\frac{\hat{\pi}_1}{\hat{\pi}_2}\in\mathbb{C}:|\frac{\hat{\pi}_1}{\hat{\pi}_2}|<R\right\}$, respectively. Therefore, term-by-term integration is valid, and

$$\begin{aligned}E_t[\nu_{2,t+u}] &= \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\binom{n\frac{\gamma_1}{\gamma_2}}{n-1}E_t\left[\left(\frac{\hat{\pi}_{2,t+u}}{\hat{\pi}_{1,t+u}}\right)^{\frac{n}{\gamma_2}}1_{\left\{\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}>R\right\}}\right]\\ &\quad+E_t\left[1_{\left\{\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}<R\right\}}\right]-\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\binom{n\frac{\gamma_2}{\gamma_1}}{n-1}E_t\left[\left(\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}\right)^{\frac{n}{\gamma_1}}1_{\left\{\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}<R\right\}}\right].\end{aligned}$$

From Lemma B2 in Appendix B it follows that

$$E_t\left[A_{t+u}^n 1_{\{A_{t+u}<R\}}\right]=A_t^n e^{n\left(\mu_{A,1}-\frac{1}{2}\sigma_A^2\right)u}e^{n^2\sigma_A^2 u}\Phi\left(\frac{\ln\left(\frac{R}{A_t}\right)-\left(\mu_A-\frac{1}{2}\sigma_A^2\right)u}{\sigma\sqrt{u}}-n\sigma_A\sqrt{u}\right),$$

and

$$E_t\left[A_{t+u}^{-\frac{n}{\eta}}1_{\{A_{t+u}>R\}}\right]=A_t^{-\frac{n}{\eta}}e^{-\frac{n}{\eta}\left(\mu_A-\frac{1}{2}\sigma_A^2\right)u}e^{\left(\frac{n}{\eta}\right)^2\sigma_A^2 u}\Phi\left(\frac{-\ln\left(\frac{R}{A_t}\right)+\left(\mu_A-\frac{1}{2}\sigma_A^2\right)u}{\sigma\sqrt{u}}-\frac{n}{\eta}\sigma_A\sqrt{u}\right),$$

where A is given in (13), μ_A and σ_A are the drift (under $\mathbb{P}$) and diffusion coefficients, respectively of $\frac{dA}{A}$, i.e.

$$\begin{aligned}
\mu_A &= \frac{\beta_2 - \beta_1}{\gamma_1} + (\eta - 1)\left(\mu_Y - \frac{1}{2}\sigma_Y^2\right) - \frac{1}{2\gamma_1}\sigma_\xi^2 + \frac{1}{2}\sigma_A^2, \\
\sigma_A &= (\eta - 1)\sigma_Y - \frac{1}{\gamma_1}\sigma_\xi.
\end{aligned}$$

Since A is given by (13), we can rewrite (A18) and (A18) in the following more symmetric form:

$$E_t\left[\left(\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}\right)^{\frac{n}{\gamma_1}} 1_{\left\{\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}<R\right\}}\right] = \left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}} e^{\frac{n}{\gamma_1}\mu_\Delta u} e^{\left(\frac{n}{\gamma_1}\right)^2 \sigma_\Delta^2 u} \Phi\left(\frac{\ln\left(\frac{R}{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}\right) - \mu_\Delta u}{\sigma_\Delta\sqrt{u}} - \frac{n}{\gamma_1}\sigma_\Delta\sqrt{u}\right),$$

and

$$E_t\left[\left(\frac{\hat{\pi}_{2,t+u}}{\hat{\pi}_{1,t+u}}\right)^{\frac{n}{\gamma_2}} 1_{\left\{\frac{\hat{\pi}_{1,t+u}}{\hat{\pi}_{2,t+u}}>R\right\}}\right] = \left(\frac{\hat{\pi}_{2,t}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_2}} e^{-\frac{n}{\gamma_2}\mu_\Delta u} e^{\left(\frac{n}{\gamma_2}\right)^2 \sigma_\Delta^2 u} \Phi\left(\frac{-\ln\left(\frac{R}{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}\right) + \mu_\Delta u}{\sigma_\Delta\sqrt{u}} - \frac{n}{\gamma_2}\sigma_\Delta\sqrt{u}\right),$$

where μ_Δ and σ_Δ are the drift (under $\mathbb{P}$) and diffusion components, respectively, of $\ln\frac{\pi_{1,t}}{\pi_{2,t}}$, and are:

$$\mu_\Delta = \beta_2 - \beta_1 + (\gamma_2 - \gamma_1)\left(\mu_Y - \frac{1}{2}\sigma_Y^2\right) - \frac{1}{2}\sigma_\xi^2, \tag{A18}$$

$$\sigma_\Delta = (\gamma_2 - \gamma_1)\sigma_Y - \sigma_\xi. \tag{A19}$$

Therefore,

$$\begin{aligned}
E_t\nu_{2,t+u} &= \Phi\left(\frac{\ln\left(\frac{R}{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}\right) - \mu_\Delta u}{\sigma_\Delta\sqrt{u}}\right) \\
&\quad + \sum_{n=1}^{\infty}\frac{(-)^{-n}}{n}\binom{n\frac{\gamma_2}{\gamma_1}}{n-1}\left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}} e^{\frac{n}{\gamma_1}\mu_\Delta u} e^{\left(\frac{n}{\gamma_1}\right)^2 \sigma_\Delta^2 u} \Phi\left(\frac{\ln\left(\frac{R}{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}\right) - \mu_\Delta u}{\sigma_\Delta\sqrt{u}} - \frac{n}{\gamma_1}\sigma_\Delta\sqrt{u}\right) \\
&\quad - \sum_{n=1}^{\infty}\frac{(-)^{-n}}{n}\binom{n\frac{\gamma_1}{\gamma_2}}{n-1}\left(\frac{\hat{\pi}_{2,t}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_2}} e^{-\frac{n}{\gamma_2}\mu_\Delta u} e^{\left(\frac{n}{\gamma_2}\right)^2 \sigma_\Delta^2 u} \Phi\left(\frac{-\ln\left(\frac{R}{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}\right) + \mu_\Delta u}{\sigma_\Delta\sqrt{u}} - \frac{n}{\gamma_2}\sigma_\Delta\sqrt{u}\right).
\end{aligned}$$

We can show that $\lim_{u\to\infty}$ and $\sum_{n=1}^{\infty}$ can be interchanged (details are available upon request), which implies that

$$\lim_{u\to\infty} E_t\left[\nu_{2,t+u}\right] = \lim_{u\to\infty} \Phi\left(\frac{\ln\left(\frac{R}{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}\right) - \mu_\Delta u}{\sigma_\Delta\sqrt{u}}\right) = \begin{cases} 0 & , \mu_\Delta > 0, \\ \frac{1}{2} & , \mu_\Delta = 0, \\ 1 & , \mu_\Delta < 0. \end{cases}$$

Therefore, the economy is mean stationary under $\mathbb{P}$ if and only if $\mu_\Delta = 0$; that is,

$$\beta_1 - \beta_2 - (\gamma_2 - \gamma_1)\left(\mu_Y - \frac{1}{2}\sigma_Y^2\right) - \frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2) = 0.$$

Similarly, we can evaluate $E_t^1\nu_{2,t}$ and $E_t^2\nu_{2,t}$, and obtain necessary and sufficient conditions for mean stationarity under $\mathbb{P}^1$ and $\mathbb{P}^2$, given in (A16) and (A17), respectively.

Proof of Corollary 1: Mean stationary under identical preferences

Suppose $\beta_1 = \beta_2 = \beta$, $\gamma_1 = \gamma_2 = \gamma$, and $\mu_{Y,1} \neq \mu_{Y,2}$. Then (23) reduces to (24).

Proof of Proposition 5: Riskfree rate

Agent 1's state price density, $\pi_{1,t}$, is given in (3). Since $C_{1,t} = \nu_{1,t}Y_t$, it follows from Ito's Lemma that

$$\begin{aligned} \frac{d\pi_{1,t}}{\pi_{1,t}} &= -\left[\beta_1 + \gamma_1\left(\mu_{Y,1} + \mu_{\nu_1,t}^{\mathbb{P}^1} + \sigma_Y\sigma_{\nu_1,t}\right) - \frac{1}{2}\gamma_1\left(1+\gamma_1\right)\left(\sigma_Y + \sigma_{\nu_1,t}\right)^2\right]dt \\ &\quad - \gamma_1\left(\sigma_Y + \sigma_{\nu_1,t}\right)dZ_{1,t}. \end{aligned}$$

Hence, from (28), we have

$$r_t = \beta_1 + \gamma_1\left(\mu_{Y,1} + \mu_{\nu_1,t}^{\mathbb{P}^1} + \sigma_Y\sigma_{\nu_1,t}\right) - \frac{1}{2}\gamma_1\left(1+\gamma_1\right)\left(\sigma_Y + \sigma_{\nu_1,t}\right)^2.$$

Substituting the expressions for $\mu_{\nu_1,t}^{\mathbb{P}^1}$ and $\sigma_{\nu_1,t}$ from (A14) and (17), respectively, into (A20) and simplifying gives (29).

Proof of Corollary 2: Riskfree rate with correct beliefs or with identical risk aversions

Equation (30) follows from (29) after setting $\mu_{Y,1} = \mu_{Y,2} = \mu_Y$, and simplifying.

Proof of Proposition 6: Volatility of the risk-free rate

Applying Ito's Lemma to r_t, we obtain

$$dr_t = \mu_{r,t}dt + \sigma_{r,t}dZ_t,$$

where

$$
\begin{aligned}
\mu_{r,t} &= \nu_{1,t}\frac{\partial r_t}{\partial \nu_{1,t}}\mu_{\nu_1,t} + \frac{1}{2}\nu_{1,t}^2\frac{\partial^2 r_t}{\partial \nu_{1,t}^2}\sigma_{\nu_1,t}^2 \\
\sigma_{r,t} &= \nu_{1,t}\frac{\partial r_t}{\partial \nu_{1,t}}\sigma_{\nu_1,t}.
\end{aligned}
$$

Substituting (29) and (17) into the above expression and simplifying gives (32).

Proof of Corollary 3: Volatility of the risk-free rate if risk aversions or beliefs are identical

If the two agents have identical risk aversion, $\gamma_1 = \gamma_2 = \gamma$, then the volatility of the interest rate in (32) reduces to the expression in (33). On the other hand, if the two agents have identical beliefs, $\mu_{Y,1} = \mu_{Y,2} = \mu_Y$, then the volatility of the interest rate in (32) reduces to (34).

Proof of Proposition 7: Market price of risk

Equation (A20) gives the dynamics for Agent 1's state price density, $\pi_{1,t}$. Hence, from (28), we have

$$\theta_{1,t} = \gamma_1\left(\sigma_Y + \sigma_{\nu_{\gamma_1},t}\right), \tag{A20}$$

Substituting the expression for $\sigma_{\nu_1,t}$ from (17) into (A20) and simplifying gives (36).

Proof of Corollary 4: Market price of risk with correct beliefs or with identical risk aversions

Equation (38) follows from (36) after setting $\mu_{Y,1} = \mu_{Y,2} = \mu_Y$, and simplifying. Equations (39) and (40) follow from (36) after setting $\gamma_1 = \gamma_2 = \gamma$, and simplifying.

Proof of Proposition 8: State-price density

The equilibrium state price density is given by (26). To find a closed-form expression for the equilibrium state-price density, we find series expansions for $\nu_{k,t}^{-\gamma_k}$, $k \in \{1,2\}$. To find a series expansion for $\nu_{2,t}^{-\gamma_2}$, note that

$$\nu_{2,t}^{-\gamma_2} = (1-\nu_{1,t})^{-\gamma_2},$$

and use Theorem C2 to expand around the point $\nu_{1,t} = 0$. To do this we define

$$g(z) = (1-z)^{-\gamma_2},$$

which is complex analytic in the open ball $\{z \in \mathbb{C} : |z| < 1\}$. Hence, with f and φ defined as in (A1) and (A2), respectively, Theorem C2 implies that

$$\begin{aligned} g(\nu_{1,t}) &= (1-\nu_{1,t})^{-\gamma_2} \\ &= g(0) + \sum_{n=1}^{\infty} \frac{A_t^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[g'(x)\varphi(x)^n \right]_{x=0} \\ &= 1 + \sum_{n=1}^{\infty} \frac{A_t^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[\gamma_2 (1-x)^{n\eta-\gamma_2-1} \right]_{x=0}. \end{aligned}$$

Since,

$$\begin{aligned} &\frac{d^{n-1}}{dx^{n-1}} \gamma_2 (1-x)^{n\eta-\gamma_2-1} \\ &= \gamma_2 (-)^{n-1} (n\eta-\gamma_2-1)(n\eta-\gamma_2-2)\dots(n\eta-\gamma_2-(n-1))(1-x)^{n\eta-\gamma_2-(n-1)}, \end{aligned}$$

it follows that

$$\nu_{2,t}^{-\gamma_2} = 1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta-\gamma_2-1}{n-1}.$$

D'Alembert's ratio test implies that the above series converges absolutely for all $A \in \mathbb{C}$ such that $|A| < \overline{R}$, where

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\binom{\eta n-\gamma_2-1}{n-1}}{\binom{\eta(n+1)-\gamma_2-1}{n}}.$$

Using (A7), we rewrite the above expression as

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\eta(n+1)-\gamma_2}{\eta n-\gamma_2} \frac{B((\eta-1)(n+1)-\gamma_2-1, n+1)}{B((\eta-1)n-\gamma_2-1, n)}.$$

Hence, using (A6) and (A8), we obtain

$$\overline{R} = \lim_{n\to\infty} \frac{n+1}{n} \frac{\eta(n+1)-\gamma_2}{\eta n-\gamma_2} \frac{\frac{[(\eta-1)(n+1)-(1+\gamma_2)]^{(\eta-1)(n+1)-(1+\gamma_2)-1/2}(n+1)^{n+1-1/2}}{[\eta(n+1)-(1+\gamma_2)]^{\eta(n+1)-(1+\gamma_2)-1/2}}}{\frac{[(\eta-1)n-(1+\gamma_2)]^{(\eta-1)n-(1+\gamma_2)-1/2}n^{n-1/2}}{[\eta n-(1+\gamma_2)]^{\eta n-(1+\gamma_2)-1/2}}}.$$

We now simplify the expression

$$\frac{\frac{[(\eta-1)(n+1)-(1+\gamma_2)]^{(\eta-1)(n+1)-(1+\gamma_2)-1/2}(n+1)^{n+1-1/2}}{[\eta(n+1)-(1+\gamma_2)]^{\eta(n+1)-(1+\gamma_2)-1/2}}}{\frac{[(\eta-1)n-(1+\gamma_2)]^{(\eta-1)n-(1+\gamma_2)-1/2}n^{n-1/2}}{[\eta n-(1+\gamma_2)]^{\eta n-(1+\gamma_2)-1/2}}}. \tag{A21}$$

Simplifying the numerator of the above expression gives

$$\begin{aligned} &\frac{[(\eta-1)(n+1)-(1+\gamma_2)]^{(\eta-1)(n+1)-(1+\gamma_2)-\frac{1}{2}}(n+1)^{n+1-1/2}}{[\eta(n+1)-(1+\gamma_2)]^{\eta(n+1)-(1+\gamma_2)-\frac{1}{2}}} \\ &= \frac{(\eta-1)^{(\eta-1)(n+1)-(\gamma_2+1)-\frac{1}{2}} \left(\left[1-\frac{\gamma_2+1}{(\eta-1)(n+1)}\right]^{(n+1)} \right)^{(\eta-1)} \left[1-\frac{\gamma_2+1}{(\eta-1)(n+1)}\right]^{-(1+\gamma_2+\frac{1}{2})}}{\sqrt{n+1}\,\eta^{\eta(n+1)-(\gamma_2+1)-\frac{1}{2}} \left(\left[1-\frac{\gamma_2+1}{\eta(n+1)}\right]^{(n+1)} \right)^{\eta} \left[1-\frac{\gamma_2+1}{\eta(n+1)}\right]^{-(1+\gamma_2+\frac{1}{2})}}. \end{aligned}$$

Similarly for the denominator of (A21)

$$\frac{[(\eta-1)n-(1+\gamma_2)]^{(\eta-1)n-(1+\gamma_2)-1/2}n^{n-1/2}}{[\eta n-(1+\gamma_2)]^{\eta n-(1+\gamma_2)-1/2}}$$
$$= \frac{(\eta-1)^{(\eta-1)n-(\gamma_2+1)-\frac{1}{2}}\left(\left[1-\frac{\gamma_2+1}{(\eta-1)n}\right]^n\right)^{(\eta-1)}\left[1-\frac{\gamma_2+1}{(\eta-1)n}\right]^{-(1+\gamma_2+\frac{1}{2})}}{\sqrt{n}\,\eta^{\eta n-(\gamma_2+1)-\frac{1}{2}}\left(\left[1-\frac{\gamma_2+1}{\eta n}\right]^n\right)^{\eta}\left[1-\frac{\gamma_2+1}{\eta n}\right]^{-(1+\gamma_2+\frac{1}{2})}}.$$

Therefore,

$$\overline{R} = \lim_{n\to\infty}\frac{n+1}{n}\frac{\eta(n+1)-\gamma_2}{\eta n-\gamma_2}\frac{\dfrac{(\eta-1)^{(\eta-1)(n+1)-(\gamma_2+1)-\frac{1}{2}}\left(\left[1-\frac{\gamma_2+1}{(\eta-1)(n+1)}\right]^{(n+1)}\right)^{(\eta-1)}\left[1-\frac{\gamma_2+1}{(\eta-1)(n+1)}\right]^{-(1+\gamma_2+\frac{1}{2})}}{\sqrt{n+1}\,\eta^{\eta(n+1)-(\gamma_2+1)-\frac{1}{2}}\left(\left[1-\frac{\gamma_2+1}{\eta(n+1)}\right]^{(n+1)}\right)^{\eta}\left[1-\frac{\gamma_2+1}{\eta(n+1)}\right]^{-(1+\gamma_2+\frac{1}{2})}}}{\dfrac{(\eta-1)^{(\eta-1)n-(\gamma_2+1)-\frac{1}{2}}\left(\left[1-\frac{\gamma_2+1}{(\eta-1)n}\right]^{n}\right)^{(\eta-1)}\left[1-\frac{\gamma_2+1}{(\eta-1)n}\right]^{-(1+\gamma_2+\frac{1}{2})}}{\sqrt{n}\,\eta^{\eta n-(\gamma_2+1)-\frac{1}{2}}\left(\left[1-\frac{\gamma_2+1}{\eta n}\right]^{n}\right)^{\eta}\left[1-\frac{\gamma_2+1}{\eta n}\right]^{-(1+\gamma_2+\frac{1}{2})}}}$$
$$= \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}\frac{\frac{e^{-(\gamma_2+1)}}{e^{-(\gamma_2+1)}}}{\frac{e^{-(\gamma_2+1)}}{e^{-(\gamma_2+1)}}},$$

since $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. Hence,

$$\overline{R} = \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}.$$

Since A_t is a geometric Brownian motion, A_t is real and positive, and so the right-hand side of (A21) is absolutely convergent if $A_t < \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta} = \overline{R}$. Hence,

$$\nu_{2,t}^{-\gamma_2} = 1-\gamma_2\sum_{n=1}^{\infty}\frac{(-A_t)^n}{n}\left(\begin{array}{c} n\eta-\gamma_2-1 \\ n-1\end{array}\right),\ A_t<\overline{R}.$$

Using (9) and (14), we can rewrite the above expression as

$$\nu_{2,t}^{-\gamma_2} = \sum_{n=0}^{\infty}a_{n,2}^{\pi}\left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{\frac{n}{\gamma_1}},\ \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < \frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}}\left(\frac{\gamma_2}{\gamma_1}-1\right)^{\gamma_2-\gamma_1},$$

where $a_{n,2}^{\pi}$ is defined in (43). Therefore, the equilibrium state-price density is given by

$$\pi_t = \sum_{n=0}^{\infty}a_{n,2}^{\pi}\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}},\ \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < \frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}}\left(\frac{\gamma_2}{\gamma_1}-1\right)^{\gamma_2-\gamma_1}.$$

To find an expression for the state-price density when $A_t > \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}$, we find a series expansion for $\nu_{1,t}^{-\gamma_1}$, which is absolutely convergent for $A_t > \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}$. Note that

$$\nu_{1,t}^{-\gamma_1} = (1-\nu_{2,t})^{-\gamma_1},$$

and use Theorem C2 to expand around the point $\nu_{2,t}=0$. To do this, we define

$$g(z) = (1-z)^{-\gamma_1},$$

which is complex analytic in the open ball $\{z \in \mathbb{C} : |z| < 1\}$. Hence, with f and φ defined as in (A10) and (A11), respectively, Theorem C2 implies that

$$\begin{aligned} g(\nu_{2,t}) &= (1-\nu_{2,t})^{-\gamma_1} \\ &= g(0) + \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[g'(x)\varphi(x)^n \right]_{x=0} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[\gamma_1 (1-x)^{\frac{n}{\eta}-\gamma_1-1} \right]_{x=0}. \end{aligned}$$

Because,

$$\gamma_1(1-x)^{\frac{n}{\eta}-\gamma_1-1} = \gamma_1(-)^{n-1} \left(\frac{n}{\eta}-\gamma_1-1\right)\left(\frac{n}{\eta}-\gamma_1-2\right)\ldots\left(\frac{n}{\eta}-\gamma_1-(n-1)\right)(1-x)^{\frac{n}{\eta}-\gamma_1-(n-1)},$$

it follows that

$$\nu_{1,t}^{-\gamma_1} = 1 - \gamma_1 \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \begin{pmatrix} \frac{n}{\eta}-\gamma_1-1 \\ n-1 \end{pmatrix}.$$

By comparing the above expression with (A21), we can see that (A22) is absolutely convergent if $A_t^{-1/\eta} < \frac{(\frac{1}{\eta}-1)^{\frac{1}{\eta}-1}}{\frac{1}{\eta}^{\frac{1}{\eta}}}$, i.e. if $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta} = \overline{R}$. Thus,

$$\nu_{1,t}^{-\gamma_1} = 1 - \gamma_1 \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \begin{pmatrix} \frac{n}{\eta}-\gamma_1-1 \\ n-1 \end{pmatrix}, \; A_t > \overline{R}.$$

Using (9) and (14), we can rewrite the above expression as

$$\nu_{1,t}^{-\gamma_1} = \sum_{n=0}^{\infty} a_{n,1}^{\pi} \left(\frac{\hat{\pi}_{2,t}}{\hat{\pi}_{1,t}}\right)^{\frac{n}{\gamma_2}}, \; \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > \frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}} \left(\frac{\gamma_2}{\gamma_1}-1\right)^{\gamma_2-\gamma_1},$$

where $a_{n,1}^{\pi}$ is defined in (42). Therefore, the equilibrium state-price density is given by

$$\pi_t = \sum_{n=0}^{\infty} a_{n,2}^{\pi} \hat{\pi}_{1,t}^{\frac{n}{\gamma_1}} \hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}, \; \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > \frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}} \left(\frac{\gamma_2}{\gamma_1}-1\right)^{\gamma_2-\gamma_1}.$$

The expressions in (41) follow from (A22) and (A22).

Now observe that

$$\begin{aligned} \hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}} &= e^{-\left(1-\frac{n}{\gamma_2}\right)\hat{r}_1 t} e^{-\frac{1}{2}\left(1-\frac{n}{\gamma_2}\right)\left(\hat{\theta}_1\right)^2 t - \left(1-\frac{n}{\gamma_2}\right)\hat{\theta}_1 Z_t} e^{-\frac{n}{\gamma_2}\hat{r}_2 t} e^{-\frac{1}{2}\frac{n}{\gamma_2}\left(\hat{\theta}_2\right)^2 t - \frac{n}{\gamma_2}\hat{\theta}_2 Z_t} \\ &= e^{-\left(\left[\left(1-\frac{n}{\gamma_2}\right)\hat{r}_1 + \frac{n}{\gamma_2}\hat{r}_2\right] + \frac{1}{2}\left\{\left(1-\frac{n}{\gamma_2}\right)\left(\hat{\theta}_1\right)^2 + \frac{n}{\gamma_2}\left(\hat{\theta}_2\right)^2 - \left[\left(1-\frac{n}{\gamma_2}\right)\hat{\theta}_1 + \frac{n}{\gamma_2}\hat{\theta}_2\right]^2\right\}\right)t} \\ &\quad \times e^{-\frac{1}{2}\left[\left(1-\frac{n}{\gamma_2}\right)\hat{\theta}_1 + \frac{n}{\gamma_2}\hat{\theta}_2\right]^2 t - \left[\left(1-\frac{n}{\gamma_2}\right)\hat{\theta}_1 + \frac{n}{\gamma_2}\hat{\theta}_2\right] Z_t} \end{aligned} \tag{A22}$$

Therefore

$$\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{n}{\gamma_2}} = e^{-r^{n,1}t} e^{-\frac{1}{2}\left(\theta^{n,1}\right)^2 t - \theta^{n,1} Z_t},$$

where

$$\theta^{n,1} = \left(1-\frac{n}{\gamma_2}\right)\hat{\theta}_1 + \frac{n}{\gamma_2}\hat{\theta}_2$$

$$r^{n,1} = \left(1-\frac{n}{\gamma_2}\right)\hat{r}_1 + \frac{n}{\gamma_2}\hat{r}_2 + \frac{1}{2}\left\{\left(1-\frac{n}{\gamma_2}\right)\left(\hat{\theta}_1\right)^2 + \frac{n}{\gamma_2}\left(\hat{\theta}_2\right)^2 - \left[\left(1-\frac{n}{\gamma_2}\right)\hat{\theta}_1 + \frac{n}{\gamma_2}\hat{\theta}_2\right]^2\right\}.$$

Since,

$$\left(1-\frac{n}{\gamma_2}\right)\left(\hat{\theta}_1\right)^2 + \frac{n}{\gamma_2}\left(\hat{\theta}_2\right)^2 - \left[\left(1-\frac{n}{\gamma_2}\right)\hat{\theta}_1 + \frac{n}{\gamma_2}\hat{\theta}_2\right]^2 = \left(1-\frac{n}{\gamma_2}\right)\frac{n}{\gamma_2}(\hat{\theta}_1-\hat{\theta}_2)^2$$

it follows that

$$r^{n,1} = \left(1-\frac{n}{\gamma_2}\right)\hat{r}_1 + \frac{n}{\gamma_2}\hat{r}_2 + \frac{1}{2}\left(1-\frac{n}{\gamma_2}\right)\frac{n}{\gamma_2}(\hat{\theta}_1-\hat{\theta}_2)^2.$$

Thus, we obtain (44) and (45). Similarly, the term on the second line of (41) is

$$\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_2}} = \lambda_{1,0}^{\frac{n}{\gamma_1}}\lambda_{2,0}^{1-\frac{n}{\gamma_2}}e^{-r^{n,2}t}e^{-\frac{1}{2}\left(\theta^{n,2}\right)^2t-\theta^{n,2}Z_t},$$

where

$$\theta^{n,2} = \frac{n}{\gamma_1}\hat{\theta}_1 + \left(1-\frac{n}{\gamma_1}\right)\hat{\theta}_2, \tag{A23}$$

$$r^{n,2} = \frac{n}{\gamma_1}\hat{r}_1 + \left(1-\frac{n}{\gamma_1}\right)\hat{r}_2 + \frac{1}{2}\left(1-\frac{n}{\gamma_1}\right)\frac{n}{\gamma_1}(\hat{\theta}_1-\hat{\theta}_2)^2. \tag{A24}$$

with $\hat{r}_k$ and $\hat{\theta}_k$ defined in (10) and (11).

Proof of Corollary 5: State-price density under identical risk aversion

First we note that

$$\lim_{a\to 0}\left(\frac{\gamma+a}{\gamma}-1\right)^a = 1.$$

Therefore, setting $\gamma_1 = \gamma_2 = \gamma$ implies that

$$\frac{\gamma_1^{\gamma_2}}{\gamma_2^{\gamma_2}}\left(\frac{\gamma_2}{\gamma_1}-1\right)^{\gamma_2-\gamma_1} = 1.$$

Also note that after some tedious algebra, we can show that

$$\gamma\binom{n-\gamma-1}{n}\frac{(-)^{n+1}}{n} = \binom{\gamma}{n}.$$

Therefore, when $\gamma_1 = \gamma_2 = \gamma$, (42) and (43) reduce to (47).

Proof of Proposition 9: Risk premium and volatility of risky assets

Rather than considering equity, we shall derive results for a more general risky asset, which is a perpetual claim to the cash flow process, X, where the evolution of X is given by

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X^{sys} dZ_t + \sigma_X^{id} dZ_t^{id},$$

where Z_t^{id} is a standard Brownian motion under $\mathbb{P}$, orthogonal to Z_t. Under measure $\mathbb{P}^k$, $k \in \{1, 2\}$, the dynamics of the cash flow process are given by

$$\frac{dX_t}{X_t} = \mu_{X,k} dt + \sigma_X^{sys} dZ_{k,t} + \sigma_X^{id} dZ_t^{id},$$

where $\mu_{X,k}$ is given by

$$\frac{\mu_{X,k} - \mu_X}{\sigma_X^{sys}} = \frac{\mu_{Y,k} - \mu_Y}{\sigma_Y}.$$

Then, to get the risk premium and the volatility of the stock market, we will set $\mu_X = \mu_Y$, $\sigma_X^{sys} = \sigma_Y$, and $\sigma_X^{id} = 0$.

The risk premium for the claim paying X in perpetuity is given by the standard asset pricing equation:

$$E_t\left[\frac{dP_t^X + X_t dt}{P_t^X} - r_t dt\right] = -E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t^X}{P_t^X}\right]. \tag{A25}$$

Applying Ito's Lemma to $P_t^X = X_t p_t^X$ gives

$$\begin{aligned}
\frac{dP_t^X}{P_t^X} &= \frac{dX_t}{X_t} + \frac{dp_t^X}{p_t^X} + \frac{dX_t}{X_t}\frac{dp_t^X}{p_t^X} \\
&= \mu_X dt + \sigma_X^{sys} dZ_t + \sigma_X^{id} dZ_t^{id} + \frac{1}{p_t^X}\frac{\partial p_t^X}{\partial \nu_{1,t}}\nu_{1,t}(\mu_{\nu_{1,t}} dt + \sigma_{\nu_{1,t}} dZ_t) \\
&\quad + \frac{1}{2}\frac{1}{p_t^X}\frac{\partial^2 p_t^X}{\partial \nu_{1,t}^2}\nu_{1,t}^2 \sigma_{\nu_{1,t}}^2 dt + \sigma_X^{sys}\frac{1}{p_t^X}\frac{\partial p_t^X}{\partial \nu_{1,t}}\nu_{1,t}\sigma_{\nu_{1,t}} dt.
\end{aligned}$$

Thus, the total volatility of the return on the claim that pays X in perpetuity, $\sigma_{R,t}^X$, is given by

$$\sigma_{R,t}^X = \sqrt{\left(\sigma_{R,t}^{X,sys}\right)^2 + \left(\sigma_{R,t}^{X,id}\right)^2}.$$

where the idiosyncratic component of the volatility of the claim's returns is given by

$$\sigma_{R,t}^{X,id} = \sigma_X^{id},$$

and the systematic component of the volatility of the claim's returns is given by

$$\sigma_{R,t}^{X,sys} = \sigma_X^{sys} + \sigma_{\nu_{1,t}}\frac{\nu_{1,t}}{p_t^X}\frac{\partial p_t^X}{\partial \nu_{1,t}}.$$

Hence, substituting (A26) into (A25) gives

$$\mu_{R,t}^X - r_t = \theta_t \sigma_{R,t}^{X,sys}, \tag{A26}$$

where

$$\mu^X_{R,t}dt = E_t\left[\frac{dP^X_t + X_t dt}{P^X_t}\right].$$

Substituting (35) into (A26) gives

$$\mu^X_{R,t} - r_t = \left(\mathbf{R}_t\, \sigma_Y + \left[\frac{\mu_Y - \boldsymbol{\mu}_{Y,t}}{\sigma_Y}\right]\right)\sigma^{X,sys}_{R,t}.$$

Also, Agent k's perception of the risk premium for the claim paying X in perpetuity is given by the standard asset pricing equation:

$$E^k_t\left[\frac{dP^X_t + X_t dt}{P^X_t} - r_t dt\right] = -E^k_t\left[\frac{d\pi_{k,t}}{\pi_{k,t}}\frac{dP^X_t}{P^X_t}\right].$$

Hence,

$$\mu^X_{R,k,t} - r_t = \theta_{k,t}\sigma^{X,sys}_{R,t}, \tag{A27}$$

where

$$\mu^X_{R,k,t}dt = E^k_t\left[\frac{dP^X_t + X_t dt}{P^X_t}\right].$$

Substituting (36) and (37) into (A27) gives

$$\mu^X_{R,1,t} - r_t = \mathbf{R}_t\left(\sigma_Y + \frac{\nu_{2,t}}{\gamma_2}\left[\frac{\mu_{Y,1} - \mu_{Y,2}}{\sigma_Y}\right]\right)\sigma^{X,sys}_{R,t},$$

Agent 2's perception of the risk premium is given by

$$\mu^X_{R,2,t} - r_t = \mathbf{R}_t\left(\sigma_Y + \frac{\nu_{1,t}}{\gamma_1}\left[\frac{\mu_{Y,2} - \mu_{Y,1}}{\sigma_Y}\right]\right)\sigma^{X,sys}_{R,t},$$

Setting $\mu_X = \mu_Y$, $\sigma^{sys}_X = \sigma_Y$, and $\sigma^{id}_X = 0$ in the above expressions gives the results in the proposition.

Proof of Proposition 10: Prices of risky assets

Again, rather than considering equity, we shall derive results for a more general risky asset, which is a perpetual claim to the cash flow process, X, where the evolution of X is given by (A25). Then, to get the equations giving the price of equity in the proposition, we will set $\mu_X = \mu_Y$, $\sigma^{sys}_X = \sigma_Y$, and $\sigma^{id}_X = 0$.

To derive a closed-form expression for (50), we use (41) to write the equilibrium state-price density as

$$\pi_t = \sum_{n=0}^{\infty} a^{\pi}_{n,1}\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}1_{\left\{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}>R\right\}} + \sum_{n=0}^{\infty} a^{\pi}_{n,2}\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}1_{\left\{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}<R\right\}}.$$

Since the event $\left\{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} = R\right\}$ is of measure zero, it follows from (50) that

$$p^X_t \quad = \quad (\pi_t X_t)^{-1} j_t, \tag{A28}$$

where

$$j_t = E_t\left[\int_t^\infty \left(\sum_{n=0}^\infty a^\pi_{n,1}\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,u}^{\frac{n}{\gamma_2}} X_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}} + \sum_{n=0}^\infty a^\pi_{n,2}\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}} X_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<R\right\}}\right) du\right]. \quad \text{(A29)}$$

Since the two infinite series in the above expression stem from $\nu_{2,t}^{-\gamma_2}$ in (A21), and $\nu_{1,t}^{-\gamma_1}$ in (A22), which are complex analytic for $A \in \mathbb{C}$ such that $|A| < \overline{R}$, and $|A| > \overline{R}$, respectively, we can interchange both the conditional expectation and integral with the infinite sum to obtain

$$j_t = \sum_{n=0}^\infty a^\pi_{n,1} E_t\left[\int_t^\infty \hat{\pi}_{1,u}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,u}^{\frac{n}{\gamma_2}} X_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}} du\right] + \sum_{n=0}^\infty a^\pi_{n,2} E_t\left[\int_t^\infty \hat{\pi}_{1,u}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}} X_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<R\right\}} du\right].$$

We now rewrite the above expression as follows:

$$j_t = \pi_t X_t \left(\sum_{n=0}^\infty \omega_{n,1,t}\zeta^X_{n,1,t} + \sum_{n=0}^\infty \omega_{n,2,t}\zeta^X_{n,2,t}\right),$$

where $\omega_{n,1,t}$ and $\omega_{n,2,t}$ are given by

$$\omega_{n,1,t} = a^\pi_{n,1}\frac{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}}{\pi_t},\ n \in \mathbb{N}_0, \quad \text{(A30)}$$

$$\omega_{n,2,t} = a^\pi_{n,2}\frac{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}{\pi_t},\ n \in \mathbb{N}_0, \quad \text{(A31)}$$

and $\zeta^X_{n,1,t}$ and $\zeta^X_{n,2,t}$ are given by

$$\zeta^X_{n,1,t} = E_t\left[\int_t^\infty \frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,u}^{\frac{n}{\gamma_2}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}}\frac{X_u}{X_t} 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}} du\right],\ n \in \mathbb{N}_0,$$

$$\zeta^X_{n,2,t} = E_t\left[\int_t^\infty \frac{\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}\frac{X_u}{X_t} 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<R\right\}} du\right],\ n \in \mathbb{N}_0.$$

Equation (56) follows from (A28), and (58) follows from (41), (A30), and (A31).

We now express the weights, $\omega_{n,1,t}$ and $\omega_{n,2,t}$, in terms of the consumption shares, $\nu_{1,t}$ and $\nu_{2,t}$. From (9) and (26) it follows that

$$\pi_t = \hat{\pi}_{1,t}\nu_{1,t}^{-\gamma_1} = \hat{\pi}_{2,t}\nu_{2,t}^{-\gamma_2}.$$

Hence, for all $a \in \mathbb{R}$

$$\pi_t = \hat{\pi}^a_{1,t}\nu_{1,t}^{-a\gamma_1}\hat{\pi}_{2,t}^{1-a}\nu_{2,t}^{-(1-a)\gamma_2},$$

which implies that

$$\hat{\pi}^a_{1,t}\hat{\pi}^{1-a}_{2,t} = \pi_t\nu_{1,t}^{a\gamma_1}\nu_{2,t}^{(1-a)\gamma_2}. \quad \text{(A32)}$$

Therefore, we can rewrite the weights, $\omega_{n,1,t}$ and $\omega_{n,2,t}$, given in (A30) and (A31) as (56) and (57), respectively.

We now derive exact closed-form expressions for $\zeta^X_{n,1,t}$ and $\zeta^X_{n,2,t}$. Note that

$$\frac{\hat{\pi}_{k,u}}{\hat{\pi}_{k,u}}\frac{X_u}{X_t} = e^{-(\hat{r}_k+\gamma_k\sigma_X^{sys}\sigma_Y-\mu_{X,k})(u-t)}\frac{M_{k,u}}{M_{k,t}},$$

where $M_{k,t}$ is the following exponential martingale under $\mathbb{P}^k$:

$$\frac{dM_{k,t}}{M_{k,t}} = \sigma_X^{id}dZ_t^{id} + (\sigma_X^{sys}+\sigma_{\xi,k}-\gamma_k\sigma_Y)dZ_{k,t},\ M_{k,t}=1. \tag{A33}$$

We can thus define the new probability measures $\hat{\mathbb{P}}^k$ on $(\Omega,\mathcal{F})$ via

$$\hat{\mathbb{P}}^k(A) = E(1_A M_{k,T}),\ A\in\mathcal{F}_T,\ k\in\{1,2\}.$$

It follows that

$$\begin{aligned}
\zeta^X_{n,1,t} &= \hat{E}^1_t\int_t^\infty e^{-k_1(u-t)}\left(\frac{A_u}{A_t}\right)^{-n/\eta}1_{\{A_u>\overline{R}\}}du,\ n\in\mathbb{N}_0,\\
\zeta^X_{n,2,t} &= \hat{E}^2_t\int_t^\infty e^{-k_2(u-t)}\left(\frac{A_u}{A_t}\right)^{n}1_{\{A_u<\overline{R}\}}du,\ n\in\mathbb{N}_0,
\end{aligned}$$

where $\hat{E}^i_t[\cdot]$ is the time-t conditional expectation operator under $\hat{\mathbb{P}}^i$ and

$$k_i = \hat{r}_i + \gamma_i\sigma_X^{sys}\sigma_Y - \mu_{X,i}, \tag{A34}$$

is the discount rate used to value a security paying X units of consumption per unit time in perpetuity, when Agent i is the sole agent in the economy. From Lemma B1, it follows that

$$\zeta^X_{n,1,t} = \begin{cases} \frac{1}{\frac{1}{2}\sigma_A^2\left(\frac{n}{\eta}+a_+^*(k_1)\right)\left(a_+^*(k_1)-a_-^*(k_1)\right)}\left(\frac{A_t}{\overline{R}}\right)^{a_+^*(k_1)+\frac{n}{\eta}} & ,\ A_t<\overline{R},\\ \frac{1}{\frac{1}{2}\sigma_A^2\left(\frac{n}{\eta}+a_-^*(k_1)\right)\left(a_+^*(k_1)-a_-^*(k_1)\right)}\left(\frac{A_t}{\overline{R}}\right)^{a_-^*(k_1)+\frac{n}{\eta}} - \frac{1}{\frac{1}{2}\sigma_A^2\left(\frac{n}{\eta}+a_+^*(k_1)\right)\left(\frac{n}{\eta}+a_-^*(k_1)\right)} & ,\ A_t\geq\overline{R}, \end{cases}$$

and

$$\zeta^X_{n,2,t} = \begin{cases} -\frac{1}{\frac{1}{2}\sigma_A^2(n-a_-^*(k_2))(n-a_+^*(k_2))} + \frac{1}{\frac{1}{2}\sigma_A^2(n-a_+^*(k_2))(a_+^*(k_2)-a_-^*(k_2))}\left(\frac{A_t}{\overline{R}}\right)^{a_+^*(k_2)-n} & ,\ A_t<\overline{R},\\ \frac{1}{\frac{1}{2}\sigma_A^2(n-a_-^*(k_2))(a_+^*(k_2)-a_-^*(k_2))}\left(\frac{A_t}{\overline{R}}\right)^{a_-^*(k_2)-n} & ,\ A_t\geq\overline{R}, \end{cases}$$

where μ_A and σ_A are defined in (A18) and (A18), respectively, and $\hat{\mu}^i_A$ is the drift of $\ln\frac{\hat{\pi}_1}{\hat{\pi}_2}$ under $\hat{\mathbb{P}}^i$, i.e.

$$\hat{\mu}^i_A = \mu_A + (\sigma_X^{sys}+\sigma_{\xi,i}-\gamma_i\sigma_Y)\sigma_A,$$

and

$$a^*_\pm(k_i) = \frac{-(\hat{\mu}^i_A - \frac{1}{2}\sigma_A^2 \pm \sqrt{(\hat{\mu}^i_A-\frac{1}{2}\sigma_A^2)^2+2k_i\sigma_A^2}}{\sigma_A^2},$$

We can rewrite the above expressions in the following more symmetric form

$$\zeta^X_{n,1,t} = \begin{cases} \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma_2}+a_+(k_1)\right)(a_+(k_1)-a_-(k_1))}\left(\frac{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}{R}\right)^{a_+(k_1)+\frac{n}{\gamma_2}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R, \\ \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma_2}+a_-(k_1)\right)(a_+(k_1)-a_-(k_1))}\left(\frac{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}{R}\right)^{a_-(k_1)+\frac{n}{\gamma_2}} - \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma_2}+a_+(k_1)\right)\left(\frac{n}{\gamma_2}+a_-(k_1)\right)} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} \geq R, \end{cases} \tag{A35}$$

and

$$\zeta^X_{n,2,t} = \begin{cases} -\frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma_1}-a_-(k_2)\right)\left(\frac{n}{\gamma_1}-a_+(k_2)\right)} + \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma_1}-a_+(k_2)\right)(a_+(k_2)-a_-(k_2))}\left(\frac{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}{R}\right)^{a_+(k_2)-\frac{n}{\gamma_1}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R, \\ \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma_1}-a_+(k_2)\right)(a_+(k_2)-a_-(k_2))}\left(\frac{\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}}{R}\right)^{a_-(k_2)-\frac{n}{\gamma_1}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} \geq R, \end{cases} \tag{A36}$$

where μ_Δ and σ_Δ are given by (A18) and (A19), respectively, and $\hat{\mu}^i_\Delta$ is the drift of $\ln \frac{\hat{\pi}_1}{\hat{\pi}_2}$ under $\hat{\mathbb{P}}^i$, i.e.

$$\hat{\mu}^i_\Delta = \mu_\Delta + (\sigma^{sys}_X + \sigma_{\xi,i} - \gamma_i\sigma_Y)\sigma_\Delta, ,$$

and

$$a_\pm(k_i) = \frac{-\hat{\mu}^i_\Delta \pm \sqrt{(\hat{\mu}^i_\Delta)^2 + 2k_i\sigma^2_\Delta}}{\sigma^2_\Delta}.$$

Proof of Corollary 6: Prices of risky assets under identical risk aversions

Again, rather than considering equity, we shall derive results for a more general risky asset, which is a perpetual claim to the cash flow process, X, where the evolution of X is given by (A25). Then, to get the price of equity, we will set $\mu_X = \mu_Y$, $\sigma^{sys}_X = \sigma_Y$, and $\sigma^{id}_X = 0$.

By setting $\gamma_1 = \gamma_1 = \gamma$, (56) and (57) reduce to (60) and (61), respectively, and (54) and (55) reduce to (58) and (59), respectively. Also, the closed-form expressions for $\zeta^X_{n,1,t}$ and $\zeta^X_{n,2,t}$ in (A35) and (A36) reduce to

$$\zeta^X_{n,1,t} = \begin{cases} \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma}+a_+(k_1)\right)(a_+(k_1)-a_-(k_1))}\left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{a_+(k_1)+\frac{n}{\gamma}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < 1, \\ \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma}+a_-(k_1)\right)(a_+(k_1)-a_-(k_1))}\left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{a_-(k_1)+\frac{n}{\gamma}} - \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma}+a_+(k_1)\right)\left(\frac{n}{\gamma}+a_-(k_1)\right)} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} \geq 1, \end{cases} \tag{A37}$$

and

$$\zeta^X_{n,2,t} = \begin{cases} -\frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma}-a_-(k_2)\right)\left(\frac{n}{\gamma}-a_+(k_2)\right)} + \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma}-a_+(k_2)\right)(a_+(k_2)-a_-(k_2))}\left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{a_+(k_2)-\frac{n}{\gamma}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < 1, \\ \frac{1}{\frac{1}{2}\sigma^2_\Delta\left(\frac{n}{\gamma}-a(k_2)\right)(a_+(k_2)-a_-(k_2))}\left(\frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}}\right)^{a_-(k_2)-\frac{n}{\gamma}} & , \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} \geq 1, \end{cases} \tag{A38}$$

where μ_Δ, $\hat{\mu}^i_\Delta$ and σ_Δ are now given by

$$\begin{aligned}
\mu_\Delta &= \beta_2 - \beta_1 - \frac{1}{2}\sigma_\xi^2, \\
\hat{\mu}^i_\Delta &= \mu_\Delta + (\sigma_X^{sys} + \sigma_{\xi,i} - \gamma\sigma_Y)\sigma_\Delta, \\
\sigma_\Delta &= \sigma_\xi.
\end{aligned}$$

When $\gamma \in \mathbb{N}$, (56) reduces to

$$\begin{aligned}
p_t^X &= \sum_{n=0}^{\gamma} \omega_{n,1,t}\zeta_{n,1,t}^X + \sum_{n=0}^{\gamma} \omega_{n,2,t}\zeta_{n,2,t}^X \\
&= \sum_{n=0}^{\gamma} \omega_{n,t}\left(\zeta_{n,1,t}^X + \zeta_{\gamma-n,2,t}^X\right), \qquad \text{(A39)}
\end{aligned}$$

where $\omega_{n,t}$ is given in (63). It follows from (58) and (59) that

$$\zeta_{n,1,t}^X + \zeta_{\gamma-n,2,t}^X = E_t\left[\int_t^\infty \frac{\hat{\pi}_{1,u}^{1-\frac{n}{\gamma}}\hat{\pi}_{2,u}^{\frac{n}{\gamma}}}{\hat{\pi}_{1,t}^{1-\frac{n}{\gamma}}\hat{\pi}_{2,t}^{\frac{n}{\gamma}}}\frac{X_u}{X_t}du\right],\ n \in \mathbb{N}_0 \,\text{and}\, n \leq \gamma.$$

From (44) it follows that

$$\hat{\pi}_{1,t}^{1-\frac{n}{\gamma}}\hat{\pi}_{2,t}^{\frac{n}{\gamma}} = \lambda_{1,0}^{1-\frac{n}{\gamma}}\lambda_{2,0}^{\frac{n}{\gamma}}e_{-r^n t}e^{-\frac{1}{2}(\theta^n)^2 t - \theta^n Z_t},$$

where r^n is given in (63), and

$$\begin{aligned}
\theta^n &= \left(1-\frac{n}{\gamma}\right)\hat{\theta}_1 + \frac{n}{\gamma}\hat{\theta}_2 \\
&= \gamma\sigma_Y - \left(1-\frac{n}{\gamma}\right)\sigma_{\xi,1} - \frac{n}{\gamma}\sigma_{\xi,2} \\
&= \gamma\sigma_Y + \frac{\mu_Y - \mu_Y^n}{\sigma_Y},
\end{aligned}$$

where

$$\mu_Y^n = \left(1-\frac{n}{\gamma}\right)\mu_{Y,1} + \frac{n}{\gamma}\mu_{Y,2}.$$

Hence,

$$\begin{aligned}
\zeta_{n,1,t}^X + \zeta_{\gamma-n,2,t}^X &= E_t\left[\int_t^\infty e^{-r^n(u-t)}e^{-\frac{1}{2}(\theta^n)^2(u-t)-\theta^n(Z_u-Z_t)}\frac{X_u}{X_t}du\right] \\
&= E_t\left[\int_t^\infty e^{-\left(r^n-\mu_X+\frac{1}{2}\left((\theta^n)^2+\sigma_X^2\right)\right)(u-t)-(\theta^n-\sigma_X^{sys})(Z_u-Z_t)+\sigma_X^{id}(Z_u^{id}-Z_t^{id})}du\right] \\
&= \int_t^\infty e^{-\left(r^n-\mu_X+\frac{1}{2}\left((\theta^n)^2+\sigma_X^2\right)\right)(u-t)}E_t\left[e^{-(\theta^n-\sigma_X^{sys})(Z_u-Z_t)+\sigma_X^{id}(Z_u^{id}-Z_t^{id})}\right]du,
\end{aligned}$$

where the last step is valid, because of Fubini's Theorem. Now note that

$$E_t\left[e^{-(\theta^n-\sigma_X^{sys})(Z_u-Z_t)+\sigma_X^{id}(Z_u^{id}-Z_t^{id})}\right] = e^{\frac{1}{2}\left[(\theta^n-\sigma_X^{sys})^2+(\sigma_X^{id})^2\right](u-t)},$$

and so

$$\begin{aligned}
\zeta^X_{n,1,t} + \zeta^X_{\gamma-n,2,t} &= \int_t^\infty e^{-\left(r^n + \theta^n \sigma_X^{sys} - \mu_X\right)(u-t)} du \\
&= \left(r^n + \theta^n \sigma_X^{sys} - \mu_X\right)^{-1}.
\end{aligned}$$

From (A25) it follows that

$$\frac{\mu_Y - \mu_Y^n}{\sigma_Y} = \frac{\mu_X - \mu_X^n}{\sigma_X^{sys}},$$

and so

$$\zeta^X_{n,1,t} + \zeta^X_{\gamma-n,2,t} = p_n^X.$$

where p_n^X is given in (63). Thus, (A39) implies (62).

Proof of Proposition 11: Long-term yield

Note that

$$\nu_{1,t} + \nu_{2,t} = 1.$$

Hence,

$$\nu_{1,t} = 1 - \nu_{2,t}.$$

Because $\gamma_1 < \gamma_2$, we have

$$\nu_{1,t}^{\frac{\gamma_1}{\gamma_2}} \geq 1 - \nu_{2,t}.$$

Also note that since

$$\pi_t = \hat{\pi}_{k,t} \nu_{k,t}^{-\gamma_k} = \lambda_{k,0} e^{-\hat{r}_k t} e^{-\frac{1}{2}\hat{\theta}_k^2 t - \hat{\theta}_k Z_t} \nu_{k,t}^{-\gamma_k},$$

we have

$$\begin{aligned}
\pi_t^{-\frac{1}{\gamma_k}} &= \hat{\pi}_{k,t}^{-\frac{1}{\gamma_k}} \nu_{k,t} \\
\nu_{k,t} &= \pi_t^{-\frac{1}{\gamma_k}} \hat{\pi}_{k,t}^{\frac{1}{\gamma_k}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\pi_t^{-\frac{1}{\gamma_1}} \hat{\pi}_{1,t}^{\frac{1}{\gamma_1}}\right)^{\frac{\gamma_1}{\gamma_2}} &\geq 1 - \lambda_{2,0}^{\frac{1}{\gamma_2}} \pi_t^{-\frac{1}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{1}{\gamma_2}} \\
\left(\pi_t^{-1} \hat{\pi}_{1,t}\right)^{\frac{1}{\gamma_2}} &\geq 1 - \lambda_{2,0}^{\frac{1}{\gamma_2}} \pi_t^{-\frac{1}{\gamma_2}} \hat{\pi}_{2,t}^{\frac{1}{\gamma_2}} \\
\sum_{k=1}^{2} \hat{\pi}_{k,t}^{\frac{1}{\gamma_2}} &\geq \pi_t^{\frac{1}{\gamma_2}} \\
\left(\sum_{k=1}^{2} \hat{\pi}_{k,t}^{\frac{1}{\gamma_2}}\right)^{\gamma_2} &\geq \pi_t
\end{aligned}$$

If we define $\overline{\gamma}_2 = \max[1, \gamma_2]$, then

$$\nu_{1,t}^{\frac{\gamma_1}{\overline{\gamma}_2}} \geq 1 - \nu_{2,t},$$

and so

$$\left(\sum_{k=1}^{2}\hat{\pi}_{k,t}^{\frac{1}{\overline{\gamma}_2}}\right)^{\overline{\gamma}_2} \quad \geq \quad \pi_t$$

Now note that

$$\nu_{2,t}^{\frac{\gamma_2}{\gamma_1}} \leq 1-\nu_{1,t}.$$

Therefore,

$$\begin{aligned}
\left(\pi_t^{-\frac{1}{\gamma_2}}\hat{\pi}_{2,t}^{\frac{1}{\gamma_2}}\right)^{\frac{\gamma_2}{\gamma_1}} &\leq \quad 1-\hat{\pi}_{1,t}^{\frac{1}{\gamma_1}}\pi_t^{-\frac{1}{\gamma_1}} \\
\lambda_{2,0}^{\frac{1}{\gamma_1}}\pi_t^{-\frac{1}{\gamma_1}}\hat{\pi}_{2,t}^{\frac{1}{\gamma_1}} &\leq \quad 1-\hat{\pi}_{1,t}^{\frac{1}{\gamma_1}}\pi_t^{-\frac{1}{\gamma_1}} \\
\pi_t &\geq \quad \left(\sum_{k=1}^{2}\hat{\pi}_{k,t}^{\frac{1}{\gamma_1}}\right)^{\gamma_1}.
\end{aligned}$$

If we define $\underline{\gamma}_1 = \min[1,\gamma_1]$, then

$$\nu_{2,t}^{\frac{\gamma_2}{\underline{\gamma}_1}} \leq 1-\nu_{1,t}.$$

Then,

$$\pi_t \quad \geq \quad \left(\sum_{k=1}^{2}\hat{\pi}_{k,t}^{\frac{1}{\underline{\gamma}_1}}\right)^{\underline{\gamma}_1}.$$

Therefore,

$$\left(\sum_{k=1}^{2}\hat{\pi}_{k,t}^{\frac{1}{\gamma_2}}\right)^{\gamma_2} \geq \pi_t \geq \left(\sum_{k=1}^{2}\hat{\pi}_{k,t}^{\frac{1}{\gamma_1}}\right)^{\gamma_1},$$

and

$$\left(\sum_{k=1}^{2}\hat{\pi}_{k,t}^{\frac{1}{\overline{\gamma}_2}}\right)^{\overline{\gamma}_2} \geq \pi_t \geq \left(\sum_{k=1}^{2}\hat{\pi}_{k,t}^{\frac{1}{\underline{\gamma}_1}}\right)^{\underline{\gamma}_1}.$$

The latter inequality implies that

$$\left(\sum_{k=1}^{2}\hat{\pi}_{k,T}^{\frac{1}{\overline{\gamma}_2}}\right)^{\overline{\gamma}_2} \geq \pi_T \geq \left(\sum_{k=1}^{2}\hat{\pi}_{k,T}^{\frac{1}{\underline{\gamma}_1}}\right)^{\underline{\gamma}_1},$$

which implies that

$$\left(\sum_{k=1}^{2}(\hat{\pi}_{k,T}X_T)^{\frac{1}{\overline{\gamma}_2}}\right)^{\overline{\gamma}_2} \geq \pi_T X_T \geq \left(\sum_{k=1}^{2}(\hat{\pi}_{k,T}X_T)^{\frac{1}{\underline{\gamma}_1}}\right)^{\underline{\gamma}_1}.$$

Since $f(x,y) = (x^{1/\gamma}+y^{1/\gamma})^{\gamma}$ is strictly convex (concave) iff $\gamma < 1$ ($\gamma > 1$), it follows from Jensen's Inequality that

$$\left(\sum_{k=1}^{2}\left(E_t\left[\hat{\pi}_{k,T}X_T\right]\right)^{\frac{1}{\overline{\gamma}_2}}\right)^{\overline{\gamma}_2} \geq E_t[\pi_T X_T] \geq \left(\sum_{k=1}^{2}\left(E_t\left[\hat{\pi}_{k,T}X_T\right]\right)^{\frac{1}{\underline{\gamma}_1}}\right)^{\underline{\gamma}_1}. \tag{A40}$$

Since, $E_t\left[\hat{\pi}_{k,T}X_T\right] = e^{-(\hat{r}_k+\gamma_k\sigma_X^{sys}\sigma_X-\mu_{X,k})T}\hat{E}_t^k[M_{k,T}] = e^{-(\hat{r}_k+\gamma_k\sigma_X^{sys}\sigma_X-\mu_{X,k})T}M_{k,t}$, where $M_{k,t}$ is the exponential martingale under $\mathbb{P}^k$ defined in (A33), it follows that

$$\left(\sum_{k=1}^{2}\left(e^{-(\hat{r}_k+\gamma_k\sigma_X^{sys}\sigma_X-\mu_{X,k})T}M_{k,t}\right)^{\frac{1}{\overline{\gamma}_2}}\right)^{\overline{\gamma}_2} \geq E_t[\pi_T X_T] \geq \left(\sum_{k=1}^{2}\left(e^{-(\hat{r}_k+\gamma_k\sigma_X^{sys}\sigma_X-\mu_{X,k})T}M_{k,t}\right)^{\frac{1}{\overline{\gamma}_1}}\right)^{\overline{\gamma}_1},$$

which can be rewritten as

$$\left(\sum_{i=1}^{2}\left(e^{-k_iT}M_{i,t}\right)^{\frac{1}{\overline{\gamma}_2}}\right)^{\overline{\gamma}_2} \geq E_t[\pi_T X_T] \geq \left(\sum_{i=1}^{2}\left(e^{-k_iT}M_{i,t}\right)^{\frac{1}{\overline{\gamma}_1}}\right)^{\overline{\gamma}_1},$$

where k_i is

$$k_i = \hat{r}_i + \gamma_i\sigma_Y^2 - \mu_{Y,i}.$$

We can rewrite V_{T-t}^X as

$$V_{T-t}^X = \pi_t^{-1}E_t[\pi_T X_T],$$

and so, from (63), we obtain

$$y_{T-t}^X = -\frac{1}{T-t}\ln\frac{V_{T-t}}{X_t} = \frac{1}{T-t}\ln(\pi_t X_t) - \frac{1}{T-t}\ln E_t[\pi_T X_T].$$

Therefore,

$$\lim_{T\to\infty} y_{T-t}^X = -\lim_{T\to\infty}\frac{1}{T-t}\ln E_t[\pi_T X_T].$$

From (A40) it follows that

$$\begin{aligned} -\frac{1}{T-t}\ln\left(\sum_{i=1}^{2}\left(e^{-k_iT}M_{i,t}\right)^{\frac{1}{\overline{\gamma}_2}}\right)^{\overline{\gamma}_2} &\leq -\frac{1}{T-t}\ln E_t[X_T\pi_T] \\ \leq \quad -\frac{1}{T-t}\ln\left(\sum_{i=1}^{2}\left(e^{-k_iT}M_{i,t}\right)^{\frac{1}{\overline{\gamma}_1}}\right)^{\overline{\gamma}_1}&. \end{aligned}$$

Letting $T\to\infty$ gives

$$\min(k_1,k_2) \leq -\frac{1}{T-t}\ln E_t^X[\pi_T] \leq \min(k_1,k_2),$$

and so

$$\lim_{T\to\infty} y_{T-t}^X = \min(k_1,k_2).$$

The other results in the proposition, for the yield on riskless bonds and the term premium, follow once we set $\sigma_Y^2 = \mu_{Y,i} = 0$ in the equation above.

Proof of Corollary 7: Survival and price impact under identical preferences and different beliefs.

The corollary follows immediately from Propositions 3 and 11, after setting $\beta_1 = \beta_2 = \beta$ and $\gamma_1 = \gamma_2 = \gamma$.

B Appendix: Two Lemmas for Valuing Contingent Cashflows

The following two lemmas, on the valuation of contingent cashflows, are used in Appendix A.

Lemma B1 *The date-t price of the claim which pays out D_t^n units of consumption per unit time in perpetuity as long as $D_u < B$, where $D_u = D_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)(u-t)+\sigma(Z_u - Z_t)}$ and the discount rate is assumed to be k_2, is given by $V_{2,n,t} = V_{2,n}(D_t)$, where*

$$V_{2,n}(D_t) = E_t \int_t^{\infty} e^{-k_2(u-t)} D_u^n 1_{\{D_u < B\}}. \tag{B1}$$

The date-t price of the claim which pays out $D_t^{-n/\eta}$ units of consumption per unit time in perpetuity as long as $D_u > B$, where $D_u = D_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)(u-t)+\sigma(Z_u - Z_t)}$ and the discount rate is assumed to be k_1, is given by $V_{1,n,t} = V_{1,n}(D_t)$, where

$$V_{1,n}(D_t) = E_t \int_t^{\infty} e^{-k_1(u-t)} D_u^{-n/\eta} 1_{\{D_u > B\}}. \tag{B2}$$

Observe that k_i, defined in (A34), *is the discount rate used to value the security paying X units of consumption per unit time in perpetuity, when Agent i is the sole agent in the economy. The prices of the above claims are given by*

$$V_{2,n}(D) = \begin{cases} -\frac{D^n}{\frac{1}{2}\sigma^2(n-a_-(k_2))(n-a_+(k_2))} + \frac{B^n}{\frac{1}{2}\sigma^2(n-a_+(k_2))(a_+(k_2)-a_-(k_2))}\left(\frac{D}{B}\right)^{a_+(k_2)} & , \ D < B \\ \frac{B^n}{\frac{1}{2}\sigma^2(n-a_-(k_2))(a_+(k_2)-a_-(k_2))}\left(\frac{D}{B}\right)^{a_-(k_2)} & , \ D \geq B \end{cases},$$

and

$$V_{1,n}(D) = \begin{cases} \frac{B^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+(k_1)\right)(a_+(k_1)-a_-(k_1))}\left(\frac{D}{B}\right)^{a_+(k_1)} & , \ D < B \\ \frac{B^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_-\right)(a_+(k_1)-a_-(k_1))}\left(\frac{D}{B}\right)^{a_-(k_1)} - \frac{D^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+(k_1)\right)\left(\frac{n}{\eta}+a_-(k_1)\right)} & , \ D \geq B \end{cases},$$

where

$$a_{\pm}(k) = \frac{-(\mu - \frac{1}{2}\sigma^2) \pm \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2k\sigma^2}}{\sigma^2}.$$

Proof

We start by defining $\delta = \ln D$, $b = \ln B$, and so (B1) can be rewritten in terms of the arithmetic Brownian motion e, i.e.

$$V_{2,n}(\delta_t) = E_t \int_t^{\infty} \exp(-k_2(u-t)) \exp(n\delta_u) 1_{\{\delta_u < b\}} du.$$

The Feynman-Kac Theorem implies that $V_{2,n}(\delta)$ satisfies the following set of ordinary differential equations

$$\frac{1}{2}\sigma^2 V_{2,n}'' + \left(\mu - \frac{1}{2}\sigma^2\right) V_{2,n}' - k_2 V_{2,n} = 0, \ \delta \geq b, \tag{B3}$$

$$\frac{1}{2}\sigma^2 V_{2,n}'' + \left(\mu - \frac{1}{2}\sigma^2\right) V_{2,n}' - k_2 V_{2,n} + \exp(n\delta) = 0, \ \delta < b. \tag{B4}$$

We also have the following boundary conditions

$$
\begin{aligned}
0 &< \lim_{\delta\to\infty} V_{2,n}(\delta) < \infty, && \text{(B5)}\\
\lim_{\delta\to b+} V_{2,n}(\delta) &= \lim_{\delta\to b-} V_{2,n}(\delta), && \text{(B6)}\\
\lim_{\delta\to b+} V'_{2,n}(\delta) &= \lim_{\delta\to b-} V'_{2,n}(\delta), && \text{(B7)}\\
0 &< \lim_{\delta\to -\infty} V_{2,n}(\delta) < \infty. && \text{(B8)}
\end{aligned}
$$

The general solution of (B3) is

$$V_{2,n}(\delta) = K_{u,-}\exp(a_-\delta) + K_{u,+}\exp(a_+\delta),$$

where $K_{u,\pm}$ are constants of integration and $a_\pm$ are the roots of the characteristic equation $\frac{1}{2}\sigma^2 a^2 + \left(\mu - \frac{1}{2}\sigma^2\right) a - k_2 = 0$:

$$a_\pm = \frac{-(\mu - \frac{1}{2}\sigma^2) \pm \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2k_2\sigma^2}}{\sigma^2}.$$

The general solution of (B4) is

$$V_{2,n}(\delta) = K_{d,-}\exp(a_-\delta) + K_{d,+}\exp(a_+\delta) - \left(\frac{1}{2}\sigma^2 n^2 + \left(\mu - \frac{1}{2}\sigma^2\right) n - k\right)^{-1} \exp(n\delta),$$

where $K_{d,\pm}$ constants of integration. The boundary conditions (B5) and (B8) imply that $K_{u,+} = 0$ and $K_{d,-} = 0$, respectively. The boundary conditions (B6) and (B7) imply that

$$
\begin{aligned}
K_{u,-}e^{a_- b} &= -\left(\frac{1}{2}\sigma^2 n^2 + \left(\mu - \frac{1}{2}\sigma^2\right) n - k\right)^{-1} e^{nb} + K_{d,+}e^{a_+ b},\\
a_- K_{u,-}e^{a_- b} &= -n\left(\frac{1}{2}\sigma^2 n^2 + \left(\mu - \frac{1}{2}\sigma^2\right) n - k\right)^{-1} e^{nb} + a_+ K_{d,+}e^{a_+ b},
\end{aligned}
$$

respectively. Writing the above linear equation system in matrix form, we obtain

$$\begin{pmatrix} e^{ba_-} & -e^{ba_+} \\ a_- e^{ba_-} & -a_+ e^{ba_+} \end{pmatrix} \begin{pmatrix} K_{u,-} \\ K_{d,+} \end{pmatrix} = -\begin{pmatrix} 1 \\ n \end{pmatrix} \left(\frac{1}{2}\sigma^2 n^2 + \left(\mu - \frac{1}{2}\sigma^2\right) n - k\right)^{-1} e^{nb}.$$

Hence,

$$
\begin{aligned}
\begin{pmatrix} K_{u,-} \\ K_{d,+} \end{pmatrix} &= -\begin{pmatrix} e^{ba_-} & -e^{ba_+} \\ a_- e^{ba_-} & -a_+ e^{ba_+} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ n \end{pmatrix} \left(\frac{1}{2}\sigma^2 n^2 + \left(\mu - \frac{1}{2}\sigma^2\right) n - k\right)^{-1} e^{nb}\\
&= \frac{1}{\frac{1}{2}\sigma^2(a_+ - a_-)} \begin{pmatrix} \frac{e^{(n-a_-)b}}{n-a_-} \\ \frac{e^{(n-a_+)b}}{n-a_+} \end{pmatrix}.
\end{aligned}
$$

Therefore,

$$V_{2,n}(\delta) = \begin{cases} -\frac{e^{n\delta}}{\frac{1}{2}\sigma^2(n-a_-)(n-a_+)} + \frac{e^{nb}}{\frac{1}{2}\sigma^2(n-a_+)(a_+-a_-)} e^{a_+(\delta-b)} & , \delta < b, \\ \frac{e^{nb}}{\frac{1}{2}\sigma^2(n-a_-)(a_+-a_-)} e^{a_-(\delta-b)} & , \delta \geq b. \end{cases} \tag{B9}$$

Hence,

$$V_{2,n}(D) = \begin{cases} -\frac{D^n}{\frac{1}{2}\sigma^2(n-a_-)(n-a_+)} + \frac{B^n}{\frac{1}{2}\sigma^2(n-a_+)(a_+-a_-)}\left(\frac{D}{B}\right)^{a_+} & , D < B, \\ \frac{B^n}{\frac{1}{2}\sigma^2(n-a_-)(a_+-a_-)}\left(\frac{D}{B}\right)^{a_-} & , D \geq B. \end{cases} \tag{B10}$$

The expression in (B2) can be rewritten in terms of the arithmetic Brownian motion δ:

$$V_{1,n}(\delta_t) = E_t \int_t^\infty \exp(-k_1(u-t)) \exp\left(-\frac{n}{\eta}\delta_u\right) 1_{\{\delta_u > b\}} du. \tag{B11}$$

The Feynman-Kac Theorem implies that $V_{1,n}(e)$ satisfies the following set of ordinary differential equations

$$\frac{1}{2}\sigma^2 V_{1,n}'' + \left(\mu - \frac{1}{2}\sigma^2\right) V_{1,n}' - k_1 V_{1,n} + \exp\left(-\frac{n}{\eta}\delta\right) = 0,\ \delta \geq b, \tag{B12}$$

$$\frac{1}{2}\sigma^2 V_{1,n}'' + \left(\mu - \frac{1}{2}\sigma^2\right) V_{1,n}' - k_2 V_{1,n} = 0,\ \delta < b. \tag{B13}$$

We also have the following boundary conditions

$$\begin{aligned} 0 &< \lim_{\delta\to\infty} V_{1,n}(\delta) < \infty, \\ \lim_{\delta\to b+} V_{1,n}(\delta) &= \lim_{\delta\to b-} V_{2,n}(\delta), \\ \lim_{e\to b+} V_{1,n}'(\delta) &= \lim_{\delta\to b-} V_{2,n}'(\delta), \\ 0 &< \lim_{\delta\to-\infty} V_{1,n}(\delta) < \infty. \end{aligned}$$

The general solution of (B12) is

$$V_{1,n}(\delta) = K_{u,-}\exp(a_-\delta) + K_{u,+}\exp(a_+\delta) - \left(\frac{1}{2}\sigma^2\left(\frac{n}{\eta}\right)^2 - \left(\mu - \frac{1}{2}\sigma^2\right)\frac{n}{\eta} - k_1\right)^{-1} \exp\left(-\frac{n}{\eta}\delta\right),$$

where $K_{u,\pm}$ are constants of integration and $a_\pm$ are the roots of the characteristic equation $\frac{1}{2}\sigma^2 a^2 + \left(\mu - \frac{1}{2}\sigma^2\right)a - k_1 = 0$:

$$a_\pm = \frac{-(\mu - \frac{1}{2}\sigma^2) \pm \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2k_1\sigma^2}}{\sigma^2}.$$

Thus, we can rewrite the general solution of (B12) as

$$V_{1,n}(\delta) = K_{u,-}\exp(a_-\delta) + K_{u,+}\exp(a_+\delta) - \frac{1}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta} + a_+\right)\left(\frac{n}{\eta} + a_-\right)} \exp\left(-\frac{n}{\eta}\delta\right).$$

The general solution of (B13) is

$$V_{1,n}(\delta) = K_{d,-}\exp(a_-\delta) + K_{d,+}\exp(a_+\delta),$$

where $K_{d,\pm}$ constants of integration. The boundary conditions (B5) and (B8) imply that $K_{u,+} = 0$ and $K_{d,-} = 0$, respectively. The boundary conditions (B6) and (B7) imply that

$$\begin{aligned} K_{d,+}e^{a_+b} &= -\frac{1}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta} + a_+\right)\left(\frac{n}{\eta} + a_-\right)} \exp\left(-\frac{n}{\eta}b\right) + K_{u,-}e^{a_-b}, \\ a_+K_{d,+}e^{a_+b} &= \frac{n}{\eta}\frac{1}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta} + a_+\right)\left(\frac{n}{\eta} + a_-\right)} \exp\left(-\frac{n}{\eta}b\right) + a_-K_{u,-}e^{a_-b}, \end{aligned}$$

respectively. Writing the above linear equation system in matrix form, we obtain

$$\begin{pmatrix} e^{ba_+} & -e^{ba_-} \\ a_+ e^{ba_+} & -a_- e^{ba_-} \end{pmatrix} \begin{pmatrix} K_{d,+} \\ K_{u,-} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{n}{\eta} \end{pmatrix} \frac{1}{\frac{1}{2}\sigma^2 \left(\frac{n}{\eta} + a_+\right)\left(\frac{n}{\eta} + a_-\right)} e^{-\frac{n}{\eta}b}.$$

Hence,

$$\begin{pmatrix} K_{d,+} \\ K_{u,-} \end{pmatrix} = \begin{pmatrix} \frac{1}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+\right)} e^{-a_+ b} \\ \frac{1}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_-\right)} e^{-a_- b} \end{pmatrix} \frac{e^{-\frac{n}{\eta}b}}{a_+ - a_-}.$$

Therefore,

$$V_{1,n}(\delta) = \begin{cases} \frac{\exp\left(-\frac{n}{\eta}b\right)}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+\right)(a_+-a_-)} \exp\left(a_+(\delta - b)\right) & , \ \delta < b, \\ \frac{\exp\left(-\frac{n}{\eta}b\right)}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_-\right)(a_+-a_-)} \exp\left(a_-(\delta - b)\right) - \frac{\exp(-\frac{n}{\eta}\delta)}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+\right)\left(\frac{n}{\eta}+a_-\right)} & , \ \delta \geq b. \end{cases}$$

Hence,

$$V_{1,n}(D) = \begin{cases} \frac{B^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+\right)(a_+-a_-)} \left(\frac{D}{B}\right)^{a_+} & , \ D < B, \\ \frac{B^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_-\right)(a_+-a_-)} \left(\frac{D}{B}\right)^{a_-} - \frac{D^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+\right)\left(\frac{n}{\eta}+a_-\right)} & , \ D \geq B. \end{cases}$$

Lemma B2 *The date-t price of the zero-coupon claim which pays out D_T^n units of consumption at time T if $D_T < B$, where $D_T = D_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(Z_T - Z_t)}$ and the discount rate is assumed to be k_2, is given by $L_{2,n,t} = L_{2,n}(D_t)$, where*

$$L_{2,n}(D_t) = E_t e^{-k_2(T-t)} D_T^n 1_{\{D_T < B\}}. \tag{B14}$$

The date-t price of the fundamental financial security which pays out $D_T^{-n/\eta}$ units of consumption at time T if $D_T > B$, where $D_T = D_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(Z_T - Z_t)}$ and the discount rate is assumed to be k_1, is given by $L_{1,n,t} = L_{1,n}(D_t)$, where

$$L_{1,n}(D_t) = E_t e^{-k_1(T-t)} D_T^{-n/\eta} 1_{\{D_T > B\}}.$$

The prices of the above zero-coupon claims are given by

$$L_{2,n}(D_t) = D_t^n e^{-\left[k_2 - n\mu - \frac{1}{2}n(n-1)\sigma^2\right](T-t)} \Phi\left(\frac{\ln\left(\frac{B}{D_t}\right) - \left(\mu + \frac{1}{2}(2n-1)\sigma^2\right)(T-t)}{\sigma(T-t)^{1/2}}\right),$$

and

$$L_{1,n}(D_t) = D_t^{-\frac{n}{\eta}} e^{-\left[k_2 + \frac{n}{\eta}\left(\mu - \frac{1}{2}\left(1+\frac{n}{\eta}\right)\sigma^2\right)\right](T-t)} \left[1 - \Phi\left(\frac{\ln\left(\frac{B}{D_t}\right) - \left(\mu - \frac{1}{2}\left(1 + 2\frac{n}{\eta}\right)\sigma^2\right)(T-t)}{\sigma(T-t)^{1/2}}\right)\right].$$

Proof
We start by defining $\delta = \ln D$, $b = \ln B$, and so (B14) can be rewritten in terms of the arithmetic Brownian motion δ:

$$L_{2,n}(\delta_t) = E_t \exp(-k_2(T-t)) \exp(n\delta_T) 1_{\{\delta_T < b\}}.$$

We now evaluate the above expectation directly.

$$\begin{aligned}
L_{2,n}(\delta_t) &= E_t e^{-k_2(T-t)} e^{n\delta_T} 1_{\{\delta_T<b\}} \\
&= E_t e^{-k_2(T-t)} e^{\left(n\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)+\sigma(Z_T-Z_t)\right)\right.} 1_{\left\{\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)+\sigma(Z_T-Z_t)<b\right\}} \\
&= e^{-k_2(T-t)} e^{n\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)} E_t e^{n(\sigma(Z_T-Z_t))} 1_{\left\{\sigma(Z_T-Z_t)<b-\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)\right\}}.
\end{aligned}$$

Now note that

$$\begin{aligned}
& E_t e^{n(\sigma(Z_T-Z_t))} 1_{\left\{\sigma(Z_T-Z_t)<b-\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)\right\}} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\epsilon^2} e^{n\sigma(T-t)^{1/2}\epsilon} 1_{\left\{\sigma(T-t)^{1/2}\epsilon<b-\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)\right\}} d\epsilon \\
&= e^{\frac{1}{2}n^2\sigma^2(T-t)} \Phi\left(\frac{b-\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)}{\sigma(T-t)^{1/2}} - n\sigma(T-t)^{1/2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
L_{2,n}(\delta_t) &= e^{-k_2(T-t)+n\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)+\frac{1}{2}n^2\sigma^2(T-t)} \Phi\left(\frac{b-\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)}{\sigma(T-t)^{1/2}} - n\sigma(T-t)^{1/2}\right) \\
&= e^{n\delta_t} e^{-\left[k_2-n\mu-\frac{1}{2}n(n-1)\sigma^2\right](T-t)} \Phi\left(\frac{b-\delta_t-\left(\mu+\frac{1}{2}(2n-1)\sigma^2\right)(T-t)}{\sigma(T-t)^{1/2}}\right).
\end{aligned}$$

Hence,

$$L_{2,n}(D_t) = E_t^n e^{-\left[k_2-n\mu-\frac{1}{2}n(n-1)\sigma^2\right](T-t)} \Phi\left(\frac{\ln(\frac{B}{D_t})-\left(\mu+\frac{1}{2}(2n-1)\sigma^2\right)(T-t)}{\sigma(T-t)^{1/2}}\right).$$

Also,

$$\begin{aligned}
L_{1,n}(D_t) &= E_t e^{-k_1(T-t)} D_T^{-n/\eta} 1_{\{D_T>B\}} \\
&= E_t e^{-k_2(T-t)} e^{\left(\frac{-n}{\eta}\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)+\sigma(Z_T-Z_t)\right)\right)} 1_{\left\{\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)+\sigma(Z_T-Z_t)>b\right\}} \\
&= e^{-k_2(T-t)} e^{\frac{-n}{\eta}\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)} E_t e^{-\frac{n}{\eta}\sigma(Z_T-Z_t)} 1_{\left\{\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)+\sigma(Z_T-Z_t)>b\right\}}.
\end{aligned}$$

Now note that

$$\begin{aligned}
& E_t \exp\left(-\frac{n}{\eta}\sigma(Z_T-Z_t)\right) 1_{\left\{\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)+\sigma(Z_T-Z_t)>b\right\}} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\epsilon^2} e^{-\frac{n}{\eta}\sigma(T-t)^{1/2}\epsilon} 1_{\left\{\sigma(T-t)^{1/2}\epsilon>b-\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)\right\}} d\epsilon \\
&= e^{\frac{1}{2}\left(\frac{n}{\eta}\right)^2\sigma^2(T-t)} \left[1-\Phi\left(\frac{b-\delta_t-(T-t)\left(\mu-\frac{1}{2}\left(1+2\frac{n}{\eta}\right)\sigma^2\right)}{\sigma(T-t)^{1/2}}\right)\right].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&L_{1,n}(D_t)\\
&= e^{-k_2(T-t)} e^{\frac{-n}{\eta}\left(\delta_t+(T-t)\left(\mu-\frac{1}{2}\sigma^2\right)\right)} e^{\frac{1}{2}\left(\frac{n}{\eta}\right)^2\sigma^2(T-t)} \left[1-\Phi\left(\frac{b-\delta_t-(T-t)\left(\mu-\frac{1}{2}\left(1+2\frac{n}{\eta}\right)\sigma^2\right)}{\sigma(T-t)^{1/2}}\right)\right]\\
&= D_t^{-\frac{n}{\eta}} e^{-\left[k_2+\frac{n}{\eta}\left(\mu-\frac{1}{2}\left(1+\frac{n}{\eta}\right)\sigma^2\right)\right](T-t)} \left[1-\Phi\left(\frac{\ln\left(\frac{B}{D_T}\right)-\left(\mu-\frac{1}{2}\left(1+2\frac{n}{\eta}\right)\sigma^2\right)(T-t)}{\sigma(T-t)^{1/2}}\right)\right].
\end{aligned}
$$

C Appendix: Lagrange's Theorem

In this section, we state a number of definitions and theorems from complex analysis that are used to derive results in the paper. In particular, the insight from Lagrange that is central to the analysis in the paper is given in Theorem C2 of this appendix.

Definition C1 *If U is an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a complex function on U, we say that f is* complex differentiable *at a point z_0 of U if the limit*

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists. The limit here is taken over all sequences of complex numbers approaching z_0, and for all such sequences the difference quotient has to approach the same number $f'(z_0)$.

Definition C2 *If f is complex differentiable at every point z_0 in U, we say that f is* holomorphic *on U. We say that f is holomorphic at the point z_0 if it is holomorphic on some neighborhood of z_0. We say that f is holomorphic on some non-open set A if it is holomorphic in an open set containing A.*

Definition C3 *A function f is* complex analytic *on an open set D in the complex plane if for any z_0 in D one can write*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

in which the coefficients a_0, a_1, ... are complex numbers and the series is convergent for z in a neighborhood of z_0.

Theorem C1 *A function f is complex analytic on an open set D in the complex plane if and only if it is holomorphic in D.*

We are now ready to state the theorem that allows us to find closed-form series expansions for the sharing rule and complex analytic functions of the sharing rule.

Theorem C2 (Lagrange) *Suppose the dependence between the variables w and z is implicitly defined by an equation of the form*

$$w = f(z),$$

where f is complex analytic in a neighborhood of 0 and $f'(0) \neq 0$. Then for any function g which is complex analytic in a neighborhood of 0,

$$g(z) = g(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} g'(x)[\varphi(x)^n] \right]_{x=0},$$

where $\varphi(z) = \frac{z}{f(z)}$.

Note that the above theorem does not provide a radius of convergence for the series (C1). While the original proof of Theorem C2 due to Lagrange is not very straightforward, a relatively easier proof can be obtained by using Cauchy's Integral Formula.

D Appendix: Wealth and Portfolio Holdings of Individual Agents

The approach we have used to identify the equilibrium prices in this model is to first identify the utility function of a "central planner" or "representative agent" (see Equation (4)), then solve for each agent's share of optimal consumption (Proposition 1), and then use this to identify the state price density (Proposition 8), and finally use that to identify asset prices (Proposition 10). Alternatively, one could have solved for the competitive market equilibrium, where each agent solved recursively the problem of maximizing lifetime utility by choosing at each instant the optimal consumption and portfolio policies subject to the dynamic wealth constraint. In this section, we show how one can still determine the wealth and optimal portfolio policy of each agent by applying the insight from Cox and Huang (1989) and using the already identified consumption-sharing rule and state-price density.

Observe that the financial wealth of each agent at date t, $W_{k,t}$ for $k \in \{1, 2\}$, is the present value of that agent's future consumption:

$$W_{k,t} = E_t \left[\int_t^{\infty} \frac{\pi_u}{\pi_t} C_{k,u} du \right] = E_t \left[\int_t^{\infty} \frac{\pi_u}{\pi_t} \nu_{k,u} Y_u du \right].$$

Now, this looks very much like the problem of finding the value of a claim with payout $C_{k,t} = \nu_{k,u} Y_u$, and we can use the same approach as the one we used in Proposition 10 to obtain the price of a risky asset, which leads to the following result.

Proposition D1 *Agent k's wealth at time t is given by $W_{k,t} = w^Y_{k,t} Y_t$, where*

$$w^Y_{1,t} = \nu_{1,t} \left(\sum_{n=0}^{\infty} \epsilon_{n,1,t} \zeta^Y_{n,1,t} + \sum_{n=0}^{\infty} \epsilon_{n,2,t} \zeta^Y_{n,2,t} \right) + \sum_{n=0}^{\infty} (\omega_{n,2,t} - \epsilon_{n,2,t}) \zeta^Y_{n,2,t} \tag{D1}$$

$$w^Y_{2,t} = \nu_{2,t} \left(\sum_{n=0}^{\infty} \epsilon_{n,1,t} \zeta^Y_{n,1,t} + \sum_{n=0}^{\infty} \epsilon_{n,2,t} \zeta^Y_{n,2,t} \right) + \sum_{n=0}^{\infty} (\omega_{n,1,t} - \epsilon_{n,1,t}) \zeta^Y_{n,1,t}, \tag{D2}$$

where the weights $\epsilon_{n,1,t}$, $n \in \mathbb{N}_0$, are given by

$$\epsilon_{n,1,t} = \frac{\left(\nu_{2,t}^{\gamma_2}\right)^{1-\frac{n}{\gamma_2}} \left(\nu_{1,t}^{\gamma_1}\right)^{\frac{n}{\gamma_2}}}{\nu_{1,t}} b_{n,1}^{\pi}, \tag{D3}$$

$$\epsilon_{n,2,t} = \frac{\left(\nu_{1,t}^{\gamma_1}\right)^{\frac{n}{\gamma_1}} \left(\nu_{2,t}^{\gamma_2}\right)^{1-\frac{n}{\gamma_1}}}{\nu_{2,t}} b_{n,2}^{\pi}, \tag{D4}$$

and $b_{n,1} = b_{n,2} = 0$,

$$b_{n,1}^{\pi} = \frac{(-)^{n+1}}{n} (\gamma_1 - 1) \binom{n\frac{\gamma_1}{\gamma_2} - \gamma_2}{n-1}, \; n \in \mathbb{N}, \tag{D5}$$

$$b_{n,2}^{\pi} = \frac{(-)^{n+1}}{n} (\gamma_2 - 1) \binom{n\frac{\gamma_2}{\gamma_1} - \gamma_1}{n-1}, \; n \in \mathbb{N} \tag{D6}$$

and where both sets of weights sum to one:

$$\sum_{n=0}^{\infty} \epsilon_{n,1,t} = \sum_{n=0}^{\infty} \epsilon_{n,2,t} = 1. \tag{D7}$$

Proof

We start by deriving expressions for each agent's financial wealth at date t, denoted by $W_{k,t}$ for $k \in \{1, 2\}$. Since $W_{1,t} + W_{2,t} = P_t^Y$, we need only derive an expression for $W_{1,t}$. We know that

$$W_{1,t} = E_t \left[\int_t^{\infty} \frac{\pi_u}{\pi_t} C_{1,u} du \right].$$

Hence,

$$W_{1,t} = \pi_t^{-1} \left(E_t \left[\int_t^{\infty} \hat{\pi}_{2,u} \nu_{2,u}^{-\gamma_2} C_{1,u} 1_{\{A_u < \overline{R}\}} du \right] + E_t \left[\int_t^{\infty} \hat{\pi}_{1,u} \nu_{1,u}^{-\gamma_1} C_{1,u} 1_{\{A_u > \overline{R}\}} du \right] \right),$$

which can be rewritten as

$$\begin{aligned} W_{1,t} &= \pi_t^{-1} \left(E_t \left[\int_t^{\infty} \hat{\pi}_{2,u} \nu_{2,u}^{-\gamma_2} \nu_{1,u} Y_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R\right\}} du \right] + E_t \left[\int_t^{\infty} \hat{\pi}_{1,u} \nu_{1,u}^{1-\gamma_1} Y_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R\right\}} du \right] \right). \\ &= \pi_t^{-1} \left(E_t \left[\int_t^{\infty} \hat{\pi}_{2,u} \nu_{2,u}^{-\gamma_2} (1 - \nu_{2,u}) Y_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} < R\right\}} du \right] + E_t \left[\int_t^{\infty} \hat{\pi}_{1,u} \nu_{1,u}^{1-\gamma_1} Y_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}} > R\right\}} du \right] \right). \end{aligned}$$

Since the series expression in (A22) is valid for all real γ_1, it follows that

$$\nu_{1,t}^{1-\gamma_1} = 1 - (1 - \gamma_1) \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1}{n-1}, \; |A_t| > \overline{R}.$$

We already know that (A21) provides a convergent series expansion for $|A_t| < \overline{R}$ for all real γ_2. Hence,

$$\nu_{2,t}^{1-\gamma_2} = 1 - (1 - \gamma_2) \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2}{n-1}, \; |A_t| < \overline{R}.$$

Therefore,

$$\pi_t\nu_{1,t} = \hat{\pi}_{1,t}\nu_{1,t}^{1-\gamma_1} \;=\; \sum_{n=0}^{\infty} b_{n,1}^{\pi}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}},\; \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} > R, \tag{D8}$$

$$\pi_t\nu_{2,t} = \hat{\pi}_{2,t}\nu_{2,t}^{1-\gamma_2} \;=\; \sum_{n=0}^{\infty} b_{n,2}^{\pi}\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}},\; \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R, \tag{D9}$$

where $b_{n,1}^{\pi}$ and $b_{n,2}^{\pi}$ are given by (D5) and (D6), respectively. Note also that

$$\pi_t = \hat{\pi}_{2,t}\nu_{2,t}^{-\gamma_2} = \sum_{n=0}^{\infty} a_{n,2}^{\pi}\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}},\; \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} < R.$$

Therefore

$$\begin{aligned} W_{1,t} \;=\; & \pi_t^{-1}\left(E_t\left[\int_t^{\infty}\left(\sum_{n=0}^{\infty} a_{n,2}^{\pi}\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}} - \sum_{n=0}^{\infty} b_{n,2}^{\pi}\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}}\right)Y_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<R\right\}}du\right]\right. \\ & \left. + E_t\left[\int_t^{\infty}\sum_{n=0}^{\infty} b_{n,1}^{\pi}\hat{\pi}_{2,u}^{\frac{n}{\gamma_2}}\hat{\pi}_{1,u}^{1-\frac{n}{\gamma_2}}Y_u 1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}}du\right]\right) \end{aligned}$$

Since the expressions for $\nu_{k,t}^{1-\gamma_k}$, $k \in \{1,2\}$, and $\nu_{1,t}^{-\gamma_1}$ are complex analytic functions of A_t, term-by-term integration is valid, and we obtain

$$\begin{aligned} \frac{W_{1,t}}{Y_t} \;=\; & \pi_t^{-1}\left(\sum_{n=0}^{\infty}(a_{n,2}^{\pi} - b_{n,2}^{\pi})\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}} E_t\left[\int_t^{\infty}\frac{\hat{\pi}_{1,u}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,u}^{1-\frac{n}{\gamma_1}}}{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}\frac{Y_u}{Y_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}<R\right\}}du\right]\right. \\ & \left. + \sum_{n=0}^{\infty} b_{n,1}^{\pi}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}} E_t\left[\int_t^{\infty}\frac{\hat{\pi}_{2,u}^{\frac{n}{\gamma_2}}\hat{\pi}_{1,u}^{1-\frac{n}{\gamma_2}}}{\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}}\frac{Y_u}{Y_t}1_{\left\{\frac{\hat{\pi}_{1,u}}{\hat{\pi}_{2,u}}>R\right\}}du\right]\right), \end{aligned}$$

i.e.

$$w_{1,t}^{Y} \;=\; \pi_t^{-1}\left(\sum_{n=0}^{\infty}(a_{n,2}^{\pi} - b_{n,2}^{\pi})\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}\zeta_{n,2,t}^{Y} + \sum_{n=0}^{\infty} b_{n,1}^{\pi}\hat{\pi}_{2,t}^{\frac{n}{\gamma_2}}\hat{\pi}_{1,t}^{1-\frac{n}{\gamma_2}}\zeta_{n,1,t}^{Y}\right),$$

where $w_{1,t} = \frac{W_{1,t}}{Y_t}$. Hence,

$$w_{1,t}^{Y} = \sum_{n=0}^{\infty}(\omega_{n,2,t} - \nu_{2,t}\epsilon_{n,2,t})\zeta_{n,2,t}^{Y} + \nu_{1,t}\sum_{n=0}^{\infty}\epsilon_{n,1,t}\zeta_{n,1,t}^{Y},$$

which implies (D1), where

$$\epsilon_{n,1,t} = \frac{\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_2}}\hat{\pi}_{1,t}^{\frac{n}{\gamma_2}}}{\pi_t\nu_{1,t}}b_{n,1}^{\pi},\; n \in \mathbb{N}_0, \tag{D10}$$

$$\epsilon_{n,2,t} = \frac{\hat{\pi}_{1,t}^{\frac{n}{\gamma_1}}\hat{\pi}_{2,t}^{1-\frac{n}{\gamma_1}}}{\pi_t\nu_{2,t}}b_{n,2}^{\pi},\; n \in \mathbb{N}_0. \tag{D11}$$

Note that (D8) and (D9) imply that the weights $\epsilon_{n,1,t}$, $n \in \mathbb{N}_0$ and $\epsilon_{n,2,t}$, $n \in \mathbb{N}_0$ each sum to one, i.e. (D7). Using (A32), we can rewrite (D10) and (D11) as (D3) and (D4), respectively. Since the bond is in zero net supply $\sum_{k=1}^2 W_{k,t} = P_t^Y$, and so $\sum_{k=1}^2 w_{k,t}^Y = p_t^Y$. Thus,

$$\begin{aligned} w_{2,t}^Y &= \sum_{n=0} \omega_{n,1,t}\zeta_{n,1,t}^Y + \sum_{n=0} \omega_{n,2,t}\zeta_{n,2,t}^Y - \sum_{n=0}^{\infty}(\omega_{n,2,t} - \nu_{2,t}\epsilon_{n,2,t})\zeta_{n,2,t}^Y - \nu_{1,t}\sum_{n=0}^{\infty}\epsilon_{n,1,t}\zeta_{n,1,t}^Y \\ &= \sum_{n=0} \omega_{n,1,t}\zeta_{n,1,t}^Y + \nu_{2,t}\sum_{n=0}^{\infty}\epsilon_{n,2,t}\zeta_{n,2,t}^Y - \nu_{1,t}\sum_{n=0}^{\infty}\epsilon_{n,1,t}\zeta_{n,1,t}^Y, \end{aligned}$$

which implies (D2).

Finally, we wish to determine the proportion of investor k's wealth invested in the risky stock and the proportion invested in the instantaneously riskless asset. Denoting by $N_{k,t}^B$ and $N_{k,t}^P$ the number of bonds and units of stock, respectively, held by Agent k, we have that the financial wealth of the agent is the sum of the wealth invested in bonds and stocks:

$$W_{k,t} = N_{k,t}^B B_t + N_{k,t}^P P_t^Y.$$

Moreover, because there is only a single risky asset available in this market, the volatility of each investor's wealth will depend only on the proportion of that investor's wealth invested in the stock market. We exploit this observation to determine the share of each agent's wealth that is invested in the stock market.

Proposition D2 *The proportion of Agent k's wealth invested in the stock market,* $\Pi_{k,t}$*, is given by*

$$\Pi_{k,t} = \frac{\sigma_{W_k,t}}{\sigma_{R,t}^Y},\ k \in \{1,2\}, \tag{D12}$$

where $\sigma_{R,t}^Y$ *is the volatility of stock returns on the claim to the aggregate endowment,* Y *and is given in* (52) *and* $\sigma_{W_k,t}$ *is the volatility of Agent k's portfolio return:*

$$\sigma_{W_k,t} = \sigma_Y + \sigma_{\nu_1,t}\frac{\nu_{1,t}}{w_{k,t}^Y}\frac{\partial w_{k,t}^Y}{\partial \nu_{1,t}},\ k \in \{1,2\}. \tag{D13}$$

The proportion of Agent k's wealth invested in the locally riskfree bond is $1 - \Pi_{k,t}$.

Proof
To find the optimal portfolio policies note that

$$W_{k,t} = N_{k,t}^B B_t + N_{k,t}^P P_t^Y, \tag{D14}$$

where $N_{k,t}^B$ and $N_{k,t}^P$ are the number of bonds and units of stock, respectively, held by Agent k. Market clearing implies that

$$\begin{aligned} 0 &= \sum_{k=1}^{2} N_{k,t}^B, \\ 1 &= \sum_{k=1}^{2} N_{k,t}^P. \end{aligned}$$

Thus, we need to determine only $N^P_{1,t}$, and given this, it follows that

$$\begin{aligned} N^P_{2,t} &= 1 - N^P_{1,t} \\ N^B_{1,t} &= -N^B_{2,t} = \frac{W_{1,t} - N^P_{1,t} P^Y_t}{B_t}. \end{aligned}$$

Applying Ito's Lemma to (D14) when $k = 1$, gives

$$dW_{1,t} = B_t dN^B_{1,t} + P_t dN^P_{1,t} + N^B_{1,t} dB_t + N^P_{1,t} dP^Y_t.$$

The self-financing condition

$$B_t dN^B_{1,t} + P_t dN^P_{1,t} + N^B_{1,t} dB_t = 0,$$

implies that

$$dW_{1,t} = N^P_{1,t} dP^Y_t,$$

and hence,

$$\frac{dW_{1,t}}{W_{1,t}} = \Pi_{1,t} \frac{dP^Y_t}{P^Y_t},$$

where

$$\Pi_{k,t} = \frac{N^P_{k,t} P^Y_t}{W_{k,t}}$$

is the proportion of Agent k's wealth held in the stock market. Hence,

$$\Pi_{1,t} = \frac{\sigma_{W_{1,t}}}{\sigma^Y_{R,t}},$$

where $\sigma_{W_{1,t}}$ is given by

$$\frac{dW_{1,t}}{W_{1,t}} = \mu_{W_{1,t}} dt + \sigma_{W_{1,t}} dZ_t,$$

and $\sigma^Y_{R,t}$ is given by (52). It follows from Ito's Lemma that

$$\sigma_{W_{1,t}} = \sigma_Y + \sigma_{\nu_{1,t}} \frac{\nu_{1,t}}{w^Y_{1,t}} \frac{\partial w^Y_{1,t}}{\partial \nu_{1,t}}.$$

Similarly,

$$\Pi_{2,t} = \frac{\sigma_{W_{2,t}}}{\sigma_{R,t}},$$

where

$$\sigma_{W_{2,t}} = \sigma_Y + \sigma_{\nu_{1,t}} \frac{\nu_{1,t}}{w^Y_{2,t}} \frac{\partial w^Y_{2,t}}{\partial \nu_{1,t}}.$$

Thus, we obtain (D12) and (D13).

Table 1: Parameter Values

This table gives the parameter values we use to evaluate the quantitative implications of our model for asset prices. There are five cases we consider: (i) the base case, in which the two agents are assumed to be identical; (ii) the case with identical preferences but heterogeneous beliefs that are pessimistic; (iii) the case with identical beliefs and subjective discount rates but heterogeneous risk aversions; (iv) the case with heterogeneous beliefs and heterogeneous risk aversions; and, (v) the case where the parameters are chosen in such a way that they satisfy the stationarity condition.

Description of parameter	Symbol	Value
Expected growth rate of aggregate endowment	μ_Y	0.02
Volatility of growth rate of aggregate endowment	σ_Y	0.03
Case (i): Both agents identical		
Belief of both agents about expected growth rate of endowment	$\mu_{Y,k}$	0.02
Subjective discount rate for both agents	β_k	0.01
Relative risk aversion for both agents	γ_k	3.00
Case (ii): Heterogeneous beliefs that are also pessimistic		
Agent 1's belief about expected growth rate of aggregate endowment	$\mu_{Y,1}$	0.0125
Agent 2's belief about expected growth rate of aggregate endowment	$\mu_{Y,2}$	0.01
Case (iii): Heterogeneity only in risk aversions		
Relative risk aversion for Agent 1	γ_1	0.50
Relative risk aversion for Agent 2	γ_2	5.50
Case (iv): Heterogeneity in both beliefs and risk aversions		
Agent 1's belief about the expected growth rate of aggregate endowment	$\mu_{Y,1}$	0.0125
Agent 2's belief about the expected growth rate of aggregate endowment	$\mu_{Y,2}$	0.01
Relative risk aversion for Agent 1	γ_1	0.50
Relative risk aversion for Agent 2	γ_2	5.50
Case (v): Parameters satisfying the stationarity condition		
Belief of both agents about expected growth rate of endowment	$\mu_{Y,k}$	0.01
Subjective discount rate for Agent 1	β_1	0.01956
Subjective discount rate for Agent 2	β_1	0.00001
Relative risk aversion for Agent 1	γ_1	2.50
Relative risk aversion for Agent 2	γ_2	1.50

Figure 1: The Riskfree Interest Rate

This figure plots the instantaneously riskfree interest rate, r, as a function of the consumption share of the first agent, ν_1. The figure has five plots corresponding to the following cases: (i) Identical agents; (ii) Agents with different beliefs, which are pessimistic on average; (iii) Agents with different risk aversions but identical beliefs; (iv) Agents with different beliefs and different risk aversions; and (v) Agents with beliefs and preference such that the stationarity condition is satisfied.

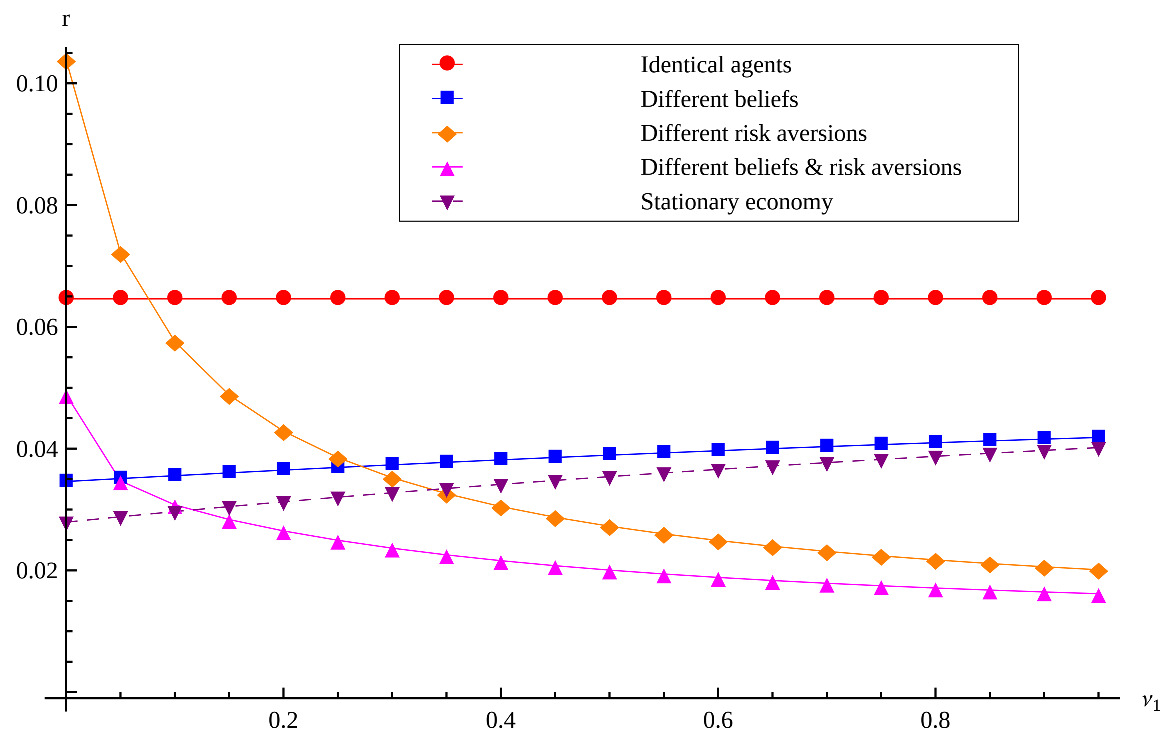


Figure 2: Volatility of the Instantaneously Riskfree Rate

This figure plots the volatility of the instantaneously riskfree rate, $|\sigma_r|$, as a function of the consumption share of the first agent, ν_1. The figure has five plots corresponding to the following cases: (i) Identical agents; (ii) Agents with different beliefs, which are pessimistic on average; (iii) Agents with different risk aversions but identical beliefs; (iv) Agents with different beliefs and different risk aversions; and (v) Agents with beliefs and preference such that the stationarity condition is satisfied.

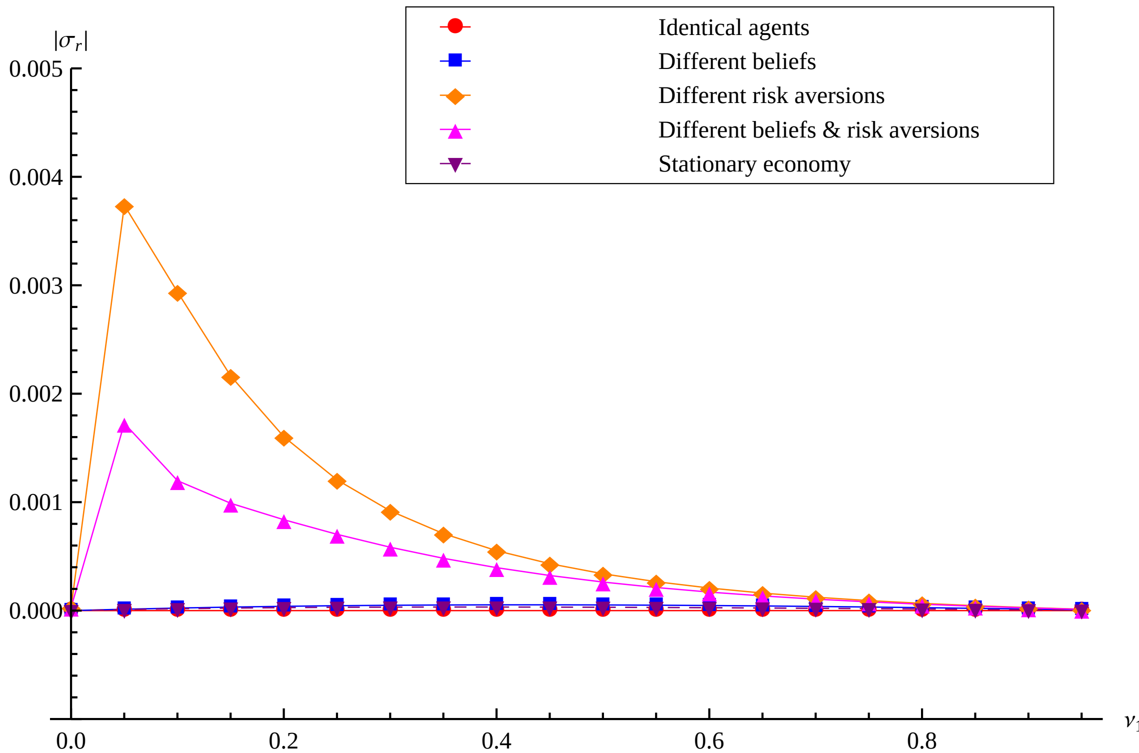


Figure 3: Market Price of Risk

This figure plots the market price of risk, θ, as a function of the consumption share of the first agent, ν_1. The figure has five plots corresponding to the following cases: (i) Identical agents; (ii) Agents with different beliefs, which are pessimistic on average; (iii) Agents with different risk aversions but identical beliefs; (iv) Agents with different beliefs and different risk aversions; and (v) Agents with beliefs and preference such that the stationarity condition is satisfied.

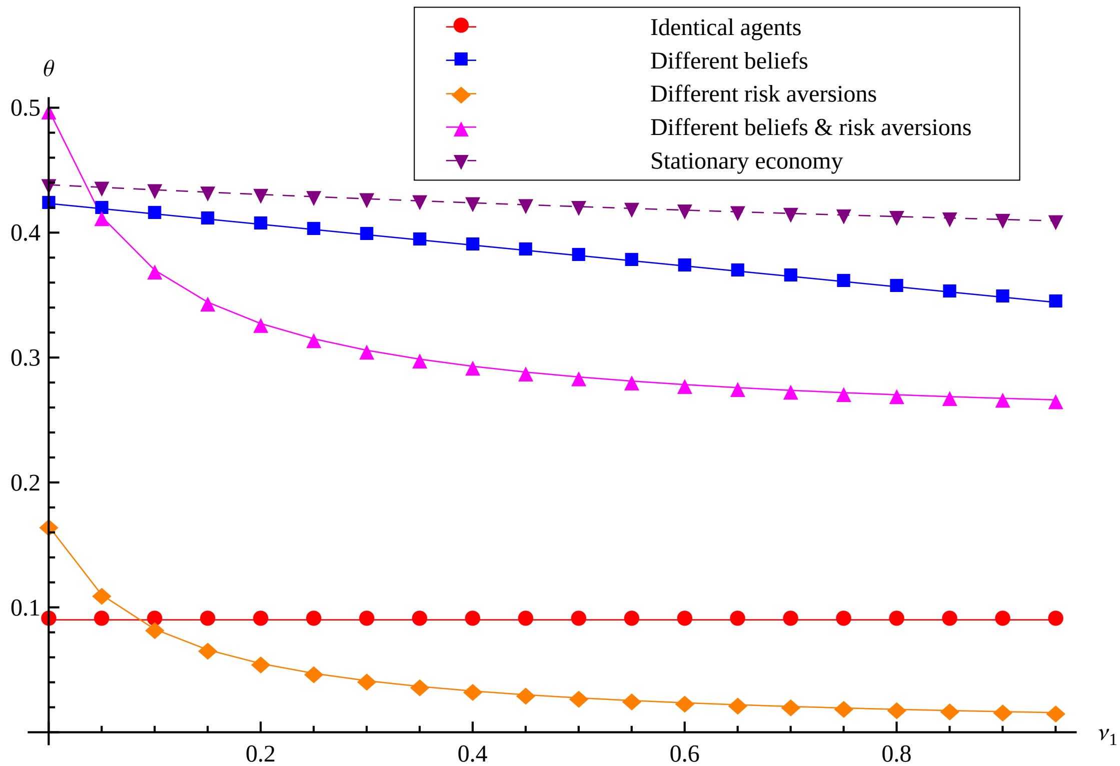


Figure 4: Volatility of Stock Market Returns

The figure has five plots corresponding to the following cases: (i) Identical agents; (ii) Agents with different beliefs, which are pessimistic on average; (iii) Agents with different risk aversions but identical beliefs; (iv) Agents with different beliefs and different risk aversions; and (v) Agents with beliefs and preference such that the stationarity condition is satisfied.

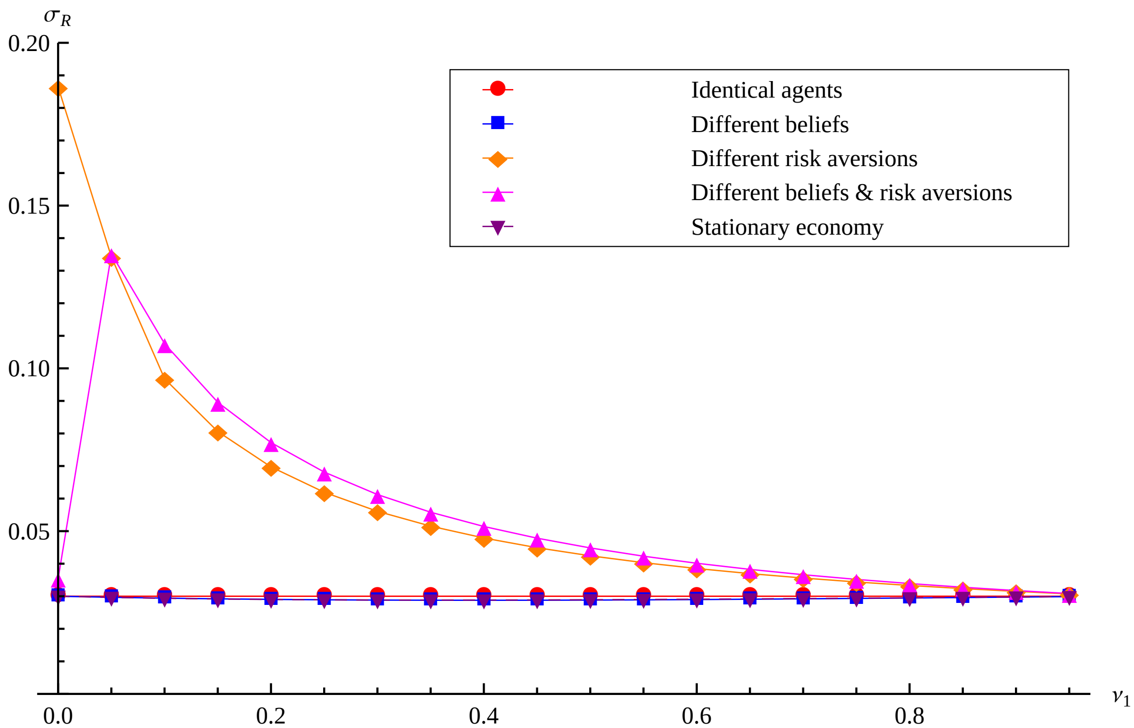


Figure 5: Equity Risk Premium

The figure has five plots corresponding to the following cases: (i) Identical agents; (ii) Agents with different beliefs, which are pessimistic on average; (iii) Agents with different risk aversions but identical beliefs; (iv) Agents with different beliefs and different risk aversions; and (v) Agents with beliefs and preference such that the stationarity condition is satisfied.

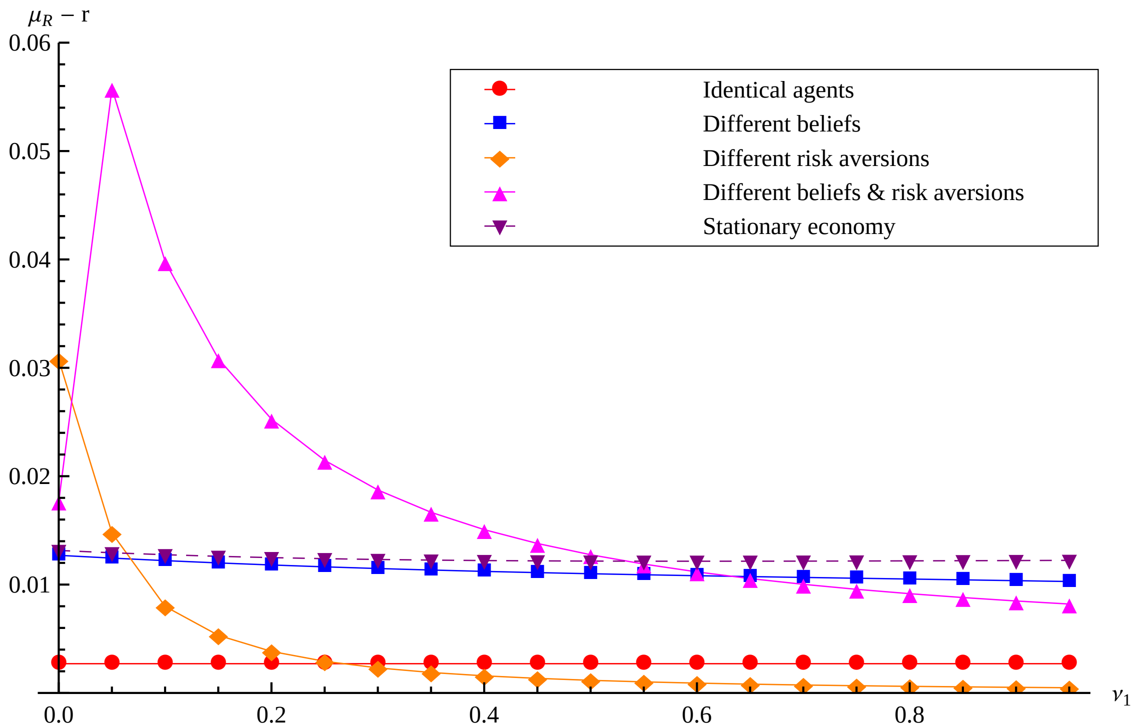
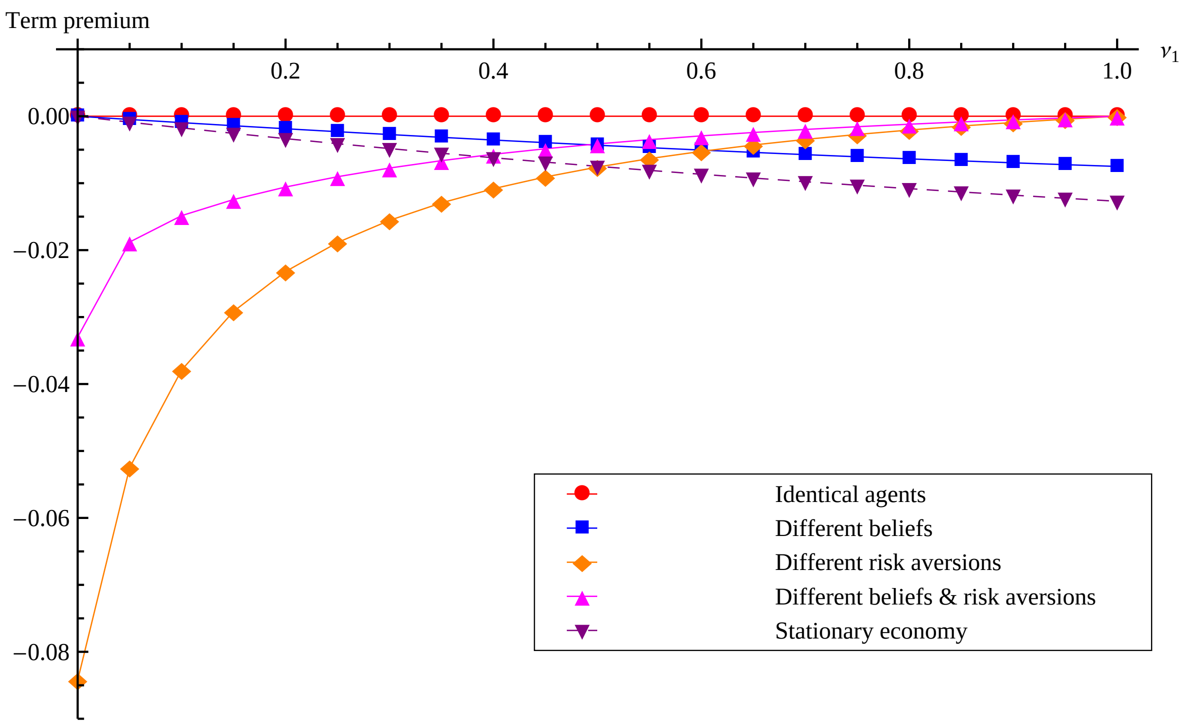


Figure 6: Term Premium

This figure plots the the limit of the *term premium*, which is the difference between the yield on a zero-coupon discount bond, y^1_{T-t}, and the instantaneous interest rate, r_t: $\lim_{T\to\infty} y^1_{T-t} - r_t$, as a function of the consumption share of the first agent, ν_1. The figure has five plots corresponding to the following cases: (i) Identical agents; (ii) Agents with different beliefs, which are pessimistic on average; (iii) Agents with different risk aversions but identical beliefs; (iv) Agents with different beliefs and different risk aversions; and (v) Agents with beliefs and preference such that the stationarity condition is satisfied.



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