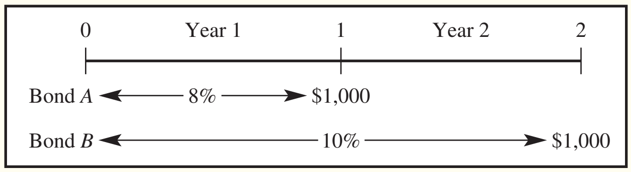
Appendix 5A The Term Structure of Interest Rates, Spot Rates, and Yield to Maturity

In the main body of this chapter, we have assumed that the interest rate is constant over all future periods. In reality, interest rates vary through time. This occurs primarily because inflation rates are expected to differ through time.

To illustrate, we consider two zero coupon bonds. Bond *A* is a one-year bond and bond *B* is a two-year bond. Both have face values of $1,000. The one-year interest rate, r_1, is 8 percent. The two-year interest rate, r_2, is 10 percent. These two rates of interest are examples of *spot rates*. Perhaps this inequality in interest rates occurs because inflation is expected to be higher over the second year than over the first year. The two bonds are depicted in the following time chart:



We can easily calculate the present value for bond *A* and bond *B* as follows:

$$\text{PV}_A = \$925.93 = \frac{\$1{,}000}{1.08}$$

$$\text{PV}_B = \$826.45 = \frac{\$1{,}000}{(1.10)^2}$$

Of course, if PV_A and PV_B were observable and the spot rates were not, we could determine the spot rates using the PV formula, because:

$$\text{PV}_A = \$925.93 = \frac{\$1{,}000}{(1 + r_1)} \rightarrow r_1 = 8\%$$

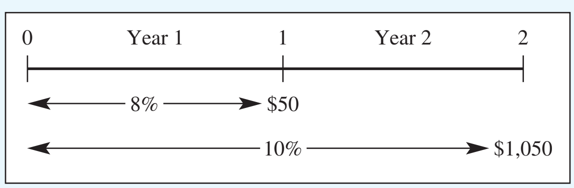
and:

$$\text{PV}_B = \$826.45 = \frac{\$1{,}000}{(1 + r_2)^2} \rightarrow r_2 = 10\%$$

Now we can see how the prices of more complicated bonds are determined. Try to do the next example. It illustrates the difference between spot rates and yields to maturity.

EXAMPLE 5A.1

On the Spot Given the spot rates r_1 equals 8 percent and r_2 equals 10 percent, what should a 5 percent coupon, two-year bond cost? The cash flows C_1 and C_2 are illustrated in the following time chart:



The bond can be viewed as a portfolio of zero coupon bonds with one- and two-year maturities. Therefore:

$$\text{PV} = \frac{\$50}{1 + 0.08} + \frac{\$1{,}050}{(1 + 0.10)^2} = \$914.06 \qquad \textbf{(A.1)}$$

(*continued*)

We now want to calculate a single rate for the bond. We do this by solving for y in the following equation:

$$\$914.06 = \frac{\$50}{1 + y} + \frac{\$1{,}050}{(1 + y)^2} \qquad \textbf{(A.2)}$$

In Equation A.2, y equals 9.95 percent. As mentioned in the chapter, we call y the *yield to maturity* on the bond. Solving for y for a multiyear bond is generally done by means of trial and error.[1] Although this can take much time with paper and pencil, it is virtually instantaneous on a handheld calculator.

It is worthwhile to contrast Equations A.1 and A.2. In A.1, we use the marketwide spot rates to determine the price of the bond. Once we get the bond price, we use A.2 to calculate its yield to maturity. Because Equation A.1 employs two spot rates whereas only one appears in A.2, we can think of yield to maturity as some sort of average of the two spot rates.[2]

Using these spot rates, the yield to maturity of a two-year coupon bond whose coupon rate is 12 percent and PV equals $1,036.73 can be determined by:

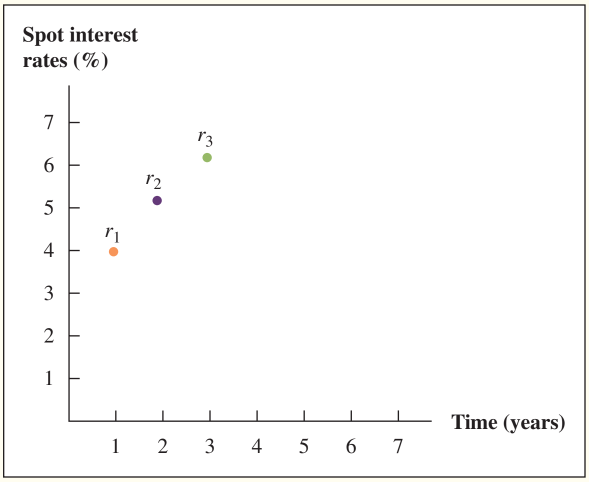
$$\$1{,}036.73 = \frac{\$120}{1 + r} + \frac{\$1{,}120}{(1 + r)^2} \rightarrow r = 9.89\%$$

As these calculations show, two bonds with the same maturity will usually have different yields to maturity if the coupons differ.

Graphing the Term Structure The *term structure* describes the relationship of spot rates with different maturities. Figure 5A.1 graphs a particular term structure. In Figure 5A.1 the spot rates are increasing with longer maturities—that is, $r_3 > r_2 > r_1$. Graphing the term structure is easy if we can observe spot rates. Unfortunately this can be done only if there are enough zero coupon government bonds.

A given term structure, such as that in Figure 5A.1, exists for only a moment in time—say 10:00 a.m., July 30, 2006. Interest rates are likely to change in the next minute, so that a different (though quite similar) term structure would exist at 10:01 a.m.

Figure 5A.1
The Term Structure of Interest Rates

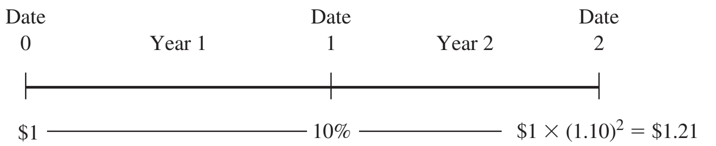


[1]The quadratic formula may be used to solve for y for a two-year bond. However, formulas generally do not apply for bonds with more than four payment dates.

[2]Yield to maturity is not a simple average of r_1 and r_2. Rather, financial economists speak of it as a time-weighted average of r_1 and r_2.

Figure 5A.2

Breakdown of a Two-Year Spot Rate into a One-Year Spot Rate and Forward Rate over the Second Year



With a two-year spot rate of 10 percent, an investor in two-year bond receives $1.21 at date 2.

This is the same return *as if* the investor received the spot rate of 8 percent over the first year and a 12.04 percent return over the second year.

\$1 ______ 8% ______ \$1.08 ______ 12.04% ______ $\$1 \times 1.08 \times 1.1204 = \$1.21$

Because both the one-year spot rate and the two-year spot rate are known at date 0, the forward rate over the second year can be calculated at date 0.

Explanations of the Term Structure

Figure 5A.1 showed one of many possible relationships between the spot rate and maturity. We now want to explore the relationship in more detail. We begin by defining a new term, the forward rate. Next, we relate this forward rate to future interest rates. Finally we consider alternative theories of the term structure.

Definition of Forward Rate Earlier in this appendix, we developed a two-year example where the spot rate over the first year is 8 percent and the spot rate over the two years is 10 percent. Here, an individual investing \$1 in a two-year zero coupon bond would have $\$1 \times (1.10)^2$ in two years.

To pursue our discussion, it is worthwhile to rewrite:[3]

$$\$1 \times (1.10)^2 = \$1 \times 1.08 \times 1.1204 \qquad \textbf{(A.3)}$$

Equation A.3 tells us something important about the relationship between one- and two-year rates. When an individual invests in a two-year zero coupon bond yielding 10 percent, his wealth at the end of two years is the same as if he received an 8 percent return over the first year and a 12.04 percent return over the second year. This hypothetical rate over the second year, 12.04 percent, is called the *forward rate.* Thus, we can think of an investor with a two-year zero coupon bond as getting the one-year spot rate of 8 percent and *locking in* 12.04 percent over the second year. This relationship is presented in Figure 5A.2.

More generally, if we are given spot rates r_1 and r_2, we can always determine the forward rate, f_2, such that:

$$(1 + r_2)^2 = (1 + r_1) \times (1 + f_2) \qquad \textbf{(A.4)}$$

We solve for f_2, yielding:

$$f_2 = \frac{(1 + r_2)^2}{1 + r_1} - 1 \qquad \textbf{(A.5)}$$

[3]12.04 percent is equal to:

$$\frac{(1.10)^2}{(1.08)} = 1$$

when rounding is performed after four digits.

EXAMPLE 5A.2

Looking Forward If the one-year spot rate is 7 percent and the two-year spot rate is 12 percent, what is f_2?

We plug in Equation A.5, yielding:

$$f_2 = \frac{(1.12)^2}{1.07} - 1 = 17.23\%$$

Consider an individual investing in a two-year zero coupon bond yielding 12 percent. We say it is as if he receives 7 percent over the first year and simultaneously locks in 17.23 percent over the second year. Note that both the one-year spot rate and the two-year spot rate are known at date 0. Because the forward rate is calculated from the one-year and two-year spot rates, it can be calculated at date 0 as well.

Forward rates can be calculated over later years as well. The general formula is:

$$f_n = \frac{(1 + r_n)^n}{(1 + r_{n-1})^{n-1}} - 1 \quad \textbf{(A.6)}$$

where f_n is the forward rate over the nth year, r_n is the n-year spot rate, and r_{n-1} is the spot rate for $n - 1$ years.

EXAMPLE 5A.3

Forward Rates Assume the following set of rates:

Year	Spot Rate
1	5%
2	6
3	7
4	6

What are the forward rates over each of the four years?

The forward rate over the first year is, *by definition*, equal to the one-year spot rate. Thus, we do not generally speak of the forward rate over the first year. The forward rates over the later years are:

$$f_2 = \frac{(1.06)^2}{1.05} - 1 = 7.01\%$$

$$f_3 = \frac{(1.07)^3}{(1.06)^2} - 1 = 9.03\%$$

$$f_4 = \frac{(1.06)^4}{(1.07)^3} - 1 = 3.06\%$$

An individual investing \$1 in the two-year zero coupon bond receives \$1.1236 [\$1 $\times$ $(1.06)^2$] at date 2. He can be viewed as receiving the one-year spot rate of 5 percent over the first year and receiving the forward rate of 7.01 percent over the second year. An individual investing \$1 in a three-year zero coupon bond receives \$1.2250 [\$1 $\times$ $(1.07)^3$] at date 3. She can be viewed as receiving the two-year spot rate of 6 percent over the first two years and receiving the forward rate of 9.03 percent over the third year.

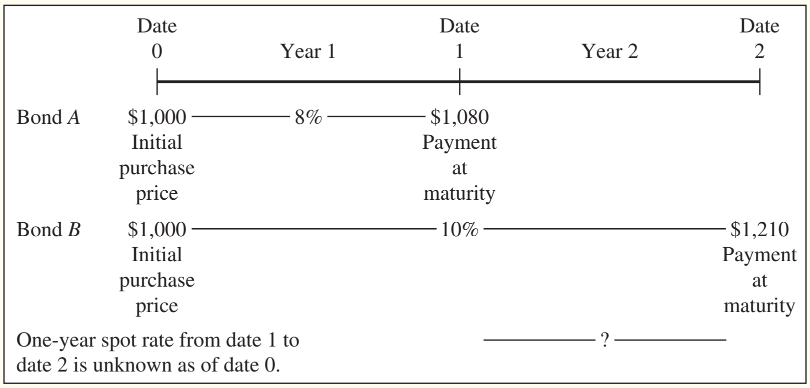
An individual investing \$1 in a four-year zero coupon bond receives \$1.2625 [\$1 $\times$ $(1.06)^4$] at date 4. He can be viewed as receiving the three-year spot rate of 7 percent over the first three years and receiving the forward rate of 3.06 percent over the fourth year.

Note that all of the four spot rates in this problem are known at date 0. Because the forward rates are calculated from the spot rates, they can be determined at date 0 as well.

The material in this appendix is likely to be difficult for a student exposed to term structure for the first time. It helps to state what the student should know at this point. Given Equations A.5 and A.6, a student should be able to calculate a set of forward rates given a set of spot rates. This can simply be viewed as a mechanical computation. In addition to the calculations, a student should understand the intuition of Figure 5A.2.

We now turn to the relationship between the forward rate and the expected spot rates in the future.

Estimating the Price of a Bond at a Future Date In the example from the body of this chapter, we considered zero coupon bonds paying \$1,000 at maturity and selling at a discount prior to maturity. We now wish to change the example slightly. Now each bond initially sells at par so that payment at maturity is above \$1,000.[4] Keeping the spot rates at 8 percent and 10 percent, we have the following:



The payments at maturity are \$1,080 and \$1,210 for the one- and two-year zero coupon bonds, respectively. The initial purchase price of \$1,000 for each bond is determined as follows:

$$\$1{,}000 = \frac{\$1{,}080}{1.08}$$

$$\$1{,}000 = \frac{\$1{,}210}{(1.10)^2}$$

We refer to the one-year bond as bond *A* and the two-year bond as bond *B*.

There will be a different one-year spot rate when date 1 arrives. This will be the spot rate from date 1 to date 2. We can also call it the spot rate over year 2. This spot rate is not known as of date 0. For example, should the rate of inflation rise between date 0 and date1, the spot rate over year 2 would likely be high. Should the rate of inflation fall between date 0 and date 1, the spot rate over year 2 would likely be low.

Now that we have determined the price of each bond at date 0, we want to determine what the price of each bond will be at date 1. The price of the one-year bond (bond *A*) must be \$1,080 at date 1 because the payment at maturity is made then. The hard part is determining what the price of the two-year bond (bond *B*) will be at that time.

Suppose we find that, on date 1, the one-year spot rate from date 1 to date 2 is 6 percent. We state that this is the one-year spot rate *over* year 2. This means that you can invest \$1,000 at date 1 and receive \$1,060 (\$1,000 × 1.06) at date 2. Because one year has already passed for bond *B*, the bond has only one year left. Because bond *B* pays \$1,210 at date 2, its value at date 1 is:

$$\$1{,}141.51 = \frac{\$1{,}210}{1.06} \quad \textbf{(A.7)}$$

[4]This change in assumptions simplifies our presentation but does not alter any of our conclusions.

Table 5A.1

Price of Bond *B* at Date 1 as a Function of Spot Rate over Year 2

Price of Bond *B* at Date 1	Spot Rate over Year 2
$\$1{,}141.51 = \dfrac{\$1{,}210}{1.06}$	6%
$\$1{,}130.84 = \dfrac{\$1{,}210}{1.07}$	7%
$\$1{,}061.40 = \dfrac{\$1{,}210}{1.14}$	14%

Note that no one *knew* ahead of time the price that bond *B* would sell for on date 1 because no one knew that the one-year spot rate over year 2 would be 6 percent.

Suppose the one-year spot rate beginning at date 1 turned out not to be 6 percent, but to be 7 percent instead. This means that you can invest \$1,000 at date 1 and receive \$1,070 (\$1,000 × 1.07) at date 2. In this case, the value of bond *B* at date 1 would be:

$$\$1{,}130.84 = \frac{\$1{,}210}{1.07} \tag{A.8}$$

Finally, suppose that the one-year spot rate at date 1 turned out to be neither 6 percent nor 7 percent, but 14 percent instead. This means that you can invest \$1,000 at date 1 and receive \$1,140 (\$1,000 × 1.14) at date 2. In this case, the value of bond *B* at date 1 would be:

$$\$1{,}061.40 = \frac{\$1{,}210}{1.14}$$

These possible bond prices are represented in Table 5A.1. The price that bond *B* will sell for on date 1 is not known before date 1 because the one-year spot rate prevailing over year 2 is not known until date 1.

It is important to reemphasize that although the forward rate is known at date 0, the one-year spot rate beginning at date 1 is *unknown* ahead of time. Thus, the price of bond *B* at date 1 is unknown ahead of time. Prior to date 1, we can speak only of the amount that bond *B* is *expected* to sell for on date 1. We write this as follows:[5]

The Amount That Bond *B* Is Expected to Sell for on Date 1:

$$\frac{\$1{,}210}{1 + \text{Spot rate expected over year 2}} \tag{A.9}$$

It is worthwhile making two points now. First, because each individual is different, the expected value of bond *B* differs across individuals. Later we will speak of a consensus expected value across investors. Second, Equation A.9 represents one's forecast of the price that the bond will be selling for on date 1. The forecast is made ahead of time—that is, on date 0.

The Relationship between Forward Rate over Second Year and Spot Rate Expected over Second Year

Given a forecast of bond *B*'s price, an investor can choose one of two strategies at date 0:

1. Buy a one-year bond. Proceeds at date 1 would be:

$$\$1{,}080 = \$1{,}000 \times 1.08 \tag{A.10}$$

[5]Technically, Equation A.9 is only an approximation due to *Jensen's inequality.* That is, expected values are:

$$\frac{\$1{,}210}{1 + \text{Spot rate}} > \frac{\$1{,}210}{1 + \text{Spot rate expected over year 2}}$$

However, we ignore this very minor issue in the rest of the analysis.

2. Buy a two-year bond but sell at date 1. *Expected* proceeds would be:

$$\frac{\$1{,}000 \times (1.10)^2}{1 + \text{Spot rate expected over year 2}} \tag{A.11}$$

Given our discussion of forward rates, we can rewrite Equation A.11 as:

$$\frac{\$1{,}000 \times 1.08 \times 1.1204}{1 + \text{Spot rate expected over year 2}} \tag{A.12}$$

(Remember that 12.04 percent was the forward rate over year 2; that is, $f_2 = 12.04\%$.)

Under what condition will the return from strategy 1 equal the expected return from strategy 2? In other words, under what condition will Equation A.10 equal Equation A.12?

The two strategies will yield the same expected return only when:

$$12.04\% = \text{Spot rate expected over year 2} \tag{A.13}$$

In other words, if the forward rate equals the expected spot rate, one would expect to earn the same return over the first year whether one

- Invested in a one-year bond.
- Invested in a two-year bond but sold after one year.

The Expectations Hypothesis

Equation A.13 seems fairly reasonable. That is, it is reasonable that investors would set interest rates in such a way that the forward rate would equal the spot rate expected by the marketplace a year from now.[6] For example, imagine that individuals in the marketplace do not concern themselves with risk. If the forward rate, f_2, is less than the spot rate expected over year 2, individuals desiring to invest for one year would always buy a one-year bond. That is, our work shows that an individual investing in a two-year bond but planning to sell at the end of one year would expect to earn less than if he simply bought a one-year bond.

Equation A.13 was stated for the specific case where the forward rate was 12.04 percent. We can generalize this as follows:

Expectations Hypothesis:

$$f_2 = \text{Spot rate expected over year 2} \tag{A.14}$$

Equation A.14 says that the forward rate over the second year is set to the spot rate that people expect to prevail over the second year. This is called the *expectations hypothesis.* It states that investors will set interest rates such that the forward rate over the second year is equal to the one-year spot rate expected over the second year.

Liquidity Preference Hypothesis

At this point, many students think that Equation A.14 *must* hold. However, note that we developed Equation A.14 by assuming that investors were risk-neutral. Suppose, alternatively, that investors are averse to risk.

Which strategy would appear more risky for an individual who wants to invest for one year?

1. Invest in a one-year bond.
2. Invest in a two-year bond but sell at the end of one year.

[6]Of course, each individual will have different expectations, so Equation A.13 cannot hold for all individuals. However, financial economists generally speak of a *consensus* expectation. This is the expectation of the market as a whole.

Strategy 1 has no risk because the investor knows that the rate of return must be r_1. Conversely, strategy 2 has much risk: The final return is dependent on what happens to interest rates.

Because strategy 2 has more risk than strategy 1, no risk-averse investor will choose strategy 2 if both strategies have the same expected return. Risk-averse investors can have no preference for one strategy over the other only when the expected return on strategy 2 is *above* the return on strategy 1. Because the two strategies have the same expected return when f_2 equals the spot rate expected over year 2, strategy 2 can have a higher rate of return only when the following condition holds:

Liquidity Preference Hypothesis:

$$f_2 > \text{Spot rate expected over year 2} \tag{A.15}$$

That is, to induce investors to hold the riskier two-year bonds, the market sets the forward rate over the second year to be above the spot rate expected over the second year. Equation A.15 is called the *liquidity preference hypothesis*.

We developed the entire discussion by assuming that individuals are planning to invest over one year. We pointed out that for these types of individuals, a two-year bond has extra risk because it must be sold prematurely. What about individuals who want to invest for two years? (We call these people investors with a two-year *time horizon.*)

They could choose one of the following strategies:

3. Buy a two-year zero coupon bond.
4. Buy a one-year bond. When the bond matures, immediately buy another one-year bond.

Strategy 3 has no risk for an investor with a two-year time horizon because the proceeds to be received at date 2 are known as of date 0. However, strategy 4 has risk because the spot rate over year 2 is unknown at date 0. It can be shown that risk-averse investors will prefer neither strategy 3 nor strategy 4 over the other when:

$$f_2 < \text{Spot rate expected over year 2} \tag{A.16}$$

Note that the assumption of risk aversion gives contrary predictions. Relationship A.15 holds for a market dominated by investors with a one-year time horizon. Relationship A.16 holds for a market dominated by investors with a two-year time horizon. Financial economists have generally argued that the time horizon of the typical investor is generally much shorter than the maturity of typical bonds in the marketplace. Thus, economists view A.15 as the best depiction of equilibrium in the bond market with *risk-averse* investors.

However, do we have a market of risk-neutral or risk-averse investors? In other words, can the expectations hypothesis of Equation A.14 or the liquidity preference hypothesis of Equation A.15 be expected to hold? As we will learn later in this book, economists view investors as being risk-averse for the most part. Yet, economists are never satisfied with a casual examination of a theory's assumptions. To them, empirical evidence of a theory's predictions must be the final arbiter.

There has been a great deal of empirical evidence about the term structure of interest rates. Unfortunately (perhaps fortunately for some students), we will not be able to present the evidence in any detail. Suffice it to say that, in our opinion, the evidence supports the liquidity preference hypothesis over the expectations hypothesis. One simple result might give students the flavor of this research. Consider an individual choosing between one of the following two strategies:

1. Invest in a one-year bond.

2′. Invest in a 20-year bond but sell at the end of one year.

[Strategy 2′ is identical to strategy 2, except that a 20-year bond is substituted for a 2-year bond.]

The expectations hypothesis states that the expected returns on both strategies are identical. The liquidity preference hypothesis states that the expected return on strategy 2′ should be above the expected return on strategy 1. Though no one knows what returns are actually expected over a particular time period, actual returns from the past may allow us to infer expectations. The results from January 1926 to December 1999 are illuminating. The average yearly return on strategy 1 is 3.8 percent and 5.5 percent on strategy 2′ over this time period.[7,8] This evidence is generally considered to be consistent with the liquidity preference hypothesis and inconsistent with the expectations hypothesis.

Questions and Problems

1. **Bond Pricing** The one-year spot rate is 8 percent and the two-year spot rate is 10 percent.
   a. What is the price of a two-year bond that pays an annual coupon of 6 percent?
   b. What is the yield to maturity of this bond?
2. **Bond Pricing** The one-year spot rate is 11 percent and the two-year spot rate is 8 percent. What is the price of a two-year bond that pays an annual coupon of 5 percent?
3. **Forward Rates** If the one-year spot rate is 7 percent and the two-year spot rate is 8.5 percent, what is the one-year forward rate over the second year?
4. **Forward Rates** Assume the following spot rates:

Year	Spot Rate (%)
1	4
2	5.5
3	6.5

   a. Calculate the one-year forward rate over the second year.
   b. Calculate the one-year forward rate over the third year.
5. **Term Structure** Assume the following forward rates:

Maturity	Forward Rates
1	4.5%
2	6.0%

   Compute the spot rates for years 1 and 2.
6. **Term Structure** Given the following two scenarios, for what range of spot rates expected over year 2 would you be better off adopting strategy 1? Explain.
   Strategy 1: Buy a two-year bond and then sell it in year 1.
   Strategy 2: Buy a one-year bond.

[7]Taken from *Stocks, Bonds, Bills and Inflation 2000 Yearbook* (Chicago: Ibbotson Associates, Inc.). Ibbotson Associates annually updates work by Roger G. Ibbotson and Rex A. Sinquefield.

[8]It is important to note that strategy 2′ does not involve buying a 20-year bond and holding it to maturity. Rather, it consists of buying a 20-year bond and selling it 1 year later—that is, when it has become a 19-year bond. This round-trip transaction occurs 74 times in the 74-year sample from January 1926 to December 1999.