ADMISSIBLE MEAN STANDARD DEVIATION INDIFFERENCE CURVES

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This note considers the question of which indifference curves in mean standard deviation space can be derived from expected utility preferences. It is shown that the well known requirement that the slope of indifference curves be zero when standard deviation is zero is not the only restriction: many families of curves satisfying this requirement which are also nicely behaved and convex downwards can be ruled out because they imply behavior inconsistent with expected utility preferences. It is therefore not possible to draw arbitrary indifference curves and assume there exist underlying expected utility preferences which correspond to these.

1. Introduction

If expected utility is a quadratic function of wealth then it is well known that mean standard deviation ($\mu - \sigma$) indifference curves can be used to investigate choices among uncertain prospects no matter how these prospects are distributed. Alternatively for prospects which have uniquely defined two parameter distributions it is possible to utilize mean standard deviation analysis no matter what the utility function is. In portfolio theory attention is usually confined to two parameter symmetric stable distributions of which the normal is a special case since these distributions replicate themselves under weighted addition; portfolios thus have the same types of distributions as the assets which they consist of.

In this note the opposite transformation is considered: namely, which mean standard deviation indifference curves can be derived from expected utility preferences.

Provided the opportunity set is linear it is well known that starting from the case where only riskless assets are held a small increase in risk accompanied by an increase in expected return always results in higher expected utility. It follows from this that the slope of $\mu - \sigma$ indifference curves must be zero when $\sigma = 0$. This result is briefly recounted in section 2.

In section 3 it is demonstrated by counterexample that this is not the only restriction that can be placed on $\mu - \sigma$ indifference curves. The counterexample is concerned with changes in investment strategies over time. In a multiperiod world, where the returns at the end of one period are simply reinvested at the beginning of the next, outcomes are multiplicative. Prospects with simple two parameter distributions which replicate themselves under multiplication are therefore considered. It is first shown that for prospects whose outcomes are independent over time preferences derived from an expected utility approach must be such that the initial investment strategy is carried through. However, it can then be shown that there exist nicely behaved, convex downwards, $\mu - \sigma$ indifference curves satisfying the zero slope at $\sigma = 0$ restriction which imply that a person will switch strategies and which therefore cannot be derived from preferences over expected utility.

Finally section 4 contains some concluding remarks.

2. The slope of $\mu - \sigma$ indifference curves at $\sigma = 0$

Consider a person whose utility U depends on his wealth at the end of the period W_1, who invests a proportion π of his current wealth W_0 (> 0) in a risky asset and the rest in a safe asset. The return on each dollar invested in the risky asset is represented by the random variable $\tilde{X}$ and on the safe asset it is R. The person's expected utility is given by

$$EU[W_1] = EU\left[\left(\pi\tilde{X} + (1 - \pi)R\right)W_0\right]. \tag{1}$$

It follows that

$$\left.\frac{dEU}{d\pi}\right|_{\pi = 0} = (E\tilde{X} - R)W_0 U'[RW_0]. \tag{2}$$
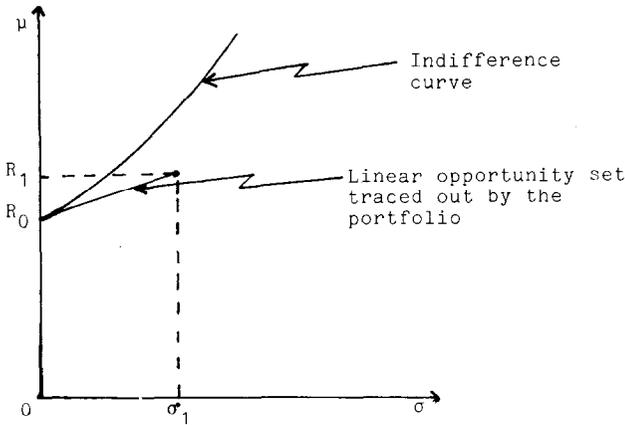
Fig. 1.

Provided the person has a positive marginal utility of wealth and $E\tilde{X} > R$ then expected utility is always increased by holding a sufficiently small amount of the risky asset in the portfolio rather than just holding the safe asset.

It follows from this that the slope of a $\mu - \sigma$ indifference curve at $\sigma = 0$ is zero. Suppose this were not the case as illustrated in fig. 1. Then it would be possible to construct a portfolio consisting of a safe asset with expected return R_0 and a risky asset with a higher expected return R_1 and standard deviation σ_1 which, no matter how small the proportion in the risky asset, would not allow the person to attain a higher indifference curve. This would contradict the result above so that the indifference curve could not be derived from an expected utility function. Hence only indifference curves with slope zero at $\sigma = 0$ can correspond to expected utility preferences.

It is important to stress that this result does not imply that a person will always be made better off by a small increase in risk which is accompanied by an increase in expected return. The result requires that the opportunity set be linear. If it is non-linear which could for example occur with certain types of non-linear income tax then it is possible for there to be increases in standard deviation accompanied by increases in expected return which do not make the person better off as illustrated in fig. 2.
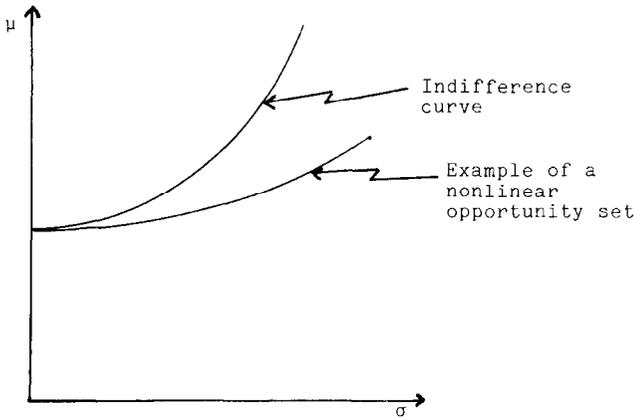
Fig. 2.

3. Other restrictions: The counterexample

Consider a security with two possible payoffs for each dollar invested: X with probability α and 0 with probability $1 - \alpha$. This is a two parameter distribution (X, α) and knowledge of μ and σ is sufficient to deduce X and α. Provided all the prospects considered have this same type of distribution it is therefore possible to represent expected utility preferences in $\mu - \sigma$ space. It can be seen that the distribution has the property that it replicates under multiplication so that if B and C have independent (X, α) distributions then so also does BC have an (X, α) distribution.

Suppose there are two assets which have this distribution: A_1 is a safe asset with parameters $(X_1,1)$ and A_2 is a risky asset with parameters (X_2, α_2). There are two periods and people's utility depends on their wealth at the end of the second period. The proceeds of investment in the first period are simply reinvested in the second period. The outcomes of the risky asset in the two periods are taken to be independent.

For simplicity consider a person who initially has wealth of $1. One possible course of action he could take is to invest this in the risky asset in both periods. The resulting prospect at the end of the second period will be denoted P_{22}^0. It consists of an uncertain payoff with a distribution of the above type with parameter (X_2^2, α_2^2). Similarly another possibility which will be denoted P_{21}^0 is to invest in the risky asset in the first period

and the safe asset in the second period. In this case the payoff has a distribution with parameters $(X_2 X_1, \alpha_2)$.

Suppose P_{21}^0 is preferred to P_{22}^0, it follows that

$$\alpha_2 U(X_2 X_1) + (1 - \alpha_2)U(0) > \alpha_2^2 U(X_2^2) + (1 - \alpha_2^2)U(0). \tag{3}$$

Dividing through by α_2 and rearranging this implies

$$U(X_2 X_1) > \alpha_2 U(X_2^2) + (1 - \alpha_2)U(0). \tag{4}$$

If the person undertakes P_{21}^0 and the return to the risky asset at the end of the first period is 0 then no further decision is possible and he will inevitably obtain 0 at the end of the second period. However, if the return is X_2 then two possibilities confronting the person are to invest in the safe asset in the second period as originally preferred or to switch and invest in the risky asset. The prospect in the former case is denoted P_{21}^1 and has a payoff with distribution $(X_2 X_1, 1)$ and in the latter case it is denoted P_{22}^1 and has a payoff with distribution (X_2^2, α_2). Eq. (4) shows that given (3), P_{21}^1 is preferred to P_{22}^1: the person will always prefer his original strategy to switching and investing in the risky asset.

To summarize expected utility preferences must be such that

$$P_{21}^0 \quad \text{is preferred to} \quad P_{22}^0 \to P_{21}^1 \quad \text{is preferred to} \quad P_{22}^1. \tag{5}$$

The four prospects P_{21}^0, P_{22}^0, P_{21}^1 and P_{22}^1 all have the same two parameter (X, α) distribution. It is therefore possible to consider the choice between them in $\mu - \sigma$ space.

Consider the family of indifference curves with the form

$$\mu = \sigma^2/8 + c, \tag{6}$$

where c is a constant representing the utility level of the indifference curve.

These indifference curves are nicely behaved, convex downwards and satisfy the requirement that the slope at $\sigma = 0$ is zero. However if $X_1 = 1$, $X_2 = 4$, and $\alpha_2 = 3/4$ it can be shown that (5) is contradicted so that the curves cannot be derived from any expected utility preferences.

With the parameter values given, the means and variances of the prospects and the value of c for the indifference curve passing through
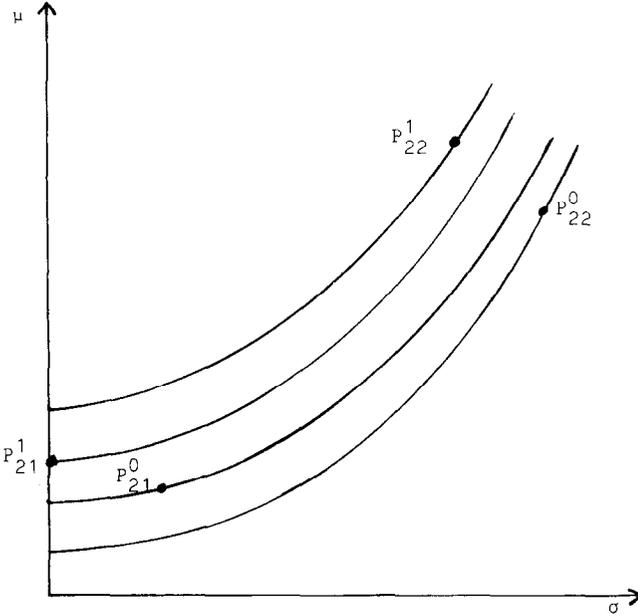
Fig. 3.

them are as follows:

	P_{21}^0	P_{22}^0	P_{21}^1	P_{22}^1
μ_2	3	9	4	12
σ^2	3	63	0	48
c	2 3/8	1 1/8	4	6

These prospects and the indifference curves passing through them are illustrated in fig. 3.

It can be seen that the indifference curves corresponding to (6) imply that

P_{21}^0 is preferred to P_{22}^0 but P_{22}^1 is preferred to P_{21}^1

so that (5) is violated and the indifference curves are not admissible.

This is only one example but clearly many other families of indifference curves could be ruled out because they imply switching in contradiction to (5).

4. Concluding remarks

This note has considered the question of which indifference curves in $\mu - \sigma$ space can be derived from expected utility preferences. It was shown that the requirement that the slope of indifference curves be zero at $\sigma = 0$ was not the only restriction: many families satisfying this requirement which are nicely behaved and convex downwards can also be ruled out because they imply behavior inconsistent with expected utility preferences. It is therefore not possible to draw arbitrary indifference curves in $\mu - \sigma$ space and assume there exist underlying expected utility preferences which correspond to these.