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NOTES

DISCOVERING PERSONAL PROBABILITIES WHEN UTILITY FUNCTIONS ARE UNKNOWN*

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Standard scoring rules require knowledge of the expert's utility function. This note shows how basic reference lottery tickets can be used to develop scoring rules which do not require this information.
(SCORING RULES; UNOBSERVABLE PREFERENCES)

For many decisions where risks are involved, the opinions of an expert about the distributions of random variables are helpful. A large literature on the elicitation of personal probabilities by means of scoring rules exists (see, e.g., Winkler 1967, Murphy and Winkler 1970, Stael von Holstein 1970, Savage 1971, and Matheson and Winkler 1976). Much of this literature assumes that the expert's utility is a linear function of money. If utility is nonlinear, then as Winkler and Murphy (1970) have shown, the standard scoring rules can be transformed provided the form of the function is known. However, the expert's utility function is often unknown. Murphy and Winkler (1970) have argued that in such cases the expert's utility function should be found either by direct interrogation or from observations of past behavior. Unfortunately, in many cases the information provided by these approaches may not be very reliable.

All this presumes that knowledge of the expert's utility function is necessary to elicit probabilities. An alternative solution is to design scoring rules which do not require this type of information. For example, Bhattacharya and Pfleiderer (1985) have shown that the use of an appropriate quadratic scoring rule gives incentives for a person to correctly announce the mean of any symmetric distribution, irrespective of their preferences.

It is well known that utility can be 'linearized' by the use of basic reference lottery tickets (Raiffa 1968). This involves making the final monetary reward have only two possible distinct values. Utility is then a linear function of payoffs defined in terms of probabilities of the high reward. The purpose of this note is to show how these techniques can be used to construct general scoring rules for eliciting experts' assessments even when their utility functions are unknown and cannot be observed.

The framework considered is as follows. A decision-maker requires an accurate assessment of the distribution of a discrete random variable X. The expert's assessment of the density function of this variable is denoted $f_E(x)$.

The decision-maker uses a scoring rule which consists of a payment $\pi(\cdot)$ to elicit the expert's assessment of the distribution. He knows the expert is an expected utility maximizer and has a utility function of the form

$$V = V(\pi(\cdot)) \qquad \text{with} \qquad V' > 0. \tag{1}$$

However, he does not know the exact form of this and cannot observe it.

Suppose initially there are just two possible outcomes x_1 and x_2 with

$$f_E(x_1) = p, \qquad f_E(x_2) = 1 - p. \tag{2}$$

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Let A be the event that x_1 is realized and B the event that x_2 is realized.

The decision-maker elicits the expert's views in the following way. First the expert announces his assessment of p. This announcement is denoted $\hat{p}$. If A occurs, a drawing is made from the reference distribution with density function

$$f_{RA}(r_A) = 2(1 - r_A) \quad \text{for} \quad 0 \le r_A \le 1, \qquad = 0 \quad \text{otherwise.} \tag{3}$$

The payment to the expert is

$$\pi(r_A) = 1 \quad \text{if} \quad 0 \le r_A \le \hat{p}, \qquad = 0 \quad \text{if} \quad \hat{p} \le r_A \le 1. \tag{4}$$

If B occurs a drawing is made from the reference distribution

$$f_{RB}(r_B) = 2r_B \quad \text{for} \quad 0 \le r_B \le 1, \qquad = 0 \quad \text{otherwise.} \tag{5}$$

The payment to the expert is

$$\pi(r_B) = 0 \quad \text{if} \quad 0 \le r \le \hat{p}, \qquad = 1 \quad \text{if} \quad \hat{p} \le r \le 1. \tag{6}$$

The reference distributions are independent of each other and of X. The expert's expected utility is then

$$EV(\hat{p}) = p\{V(1)F_{RA}(\hat{p}) + V(0)[1 - F_{RA}(\hat{p})]\} + (1 - p)\{V(0)F_{RB}(\hat{p}) + V(1)[1 - F_{RB}(\hat{p})]\} \tag{7}$$

where F_{RA} and F_{RB} are the distribution functions corresponding to (3) and (5).

The first order condition for choosing $\hat{p}$ to maximize expected utility is therefore

$$pf_{RA}(\hat{p}) = (1 - p)f_{RB}(\hat{p}). \tag{8}$$

Using (3) and (5) it follows this simplifies to

$$\hat{p} = p. \tag{9}$$

It can be seen that since (1) implies $V(1) > V(0)$, $EV''(\hat{p}) < 0$. Thus (9) is sufficient and gives the unique optimal solution. The scoring rule is strictly proper.

This method can be straightforwardly extended to the case where there are more than two x_i. Suppose there are n possible outcomes with $f_E(x_i) = p_i$ $(i = 1, \ldots, n)$. The procedure is now as follows. First the expert announces his assessment of the probabilities $\hat{p}_i$. Using the terminology of Matheson and Winkler (1976), a drawing is made from an independent weighting distribution defined over the random variable W. This has n possible outcomes w_j $(j = 1, \ldots, n)$ and density $f_W(w_j)$. The form of this distribution is arbitrary except that $f_W(w_j) > 0$ for all w_j. It is specified by the decision maker. The random variable X is then determined. If the outcomes of X and W are such that $i = j$, event A is said to have occurred. A drawing from the reference distribution with density function (3) is made. The payments are as in (4) with $\hat{p}_i$ replacing $\hat{p}$. If $i \neq j$ event B is said to have occurred and a drawing is made from the reference distribution with density function (5). The payments are as in (6) with $\hat{p}_i$ again replacing $\hat{p}$. It can then be shown as above that $\hat{p}_i = p_i$ $(i = 1, \ldots, n)$ is the unique optimal solution to the expert's optimization problem so that the scoring rule is again strictly proper.[1]

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