Interbank Market Liquidity and Central Bank Intervention*

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Abstract

We develop a simple model of the interbank market where banks trade a long term, safe asset. When there is a lack of opportunities for banks to hedge idiosyncratic and aggregate liquidity shocks, the interbank market is characterized by excessive price volatility. In such a situation, a central bank can implement the constrained efficient allocation by using open market operations to fix the short term interest rate. It can be constrained efficient for banks to hoard liquidity and stop trading with each other if there is sufficient uncertainty about aggregate liquidity demand compared to idiosyncratic liquidity demand.

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1 Introduction

Interbank markets are among the most important in the financial system. They allow liquidity to be readily transferred from banks with a surplus to banks with a deficit. They are the focus of central banks' implementation of monetary policy and have a significant effect on the whole economy. Under normal circumstances the interbank markets, especially the short term ones, work rather well. On occasion, however, such as in the crisis that started in the summer of 2007, interbank markets stop functioning well inducing central banks to intervene massively in order to try to restore normal conditions.

Despite their apparent importance, interbank markets have received relatively little attention in the academic literature. The purpose of this paper is to develop a simple theoretical framework for analyzing interbank markets and how the central bank should intervene. Our analysis is based on a standard banking model developed in Allen and Gale (2004a, 2004b) and Allen and Carletti (2006, 2008). There are two periods in the usual way. Banks can hold one-period liquid assets or two-period long term assets with a higher return. All assets are risk free in the sense that their promised payoffs are always paid. Banks face uncertain liquidity demands from their customers at the end of the first period. We distinguish between two types of uncertainty concerning banks' liquidity needs. The first is *idiosyncratic uncertainty* that arises from the fact that for any given level of aggregate demand for liquidity there is uncertainty about which banks will face the demand. The basic role of interbank markets is to allow reallocations of liquidity from banks with an excess to banks with a deficit. The second is the *aggregate uncertainty* that is due to the fact that the overall level of the demand for liquidity that banks face is stochastic.

We start with the analysis of the optimal portfolio of assets and payments that a planner who can transfer liquidity costlessly would implement. The planner is constrained in the same way as banks to offer deposit contracts where the payment at the end of the first period cannot be made contingent on the aggregate demand for liquidity in the banking system or the bank's individual liquidity demand. The resulting optimal allocation is termed the *constrained efficient allocation* because of this constraint to use deposit contracts.

We next consider the operation of an interbank market where banks can buy and sell the long term asset at the end of the first period. Since all assets are risk free in our model, there is no difference between selling the long asset and using it as collateral in a repurchase agreement. For ease of exposition, we consider outright sales of assets. The interbank market allows reallocations of liquidity between banks that depend on the realizations of the idiosyncratic and aggregate liquidity shocks. We focus on situations where the uncertainty concerning liquidity demand is not sufficient to cause banks to fail. In other words, banks find it optimal to keep enough liquidity to insure themselves against the high aggregate liquidity shock. The aggregate uncertainty about liquidity demand leads to volatile equilibrium prices for the long asset at the end of the first period, or equivalently interest rates. The intuition hinges on the simple fact that prices in the interbank market have to adjust to satisfy the market clearing condition and to provide banks with the appropriate incentives to keep the necessary liquidity initially. When the aggregate liquidity demand turns out to be low (that is, in the good state), there is an excess supply of aggregate liquidity at the end of the first period. The price of the long term asset is bid up to the level where the return during the second period is the same for both assets so that banks will be willing to hold both of them. The high price in the good state implies that prices have to fall in the bad state, that is when the high aggregate liquidity shock is realized, in order for banks to be willing to hold both the short and the long term assets initially. If this was not the case, the long asset would dominate the short asset and banks would not hold any liquidity to start with. Given that consumers are risk averse, this price volatility is inefficient because it leads to consumption volatility thus preventing the implementation of the constrained efficient allocation.

The main result of the paper is to show that the introduction of a central bank that engages in open market operations to fix the price of the long asset at the end of the first period (or equivalently fix the short term interest rate) removes the inefficiency associated with a lack of hedging opportunities. This intervention allows the banks to implement the constrained efficient allocation.

To see how this occurs it is helpful to consider two special cases. The first is where there is just idiosyncratic liquidity risk and no aggregate risk. Provided the central bank engages in the right open market operations and fixes the price in the interbank market at the end of the first period at the appropriate level, banks with a high liquidity demand will be able to sell their holdings of the long term asset to raise liquidity. The banks with low liquidity demand at the end of the first period are happy to buy the long asset and provide liquidity to the market because they need payoffs at the end of the second period to meet their needs then. The second special case is where there is no idiosyncratic uncertainty but there is aggregate uncertainty about liquidity demand. Here the central bank must fix the price by engaging in open market operations. In particular, it needs to remove excess liquidity from the banks by selling the long asset when aggregate liquidity demand is low. It can do this by selling government securities that replicate the long asset that are funded through lump sum taxes on late consumers at the final date. The optimal intervention by the central bank when there is both idiosyncratic and aggregate uncertainty combines the two policies in the special cases. The central bank must fix the price at the appropriate level that allows banks to reallocate liquidity from those with low idiosyncratic shocks to those with high ones. At the same time it must use open market operations to control the aggregate liquidity in the market to fix the price. We show that achieving both objectives simultaneously is possible and the constrained efficient allocation can be implemented. This result is in line with the argument of Goodfriend and King (1988) that open market operations are sufficient to address pure liquidity risk on the interbank market.

One of the implications of our model is that even when the constrained efficient allocation is being implemented by the policies of the central bank, an increase in aggregate uncertainty can cause banks to stop using the interbank markets to trade with each other. The banks hoard liquidity because they may need it to meet high aggregate demand. When aggregate demand is low, however, they have enough liquidity to deal with variations in idiosyncratic demand and as a result the banks stop trading with each other. At least in the context of the model considered here, this cessation of trade does not have consequences on the banks' ability to remain active. There is no need for central banks to intervene since the liquidity hoarding is consistent with constrained efficiency.

The basic problem in our model that leads to a need for central bank intervention is that financial markets are incomplete. In particular, banks are unable to hedge the idiosyncratic and aggregate liquidity shocks that they face. We consider how complete markets would operate and allow these risks to be hedged. There are many forms that such complete markets could take. We consider how markets for Arrow securities where all trades are made at the initial date allow the constrained efficient allocation to be implemented. This involves a large number of securities being issued and traded. In practice, the costs of issuance and of the infrastructure for trading securities to implement this would be prohibitive.

Finally, we consider the multi-period case. Instead of three dates there is now an infinite horizon. We show how central bank intervention combined with a tax and transfer scheme can implement the constrained efficient allocation for the tractable case of idiosyncratic risk.

A number of other papers focus on inefficiencies due to the incompleteness of interbank markets. Freixas et al. (2009) assume that there are two possible distributions of idiosyncratic shocks across the banking system. This leads to a multiplicity of equilibria. Their main result is to show that the government can implement the constrained efficient allocation by setting interest rates that depend on the pattern of idiosyncratic risk shocks. Allen and Gale (2000) show how incompleteness in links between banks in the interbank markets can lead to contagion.

In addition to the incompleteness of markets that we focus on, asymmetric information, monopoly power, and various other imperfections can lead to problems in interbank markets. Heider et al. (2008) focus on the credit risk problem that asymmetric information introduces when a bank's probability of default cannot be directly observed by outsiders. If the problem is small there is full participation. If credit risk is significant for some banks and this is reflected in a higher interest rate for all banks, safer borrowers drop out of the market. When the adverse selection problem is severe the market breaks down either because lenders prefer to hold on to their funds or because borrowers find it too expensive to borrow. Other analyses of problems in the interbank market arising from asymmetric information of various kinds include Bhattacharya and Gale (1987), Flannery (1996), and Freixas and Jorge (2008).

Acharya et al. (2008) model the interbank markets as being characterized in times of crisis by moral hazard, asymmetric information, and monopoly power. In their model, a bank with surplus liquidity is able to bargain with a bank that needs liquidity to keep funding projects and is able to extract all the surplus. The authors provide a number of historical examples where some banks had monopoly power over others during a crisis. Repullo (2005) considers the poor functioning of interbank markets due to banks' free riding on central bank liquidity. Other papers where markets for liquidation do not work properly or are absent and some form of government intervention may improve efficiency are Holmstrom and Tirole (1998), Gorton and Huang (2004, 2006), Diamond and Rajan (2005, 2008), and Acharya and Yorulmazer (2008).

The paper proceeds as follows. Section 2 describes the model. The constrained efficient allocation is derived in Section 3. We then consider the operation of an interbank market for the long asset in Section 4. The role of the central bank is analyzed in Section 5. Section 6 considers how complete markets would implement the constrained efficient allocation. A multi-period version of the model is presented in Section 7. Finally, Section 8 concludes.

2 The model

The model is based on Allen and Gale (2004a, 2004b) and Allen and Carletti (2006, 2008). There are three dates $t = 0, 1, 2$ and a single good that can be used for consumption or investment at each date. The banking sector consists of a large number of competitive institutions.

There are two securities, one *short* and one *long*. Both are risk free. The short security is represented by a storage technology: one unit at date t produces one unit at date $t+1$. The long security is a simple constant-returns-to-scale investment technology that takes two periods to mature: one unit invested in the long security at date 0 produces $R > 1$ units of the good at date 2 so it is more productive than the short security.

We assume there is a market for liquidating the long asset at date 1. Each unit can be sold for P. Participation in this market is limited: financial institutions such as banks can buy and sell in the asset market at date 1 but individual consumers cannot.

Banks raise funds from depositors, who have an endowment of one unit of the good at date 0 and none at dates 1 and 2. Depositors are uncertain about their preferences: with probability λ they are *early consumers*, who only value the good at date 1, and with probability $1-\lambda$ they are *late consumers*, who only value the good at date 2. There are two types of uncertainty that determine λ:

$$\lambda_{\theta i} = \alpha_i + \varepsilon\theta$$

where $\alpha_i, i = H, L$ is an *idiosyncratic* bank-specific shock and $\theta = 0, 1$ is an *aggregate shock*. Except where otherwise stated we assume $\varepsilon > 0$. For simplicity, we assume that the random variables α_i and θ have two-point supports. That is:

$$\alpha_H = \bar{\alpha} + \eta \quad \text{w. pr. } \tfrac{1}{2},$$
$$\alpha_L = \bar{\alpha} - \eta \quad \text{w. pr. } \tfrac{1}{2},$$

where $0 < \alpha_L \leq \alpha_H < 1$; and

$$\theta = \begin{cases} 0 & \text{w. pr. } \pi, \\ 1 & \text{w. pr. } (1-\pi), \end{cases}$$

where $0 < \pi < 1$. Because there are only two values of θ, the price at which the long asset can be sold at date 1 takes at most two values, P_θ, where $\theta = 0, 1$.

Uncertainty about time preferences generates a preference for liquidity and a role for the intermediary as a provider of liquidity insurance. The utility of consumption is represented by a utility function $u(c)$ with the usual properties. Expected utility at date 0 is given by

$$EU = E\left[\lambda_{\theta i} u(c_1) + (1 - \lambda_{\theta i}) u(c_2)\right],$$

where c_t denotes consumption at date $t = 1, 2$.

Banks compete by offering deposit contracts to consumers in exchange for their endowments and consumers respond by choosing the most attractive of the contracts offered. Free entry ensures that banks offer deposit contracts that maximize consumers' welfare and earn zero profits in equilibrium. Otherwise, a bank could enter and make a positive profit by offering a more attractive contract.

There is no loss of generality in assuming that consumers deposit their entire endowment in a bank at date 0 since the bank can do anything the consumers can do. The bank invests y units per capita in the short asset and $1 - y$ units per capita in the long asset and offers each consumer a deposit contract, which allows the consumer to withdraw either d units at date 1 or the residue of the bank's assets at date 2 divided equally among the remaining depositors.

A consumer's type is private information. An early consumer cannot misrepresent his type because he needs to consume at date 1; but a late consumer can claim to be an early consumer, withdraw d at date 1, store it until date 2 and then consume it. The deposit contract is *incentive compatible* if and only if the residual payment to late consumers at date 2 is at least d. Since the late consumers are residual claimants at date 2, it is possible to give them at least d units of consumption if and only if

$$\lambda d + (1 - \lambda) d \frac{P_\theta}{R} \leq y + P_\theta (1 - y). \tag{1}$$

The left hand side is a lower bound for the present value of consumption at date 1 when early consumers are given d and late consumers are given at least d. The first term is the consumption given to the early consumers. The second term is the present value of the $(1-\lambda)d$ given to the late consumers. The price of the long asset at date 1 is P_θ and this long asset pays off R at date 2 so the date 1 present value of 1 unit of consumption at date 2 is P_θ/R. The right hand side is the value of the bank's portfolio. The bank has y in the short asset and $(1-y)$ of the long asset worth P_θ per unit. Thus, condition (1) is necessary and sufficient for the deposit contract d to satisfy incentive compatibility and the budget constraint simultaneously. If (1) was not satisfied the late consumers would receive less than the early consumers if they left their funds in the bank so they would find it optimal to withdraw and there would be a run. The inequality in (1) is referred to as the *incentive constraint* for short. We restrict our analysis to the set of parameters where this constraint is satisfied for the optimal contract. We also assume that bank runs do not occur when the constraint is satisfied. In other words, late consumers will withdraw at date 2 as long as the bank can satisfy the incentive constraint.

All uncertainty is resolved at the beginning of date 1. In particular, depositors learn whether they are early or late consumers and the values of α and θ are determined. While each depositor's individual realization of liquidity demand is observed only by them, α and θ are publicly observed.

3 The constrained efficient allocation

The planner invests in a portfolio of the short and long asset. The proceeds are distributed directly to early and late consumers. The planner does not need to worry about idiosyncratic liquidity risk since the H group with α_H early consumers will be balanced by the L group with α_L early consumers. It is possible to just plan for $\bar{\alpha}$ early consumers in total.

The planner provides early consumers with consumption d and late consumers receive c_{20} when $\theta = 0$ and c_{21} when $\theta = 1$. Using the notation $\lambda_0 = \bar{\alpha}$ and $\lambda_1 = \bar{\alpha} + \varepsilon$ the planner's problem can be written

$$\max_{y,d} \quad \pi\left[\lambda_0 u(d) + (1-\lambda_0)u(c_{20})\right] + (1-\pi)\left[\lambda_1 u(d) + (1-\lambda_1)u(c_{21})\right]$$

$$\text{s.t.} \quad \begin{aligned} &\lambda_0 d \leq y \\ &(1-\lambda_0)c_{20} = y - \lambda_0 d + (1-y)R \\ &\lambda_1 d \leq y \\ &(1-\lambda_1)c_{21} = y - \lambda_1 d + (1-y)R \\ &0 \leq d, 0 \leq y \leq 1. \end{aligned} \tag{2}$$

The first two constraints represent the physical constraints on consumption at the two dates in state $\theta = 0$. At date 1 it is not possible to consume more output than exists. At date 2 the $(1-\lambda_0)$ late consumers consume c_{20}. The total amount available for them is whatever is not consumed at date 1, $y - \lambda_0 d$, together with what is produced at date 2, $(1-y)R$. Similarly for the next two constraints for state $\theta = 1$. Finally, we have the usual constraints on d and y.

We denote the optimal solution to this problem y^* and d^*. Note that it cannot be the case at the optimum that $y^* > \lambda_1 d^*$. If this inequality held, it would be possible to increase expected utility by holding d constant and reducing y since $R > 1$. Hence at the optimum

$$y^* = \lambda_1 d^* = (\bar{\alpha} + \varepsilon)d^* > \lambda_0 d^*. \tag{3}$$

Thus the planner's problem is to choose d to

$$\max \quad \pi\left[\lambda_0 u(d) + (1-\lambda_0)u(\tfrac{\varepsilon d + (1-\lambda_1 d)R}{1-\lambda_0})\right] + (1-\pi)\left[\lambda_1 u(d) + (1-\lambda_1)u(\tfrac{(1-\lambda_1 d)R}{1-\lambda_1})\right].$$

This gives the first order condition that determines d^* as

$$\pi\left[\lambda_0 u'(d^*) + u'(\frac{\varepsilon d^* + (1-\lambda_1 d^*)R}{1-\lambda_0})(\varepsilon - \lambda_1 R)\right] + (1-\pi)\left[\lambda_1 u'(d^*) + u'(\frac{(1-\lambda_1 d^*)R}{1-\lambda_1})(-\lambda_1 R)\right] = 0. \tag{4}$$

Differentiating a second time with respect to d it can be easily checked that the second derivative is negative since $u'' < 0$. Thus the constrained efficient allocation is unique.

We turn next to consider the allocation when there is an interbank market at date 1 that allows banks to buy and sell the long asset.

4 Interbank markets

Suppose there is an interbank market at date 1 for trading the long asset at price P_θ. Banks can buy the long and short assets at date 0 for a price of 1 and at date 1 it is also possible to buy the short asset at a price of 1. This set of markets is *incomplete* in that it is not possible to completely hedge the risk of aggregate and idiosyncratic liquidity shocks. It is shown that this incompleteness leads to price volatility.

Once the banks have received the funds of depositors at date 0 they can use them to obtain the two assets. In addition to choosing their portfolio of y in the safe asset and $1 - y$ in the long asset at date 0, they must also set the amount d that depositors can withdraw at date 1. When they know the level of aggregate liquidity demand and their own idiosyncratic liquidity shock at date 1, they can use the interbank market to buy or sell the long asset.

The consumption of a bank's depositors at date 2 depends on the aggregate state since this determines P_θ. It also depends on the idiosyncratic shock that strikes the bank since this determines the proportions λ of early and $1 - \lambda$ of late consumers. In particular, for y and d such that the incentive constraint (1) is satisfied so bankruptcy is avoided

$$c_{2\theta i} = \frac{\left[1 - y + \frac{y - \lambda_{\theta i} d}{P_\theta}\right] R}{1 - \lambda_{\theta i}}, \tag{5}$$

for $\theta = 0, 1$ and $i = H, L$. The term in square brackets represents the amount of long asset held by the bank at date 2. The $(1 - y)$ term is the initial holding of the long asset purchased at date 0. If $y - \lambda_{\theta i} d > 0$ then excess liquidity at date 1 can be used to purchase the long asset. The amount of the long asset that can be purchased is $(y - \lambda_{\theta i} d)/P_\theta$. If $y - \lambda_{\theta i} d < 0$ then it is necessary to sell the long asset held by the bank in the market at date 1 to fund the shortfall of liquidity. In this case $(y - \lambda_{\theta i} d)/P_\theta$ represents the amount that must be sold. Each unit of the long asset pays off R and the total payoff must be split between the $(1 - \lambda_{\theta i})$ late consumers.

The problem each bank solves at date 0 is to choose y and d to

$$\begin{array}{ll} \max & \frac{1}{2}\{\pi[\lambda_{0H} u(d) + (1-\lambda_{0H})u(c_{20H}) + \lambda_{0L} u(d) + (1-\lambda_{0L})u(c_{20L})] \\ & +(1-\pi)[\lambda_{1H} u(d) + (1-\lambda_{1H})u(c_{21H}) + \lambda_{1L} u(d) + (1-\lambda_{1L})u(c_{21L})]\} \\ \text{s.t.} & 0 \leq d, 0 \leq y \leq 1, \end{array} \tag{6}$$

taking prices P_0 and P_1 as given. The first order conditions for this with respect to the choice of y and d are:

$$\pi\left(\frac{1}{P_0} - 1\right)[u'(c_{20H}) + u'(c_{20L})] + (1-\pi)\left(\frac{1}{P_1} - 1\right)[u'(c_{21H}) + u'(c_{21L})] = 0 \tag{7}$$

$$\begin{aligned} &[\bar{\alpha} + (1-\pi)\varepsilon]u'(d) - \frac{1}{2}R[\frac{\pi}{P_0}\left(\alpha_H u'(c_{20H}) + \alpha_L u'(c_{20L})\right) \\ &+\frac{1-\pi}{P_1}\left((\alpha_H + \varepsilon)u'(c_{21H}) + (\alpha_L + \varepsilon)u'(c_{21L})\right)] = 0. \end{aligned} \tag{8}$$

Now since the aggregate measure of banks is 1 the aggregate amount of liquidity is y. There are two aggregate states of demand for liquidity, $\theta = 0$ where $\lambda_0 = \bar{\alpha}$ and $\theta = 1$ where $\lambda_1 = \bar{\alpha} + \varepsilon$. Within each of these states, half of the banks have high idiosyncratic demand, α_H, for liquidity. In this case they can liquidate part of their holdings of the long asset in the interbank market to meet the high demand for liquidity from their customers. The other half of the banks have low liquidity demand, α_L. They are willing to use their excess liquidity to buy the long asset in the interbank market. Since, we are assuming bankruptcy is not optimal, we know that the aggregate amount of liquidity y must be sufficient to cover demand in state $\theta = 1$ so we have $y \geq \lambda_1 d = (\bar{\alpha} + \varepsilon)d$. Since $\varepsilon > 0$ this implies that $y > \lambda_0 d = \bar{\alpha} d$. As a result there is excess liquidity at date 1 in state $\theta = 0$. In order for the interbank market to clear it is necessary that

$$P_0 = R. \tag{9}$$

In this case banks are willing to hold both the long asset and the excess liquidity between dates 1 and 2. If $P_0 < R$ they will be willing to hold only the long asset while if $P_0 > R$ they will be willing to hold only the short asset. Hence P_0 must be given by (9).

Notice that if $y > (\bar{\alpha} + \varepsilon)d$ a similar argument would hold for state $\theta = 1$ and we would have $P_1 = R$. But this cannot be an equilibrium given $P_0 = R$ because then the long asset would dominate the short asset between dates 0 and 1 and there would be no investment in the short asset at all. Hence equilibrium requires

$$y = \lambda_1 d = (\bar{\alpha} + \varepsilon)d. \tag{10}$$

It then follows that P_1 must be such that banks are willing to hold both the long and short asset between dates 0 and 1. To find the equilibrium value of P_1we substitute for P_0 and y using (9) and (10) and solve the first order conditions (7) and (8) for P_1 and d. Note that it must be the case that $P_1 < 1$ otherwise the long asset would dominate the short asset. The prices P_0 and P_1 ensure that the banks are willing to hold both the long and the short asset between dates 0 and 1. Price volatility is necessary to provide the correct incentives in both periods.

One issue concerns the circumstances under which banks will stop trading with each other in the interbank market. The essential purpose of the interbank market is to allow banks with high liquidity needs to sell the long asset and obtain liquidity from banks with low liquidity needs. If the amount of liquidity the banks hold to deal with aggregate uncertainty is large enough then in state $\theta = 0$ when aggregate liquidity demand is low, they may not need to go to the interbank market to raise liquidity since they hold so much internally anyway. In particular, they will not need to enter the market in state $\theta = 0$ when they are an H bank if $\lambda_1 d > \lambda_{0H} d$. Using $\lambda_1 = \bar{\alpha} + \varepsilon$ and $\lambda_{0H} = \bar{\alpha} + \eta$ it can be seen that this simplifies to

$$\varepsilon > \eta.$$

Thus banks will stop trading with each other if aggregate uncertainty is large enough relative to idiosyncratic uncertainty.

5 Central bank intervention

In this section we introduce a central bank that can engage in open market operations. In practice central banks hold large portfolios of securities that they use to intervene in the markets. They buy or sell securities to affect the amount of liquidity held by banks. In recent years the focus of most central banks has been to use open market operations to target the interest rate in the overnight interbank market. In order to explain how the central bank can implement the constrained efficient allocation, we proceed in three steps. The first is to show how this can be done when there is only idiosyncratic risk. The second is to show how open market operations can be used when there is just aggregate risk. Finally, we consider the two types of risk together.

5.1 Idiosyncratic liquidity risk alone: $\eta > 0, \varepsilon = 0$

We start with the simplest case where there is only idiosyncratic risk in liquidity demand, and no aggregate risk so $\eta > 0, \varepsilon = 0$. We show that by holding an appropriate portfolio of securities and engaging in open market operations and fixing the price of the long asset at $P = 1$, the central bank can ensure that the constrained efficient allocation y^*, d^*, and c_2^* can be implemented.

Since there is no aggregate uncertainty we know that it is efficient to use the short asset to provide early consumption and the long asset to provide late consumption so

$$y^* = \bar{\alpha}d^* = \lambda d^*; c_2^* = \frac{(1-y^*)R}{1-\lambda}.$$

Our approach is to show that the banks can provide their depositors with this allocation provided the central bank adopts the optimal policy.

Let X_0 denote the lump sum tax that is imposed by the government at date 0 to fund the portfolio for open market operations of the central bank. The central bank uses these funds to buy the short term asset at date 0. Depositors then have $1 - X_0$ remaining that they put in the banks. Suppose the banks hold $y^* - X_0$ in the short asset and $1 - y^*$ in the long asset between dates 0 and 1.

At date 1, half the banks have $\lambda_H = \bar{\alpha} + \eta$ early consumers while the other half have $\lambda_L = \bar{\alpha} - \eta$. Banks of type $i, i = H, L$, require total liquidity of $\lambda_i d^*$. They have liquidity $y^* - X_0$ so their net need is $y^* - X_0 - \lambda_i d^*$. If this is positive they use it to buy the long term asset. If it is negative they sell the long term asset to raise the needed liquidity. The central bank sets $P = 1$ and supplies its holding of the short asset X_0 to the market and receives X_0 of the long asset. The interbank market clears since

$$\frac{1}{2}(y^* - X_0 - \lambda_H d^*) + \frac{1}{2}(y^* - X_0 - \lambda_L d^*) + X_0 = y^* - \bar{\alpha}d^* = 0.$$

A bank of type i now has $1 - y^* + y^* - X_0 - \lambda_i d^* = 1 - X_0 - \lambda_i d^*$ in the long asset. At date 2 these holdings allow the banks to provide a payout to their late consumers of

$$\beta_{2i} = \frac{(1 - X_0 - \lambda_i d^*)R}{1 - \lambda_i}.$$

At date 2, the central bank has $X_0 R$. These funds are remitted to the government and the government then distributes them as a lump sum grant to all the $1 - \lambda$ late consumers of

$$\gamma_2 = \frac{X_0 R}{1-\lambda}.$$

In order to implement the constrained efficient allocation, it is necessary that the sum of these payouts is equal to c_2^*. Thus we need

$$\beta_{2i} + \gamma_2 = \frac{(1 - X_0 - \lambda_i d^*)R}{1-\lambda_i} + \frac{X_0 R}{1-\lambda} = c_2^* = \frac{(1-y^*)R}{1-\lambda}.$$

It can easily be checked that

$$X_0 = 1 - d^*$$

solves this equation for any value of λ_i since $y^* = \lambda d^*$. The central bank policy described allows banks to implement the constrained efficient allocation. By holding a portfolio of $1 - d^*$ of the short asset and setting $P = 1$ the central bank effectively allows the banks to be indifferent to having early or late consumers. They give early consumers d^* and late consumers

$$\beta_{2i} = \frac{(1 - X_0 - \lambda_i d^*)R}{1-\lambda_i} = d^* R.$$

The present value of the payments to early and late consumers is the same so that the size of a bank's idiosyncratic shock becomes irrelevant. It can be demonstrated that y^* and d^* maximize the expected utility of the bank's depositors subject to the usual constraints by checking the first order conditions. Since this is the best feasible allocation, it is individually optimal for each bank to choose this. Note also that we have demonstrated the optimality of central bank intervention for two groups H and L but it is clear that the same result will hold for an arbitrary distribution of idiosyncratic shocks.

In the discussion so far, we have used the terminology that $X_0 = 1 - d^*$ is a lump sum tax that finances the central bank's holdings of the short asset from date 0 to date 1. The central bank uses these holdings to purchase the long asset and holds it between date 1 and date 2. At date 2 the payoffs from this long asset are paid to the government and they are used to finance a lump sum grant to late consumers. This terminology presumes $d^* < 1$. This will be the case if, for example, constant relative risk aversion is sufficiently below 1. However, if, for example, constant relative risk aversion is sufficiently above 1, then $d^* > 1$. In this case all the signs are reversed. Instead of a lump sum tax, X_0 represents a lump sum grant of an asset that replicates the short asset. In other words, the asset is "money". The central bank has a liability rather than an asset at date 0. At date 1 the central bank sells an asset that replicates the long asset to remove the money from the banking system. At date 2 there is a lump sum tax to make the payment on the government asset replicating the long asset. Thus the theory presented provides a rich model of central bank intervention in the interbank markets.

5.2 Aggregate liquidity risk alone: $\eta = 0, \varepsilon > 0$

Next consider what happens with no idiosyncratic risk but with positive aggregate risk. In this case the constrained efficient allocation can be implemented by having the central bank engage in open market operations to fix $P_0 = P_1 = 1$.

Let y^*, d^*, and $c^*_{2\theta}$ denote the allocation. We know from Section 3 that

$$
\begin{aligned}
y^* &= \lambda_1 d^* = (\lambda_0 + \varepsilon) d^*, && (11)\\
c^*_{20} &= \frac{y^* - \lambda_0 d^* + (1 - y^*)R}{1 - \lambda_0} = \frac{\varepsilon d^* + (1 - y^*)R}{1 - \lambda_0}, && (12)\\
c^*_{21} &= \frac{(1 - y^*)R}{1 - \lambda_1}. && (13)
\end{aligned}
$$

As usual we show that it is feasible for the banks to implement the constrained efficient allocation. Given that this provides the highest level of expected utility that is feasible, then it will be optimal for them to do so.

At date 0 the banks hold y^* of the short asset and $1 - y^*$ of the long asset.

At date 1 in state $\theta = 0$ the central bank needs to drain liquidity to ensure $P_0 = 1$. The government issues $X_1 = \varepsilon d^*$ of debt at date 1 that pays R at date 2. Thus the total owed by the government on its debt at date 2 is $\varepsilon d^* R$. The debt is given to the central bank at date 1 to allow it to conduct open market operations. The central bank sells the government debt, which is equivalent to the long asset, for $P_0 = 1$. This removes the excess liquidity from the market and prevents the price of the long asset being bid up to $P_0 = R$. The central bank holds the liquidity of εd^* that it acquires until date 2.

After the central bank's open market operations, the banks own $1 - y^* + \varepsilon d^*$ of the long asset. At date 2 this allows them to pay to each of their $1 - \lambda_0$ late consumers

$$\beta_{20} = \frac{(1 - y^* + \varepsilon d^*)R}{1 - \lambda_0}.$$

The central bank ends up at date 2 with $X_1 = \varepsilon d^*$ of the short asset. The proceeds from these assets are assumed to be returned to the government. The government has resources of εd^* and owes $\varepsilon d^* R$ on its long term debt so it needs to impose a lump sum tax on each of the $1 - \lambda_0$ late consumers of

$$\gamma_{20} = \frac{\varepsilon d^*(1 - R)}{1 - \lambda_0}.$$

Hence late consumers receive

$$\beta_{20} + \gamma_{20} = \frac{\varepsilon d^* + (1 - y^*)R}{1 - \lambda_0} = c^*_{20}$$

as required to implement the constrained efficient allocation.

In state $\theta = 1$ each bank pays d^* to λ_1 early consumers at date 1. This implies that the banks have no short asset but only $1 - y^*$ of the long asset after this. The central bank announces that it is setting $P_1 = 1$ but does not need to actively conduct open market operations to ensure this. Each bank pays $(1 - y^*)R$ to the $1 - \lambda_1$ late consumers so each receives a payoff of

$$\beta_{21} = \frac{(1 - y^*)R}{1 - \lambda_1} = c^*_{21}.$$

This demonstrates that the banks can implement the constrained efficient allocation in the case of aggregate liquidity shocks given the open market operations of the central bank described. It can again be shown that y^* and d^* satisfy each bank's optimization problem. Since this allocation gives the highest expected utility that is feasible it is optimal for the banks to implement it.

5.3 Idiosyncratic and aggregate liquidity risk: $\eta > 0, \varepsilon > 0$

We continue to denote the constrained efficient allocation y^*, d^*, and $c^*_{2\theta}$ as in (11)-(13). With both idiosyncratic and aggregate risk the open market operations of the central bank necessary to implement the constrained efficient allocation combine the elements from the two cases alone. At date 0 the government imposes a lump sum tax of X_0 and gives it to the central bank. The central bank uses it to fund a portfolio of the short asset. At date 1 in state $\theta = 0$ the central bank fixes the price of the long asset at $P_0 = 1$ by removing liquidity from the market. In order to do this it uses government securities that pay R at date 2 and sells them at $P_0 = 1$. The quantity of government securities issued at date 1 is denoted X_1. In order to ensure the price of the long asset can be successfully fixed, it is necessary that

$$X_0 + X_1 = \lambda_1 d^* - \lambda_0 d^* = \varepsilon d^*. \tag{14}$$

This ensures that all of the excess liquidity is drained from the banks into the central bank and there is no pressure to push up P_0 in state $\theta = 0$.

The date 2 interest paid on the securities issued at date 1 is paid from the short asset held by the central bank. If any is left over then this is paid out as a lump sum grant to late consumers. If the resources of the central bank are insufficient then the shortfall is covered by a lump sum tax. In state $\theta = 1$ the central bank needs to supply liquidity to the market because there is just enough liquidity in the financial system $y^* = \lambda_1 d^*$ to satisfy the aggregate demand in state $\theta = 1$. If the central bank did not release this liquidity the banks would not have enough to satisfy the early consumers' demands. It does this by using the short asset it holds to buy the long asset. This enables it to fix the price at $P_1 = 1$.

We next determine the choice of X_0 and X_1 and the banks' portfolio that implements the constrained efficient allocation. At date 0 after the lump sum tax of X_0 the depositors have $1 - X_0$ remaining and they deposit this in the banks. The banks choose a portfolio of $y^* - X_0$ in the short asset and $1 - y^*$ in the long asset.

At date 1 in state $\theta = 0$ the i banks need liquidity $\lambda_{0i} d^*$ to satisfy the demands of their early consumers. They have $y^* - X_0$. They therefore sell $\lambda_{0i} d^* - (y^* - X_0)$ of the long asset. (Note that if $\lambda_{0i} d^* < (y^* - X_0)$, they are buying the long asset, the total supply of which includes that issued by the central bank.) The amount of the long asset they have remaining is $1 - y^* - [\lambda_{0i} d^* - (y^* - X_0)] = 1 - \lambda_{0i} d^* - X_0$. At date 2 they are able to use the payoffs of these long term assets to give each of their $1 - \lambda_{0i}$ late consumers to provide a payout of

$$\beta_{20i} = \frac{(1 - \lambda_{0i} d^* - X_0)R}{1 - \lambda_{0i}}.$$

In addition to this payoff from their bank the late consumers receive a lump sum grant (or tax) from the government. The central bank has X_0 in cash from date 0. As explained above, at date 1 they issue $X_1 = \varepsilon d^* - X_0$ of securities that pay R at date 2. Thus at date 2 the total amount owed in interest is $X_1 R = (\varepsilon d^* - X_0)R$. The central bank holds the proceeds of the debt issue $X_1 = \varepsilon d^* - X_0$ in the short asset. In total they have $X_0 + \varepsilon d^* - X_0 = \varepsilon d^*$ of the short asset. This allows a lump sum grant to each of the $1 - \lambda_0$ late consumers of

$$\gamma_{20} = \frac{\varepsilon d^* - (\varepsilon d^* - X_0)R}{1 - \lambda_0} = \frac{X_0 R - \varepsilon d^*(R - 1)}{1 - \lambda_0}.$$

The amount received by each of the late consumers in the $i = H, L$ banks equals then $\beta_{20i} + \gamma_{20}$. In order to implement the constrained efficient allocation, it is necessary that this is equal to the constrained efficient allocation c^*_{20} so using (12) we have

$$\beta_{20i} + \gamma_{20} = \frac{(1 - \lambda_{0i} d^* - X_0)R}{1 - \lambda_{0i}} + \frac{X_0 R - \varepsilon d^*(R - 1)}{1 - \lambda_0} = c^*_{20} = \frac{\varepsilon d^* + (1 - y^*)R}{1 - \lambda_0}. \qquad (15)$$

As with just idiosyncratic risk it can be seen that $X_0 = 1 - d^*$ allows both H and L banks to implement the constrained efficient allocation.

It remains to show that $X_0 = 1 - d^*$ allows the banks to ensure early consumers receive d^* and late consumers receive c_{21}^* in state $\theta = 1$. Similarly to (15) it can be shown that late consumers receive

$$\beta_{21i} + \gamma_{21} = \frac{(1 - \lambda_{1i} d^* - X_0)R}{1 - \lambda_{1i}} + \frac{X_0 R}{1 - \lambda_1}. \tag{16}$$

The main difference here is in last term, which is the lump sum grant. As explained above, in state $\theta = 1$ the central bank at date 1 uses the short term asset to purchase X_0 of the long asset. This pays off a total of $X_0 R$ at date 2 to be distributed among the $1 - \lambda_1$ late consumers.

Again substituting $X_0 = 1 - d^*$ and using $y^* = \lambda_1 d^*$ it follows that

$$\beta_{21i} + \gamma_{21} = \frac{(1 - y^*)R}{1 - \lambda_1} = c_{21}^*.$$

Thus the central bank policy described allows banks to implement the constrained efficient allocation. Each bank's optimization problem is again satisfied.

Just as in Section 4, if aggregate uncertainty is sufficiently large relative to idiosyncratic uncertainty the banks will stop trading with each other in state $\theta = 0$. As there, the condition is that $\lambda_1 d > \lambda_{0H} d$ or equivalently $\varepsilon > \eta$. The difference here is that the H banks continue to trade with the central bank. The central bank sells long securities to the banks but that is the only trade that takes place. Since now these allocations where banks do not trade with each other in the interbank market are constrained efficient they cannot be improved on. Thus the observation that banks stop lending to each other does not necessarily mean there is a market failure or inefficiency.

6 Complete markets

In the model analyzed so far, markets are incomplete because it is not possible to hedge aggregate or idiosyncratic liquidity risk. In this section we consider the allocation that would occur with complete markets where liquidity risk can be hedged. This version of the model is a special case of that considered in Allen and Gale (2004a). They show that with complete markets and incomplete contracts of the type considered here the allocation is constrained efficient. In other words, a planner subject to the constraint of using a fixed payment in the first period cannot improve upon the complete markets allocation. Institutionally there are a number of ways that complete markets can be implemented. We focus on the simplest institutional structure where all trades occur at date 0. Other possibilities are discussed briefly at the end of the section.

Initially we will focus on aggregate risk and will introduce idiosyncratic risk at a later stage. For the moment, $\lambda_0 = \bar{\alpha}$ and $\lambda_1 = \bar{\alpha} + \varepsilon$.

So far we have assumed that assets are held by the bank. Since the assets are produced with constant returns to scale, with complete markets there will be zero profits associated with producing them. Therefore it does not matter which agents hold them. Let's suppose initially firms hold the assets and issue securities against them. Banks use the funds from deposits to buy these securities. We will model these securities in the form of Arrow securities where each security pays off 1 in a particular state and nothing in any of the other states. All of these Arrow securities are traded at date 0.

There are five aggregate states in total. At date 0 there is one state. There are two states $\theta = 0, 1$ at the two subsequent dates $t = 1, 2$ to give four further states (t, θ) for a total of five. We take consumption at date 0 as the numeraire with the price of the Arrow security paying off 1 unit of consumption in that state normalized at 1. The prices of the Arrow securities that pay off in the other states (t, θ) are denoted $p_{t\theta}$.

We can represent the short assets and the long asset by their payoffs in the five states $(0, 10, 11, 20, 21)$ as follows:

Asset	Payoffs	Zero-profit condition
Short asset from date 0 to 1	$(-1,1,1,0,0)$	$-1+p_{10}+p_{11}\leq 0$
" " " " 1 to 2 in state $\theta=0$	$(0,-1,0,1,0)$	$-p_{10}+p_{20}\leq 0$
" " " " 1 to 2 in state $\theta=1$	$(0,0,-1,0,1)$	$-p_{11}+p_{21}\leq 0$
Long asset from date 0 to 2	$(-1,0,0,1,1)$	$-1+p_{20}R+p_{21}R\leq 0$

If the zero profit condition is satisfied with an equality the asset is produced. If it is satisfied with a strict inequality it is not produced. To implement the constrained efficient allocation we have

$$-1+p_{10}+p_{11}=0; -p_{10}+p_{20}=0; -p_{11}+p_{21}<0; -1+p_{20}R+p_{21}R=0.$$

The problem of the representative bank is to use the Arrow security markets at date 0 to purchase the units of consumption to maximize its depositors' expected utility. The total amount of consumption it purchases is $\lambda_\theta d$ at date 1 and $(1-\lambda_\theta)c_{2\theta}$ at date 2 for $\theta=0,1$. The bank chooses d, c_{20}, and c_{21} to

$$\begin{array}{ll} \max & \pi\left[\lambda_0 u(d)+(1-\lambda_0)u(c_{20})\right]+(1-\pi)\left[\lambda_1 u(d)+(1-\lambda_1)u(c_{21})\right] \\ \text{s.t.} & \begin{array}{l} p_{10}\lambda_0 d+p_{20}(1-\lambda_0)c_{20}+p_{11}\lambda_1 d+p_{21}(1-\lambda_1)c_{21}=1 \\ 0\leq d, c_{20}, c_{21}. \end{array} \end{array} \tag{17}$$

The first line is depositors' expected utility. The second is the budget constraint in the date 0 markets. There is a single budget constraint because all transactions take place at date 0. The third line has the usual non-negativity constraints.

Denoting the Lagrange multiplier for the budget constraint μ, the first order conditions for the choice of d, c_{20}, and c_{21} are:

$$\pi\lambda_0 u'(d)+(1-\pi)\lambda_1 u'(d)+\mu(p_{10}\lambda_0+p_{11}\lambda_1)=0$$

$$\pi u'(c_{20})+\mu p_{20}=0$$

$$(1-\pi)u'(c_{21}) + \mu p_{21} = 0.$$

Substituting the constrained efficient values of d, c_{20}, and c_{21} into these, and using the budget constraint and the zero profit conditions, it is possible to derive the prices that implement the constrained efficient allocation. These prices allow the firms to produce the assets at zero profits, and the banks to maximize depositors' welfare.

So far we have abstracted from idiosyncratic risk. We next consider how this can be accommodated. Suppose each firm issues state-contingent Arrow securities based on the shock H and L experienced by the purchasing bank. They issue these securities in small amounts to all other banks so that the idiosyncratic risk is diversified away. Each bank will buy enough of the H and L securities to cover their needs in each of the states. As usual we denote $\lambda_{\theta i} = \alpha_i + \varepsilon\theta$ for $\theta = 0, 1$ and $i = H, L$. The Arrow securities each bank buys are $\lambda_{\theta i} d$ at date 1 and $(1 - \lambda_{\theta i})c_{2\theta}$ at date 2 for $\theta = 0, 1$ and $i = H, L$. The price of these securities are $p_{t\theta i}$ for $t = 1, 2, \theta = 0, 1$ and $i = H, L$.

In order for the banks to be able to afford the optimal state contingent securities it is necessary that

$$\lambda_{\theta H} p_{t\theta H} + \lambda_{\theta L} p_{t\theta L} = \lambda_\theta p_{t\theta} \text{ for } t = 1, 2 \text{ and } \theta = 0, 1.$$

Since the aggregate state (t, θ) is the same for each H and L, and $\frac{1}{2}$ of the banks are H and $\frac{1}{2}$ are L consider the symmetric equilibrium with

$$p_{t\theta H} = p_{t\theta L} = \frac{1}{2} p_{t\theta}.$$

This ensures that the banks can afford to purchase the constrained efficient allocation. Since this gives the highest expected utility for the depositors, it is the best that the banks can do.

In the case of incomplete markets, the banks held the assets. With complete markets we have, for simplicity, been assuming that firms hold the assets and issue the securities. Since there are zero profits from producing the assets we could just as well assume that the banks held the assets. In order to obtain the benefits of diversification, they would issue securities against the assets in the same way as the firms. They would also buy them in the same way as previously. Thus they would be on both sides of the market buying and selling large numbers of securities. Essentially each bank is issuing to and holding the securities of every other bank. Since issuing securities and maintaining the accounting and other infrastructure associated with them is costly, it would be impractical to implement this kind of arrangement. This is why the role of the central bank in implementing the constrained efficient allocation is so important.

The institutional structure where all trades take place at date 0 described above is only one institutional structure that will implement complete markets. Another structure is to have dynamic markets where firms issue state contingent Arrow securities between dates 0 and 1 that are contingent on the state $\theta = 0, 1$ and allow the banks to hedge this risk. At date 1, there are markets for date 2 consumption that the banks and firms can also trade in. However, this case also requires a large number of markets and securities to allow the idiosyncratic risk to be diversified away.

7 The multi-period case

In this section we extend the results of the simple, two-period model to an economy with a countably infinite number of dates, indexed by $t = 0, 1, ...$ We continue to assume that agents have one unit of the good at date 0 and nothing at dates $t > 0$. Agents only value consumption at a single date $t = 1, 2, ...$ At the initial date, all agents are identical. At each subsequent date t, a fraction of the agents receives a liquidity shock that makes them want to consume at that date. The agents who receive the liquidity shock consume what they can and leave the market. The liquidity shock has a constant hazard $\bar{\lambda} \in [0, 1]$. In other words, a consumer receives a liquidity shock with probability $\bar{\lambda}$ at date t if he has not received one previously. Then the probability that an agent does not receive a liquidity shock before date t is $\left(1-\bar{\lambda}\right)^{t-1}$ and the probability that he receives the liquidity shock at date t is $\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}$.

We assume that the fraction of agents receiving the shock is equal to the hazard rate $\bar{\lambda}$ in each period. So at the beginning of date t, there are $\left(1-\bar{\lambda}\right)^{t-1}$ agents left in the market and, of these, a fraction $\bar{\lambda}$ receive the liquidity shock. So the number who want to consume at date t is also $\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}$.

There is an infinite number of long assets, one corresponding to each date $t = 1, 2, ...$ One unit of the good invested in asset $t \geq 1$ produces R^t units of the good at date t and nothing at other dates. Investment is irreversible, so once the investment is made it is impossible to obtain any consumption from this asset before date t. We assume that the returns per period on investment in the long asset is a constant $R > 1$. There is also a short asset, represented by a storage technology, that can be used at any date, but since there is no aggregate uncertainty (the hazard rate $\bar{\lambda}$ is a constant), the short asset will always be dominated by one of the long assets.

7.1 The planner's problem

A planner who wants to maximize the value of the typical agent's expected utility will choose a sequence of consumption levels $\mathbf{c} = \{c_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=1}^{\infty}\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}u\left(c_t\right)$$

subject to a feasibility constraint

$$\sum_{t=1}^{\infty}\frac{1}{R^t}\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}c_t \leq 1.$$

In deriving the feasibility constraint, we make use of the fact that the most efficient way of providing consumption at date t is to invest in the long asset corresponding to date t. The total amount of consumption provided at date t is $\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}c_t$, since each agent receives

c_t and there are $\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}$ agents who want to consume at date t. To provide one unit of consumption at date t requires $\frac{1}{R^t}$ units of investment at date 0, so the total investment required at date 0 to provide consumption at date t is $\frac{1}{R^t}\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}c_t$.

Under the usual assumptions on $u(\cdot)$, there is a unique solution $\mathbf{c}^*=\{c_t^*\}$ to the planner's problem and it satisfies $c_t^*>0$ for every $t\geq 1$. The solution is determined by the first-order conditions

$$u'(c_t^*)=\frac{\mu}{R^t},\qquad \forall t=1,2,...,$$

where $\mu>0$ is the Lagrange multiplier on the feasibility constraint, and by the feasibility constraint

$$\sum_{t=1}^{\infty}\frac{1}{R^t}\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}c_t^*=1.$$

Note that the first-order conditions imply that $c_{t+1}^*>c_t^*$ for every $t\geq 1$, so the incentive compatibility condition is automatically satisfied at each date.

Example 1 *To illustrate the planner's solution, suppose that the utility function $u(\cdot)$ has constant relative risk aversion $\rho>1$. Then the first-order condition is*

$$(c_t^*)^{-\rho}=\frac{\mu}{R^t},\tag{18}$$

which implies

$$\frac{c_{t+1}^*}{c_t^*}=R^{\frac{1}{\rho}}.$$

The present value of consumption at date t is $\frac{c_t^*}{R^t}$ is declining over time: since $\rho>1$,

$$\frac{c_t^*}{R^t}=\frac{c_{t+1}^*}{R^{\frac{1}{\rho}}}\frac{1}{R^t}>\frac{c_{t+1}^*}{R^{t+1}}.$$

As we shall see this turns out to be an obstacle to the decentralization of the optimal allocation in a laisser faire system.

7.2 The banking system

Now let us consider the problem of implementing the optimal allocation when individual banks receive idiosyncratic liquidity shocks. There is a continuum of banks identified with points in $[0,1]$. At any date $t \geq 1$, bank i receives a liquidity shock λ_{it}, where λ_{it} denotes the fraction of depositors remaining who want to consume in period t. We assume the random variables $\{\lambda_{it}\}_{i,t=1}^{\infty}$ are i.i.d. with c.d.f. $F(\lambda)$ and that $F(\lambda)$ satisfies

$$\int_0^1 \lambda dF(\lambda) = \bar{\lambda}.$$

We assume the "law of large numbers" convention is satisfied, so that the average liquidity shock across all banks i is equal to $\bar{\lambda}$ at each date t:

$$\int_0^1 \lambda_{it} di = \bar{\lambda}, \qquad \forall t = 1, 2, ...$$

Although there is no aggregate uncertainty—the fraction of the remaining depositors that want to withdraw in each period is equal to the constant $\bar{\lambda}$—the amount of withdrawals at bank i is uncertain. We can assume without loss of generality that bank i starts out with a unit mass of identical depositors. At date 1, a fraction λ_{i1} withdraw, leaving $(1-\lambda_{i1})$ agents as depositors. At date 2, a fraction λ_{i2} of these depositors, i.e., $(1-\lambda_{i1})\lambda_{i2}$, choose to withdraw, leaving $(1-\lambda_{i1})(1-\lambda_{i2})$ agents as depositors. So at the beginning of date t, the number depositors is $\prod_{\tau=1}^{t-1}(1-\lambda_{i\tau})$ and the number of withdrawing depositors is $\prod_{\tau=1}^{t-1}(1-\lambda_{i\tau})\lambda_{it}$.

Example 2 *Now we can see why a decentralized banking system might have difficulty implementing the optimal allocation. Consider the example with constant relative risk aversion $\rho > 1$ and suppose that each bank promises the depositors c_t^* if they withdraw at date t. Since each depositor who withdraws at date t receives more in present value than the depositors who withdraw later, a bank with $\lambda_{it} > \bar{\lambda}$ would have to pay out more than a bank with $\lambda_{it} < \bar{\lambda}$.*

So banks that get hit with high withdrawals early on will not be able to meet their customers' demands later on.

To illustrate how the interbank market and open market operations can help banks implement the optimal allocation, we consider first a rather special policy for implementing the optimal allocation. Later we show that there are many policies that will achieve the same end.

Bank i's deposit contract promises the individual depositors d_t^* if withdrawal occurs at date $t \geq 1$. Suppose that bank i's portfolio is the same as the planner's. If y_t^* denotes the investment in the asset that pays off at date t, then

$$y_t^* = \frac{1}{R^t}\left(1 - \bar{\lambda}\right)^{t-1} \bar{\lambda} c_t^*, \qquad \forall t = 1, 2, ...$$

In other words, the amount invested in the asset that pays off at date t is sufficient to provide the consumption at date t in the optimal allocation. By assumption, $\sum_{t=1}^{\infty} y_t^* = 1$, so the bank's budget constraint at date 0 is satisfied.

At each date $t \geq 1$ there is a market in which the long assets can be traded. The central bank ensures that at each date $t \geq 1$, all the remaining assets sell for the same price

$$P_t = R^t, \qquad \forall t = 1, 2, ...$$

Given this pricing rule, the one-period holding return on every asset is R and banks will be indifferent between holding assets of different maturities at every date.

The main problem of decentralizing the optimal consumption allocation arises because banks receive different liquidity shocks and this may make it impossible for some banks to meet their commitments to depositors and satisfy their budget constraints. One way to ensure that the banks can do both is to set the contractual payment d_t^* equal to an amount that has a constant present value. In the present case, that requires

$$d_t^* = R^t, \qquad \forall t = 1, 2, ...$$

Then the bank's budget constraint is independent of the realization of $\{\lambda_{it}\}_{t=1}^{\infty}$ since each cohort withdraws the same amount in present value. This strategy will not guarantee that agents receive the optimal consumption allocation unless the government steps in to adjust their income by means of lump sum taxes and transfers. It it turns out that the government can do this while balancing its own budget.

It remains to show that there is a government policy that will implement the optimal consumption allocation, given the bank's choice of deposit contract and portfolio. By imposing the tax-transfer scheme $\mathbf{s}^* = \{s_t^*\}_{t=1}^{\infty}$ satisfying

$$s_t^* + d_t^* = c_t^*, \qquad \forall t = 1, 2, ...,$$

the government can ensure that each consumer receives c_t^* if he withdraws at date t.

The tax-transfer scheme ensures that the consumers receive the optimal amount of consumption at each date, but the government needs a source of income to pay for the transfers or a way of redistributing the taxes. We assume that the government can issue one-period bonds at each date. Then the budget constraint can be balanced in each period by issuing or retiring debt. In equilibrium, the return on government bonds must equal the return on long assets, R. At the first date, the government gives B_0^* bonds to each bank. Let B_t^* denote the per capita value of the bonds issued at date $t \geq 1$. The government's budget constraint at date $t + 1$, expressed in per capita terms, is

$$RB_{t-1}^* - B_t^* = \left(1 - \bar{\lambda}\right)^{t-1} \bar{\lambda} s_t^*, \qquad \forall t = 1, 2, ...,$$

where RB_{t-1}^* is the cost of redeeming the old bonds, B_t^* is the revenue from issuing new bonds, and $\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}s_t^*$ is the cost of the tax-transfer to the current consumers. We assume that $B_t^* \geq 0$ for any $t = 0$, and choose B_0^* large enough so that this condition can be satisfied.

Example 3 *We can illustrate the debt policy using our example of constant relative risk aversion $\rho > 1$. We start with a high present value of consumption $\frac{c_1^*}{R} > 1$, which requires $s_1^* = c_1^* - R > 0$. So the government has to raise revenue by issuing debt. Suppose that we put $B_0^* = 0$ and set $B_1^* = \bar{\lambda}s_1$. In subsequent periods, the present value of consumption falls but as long as $\frac{c_t^*}{R^t} > 1$, we have $s_t^* = c_t^* - R^t > 0$ and the government needs revenues in order to pay for the transfers. So at each date t such that $\frac{c_t^*}{R^t} \geq 1$, the debt must be increased. Eventually, we reach a date T such that $s_T^* = c_T^* - R^T < 0$. At that point, the government begins to tax the consumers and retire the debt. At subsequent dates, as the present value of consumption continues to fall, the government can retire debt at an accelerating pace.*

It remains to check the market-clearing conditions at each date. The goods market clears because the final demand for consumption comes entirely from consumers and their demand is equal to the supply of goods in each period,

$$\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}c_t^* = R^t y_t^*.$$

The securities markets clear by Walras' law. To see this, note that the net demand for assets from the private sector banks is equal to the total supply of liquidity $R^t y_t^* + RB_{t-1}^*$ minus the deposits paid out to consumers $\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}R^t$. The net supply of bonds is B_t^*, so demand equals supply if

$$R^t y_t^* + RB_{t-1}^* - \left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}R^t = B_t^*,$$

or

$$RB_{t-1}^* - B_t^* = \left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}R^t - R^t y_t^*.$$

From the government's budget constraint the left hand side is equal to $\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}s_t^*$. From the market-clearing condition for the goods market and the definition of the deposit contract, the right hand side can be rewritten as

$$\begin{aligned}\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}R^t - R^t y_t^* &= \left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}d^* - \left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}c_t^* \\ &= \left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}s_t^*,\end{aligned}$$

as required.

We can show that the deposit contract $\mathbf{d}^* = \{d_t^*\}_{t=1}^{\infty}$ maximizes the expected utility of the bank's depositors subject to the bank's budget constraint by checking the first-order conditions, so the contract is optimal for a competitive bank.

7.3 Alternative schemes

The construction used above is just one of many. There is a Modigliani-Miller theorem that allows the bank to offer different deposit contracts if the government adjusts its debt policy appropriately.

Suppose that we set

$$d_t^* = c_t^*, \qquad \forall t = 1, 2, ...$$

and leave the specification of the variables (c_t^*, y_t^*) and P_t^* unchanged. Is there a debt policy $\{B_t\}_{t=0}^{\infty}$ that will satisfy both the government's and the banks' budget constraints? From the point of view of the individual bank, the important thing is to remove the capital gains and losses caused by the liquidity shocks, in other words, to keep constant the assets per depositor held by the bank.

If B_0 is the quantity of bonds (positive or negative) given to banks initially, then the value of assets per capita at the end of date $t = 1$ is

$$a_1^* = \frac{R\left(1+B_0\right) - \lambda_1 c_1^*}{1-\lambda_1}.$$

The quantity a_1^* is independent of the idiosyncratic shock λ_1 if and only if

$$R(1+B_0)=c_1^*.$$

Similarly, at any date t, the value of assets per capita at the end of the period will be

$$a_t^* = \frac{R(1+B_{t-1})-\lambda_t c_t^*}{1-\lambda_t},$$

which is independent of λ_t if and only if $R(1+B_{t-1})=c_t^*$. This suggests that we define the sequence B_t^* by putting

$$B_t^* = \frac{c_{t+1}^*-R}{R}, \qquad \forall t=0,1,....$$

It remains to explain how the budget constraints will be satisfied.

The government must impose lump sum taxes equal to the difference between the cost of retiring the old debt and the new debt issued in each period. That is,

$$-\left(1-\bar{\lambda}\right)^{t-1}\bar{\lambda}s_t^* = RB_{t-1}^*-B_t^*, \qquad \forall t=0,1,....$$

This tax can be imposed on the remaining depositors at the end of period t. Since the depositors have no cash (and the deposit contract gives them none), it would be easiest to levy this as a tax on deposits. If the government's budget constraint is balanced, the banks' budget constraints must be balanced as well.

It remains to check that the market-clearing conditions are satisfied as well. At date 0, the government makes an ex gratia payment of bonds to the banks. This does not disturb the goods market. Next consider what happens at date 1. The government owes the banks RB_0^*. It gives them B_1^* and levies taxes totalling $-\bar{\lambda}s_1^* = RB_0^* - B_1^*$, which discharges its obligations to the banks exactly, again without disturbing the goods market. The same argument shows that the goods market is not disturbed at any future date t and, since all assets bear the same return R, the banks are willing to hold all the assets offered to it.

8 Concluding remarks

This paper has developed a simple model of the interbank market. We have shown how central bank intervention in this market can improve welfare in a variety of situations.

The model is very simple. However, it can be extended in a number of directions to consider important issues. We have so far ignored bankruptcy of financial institutions. This will likely occur if the high liquidity demand crisis state occurs with small enough probability. Incorporating this will allow open market operations to be compared with lender of last resort policies.

The model is a real one in that all the funds of the bank that are used for intervention are raised through lump sum taxes. If the bank uses seigniorage instead then this should allow some insight into the relationship between monetary policy and financial stability.

For reasons of tractability we have only considered the multi-period version of the model with idiosyncratic risk. The term premium in this case is exogenously determined by the technological returns of the assets. If aggregate liquidity risk is introduced into the model, the short asset may be used and this may lead to a spike in the liquidity term premium in the high liquidity demand state similarly to the two-period model above. The approach therefore has the potential to explain the kind of liquidity term premium that appears to be emblematic of the credit crisis that started in the summer of 2007.

Finally, we have only considered open market operations by the central bank in the interbank market. Another possibility would be for the central bank to pay banks interest on funds deposited with them as suggested by Goodfriend (2002). Considering this policy in the context of our model would be an interesting extension.

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