Derivation of Equations in "The Social Value of Asymmetric Information"

by

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The algebra for deriving some of the expressions in the paper is somewhat complex. It is therefore given below for the most cumbersome. Equations in these notes are distinguished from those in the text and the appendix by the prefix α.

Equation (34)

From (33),

$$E(V_U(W_{1i})|x)\Big|_{\lambda=0} = -\exp[-a(R\bar{M}_i + (E\theta - a\sigma_u^2 x)\bar{X}_i + \frac{a}{2}\sigma_u^2 x^2)] \qquad (33)$$

Since $\sigma_u^2 = \sigma_\theta^2 + \sigma_\varepsilon^2$,

$$= -\exp[-a(R\bar{M}_i + \bar{X}_i E\theta - \frac{a}{2}\sigma_\theta^2\bar{X}_i^2 + \frac{a}{2}\sigma_\theta^2\bar{X}_i^2$$

$$-a\sigma_\varepsilon^2 x\bar{X}_i - a\sigma_\theta^2 x\bar{X}_i + \frac{a}{2}\sigma_\varepsilon^2 x^2 + \frac{a}{2}\sigma_\theta^2 x^2)]$$

$$= -\exp[-a(R\bar{M}_i + \bar{X}_i E\theta - \frac{a}{2}\sigma_\theta^2\bar{X}_i^2$$

$$-a\sigma_\varepsilon^2 x\bar{X}_i + \frac{a}{2}\sigma_\varepsilon^2 x^2 + \frac{a}{2}\sigma_\theta^2(x - \bar{X}_i)^2)]$$

Using (28),

$$= E(V_I(W_{1i})|x)\Big|_{\lambda=1,c=0}\ \exp[\frac{-a^2\sigma_\theta^2}{2}(x - \bar{X}_i)^2]$$

which is (34).

Equation (43)

From (41),

$$E(V_U(W_{1i})|w_\lambda) = -\exp[-a(R\bar{M}_i + RP\bar{X}_i + \frac{(E(u|w_\lambda) - RP)^2}{2a\ \mathrm{Var}(u|w_\lambda)})] \qquad (41)$$

and from (42),
$$RP = \frac{\lambda w_\lambda + (1-\lambda)\beta\, E(u|w_\lambda) - a\sigma_\varepsilon^2 Ex}{\lambda + (1-\lambda)\beta} \tag{42}$$

$$= \frac{1}{D}\left[\frac{\lambda}{a\sigma_\varepsilon^2} w_\lambda + \frac{(1-\lambda)}{a\ Var(u|w_\lambda)} E(u|w_\lambda) - Ex\right] \tag{α1}$$

where
$$D = \frac{\lambda}{a\sigma_\varepsilon^2} + \frac{(1-\lambda)}{a\ Var(u|w_\lambda)} .$$

Let
$$\tilde{w}_\lambda = w_\lambda - E\theta$$

then substituting for $E(u|w_\lambda)$ using (39)

$$RP = \frac{1}{D}\left[\frac{\lambda}{a\sigma_\varepsilon^2} w_\lambda + \frac{(1-\lambda)}{a\ Var(u|w_\lambda)} (E\theta + \eta\tilde{w}_\lambda) - Ex\right]$$

$$= E\theta + \frac{1}{D}(G\tilde{w}_\lambda - Ex) \tag{α2}$$

where
$$G = \frac{\lambda}{a\sigma_\varepsilon^2} + \frac{(1-\lambda)}{a\ Var(u|w_\lambda)} \eta .$$

Using the definition of $E(u|w_\lambda)$ again

$$RP - E(u|w_\lambda) = \frac{1}{D}\left(\left[\frac{\lambda}{a\sigma_\varepsilon^2} + \frac{(1-\lambda)}{a\ Var(u|w_\lambda)} \eta\right]\tilde{w}_\lambda - D\eta\tilde{w}_\lambda - Ex\right)$$

$$= \frac{1}{D}(F\tilde{w}_\lambda - Ex) \tag{α3}$$

where
$$F = \frac{\lambda}{a\sigma_\varepsilon^2}(1-\eta) .$$

Substituting in (41) for RP and $RP - E(u|w_\lambda)$,

$$E\left(V_U(W_{1i})|w_\lambda\right) = -\exp\left[-a\left(R\overline{M}_i + (E\theta - \frac{Ex}{D})\overline{X}_i + \frac{G}{D}\overline{X}_i\tilde{w}_\lambda\right.\right.$$

$$\left.\left. + \frac{F^2\tilde{w}_\lambda^2 - 2FEx\ \tilde{w}_\lambda + (Ex)^2}{2aD^2 Var(u|w_\lambda)}\right)\right] . \tag{α4}$$

Taking expectations over $\tilde{w}_\lambda$ and rearranging

$$EV_U(W_{1i}) = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \text{ Var } w_\lambda}} \exp[-a(R\bar{M}_i + (E\theta - \frac{Ex}{D})\bar{X}_i)$$

$$-\frac{(Ex)^2}{2D^2\text{Var}(u|w_\lambda)} - \frac{1}{2}[\frac{F^2}{D^2\text{Var}(u|w_\lambda)} + \frac{1}{\text{Var } w_\lambda}]\tilde{w}_\lambda^2 + [\frac{FEx}{D^2\text{Var}(u|w_\lambda)} - a\frac{G}{D}\bar{X}_i]\tilde{w}_\lambda]d\tilde{w}_\lambda$$

$$= -\exp[-a(R\bar{M}_i + (E\theta - \frac{Ex}{D})\bar{X}_i + \frac{(Ex)^2}{2aD^2\text{Var}(u|w_\lambda)})]$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \text{ Var } w_\lambda}} \exp[-\frac{1}{2}([\frac{F^2}{D^2\text{Var}(u|w_\lambda)} + \frac{1}{\text{Var } w_\lambda}]\tilde{w}_\lambda^{\ 2} - 2[\frac{FEx}{D^2\text{Var}(u|w_\lambda)} - a\frac{G}{D}\bar{X}_i]\tilde{w}_\lambda)]d\tilde{w}_\lambda .$$

Now let

$$w_\lambda^* = \tilde{w}_\lambda + \frac{\zeta}{\xi}$$

where

$$\zeta = a\frac{G}{D}\bar{X}_i - \frac{F\ Ex}{D^2\text{Var}(u|w_\lambda)}$$

$$\xi = \frac{F^2}{D^2\text{Var}(u|w_\lambda)} + \frac{1}{\text{Var } w_\lambda}$$

Then

$$EV_U(W_{1i}) = -\exp[-a(R\bar{M}_i + (E\theta - \frac{Ex}{D})\bar{X}_i$$

$$+ \frac{1}{2a}(\frac{(Ex)^2}{D^2\text{Var}(u|w_\lambda)} - \frac{\zeta^2}{\xi}))]\ \frac{\xi^{-\frac{1}{2}}}{\sqrt{\text{Var } w_\lambda}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\ \xi^{-\frac{1}{2}}} \exp[-\frac{w_\lambda^{*2}}{2\xi^{-1}}]dw_\lambda^* .$$

Since the integral in the above expression is one

$$EV_{Ui} = B\exp[-(-a\frac{Ex\bar{X}_i}{D} + \frac{1}{2}\xi^{-1}[\frac{(Ex)^2}{D^2\text{Var}(u|w_\lambda)}\xi - \zeta^2])]$$

where

$$B = -\frac{\xi^{-\frac{1}{2}}}{\sqrt{\text{Var } w_\lambda}} \exp[-a(R\bar{M}_i + \bar{X}_i E\theta)] \ .$$

$$EV_{Ui} = B \exp[-\tfrac{1}{2}(-2a \frac{Ex\ \bar{X}_i}{D}$$

$$+ [\frac{F^2}{D^2\text{Var}(u|w_\lambda)} + \frac{1}{\text{Var } w_\lambda}]^{-1}(\frac{(Ex)^2}{D^2\text{Var}(u|w_\lambda)\text{Var } w_\lambda} - (\frac{a}{D}\frac{G}{})^2\bar{X}_i{}^2 + \frac{2FaG\ Ex\ \bar{X}_i}{D^3\text{Var}(u|w_\lambda)}))]$$

$$= B \exp[-\tfrac{1}{2}(-2a \frac{Ex\ \bar{X}_i}{D}$$

$$+ (\frac{1}{F^2\text{Var } w_\lambda + D^2\text{Var}(u|w_\lambda)})((Ex)^2 - (aG)^2\text{Var}(u|w_\lambda)\text{Var } w_\lambda\ \bar{X}_i^2$$

$$+ 2Fa \frac{G}{D} \text{Var } w_\lambda\ Ex\ \bar{X}_i))]$$

$$= B \exp[-\tfrac{1}{2}(\frac{1}{F^2\text{Var } w_\lambda + D^2\text{Var}(u|w_\lambda)})$$

$$((Ex)^2 - (aG)^2\text{Var}(u|w_\lambda)\text{Var } w_\lambda\ \bar{X}_i^2 + 2Ex\ \bar{X}_i\ a(F \frac{G}{D} \text{Var } w_\lambda$$

$$- \frac{F^2}{D}\text{Var } w_\lambda - D\ \text{Var}(u|w_\lambda)))] \ . \qquad (\alpha 5)$$

It follows from the definitions of G and F that

$$G = F + \eta D \ .$$

Hence

$$a(F \frac{G}{D} \text{Var } w_\lambda - \frac{F^2}{D}\text{Var } w_\lambda - D\ \text{Var}(u|w_\lambda)) = a(\eta F\ \text{Var } w_\lambda - D\ \text{Var}(u|w_\lambda))$$

Using the definitions of $\text{Var}(u|w_\lambda)$ and η,

$$= \sigma_\theta^2 \frac{\lambda}{\sigma_\varepsilon^2}(1 - \eta) - \left(\sigma_\varepsilon^2 + \sigma_\theta^2(1 - \eta)\right)\left[\frac{-\lambda}{\sigma_\varepsilon^2} + \frac{(1 - \lambda)}{\mathrm{Var}(u|w_\lambda)}\right]$$

$$= -\lambda - (1 - \lambda)$$

$$= -1 \qquad (\alpha 6)$$

Hence

$$EV_U(W_{1i}) = B \exp\left[-\frac{1}{2}\left(\frac{a^2\sigma_u^2}{a^2\sigma_u^2(F^2\mathrm{Var}\ w_\lambda + D^2\mathrm{Var}(u|w_\lambda))}\right)\right.$$

$$\left.\left((\bar{X}_i - Ex)^2 - (1 + a^2G^2\mathrm{Var}(u|w_\lambda)\mathrm{Var}\ w_\lambda)\bar{X}_i^2\right)\right] \qquad (\alpha 7)$$

It is next necessary to evaluate $a^2\sigma_u^2(F^2\mathrm{Var}\ w_\lambda + D^2\mathrm{Var}(u|w_\lambda))$ and $1 + a^2G^2\mathrm{Var}(u|w_\lambda)\mathrm{Var}\ w_\lambda$. Now it follows from ($\alpha$6) that

$$F^2\mathrm{Var}\ w_\lambda + D^2\mathrm{Var}(u|w_\lambda) = FG\ \mathrm{Var}\ w_\lambda + \frac{D}{a}$$

Hence,

$$a^2\sigma_u^2(F^2\mathrm{Var}\ w_\lambda + D^2\mathrm{Var}(u|w_\lambda)) = a^2\sigma_u^2\ FG\ \mathrm{Var}\ w_\lambda + a\sigma_u^2 D$$

Using the fact that $F = G - \eta D$ and the definition of η

$$= a^2\sigma_u^2G^2\mathrm{Var}\ w_\lambda - a^2\sigma_u^2\sigma_\theta^2 GD + a\sigma_u^2 D$$

Using the definition of $\mathrm{Var}(u|w_\lambda)$

$$= a^2G^2\mathrm{Var}(u|w_\lambda)\mathrm{Var}\ w_\lambda + a^2G^2\sigma_\theta^2\eta\ \mathrm{Var}\ w_\lambda$$

$$- a^2\sigma_u^2\sigma_\theta^2 GD + a\sigma_u^2 D$$

$$= a^2G^2\mathrm{Var}(u|w_\lambda)\mathrm{Var}\ w_\lambda + a\sigma_\theta^2G(a\sigma_\theta^2G - a\sigma_u^2D)$$

$$+ a\sigma_u^2 D \qquad (\alpha 8)$$

Now using the definitions of G and D

$$a\sigma_\theta^2 G - a\sigma_u^2 D = \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\lambda + \frac{(1-\lambda)}{\mathrm{Var}(u|w_\lambda)}\eta\sigma_\theta^2$$

$$- \sigma_u^2 \frac{\lambda}{\sigma_\varepsilon^2} - \sigma_u^2 \frac{(1-\lambda)}{\mathrm{Var}(u|w_\lambda)}$$

$$= -\lambda + \frac{1-\lambda}{\mathrm{Var}(u|w_\lambda)}(\eta\sigma_\theta^2 - \sigma_u^2)$$

Using the definition of $\mathrm{Var}(u|w_\lambda)$

$$= -1 \qquad (\alpha 9)$$

Hence from this and (α8)

$$a^2\sigma_u^2\left(F^2\mathrm{Var}\ w_\lambda + D^2\mathrm{Var}(u|w_\lambda)\right) = a^2G^2\mathrm{Var}(u|w_\lambda)\mathrm{Var}\ w_\lambda$$

$$- a\sigma_\theta^2 G + a\sigma_u^2 D$$

Using (α9) again

$$= 1 + a^2G^2\mathrm{Var}(u|w_\lambda)\mathrm{Var}\ w_\lambda \qquad (\alpha 10)$$

Now from the definitions of G, η and β

$$1 + a^2G^2\mathrm{Var}(u|w_\lambda)\mathrm{Var}\ w_\lambda = 1 + \frac{1}{4\sigma_\varepsilon}[\lambda + (1-\lambda)\beta\eta]^2 \frac{\sigma_\varepsilon^2\sigma_\theta^2}{\beta\eta}$$

$$= 1 + \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\left[\frac{\lambda^2}{\beta\eta} + (1-\lambda)^2\beta\eta + 2\lambda(1-\lambda)\right]$$

Using the fact that $2\lambda(1-\lambda) = 1 - \lambda^2 - (1-\lambda)^2$,

$$= 1 + \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\lambda^2\left(\frac{1}{\beta\eta} - 1\right) + \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\left[1 - (1-\lambda)^2(1-\beta\eta)\right]$$

Using the fact that $\left(\frac{1}{\beta\eta} - 1\right) = \frac{a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2}{\lambda^2\sigma_\theta^2}$

$$= 1 + a^2\sigma_u^2\sigma_x^2 + \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\left[1 - (1-\lambda)^2(1-\beta\eta)\right]$$

Thus using this and (α10) in (α7)

$$EV_{Ui} = B \exp[-\tfrac{1}{2}a^2\sigma_u{}^2(\frac{(\bar{X}_i - Ex)^2}{1 + a^2\sigma_u^2\sigma_x^2 + \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}[1 - (1-\lambda)^2(1 - \beta\eta)]} - \bar{X}_i^2)].$$

It remains to evaluate B and in particular $\xi^{-\frac{1}{2}}/\sqrt{\text{Var } w_\lambda}$. From the definition of ξ,

$$\frac{\xi^{-1}}{\text{Var } w_\lambda} = \frac{1}{\text{Var } w_\lambda[\frac{F^2}{D^2\text{Var}(u|w_\lambda)} + \frac{1}{\text{Var } w_\lambda}]}$$

$$= [\frac{F^2\text{Var } w_\lambda}{D^2\text{Var}(u|w_\lambda)} + 1]^{-1} .$$

Now

$$\frac{F^2\text{Var } w_\lambda}{D^2\text{Var}(u|w_\lambda)} = \frac{F^2(\text{Var } w_\lambda)^2}{D^2\text{Var}(u|w_\lambda)\text{Var } w_\lambda} ,$$

Using the definitions of F, η, $\text{Var}(u|w_\lambda)$ and $\text{Var } w_\lambda$

$$= \frac{\frac{a^2\sigma_\varepsilon^4\sigma_x^4}{\lambda^2}}{D^2(\sigma_u^2 \text{ Var } w_\lambda - \sigma_\theta^4)}$$

$$= \frac{a^2\sigma_\varepsilon^4\sigma_x^4}{D^2[a^2\sigma_\varepsilon^4\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\varepsilon^2\sigma_\theta^2]}$$

Using the definition of D

$$= \frac{a^4\sigma_\varepsilon^6\sigma_x^4}{[\lambda + (1-\lambda)\beta]^2[a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\theta^2]} .$$

Hence substituting back into $\xi^{-1}/$ Var w_λ and using this to evaluate B, (43) follows:

$$EV_{Ui} = -\left(1 + \frac{a^4\sigma_\varepsilon^6\sigma_x^4}{[\lambda + (1-\lambda)\beta]^2[a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\theta^2\ ]}\right)^{-\frac{1}{2}}$$

$$\exp\left[-a\left(R\bar{M}_i + \bar{X}_i E\theta + \frac{a}{2}\sigma_u^2\left(\frac{(\bar{X}_i - Ex)^2}{1 + a^2\sigma_u^2\sigma_x^2 + \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}[1 - (1-\lambda)^2(1-\beta\eta)]} - \bar{X}_i^2\ \right)\right)\right].$$

Equation (44)

Differentiating (43),

$$\frac{dEV_{Ui}}{d\lambda} = \Big\{\frac{1}{2}\left(1 + \frac{a^4\sigma_\varepsilon^6\sigma_x^4}{[\lambda + (1-\lambda)\beta]^2[a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\theta^2\ ]}\right)^{-\frac{3}{2}}$$

$$\left(\frac{-\ a^4\sigma_\varepsilon^6\sigma_x^4}{([\lambda + (1-\lambda)\beta]^2[a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\theta^2\ ])^2}\right)$$

$$\big(2(\lambda + (1-\lambda)\beta)(1 - \beta + (1-\lambda)\beta')(a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\theta^2)$$

$$+\ 2\lambda\ \sigma_\theta^2[\lambda + (1-\lambda)\beta]^2\big) + \left(1 + \frac{a^4\sigma_\varepsilon^6\sigma_x^4}{[\lambda + (1-\lambda)\beta]^2[a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\theta^2\ ]}\right)^{-\frac{1}{2}}$$

$$\frac{a^2}{2}\sigma_u^2\left(\frac{-(\sigma_\theta^2/\sigma_\varepsilon^2)[2(1-\lambda)(1-\beta\eta) + (1-\lambda)^2(\beta'\eta + \beta\eta')](\bar{X}_i - Ex)^2}{[1 + a^2\sigma_u^2\sigma_x^2 + (\sigma_\theta^2/\sigma_\varepsilon^2)(1 - (1-\lambda)^2(1-\beta\eta))]^2}\right)\Big\}$$

$$\exp\left[-a\left(R\bar{M}_i + \bar{X}_i E\theta + \frac{a}{2}\sigma_u^2\left(\frac{(\bar{X}_i - Ex)^2}{1 + a^2\sigma_u^2\sigma_x^2 + \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}(1 - (1-\lambda)^2(1-\beta\eta))} - \bar{X}_i^2\right)\right)\right]$$

$$= EV_{Ui}\{(\frac{a^4\sigma_\varepsilon^6\sigma_x^4}{[\lambda + (1-\lambda)\beta]^2[a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\theta^2] + a^4\sigma_\varepsilon^6\sigma_x^4})$$

$$(\frac{1 - \beta + (1-\lambda)\beta'}{\lambda + (1-\lambda)\beta} + \frac{\lambda\sigma_\theta^2}{a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2 + \lambda^2\sigma_\theta^2})$$

$$+ \frac{a^2}{2}\sigma_u^2(\frac{(\sigma_\theta^2/\sigma_\varepsilon^2)[2(1-\lambda)(1-\beta\eta) + (1-\lambda)^2(\beta'\eta + \beta\eta')](\bar{X}_i - Ex)^2}{[1 + a^2\sigma_u^2\sigma_x^2 + (\sigma_\theta^2/\sigma_\varepsilon^2)(1 - (1-\lambda)^2(1-\beta\eta))]^2})\}$$

which is (44).

Equation (46)

It follows from (α2) that

$$RP = E\theta - \frac{Ex}{D} + \frac{G}{D}[\theta - E\theta - \frac{a\sigma_\varepsilon^2}{\lambda}(x - Ex)]$$

Using this together with (6) and (7)

$$L(W_{0i}) = \bar{M}_i + \frac{1}{R}\{(E\theta - \frac{Ex}{D})\ell + \frac{G}{D}[\theta - E\theta - \frac{a\sigma_\varepsilon^2}{\lambda}(x - Ex)]\frac{(x - \bar{X})}{\gamma}\} \qquad (\alpha 11)$$

Taking expectations with respect to θ and x

$$EL(W_{0i}) = \bar{M}_i + \frac{1}{R}[(E\theta - \frac{Ex}{D})E\ell - \frac{G}{D}\frac{a\sigma_\varepsilon^2}{\lambda\gamma}\sigma_x^2]$$

$$= \bar{M}_i + \frac{1}{R}[E\theta E\ell - \frac{a\sigma_\varepsilon^2}{\lambda + (1-\lambda)\beta}(E\ell Ex + [1 + (\frac{1}{\lambda} - 1)\beta\eta]\frac{\sigma_x^2}{\gamma})]$$

which is (46).

Equation (47)

Differentiating (46)

$$\frac{dEL(W_{0i})}{d\lambda} = \frac{1}{R}\Big\{\frac{a\sigma_\varepsilon^2[1 - \beta + (1 - \lambda)\beta']}{[\lambda + (1 - \lambda)\beta]^2}\ \big(E\ell Ex + [1 + (\tfrac{1}{\lambda} - 1)\beta\eta]\frac{\sigma_x^2}{\gamma}\big)$$

$$- \frac{a\sigma_\varepsilon^2\sigma_x^2}{\gamma[\lambda + (1 - \lambda)\beta]}\Big[-\beta'\eta - \beta\eta' + \frac{d(\beta\eta/\lambda)}{d\lambda}\Big]\Big\} \qquad (\alpha 12)$$

Now

$$\frac{\beta\eta}{\lambda} = \frac{\sigma_\varepsilon^2}{\mathrm{Var}(u|w_\lambda)}\ \frac{\sigma_\theta^2}{\mathrm{Var}\ w_\lambda}\ \frac{1}{\lambda}$$

$$= \frac{\sigma_\theta^2\sigma_\varepsilon^2}{(\sigma_u^2\ \mathrm{Var}\ w_\lambda - \sigma_\theta^4)\lambda}$$

$$= \frac{\lambda\sigma_\theta^2\sigma_\varepsilon^2}{\lambda^2\sigma_\theta^2\sigma_\varepsilon^2 + a^2\sigma_\varepsilon^4\sigma_x^2\sigma_u^2}$$

$$\frac{d\beta\eta/\lambda}{d\lambda} = \frac{[\sigma_\theta^2\sigma_\varepsilon^2(\lambda^2\sigma_\theta^2\sigma_\varepsilon^2 + a^2\sigma_\varepsilon^4\sigma_x^2\sigma_u^2) - \lambda\sigma_\theta^2\sigma_\varepsilon^2(2\lambda\sigma_\theta^2\sigma_\varepsilon^2)]}{[\lambda^2\sigma_\theta^2\sigma_\varepsilon^2 + a^2\sigma_\varepsilon^4\sigma_x^2\sigma_u^2]^2}$$

$$= \frac{a^2\sigma_\varepsilon^6\sigma_x^2\sigma_u^2\sigma_\theta^2 - \lambda^2\sigma_\theta^4\sigma_\varepsilon^4}{[\lambda^2\sigma_\theta^2\sigma_\varepsilon^2 + a^2\sigma_\varepsilon^4\sigma_x^2\sigma_u^2]^2}$$

Substituting this in (α12) gives (47).

Equation (57)

Using the budget constraint (4)

$$[(uX_U + RM_i)|\varepsilon,\ \theta,\ x] = (u - RP)X_U + RP\bar{X}_i + R\bar{M}_i$$

Taking expectations over ε,

$$E[(uX_U + RM_i)|\theta,\ x] = (\theta - RP)X_U + RP\bar{X}_i + R\bar{M}_i$$

Using (40), (α2) and (α3)

$$= [\theta - E\theta - \frac{(G\tilde{w}_\lambda - Ex)}{D}]\ \frac{(Ex - F\tilde{w}_\lambda)}{aDVar(u|w_\lambda)}$$

$$+ [E\theta + \frac{(G\tilde{w}_\lambda - Ex)}{D}]\bar{X}_i + R\bar{M}_i$$

Now let

$$\tilde{\theta} = \theta - E\theta$$

then using the definitions of G and $\tilde{w}_\lambda$ it follows

$$E[(uX_U + RM_i)|\theta,\ x] = [(1 - \frac{G}{D})\tilde{\theta} + \frac{Ex_1}{D}]\frac{(Ex_2 - F\tilde{\theta})}{aDVar(u|w_\lambda)}$$

$$+ [E\theta + \frac{(G\tilde{\theta} - Ex_1)}{D}]\bar{X}_i + R\bar{M}_i$$

where
$$Ex_1 = x + \beta\eta\,\frac{(1 - \lambda)}{\lambda}\,(x - Ex)$$

$$= x + \frac{\lambda\sigma_\theta^2(1 - \lambda)}{\lambda^2\sigma_\theta^2 + a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2}\,(x - Ex)$$

$$Ex_2 = x - \eta(x - Ex)$$

Taking expectations over θ and putting $\bar{X}_i = \bar{X}$

$$E[(uX_U + RM_i)|x] = \frac{-F(1 - G/D)\sigma_\theta^2}{aD\ Var(u|w_\lambda)} + \frac{(Ex_1/D)Ex_2}{aD\ Var(u|w_\lambda)} + (E\theta - \frac{Ex_1}{D})\bar{X} + R\bar{M}_i$$

Differentiating with respect to λ

$$\frac{dE[(uX_U + RM_i)|x]}{d\lambda} = \frac{(-(dF/d\lambda)(1 - G/D) + F[d(G/D)/d\lambda])\sigma_\theta^2}{aD\ Var(u|w_\lambda)}$$

$$+ \frac{F(1 - G/D)\sigma_\theta^2}{[aD\ Var(u|w_\lambda)]^2} \frac{d[aD\ Var(u|w_\lambda)]}{d\lambda} + \frac{[d(Ex_1/D)/d\lambda]Ex_2 + Ex_1(dEx_2/d\lambda)}{aD\ Var(u|w_\lambda)}$$

$$- \frac{(Ex_1/D)Ex_2}{[aD\ Var(u|w_\lambda)]^2} \frac{d[aD\ Var(u|w_\lambda)]}{d\lambda} - \frac{d(Ex_1/D)\bar{X}}{d\lambda} . \tag{α13}$$

It follows from (19) - (22) and the definitions of the terms that

$$F\Big|_{\lambda=0} = \frac{G}{D}\Big|_{\lambda=0} = 0 \tag{α14}$$

$$\frac{dF}{d\lambda}\Big|_{\lambda=0} = \frac{dG}{d\lambda}\Big|_{\lambda=0} = \frac{1}{a\sigma_\varepsilon^2} \tag{α15}$$

Now

$$\frac{d(Ex_1/D)}{d\lambda} = \frac{1}{D}\frac{dEx_1}{d\lambda} - \frac{Ex_1}{D^2}\frac{dD}{d\lambda}$$

Since

$$\frac{dVar(u|w_\lambda)}{d\lambda}\Big|_{\lambda=0} = 0 , \tag{α16}$$

$$\frac{dD}{d\lambda}\Big|_{\lambda=0} = \frac{1}{a\sigma_\varepsilon^2} - \frac{1}{a\sigma_u^2} = \frac{\sigma_\theta^2}{a\sigma_\varepsilon^2\sigma_u^2} \tag{α17}$$

Also

$$\frac{dEx_1}{d\lambda}\Big|_{\lambda=0} = \frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\frac{1}{a^2\sigma_u^2\sigma_x^2}(x - Ex)$$

Hence using these and the definitions of Ex_1 and D

$$\left.\frac{d(Ex_1/D)}{d\lambda}\right|_{\lambda=0} = -\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\left[a\sigma_u^2 x - \frac{1}{a\sigma_x^2}(x - Ex)\right] \tag{α18}$$

Also it can be seen

$$\left.\frac{dEx_2}{d\lambda}\right|_{\lambda=0} = 0 \ . \tag{α19}$$

Hence using (α14), (α15), (α16) and (α19) in (α13) it follows

$$\left.\frac{dE\left[(uX_U + RM_i)|x\right]}{d\lambda}\right|_{\lambda=0} = -\frac{\sigma_\theta^2}{a\sigma_\varepsilon^2}$$

$$- (x - \bar{X})\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\left[a\sigma_u^2 x - \frac{1}{a\sigma_x^2}(x - Ex)\right] - \frac{\sigma_\theta^2}{\sigma_\varepsilon^2} a\sigma_u^2 x^2$$

$$= -\frac{\sigma_\theta^2}{\sigma_\varepsilon^2}\left[\frac{1}{a} - \frac{1}{a\sigma_x^2}(x - Ex)(x - \bar{X}) + a\sigma_u^2 x(2x - \bar{X})\right]$$

which is (57).

Equation (58)

Putting $\bar{X}_i = \bar{X}$ and taking expectations over θ it follows from (α11)

$$E\left[L(W_{0i})|x\right] = \bar{M}_i + \frac{1}{R}\left[\left(E\theta - \frac{Ex}{D}\right)\frac{(x - \bar{X})}{\gamma} - \frac{a\sigma_\varepsilon^2}{\gamma\lambda}\frac{G}{D}(x - Ex)(x - \bar{X})\right]$$

Differentiating with respect to λ

$$\frac{dE[L(W_{0i})|x]}{d\lambda} = \frac{1}{R}\{\frac{a\sigma_\varepsilon^2[1 - \beta + (1 - \lambda)\beta']}{\gamma[\lambda + (1 - \lambda)\beta]^2}$$

$$((x - \bar{X})Ex + [1 + (\frac{1}{\lambda} - 1)\beta\eta](x - Ex)(x - \bar{X}))$$

$$- \frac{a\sigma_\varepsilon^2(x - Ex)(x - \bar{X})}{\gamma[\lambda + (1 - \lambda)\beta]}[-\beta'\eta - \beta\eta' + \frac{a^2\sigma_\varepsilon^6\sigma_x^2\sigma_u^2\sigma_\theta^2 - \lambda^2\sigma_\theta^4\sigma_\varepsilon^4}{[\lambda^2\sigma_\theta^2\sigma_\varepsilon^2 + a^2\sigma_\varepsilon^4\sigma_x^2\sigma_u^2]^2}]\}$$

$$\left.\frac{dE[L(W_{0i})|x]}{d\lambda}\right|_{\lambda=0} = \frac{a\sigma_\varepsilon^2}{R\gamma}\frac{\sigma_\theta^2}{\sigma_u^2}\frac{\sigma_u^4}{\sigma_\varepsilon^4}[(x - \bar{X})Ex$$

$$+ (x - Ex)(x - \bar{X})] - \frac{a\sigma_\varepsilon^2\sigma_u^2}{R\gamma\sigma_\varepsilon^2}(\frac{\sigma_\theta^2}{a^2\sigma_\varepsilon^2\sigma_x^2\sigma_u^2})(x - \bar{X})(x - Ex)$$

$$= (x - \bar{X})\frac{\sigma_\theta^2}{R\gamma\sigma_\varepsilon^2}[a\sigma_u^2 x - \frac{(x - Ex)}{a\sigma_x^2}]$$

which is (58).

2.26.12