Synchronicity and Fragility*
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Abstract

We show that the correlation across financial institutions is a major force that increases their overall fragility. Our model features a financial system with banks individually prone to runs and interconnected through fire sales. Strategic complementarities within and across banks amplify each other, and this causes the correlation in their risks to be a key factor driving the fragility of each bank and the system as a whole. Reducing asset commonality and improving bank-specific disclosure alleviate fragility through desynchronizing runs across banks. Secondary market liquidity injections have a significant stabilizing effect through synchronicity reduction on top of their direct effect.

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1 Introduction

The global financial crisis of 2008 put the correlation across financial institutions (for short, banks) in the limelight. The concern emphasized by policymakers (Haldane, 2009; Yellen, 2013) is that the joint failure of many banks presents a big threat to the economy. Hence, several measures were developed to track the comovement in banks’ risks. What is less clear is whether the correlation across banks makes them more prone to failure overall. This is critical for evaluating the importance of tail comovement. After all, if banks were more correlated in their failures, but at the same time less likely to fail overall, then the implications of comovement for financial fragility would be much less severe. The model presented in this paper uncovers a new channel through which risk correlation makes banks overall more fragile. Based on this channel, we develop new implications for financial policy aimed at minimizing financial fragility.

Our model features banks that are individually fragile, due to the provision of liquidity transformation, and indirectly interconnected through fire-sale spillovers. When facing withdrawals, banks liquidate their assets in a common market, thereby imposing negative fire-sale externalities on one another. We define financial fragility as the unconditional probability of individual banks to suffer a run. We show that the degree of synchronicity—the extent to which runs are synchronized across banks and a measure of tail-risk comovement in our setting—is a key indicator of a high level of financial fragility of individual banks and the system as a whole. In this economy, reducing synchronicity becomes a natural goal for regulators aiming at minimizing real losses caused by financial fragility and inefficient fire sales.

Why is run synchronicity a cause of fragility? The main economic force behind this result is a two-layered coordination problem that naturally arises in our setting. First, typical to a bank-run model, there is a within-bank strategic complementarity. Investors who withdraw money from a bank early cause costly liquidations and impose negative externalities on those who stay. Second, there is a cross-bank strategic complementarity due to fire-sale spillovers. Key to our results is that these two complementarities amplify each other. An investor is more concerned about withdrawals in her bank when she expects investors of other banks to withdraw in their banks, forcing premature liquidations.

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1Popular systemic risk measures are based on tail comovement among financial institutions, e.g. Huang, Zhou, and Zhu (2009), Adrian and Brunnermeier (2016), Acharya, Pedersen, Philippon, and Richardson (2017), Brownlees and Engle (2017).
2We refer to individual institutions as banks, but our analyses can be applied to other types of institutions with runnable liabilities (e.g., corporate bond mutual funds, Goldstein, Jiang, and Ng, 2017).
and driving up the fire-sale discount. In other words, if withdrawals are synchronized across banks, cross-bank fire-sale spillovers are particularly detrimental to bank stability. Notably, in the absence of reinforcement between the two complementarities (e.g. if investors are equally concerned about fire-sale spillovers regardless of the run severity in their own banks), the degree of run synchronization across banks does not affect fragility.

Going back to the literature on systemic risk, it is important to highlight that our theory offers very distinct insights. Other theories feature a tension between systemic risk and risks of individual institutions: As interconnectedness rises, banks are more likely to fail together but weaker banks are less likely to fail.\(^3\) In our paper, such tension does not exist, since higher synchronization across banks makes each one of them more likely to fail in the first place. Hence, synchronicity is a first-order concern. This highlights the need to track synchronicity and mitigate it. Ignoring the impact of policies on run synchronicity would lead to an imprecise assessment of their effect on financial stability.

How can regulators reduce run synchronicity? To answer this question, we identify the main factors contributing to run synchronicity and point out policies that mitigate them. First, if banks hold similar assets, they are exposed to similar fundamental shocks, which naturally leads to synchronization of their withdrawals. Policies that reduce asset commonality, such as ring fencing, can thus reduce run synchronicity. Second, because investors are the ones who make withdrawal decisions, what matters for runs is perceived rather than physical asset commonality. By mandating disclosure of bank-specific information, regulators can reduce investors’ perceived asset commonality and thus run synchronicity. Third, since run decisions are shaped by investors’ beliefs about the magnitude of the fire-sale discount, liquidity conditions in the asset market also play an important role. We show that secondary market liquidity injections have a stronger positive effect on the liquidity conditions perceived by investors of banks with stronger fundamentals than those perceived by investors of weaker banks. Therefore, liquidity injections also reduce the synchronicity of withdrawals across banks.

We now describe the model and results in more detail. As is typical for models studying financial fragility, run decisions of bank investors, and thus financial fragility, are characterized by run thresholds. Specifically, run happens if and only if the aggregate fundamental falls below a bank-specific threshold. Therefore, low run thresholds indicate low fragility. In our model, all bank-specific thresholds are determined jointly in equilibrium. When deciding whether to withdraw their funds early, investors assess run

\(^3\)See, for example, Cabrales, Gottardi, and Vega-Redondo (2017) in the context of financial networks and Bouvard, Chaigneau, and Motta (2015) in the context of bank disclosure.
situations in all banks in the financial system. To pin down equilibrium run thresholds, we thus need to characterize investors’ beliefs about run decisions of other investors in her own bank and other banks in the financial system. Analytic characterization of these beliefs in a setting featuring interacting within- and cross-bank complementarities is one contribution of our paper.

There are two main factors that shape run thresholds. The first one is bank fundamentals. In the model, bank asset returns are subject to aggregate and idiosyncratic shocks. Idiosyncratic shocks separate banks into two types ex post: strong banks receiving a positive shock and weak banks receiving a negative shock.\(^4\) Strong banks are relatively more stable, and a worse aggregate shock is needed to trigger runs on these banks. Hence, they have a lower run threshold. Second, run thresholds depend on investors’ beliefs about the fire-sale discount. In the critical region, marginal strong-bank investors expect a higher fire-sale discount than marginal weak-bank investors do. This is because if strong-bank investors are on the margin of running or not, they expect that weak banks are experiencing severe run problems. Conversely, marginal investors of weak banks expect fewer runs on strong banks. Overall, reflecting both the fundamental effect and the fire-sale effect, the run thresholds of both types of banks are determined in equilibrium. The distance between the run thresholds of strong and weak banks captures run asynchronicity.

Our main theoretical result is that in the presence of reinforcing within- and cross-bank strategic complementarities, both weak and strong banks become more resilient to panic runs as run asynchronicity in the economy increases. As runs become more asynchronous, the distance between run thresholds increases. Strong banks’ stability is then challenged by the greater downward pressure that weak banks impose on liquidation prices, and weak banks’ fragility is alleviated by the lower pressure that strong banks impose. The key behind the decrease in the overall fragility is that the effect on weak banks dominates that on strong banks. This is a direct result of the fact that within- and cross-bank complementarities are mutually reinforcing. In particular, given that weak banks are more internally fragile, their investors are more strongly affected by the alleviated fire-sale pressure than strong-bank investors are affected by the intensified fire-sale pressure. Hence, fragility is lower when runs are asynchronous.

We further show that the stabilizing effect of an increase in run asynchronicity vanishes as it becomes sufficiently large. In particular, an increase in asynchronicity makes both strong and weak banks more resilient if investors in the two types of banks face strategic

\(^4\)Our main results go through if there are more than two bank types with zero-mean idiosyncratic shocks taking \(N \geq 2\) values.
uncertainty about each other. However, if asynchronicity becomes too large, strong-bank (weak-bank) investors that are on the margin of running expect all (no) weak-bank (strong-bank) investors to run, and the fire-sale pressure on their banks is maximized (minimized). In that case, an increase in asynchronicity no longer reshapes strategic beliefs and ceases to have an effect on fragility.

After establishing the key insight about synchronicity and fragility, we proceed to policy analyses. We show that in our model reducing bank fragility is welfare-improving because runs are associated with real resource losses due to fire sales. Thus, stabilizing financial sector is a natural goal for the regulator. We consider three policies, often discussed as part of financial regulation, and study their effect on fragility through the channel of synchronicity.

First, we consider a ring-fencing policy that separates bank balance sheets into different divisions according to business or geographical focuses.\(^5\) Such a policy increases the dispersion of asset returns across different bank divisions. In the model, this corresponds to an increase in the ex-post difference between weak and strong banks, making runs more asynchronous and the financial system more stable. The existing literature that examines the relation between asset commonality and financial stability has mostly focused on cascades of fundamental defaults in which investors play a passive role. This literature suggests that asset differentiation reduces systemic bank failures at the expense of more individual failures of weak banks.\(^6\) Simulation analyses suggest that purely fundamental default cascades are unlikely in modern financial systems (Elsinger, Lehar, and Summer, 2006; Upper, 2011). In contrast, we consider panic-based runs with investors actively withdrawing because of concerns about others’ withdrawals, precipitating illiquidity which then spreads across banks through fire-sale spillovers. Contagious panic-based runs can happen even if banks are quite far from being fundamentally insolvent. Importantly, we show that in our environment asset differentiation makes both strong and weak banks unequivocally more resilient to panic-based runs.

Second, we consider regulatory disclosure that affects the quality of bank-specific information available to investors. We extend the model by adding noise to investors’ information about bank-specific shocks. In this extended setting, we show that what matters for run asynchronicity is perceived differences between bank asset returns. In an opaque financial system, investors can hardly distinguish between strong and weak

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\(^5\) An example of such a policy is the U.K. Financial Services Act 2012 that requires large banks to isolate their core retail banking business from trading and riskier activities. The Volcker rule also aims to split retail and investment banking activities by prohibiting banks from conducting proprietary trading.

\(^6\) See, for example, Shaffer (1994), Wagner (2010 and 2011), and Ibragimov, Jaffee, and Walden (2011).
banks, so runs are synchronized. Disclosing bank-specific information enlarges the perceived dispersion in bank asset returns, which enlarges asynchronicity and stabilizes the financial system. While existing literature argues that disclosing bank-specific information can undermine stability of weak banks (e.g., Bouvard et al., 2015 and Goldstein and Leitner, 2018), our results suggest that disclosure can stabilize the entire financial sector, including weak banks, through alleviating fire-sale pressure on particularly fragile institutions.

The third policy we examine is secondary market liquidity injections, widely used by regulators during market turmoils in the 2007–2008 financial crisis and the COVID-19 pandemic. Different from ring-fencing and disclosure policies, a liquidity injection does not affect dispersion (or perceived dispersion) in bank fundamentals. However, it does affect run asynchronicity through reshaping investors’ beliefs about the fire-sale discount. In the model, a liquidity injection reduces the fire-sale discount for any amount of long-term assets liquidated by banks. Surprisingly, even though an injection is not targeted—that is, the regulator does not purchase assets owned by a particular group of banks—it tends to benefit strong banks more than weak banks. The reason is that strong banks start to experience runs when weak banks are already forced to liquidate a lot of their assets; that is, strong banks are under runs when the liquidity conditions are particularly dire. A liquidity injection thus provides a large implicit subsidy to strong banks, thereby further strengthening them relative to weak banks, desynchronizing runs, and stabilizing the financial system.

Importantly, different from ring-fencing and disclosure policies that influence financial fragility only through changing run asynchronicity, liquidity injections also have a direct stabilizing effect. That is, a reduction in the fire-sale discount following a liquidity injection benefits banks even if run asynchronicity is held fixed. To understand how this direct effect compares to the indirect one, we conduct a calibration exercise using the data on U.S. banks during the 2007–2008 crisis. We find that the indirect effect is substantial and accounts for roughly a quarter of the decline in the average run threshold following a liquidity injection in the asset market.

**Literature** Our model, featuring interconnected fragile banks, builds on two large strands of literature on financial fragility. The first one studies fragility of financial institutions due to strategic complementarities among investors and panic-driven runs

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5 A contemporaneous paper by Dai, Luo, and Yang (2021) argues that disclosure of banks’ systemic risk exposures—but not their idiosyncratic risks—can mitigate financial fragility. In our framework with reinforcing complementarities, disclosing bank-specific information can enlarge run asynchronicity which is beneficial for overall stability.
(Diamond and Dybvig, 1983). Settings with strategic complementarities typically feature multiple equilibria if there is no uncertainty about bank fundamentals. This may lead to ambiguous policy implications because the same policies can have different effects in different equilibria. We use the global games technique (Rochet and Vives, 2004; Goldstein and Pauzner, 2005) to pin down a unique equilibrium. Specifically, bank investors in our model are uncertain about the aggregate fundamental and thus cannot perfectly foresee run situations in their own and other banks. This modeling choice captures the reality of economy-wide financial crises since such crises are marked by heightened aggregate uncertainty and, in particular, uncertainties among market participants about others’ behaviors. The global games approach allows us to characterize the strategic beliefs of bank investors about run situations in their own banks and other banks in the financial system. This approach is particularly appealing for policy analysis as it ties the endogenous likelihood of a crisis to economic fundamentals, which allows us to make precise predictions about costs and benefits of different policy interventions.

The fact that fragilities of individual banks are interrelated through fire sales connects our paper to the second literature that studies contagion through fire-sale spillovers. Early contributions (Cifuentes, Ferrucci, and Shin, 2005; Diamond and Rajan, 2005) emphasize existence of cross-bank spillovers due to limited liquidity pool. Uhlig (2010) discusses microfoundations of fire-sale discounts faced by banks. More closely related to us are works by Liu (2016, 2018) and Eisenbach (2017) who study models in which bank runs are connected through secondary markets in the absence of uncertainty about the liquidation price. However, their focus is different from ours: Eisenbach (2017) studies the effectiveness of rollover risk as a market disciplinary device in the presence of fire-sale spillovers, and Liu (2016, 2018) shows that fire-sale spillovers might lead to instability due to equilibrium multiplicity. In contrast, we emphasize that the feedback loop between bank runs and secondary market liquidity conditions has heterogeneous effects on different banks in the financial system. We highlight that run synchronicity reflects the strength of the feedback loop and is a key driver of fragility. This also has important policy implications as reducing run synchronicity enhances financial stability.8

A few papers emphasize that regulators should target agents that impose a stronger externality on others in one-complementarity coordination games. Sákovics and Steiner (2012) consider a coordination game with heterogeneous agents and study optimal tar-

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8Settings with multilayered complementarities have been used to study other questions, e.g. capital and liquidity regulation (Carletti, Goldstein, and Leonello, 2020) and twin crises (Goldstein, 2005). Given their different focus, these papers do not talk about run synchronicity and its importance for fragility.
geted interventions. Shen and Zou (2020) propose policies that screen agents based on their heterogeneous information in global games and illustrate the efficiency of such policies in targeting agents with medium beliefs. Cong, Grenadier, and Hu (2020) argue that saving small banks is cheaper and can generate stronger informational externalities. Probably the closest to us is Choi (2014). He argues that regulators should support strong banks because fragility of strong banks affects weak banks on the margin but not vice versa. In our setting, fragilities of weak and strong banks always affect each other simultaneously.\(^9\) More importantly, our key economic mechanism—reinforcing within- and cross-bank complementarities—is absent from Choi (2014). We show that, due to reinforcing complementarities, reducing run synchronicity is beneficial for all banks. This economic mechanism is behind our novel policy analyses. In particular, we show that even untargeted policies such as liquidity injections can reduce run synchronicity and boost financial stability.

Our paper is also related to a series of studies on the role of asset commonality for systemic risk (Shaffer, 1994; Acharya, 2009; Stiglitz, 2010; Ibragimov et al., 2011; Wagner, 2010 and 2011; Allen, Babus, and Carletti, 2012; Cabrales et al., 2017; Kopytov, 2019).\(^10\) As discussed above, these papers focus on correlated fundamental bank defaults. They emphasize a trade-off between losses due to failures of individual institutions and systemic crises, which means that whether asset heterogeneity is desirable might depend on the distribution of shocks. For example, Cabrales et al. (2017) argue that homogeneous systems are socially desirable if negative shocks are expected to be small but heterogeneity is beneficial if shocks are large. In contrast, we focus on panic-driven runs. We show that what matters for financial stability is not the degree of asset commonality per se but rather run synchronicity. Furthermore, in our framework asset heterogeneity reduces run probability for all banks irrespective of distributional assumptions on shocks. This is consistent with empirical findings of Huang et al. (2009) who document that an increase in asset correlation among large U.S. banks leads to an increase in their individual default probabilities.

More generally, our paper makes a theoretical contribution to the global games literature pioneered by Carlsson and van Damme (1993) and later developed into various

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\(^9\) The difference arises because Choi (2014) uses a binary payoff structure similar to Morris and Shin (1998), while we use Diamond and Dybvig (1983)-like payoffs. In addition, he shows that, depending on the parameters, it can be optimal to support both types of banks or only weak banks. His main result emerges in the continuous-time model where the noise in investors’ signals has bounded support.

\(^10\) A recent paper by Song and Thakor (2022) shows how an endogenously arising interbank market can increase asset heterogeneity and improve intermediation efficiency.
topics such as currency attacks (Morris and Shin, 1998; Hellwig, Mukherji, and Tsyvinskii, 2006) and bank runs (Rochet and Vives, 2004; Goldstein and Pauzner, 2005). A key feature of a standard global games setting is the Laplacian property (Morris and Shin, 2003): A marginal investor is uninformed about the rank of her signal, and her belief about the mass of runners is uniform. Even when investors have heterogeneous payoffs, a version of the Laplacian property holds for the weighted average belief of marginal investors of different types (Sákovics and Steiner, 2012). In those settings, making runs less synchronous has no impact on investors’ run decisions on average, as the optimism of weak-bank investors about the actions of strong-bank investors is exactly offset by the pessimism of strong-bank investors. This is no longer true in our model featuring two types of interacting strategic complementarities. In our setting, the overall fragility depends on the weighted average of beliefs about the interaction between fire-sale pressure and runs in individual banks. These interaction terms are not symmetric across bank types and depend on the degree of run synchronicity.

The remainder of the paper is organized as follows. Section 2 lays out the model. Section 3 presents the main theoretical results about the relationship between run asynchronicity and fragility. Section 4 discusses how policies affect run asynchronicity, financial stability and welfare. Section 5 considers several extensions of the model. Section 6 concludes.

2 Model

We describe the model in Sections 2.1–2.5 and discuss several modeling choices in Section 2.6.

The economy is populated with three types of risk-neutral agents: banks, bank investors, and deep-pocketed outside investors. There are three periods, \( t = 0, 1, 2 \), and no time discounting.

2.1 Banks

There is a continuum of banks indexed by \( i \in [0, 1] \). At \( t = 0 \), bank \( i \) collects one unit of capital from a unit mass of investors in the form of demandable debt and makes long-term investment that generates a gross return of \( z_i \) at \( t = 2 \), where

\[
z_i = \theta + \zeta_i.
\]
Here $\theta$ is the aggregate component shared by all banks and $\zeta_i$ is the bank-specific component. The cumulative distribution function of the aggregate fundamental $\theta$ has a support $[\bar{\theta}, \hat{\theta}]$, where $\infty \geq \bar{\theta} > \hat{\theta} > 0$. We assume that the distribution of bank-specific shock $\zeta_i$ is

$$\zeta_i = \begin{cases} 
\eta & \text{with probability } \frac{1}{2}, \\
-\eta & \text{with probability } \frac{1}{2},
\end{cases}$$

where $\eta \geq 0$, so that the bank-specific fundamental $\zeta_i$ has a zero mean. In Section 5.2, we show that our main results hold in a setting where $\zeta_i$ is a zero-mean random variable that can take an arbitrary number of values $N \geq 2$ with arbitrary probabilities. The size of bank-specific shocks is restricted to be such that the overall productivity is always positive for all banks, $\bar{\theta} - \eta > 0$. Both $\theta$ and $\zeta_i$ are unknown at $t = 0$.

Upon shock realizations at $t = 1$, banks become heterogeneous. In particular, there are two groups of banks: strong banks with $\zeta_i = \eta$ and weak banks with $\zeta_i = -\eta$. The masses of the two groups are identical. Parameter $\eta$ governs the ex-post difference between performances of strong and weak banks and reflects banks’ asset heterogeneity. From the ex-ante perspective, $\eta$ affects the pairwise correlation between bank fundamentals,

$$\text{Corr} (z_i, z_j) = \frac{\mathbb{E} \theta + \mathbb{E} \zeta_i}{\sqrt{\mathbb{E} \theta^2 + \mathbb{E} \zeta_i^2}} = \frac{\mathbb{E} \theta}{\sqrt{\mathbb{E} \theta^2} + \eta^2}.$$

As $\eta$ increases, the relative importance of aggregate shocks declines and performances of bank assets become less correlated.

At $t = 1$, after both aggregate and bank-specific productivities are realized, bank investors may choose to withdraw their funds early. Under such circumstances, bank $i$ needs to repay one unit of capital to each runner and thus is forced to liquidate its long-term investment early in the asset market. The liquidation process and the investors’ early withdraw decisions are described in the two following subsections.

### 2.2 Outside investors and the asset market

At $t = 1$, if a mass $m_i$ of investors withdraw their funds early from bank $i$, bank $i$ needs to raise funds of amount $m_i$ by partially liquidating its long-term investment. This means that bank $i$ has to liquidate a fraction $\frac{m_i}{p_i}$ of its long-term investment, where $p_i$ is an endogenous liquidation price. Below, we describe the asset market and characterize the market-clearing prices $\mathbf{p} = \{p_i\}_{i \in [0,1]}$.

The asset market is competitive and populated with a unit mass of outside investors. Reminiscent of the cash-in-the-market pricing (Allen and Gale, 1994), liquidity is scarce in the asset market, which can cause asset prices to fall below their fundamental values. In particular, in order to purchase a portfolio $\{k_i\}_{i \in [0,1]}$ of bank assets, an outside investor
has to raise $L = \int p_ikidi$ units of cash, the opportunity cost of which is $g(L) \geq L$. We assume that $g(\cdot)$ is an increasing and convex function with $g'(0) = 1$. Therefore, $g(L) - L$ represents the cost of liquidity in the asset market or, equivalently, real losses due to fire sales. The fact that $g(L)$ is convex captures an increasing marginal cost of liquidity.

**Definition 1.** Given masses of runners $m = \{m_i\}_{i\in[0,1]}$ and bank fundamentals $z = \{z_i\}_{i\in[0,1]}$, an equilibrium in the asset market consists of outside investors’ demand functions $\{k_i(p, z)\}_{i\in[0,1]}$ and liquidation prices $p = \{p_i(m, z)\}_{i\in[0,1]}$ such that:

1. Given the liquidation prices $p$, outside investors’ demand functions $\{k_i(p, z)\}_{i\in[0,1]}$ maximize their expected payoffs:

$$\max_{\{k_i\}_{i\in[0,1]}} \int z_ikidi - g\left(\int pikidi\right).$$

2. The liquidation prices satisfy the market-clearing conditions: $m_i = p_ik_i \forall i \in [0,1]$.

The key feature of the asset market is that fire-sale externalities can spill over across banks. One interpretation of this feature is that banks face the same group of buyers of their assets (e.g. hedge funds). Even if asset markets for different banks are separated, arbitrage capital might flow across these markets, leading to comoving fire-sale discounts.

The following lemma summarizes the key properties of the liquidation prices.

**Lemma 1.** Given masses of runners $m$ and bank fundamentals $z$, the equilibrium liquidation price for bank $i$’s assets is

$$p_i = p(z_i, m) = \frac{z_i}{\lambda(m)} \forall i \in [0,1],$$

where $m \equiv \int m_idi$ is the total mass of runners in the economy and $\lambda(m) \equiv g'(m)$ is a strictly increasing function.

**Proof.** See Appendix A.1. \qed

The liquidation prices of bank assets are proportional to their productivities $z_i$’s and are subject to a common fire-sale discount factor $\lambda(m)$. The discount factor $\lambda(m)$ increases in the total mass of runners in the entire financial system. Intuitively, if more bank investors withdraw their funds early, banks have to raise more liquidity in the asset market.

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11The assumptions on $g(\cdot)$ can be relaxed to accommodate cash reserves of outside investors. See Section 4.3 for the exercise in that spirit.
market. Because the marginal cost of liquidity is an increasing function, i.e. \( g''(\cdot) > 0 \), the price discount factor \( \lambda(m) \) increases if more bank investors withdraw early. This is akin to the key property of the cash-in-the-market pricing: asset prices fall as the liquidity demand exceeds what is available in the market.\(^{12}\)

### 2.3 Bank investors and runs

This section describes the behavior of bank investors. For each bank \( i \), there is a unit mass of infinitesimal investors indexed by \( l \in [0, 1] \). At \( t = 0 \), each investor contributes one unit of capital to her bank. At that point, bank-specific fundamentals are unknown and so investors are indifferent about which bank to invest in.

At \( t = 1 \), an investor \( l \) of bank \( i \) observes the bank-specific fundamental \( \zeta_i \) and receives a noisy private signal \( s_{il} \) about the aggregate fundamental \( \theta \),\(^{13}\)

\[
s_{il} = \theta + \sigma \epsilon_{il}
\]

(1)

The signal noise \( \epsilon_{il} \) has a cumulative distribution function \( F_{\epsilon}(\cdot) \), which is differentiable and strictly increasing on its support \([\xi, \bar{\epsilon}]\). A corresponding probability density function is denoted by \( f_{\epsilon}(\cdot) \). In what follows, we work with a bounded noise support, \(-\infty < \xi < \bar{\epsilon} < \infty\), but our analyses carry through if it is unbounded. Without loss of generality, we assume that \( \mathbb{E}\epsilon_{il} = 0 \) and \( \mathbb{V}\epsilon_{il} = 1 \), so that \( \sigma \) is the standard deviation of the private signal. This information structure follows a conventional global games setup.

With probability \( \bar{m} \in (0, 1) \), investor \( l \) is “non-sleepy” and may withdraw her funds from her bank at \( t = 1 \). With probability \( 1 - \bar{m} \), investor \( l \) is “sleepy” (for example, due to limited attention) and neglects the option to withdraw early. Therefore, bank \( i \) needs to liquidate at most a fraction \( \bar{m} \frac{p_i}{\bar{p}_i} \) of its assets if all “non-sleepy” investors withdraw their funds early. We rule out bank failures by assuming that \( \bar{m} \frac{p_i}{\bar{p}_i} \leq 1 \) (as in Chen, Goldstein, and Jiang, 2010). This assumption is made for technical reasons; we discuss it in more detail in Section 2.6. Importantly, although banks do not go bankrupt, bank runs are still subject to real losses due to inefficient fire sales.

\(^{12}\)In Allen and Gale (1994), \( g(L) = \infty \) if \( L \) exceeds a certain threshold \( \bar{L} \), so outsiders are constrained by the amount of their cash reserves. We allow outsiders to raise additional funds when they run out of cash, so that banks’ demand for liquidity can always be satisfied.

\(^{13}\)Section 4.2 extends the model by allowing for partially informative signals about bank-specific fundamentals. Under imperfect information about the aggregate fundamental \( \theta \), there exist strategic uncertainties both within and across banks, which allows investors to coordinate and supports equilibrium uniqueness. Under perfect information about \( \theta \), multiplicity is possible (Liu, 2016).
In the absence of bank failures, early withdrawers are guaranteed to get their funds back at \( t = 1 \). At \( t = 2 \), the investment return to bank \( i \) is equally distributed among investors who have not withdrawn their funds at \( t = 1 \). Let \( u_i(a_{il}) \) denote a “non-sleepy” investor \( l \)'s payoff conditional on her decision to withdraw early \( a_{il} \in \{ \text{run}, \ \text{stay} \} \) from bank \( i \).

\[
u_i(a_{il}) = \begin{cases} 1 & \text{if } a_{il} = \text{run}, \\ z_i \left( \frac{1 - m_i}{p_i} \right) & \text{if } a_{il} = \text{stay}. \end{cases}\]

Plugging in the market-clearing liquidation price \( p_i \) derived in Lemma 1, we can express the incremental payoff from staying as

\[
\pi(z_i, m_i, m) \equiv u_i(\text{stay}) - u_i(\text{run}) = \frac{z_i - m_i \lambda(m)}{1 - m_i} - 1.
\]

The incremental payoff represents an investor’s incentive to stay. In particular, a “non-sleepy” investor \( l \) of bank \( i \) runs if and only if the expected incremental payoff given her signal is negative,

\[
E[\pi(z_i, m_i, m)|s_{il}] < 0.
\]

Equation (2) reveals two types of strategic complementarities featured by our model. First, there is a within-bank strategic complementarity: an investor’s incremental payoff from staying declines if more investors in her own bank run.\(^{14}\) On top of that, the fire-sale externalities in the asset market give rise to a cross-bank strategic complementarity: an investor’s incremental payoff from staying declines if more investors in other banks run. More importantly, these two complementarities amplify each other, namely,

\[
\frac{\partial^2 \pi(z_i, m_i, m)}{\partial m_i \partial m} = - \frac{\lambda(m)}{(1 - m_i)^2} < 0.
\]

Banks that encounter more runs (higher \( m_i \)'s) have to liquidate more assets to repay runners. That naturally makes the long-term payoffs of non-running investors more sensitive to fluctuations in the liquidation prices, which in turn depend on the total mass of runners in the economy \( m \). In Section 3.2, we discuss this feature in detail and explain

\(^{14}\)Note that when \( \lambda(m) < z_i \), the fire-sale discount is small relative to long-term asset return, so the incremental payoff function (2) implies within-bank strategic substitution. However, as we show in Appendix C, the incremental payoff function satisfies a single-crossing property, which, together with mild assumptions on the noise distribution, are sufficient for the uniqueness of threshold equilibrium.
its importance for our results.

2.4 Timeline and equilibrium definition

Figure 1 depicts the timeline of our model.

<table>
<thead>
<tr>
<th>Banks receive funding and invest</th>
<th>( \theta ) and ( \zeta_i )'s are realized</th>
<th>Investors receive private signals</th>
<th>“Non-sleepy” investors decide whether to run</th>
<th>Banks liquidate assets to repay runners</th>
<th>Investors who stay get repaid</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0 )</td>
<td></td>
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<td></td>
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<tr>
<td>( t = 1 )</td>
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<tr>
<td>( t = 2 )</td>
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</tbody>
</table>

Figure 1: Timeline

Denote by \( \{a_{il}(s_{il}, \zeta_i)\}_{i \in [0,1]} \) the set of strategies of “non-sleepy” investors that map bank-specific fundamentals \( \zeta_i \) and their private signals about the aggregate fundamental \( s_{il} \) to their action space \( a_{il} \in \{\text{run}, \text{stay}\} \).

**Definition 2.** Bank investors’ strategies \( \{a_{il}(s_{il}, \zeta_i)\}_{i \in [0,1]} \), outside investors’ demand functions \( \{k_i(p, z)\}_{i \in [0,1]} \), liquidation prices \( p = \{p_i(m, z)\}_{i \in [0,1]} \), and masses of runners \( m = \{m_i(\theta, \zeta_i)\}_{i \in [0,1]} \) constitute an equilibrium if

1. Given \( m \) and \( z \), \( \{k_i(p, z)\}_{i \in [0,1]} \) and \( p \) constitute an equilibrium in the asset market as in Definition 1;

2. Given her private signal and the bank-specific fundamental of her bank, each “non-sleepy” investor forms beliefs about \( p \) and \( m \) and runs if and only if (3) holds;

3. \( m_i(\theta, \zeta_i) = \int \mathbb{1} \{a_{il}(s_{il}, \zeta_i) = \text{run}\} dl \).

2.5 Global games and threshold equilibrium

We focus on threshold equilibria in which investor \( l \) of bank \( i \) runs when \( s_{il} < \theta_i^* \) and stays when \( s_{il} > \theta_i^* \). We characterize the threshold equilibrium in this section and prove its uniqueness in Appendix C. Throughout most of the paper, we focus on the limiting case of infinitely precise signals (\( \sigma \to 0 \)) as is typical in the global games literature. As such, run thresholds should be understood as limits with \( \sigma \to 0 \). However, for brevity and when it does not cause confusion, we do not explicitly write endogenous variables as

\[ \text{Because the incremental payoff function (2) does not always imply within-bank strategic complementarity, non-threshold equilibria cannot be ruled out. However, under additional assumptions on the noise distribution—e.g., if it is uniform—it can be shown that non-threshold equilibria do not exist.} \]
limits. For example, in most of the cases we write a bank $i$’s run threshold as $\theta^*_i$, not as $\lim_{\sigma \to 0} \theta^*_i(\sigma)$. We show that our results are robust to non-negligible noise in Section 5.1.

A threshold investor with a signal $s_{il} = \theta^*_i$ is indifferent between running and staying,

$$\int_0^1 \frac{\theta(x) + \zeta - \lambda (m^\text{tot}_i(x)) \bar{m}x}{1 - \bar{m}x} dx = 1,$$

(5)

where $x$ is the fraction of “non-sleepy” investors running on bank $i$. As is standard in global games models, a threshold investor with a signal $\theta^*_i$ has a Laplacian belief. That is, she believes that the mass of runners within her own bank, $\bar{m}x = \bar{m} F_\epsilon \left( \frac{\theta^*_i - \theta}{\sigma} \right)$, is uniformly distributed (Morris and Shin, 2003). Crucially, because her signal is informative about the aggregate productivity, she also makes inference about actions of investors of other banks. From the perspective of the threshold investor receiving the signal $\theta^*_i$, the aggregate fundamental is $\theta(x) = \theta^*_i - \sigma F^{-1}_\epsilon(x)$ and the amount of runs on bank $j$ is $\bar{m} F_\epsilon \left( \frac{\theta^*_j - \theta(x)}{\sigma} \right)$. The total mass of runners in the economy from the perspective of this investor is then

$$m^\text{tot}_i(x) = \bar{m} \int F_\epsilon \left( \frac{\theta^*_j - \theta(x)}{\sigma} \right) dj = \bar{m} \int F_\epsilon \left( \frac{\theta^*_j - \theta^*_i + F^{-1}_\epsilon(x)}{\sigma} \right) dj.$$

(6)

Taking the limit $\sigma \to 0$ in (5) and using (6), we can write thresholds $\theta^*_i$ for investors of all banks in the economy. Specifically, given the binary structure of bank-specific shocks, the run thresholds of strong and weak banks, denoted as $\theta^*_s$ and $\theta^*_w$, are determined by

$$\theta^*_s + \eta = \frac{1}{\int_0^1 \frac{1}{1 - \bar{m}x} dx} (1 + I_s(\Delta)),$$

(7)

$$\theta^*_w - \eta = \frac{1}{\int_0^1 \frac{1}{1 - \bar{m}x} dx} (1 + I_w(\Delta)),$$

(8)

where

$$I_s(\Delta) \equiv \int_0^1 \lambda \left( \frac{\bar{m}x}{2} + \frac{\bar{m}}{2} F_\epsilon \left( \Delta + F^{-1}_\epsilon(x) \right) \right) \frac{\bar{m}x}{1 - \bar{m}x} dx,$$

(9)

$$I_w(\Delta) \equiv \int_0^1 \lambda \left( \frac{\bar{m}x}{2} + \frac{\bar{m}}{2} F_\epsilon \left( -\Delta + F^{-1}_\epsilon(x) \right) \right) \frac{\bar{m}x}{1 - \bar{m}x} dx,$$

(10)
and

$$\Delta \equiv \lim_{\sigma \to 0} \frac{\theta^*_w - \theta^*_s}{\sigma}. \quad (11)$$

Here $I_s(\Delta)$ and $I_w(\Delta)$ represent expected fire-sale losses borne by threshold investors with Laplacian beliefs in strong and weak banks respectively. In what follows, we refer to $I_s(\Delta)$ and $I_w(\Delta)$ as fire-sale pressure on strong and weak banks. Notably, fire-sale pressure on a given bank is exerted by all banks, including banks of its own group.

The endogenous variable $\Delta$ is the distance between the two run thresholds, which measures the extent to which runs are synchronized and tail-risks comove across strong and weak banks. Hence, we call $\Delta$ the run asynchronicity. It is straightforward to verify that $\Delta \geq 0$.\footnote{Suppose not. Then $\theta^*_w < \theta^*_s$ and $I_w(\Delta) > I_s(\Delta)$, which contradict (7) and (8) because $\eta \geq 0$.} If $\Delta = 0$, runs are perfectly synchronized—that is, each threshold investor believes that all banks in the economy always face the same amount of runs. If $\Delta > 0$, threshold investors of strong banks believe that weak banks face more severe run problems. In contrast, threshold investors of weak banks believe that strong banks face less severe run problems. Therefore, if $\Delta > 0$, runs are asynchronous.

Run asynchronicity governs cross-bank interactions in this economy. Because banks impose fire-sale externalities on one another, investors need to evaluate the run situations in all banks simultaneously to make their own decision to withdraw early. Consider a threshold investor of a strong bank as an example. As shown in Equation (9), if a fraction $x$ of “non-sleepy” investors run on strong banks, she expects a fraction $F(\Delta + F^{-1}(x)) \geq x$ of “non-sleepy” investors to run on weak banks. $\Delta$ plays an important role in shaping investors’ beliefs about the total amount of runs in the economy and thus the fire-sale discount, which in turn governs their own run decisions.

2.6 Discussion of assumptions

We conclude this section by discussing a few modeling choices.

**Fire sales** In our baseline model, the fire-sale discount emerges from the scarcity of outside liquidity. In Appendix B.1, we provide another microfoundation for the fire-sale discount in which outside investors are less efficient than banks in managing assets or incur inventory costs when holding them (Shleifer and Vishny, 1992; Kiyotaki and Moore, 1997). We show that that the pricing function takes the same form as in Lemma 1. Hence, our results are robust to various sources of fire-sale externalities.
At the same time, the fact that fire sales are inefficient, in the sense that the liquidation price $p_i$ falls below its value in best use $z_i$, is crucial for our mechanism. In reality, there exist forces mitigating fire sales. For example, strong banks may acquire long-term projects of weak banks in the interbank market without liquidating them (Allen and Gale, 2000). Similarly, strong banks may provide loans to weak banks facing early withdrawals. While such mitigating forces certainly exist, they are unlikely to fully eliminate fire sales and bank runs in practice. Section 4.3 discusses how our results are affected by liquidity injections that ameliorate fire sales by increasing the amount of outside liquidity at $t = 1$.

Liquidity buffers In our model, banks can only invest in a long-term project. In reality, banks that face a possibility of early withdrawals have incentives to hold liquid assets to reduce liquidations of a long-term project in the event of a run. However, banks are unlikely to invest so much in liquid assets that the possibility of runs is fully eliminated because that would require substantially limiting their investments in productive long-term projects (Liu and Mello, 2011; Carletti et al., 2020; Kashyap, Tsomocos, and Vardoulakis, 2020). If runs are possible, our results remain unchanged qualitatively. Hence, we do not model banks’ liquidity choice explicitly.

“Sleepy” investors We follow Chen et al. (2010) and assume that the fraction of investors that can withdraw funds early is $\bar{m} < 1$ such that bank failures are ruled out. This assumption is made for technical reasons. Goldstein and Pauzner (2005) show that the possibility of bank failures creates a region of strategic substitution, making the analysis much more technically involved. We eliminate this region by assuming that investors have limited attention, modeled by a constant upper bound on the fraction of investors who can run. While this assumption rules out complete bank failures, the key property of runs—that they reduce amount of funds to stayers—is preserved. Furthermore, runs are socially undesirable in our model because they trigger fire sales and real losses. We formally define welfare in Section 4.

It is possible to model investor inattention differently, for example, by considering an extension in which $\bar{m}$ is correlated with economic fundamentals. In particular, it is plausible to assume that investors are less likely to be inattentive in bad times. Making $\bar{m}$ a decreasing function of the aggregate fundamental $\theta$ is likely to worsen the within-bank strategic complementarity. However, we view this extension to be outside the scope of this paper because it creates technical complications and is not directly related to our key mechanism of interacting within- and cross-bank complementarities.
3 Run synchronicity and financial stability

In this section, we analyze the model described in the previous section. Section 3.1 presents our main result on the relation between run asynchronicity and financial stability. Section 3.2 discusses how reinforcing complementarities contribute to this result.

3.1 Characterizing the relation

We decompose our analyses into two cases. We first focus on the interesting and realistic case featuring nontrivial cross-bank strategic interactions ($\Delta < \bar{\Delta} \equiv \bar{\epsilon} - \epsilon$). We then discuss the other case with no cross-bank strategic interactions ($\Delta \geq \bar{\Delta}$).

Nontrivial cross-bank strategic interactions If run asynchronicity is not too large, $\Delta < \bar{\Delta} \equiv \bar{\epsilon} - \epsilon$, the financial system features nontrivial cross-bank strategic uncertainties. In this case, threshold investors of strong banks are uncertain about run behavior of weak-bank investors, and vice versa. Formally speaking, it implies that fire-sale pressure terms $I_s(\Delta)$ and $I_w(\Delta)$, given by (9) and (10), depend on $\Delta$.

Using Equations (7) and (8), we can write average fragility $\theta^* \equiv \frac{1}{2} \theta^*_s + \frac{1}{2} \theta^*_w$ as

$$\theta^* = \frac{1}{\int_0^1 \frac{dx}{1 - m_0 x}} \left( 1 + \frac{1}{2} I_s(\Delta) + \frac{1}{2} I_w(\Delta) \right),$$

where run asynchronicity $\Delta$ is implicitly given by

$$\eta = \frac{1}{2 \int_0^1 \frac{dx}{1 - m_0 x}} (I_s(\Delta) - I_w(\Delta)).$$

By definition of run asynchronicity in the limit of negligible information friction (11), a finite $\Delta$ implies that the run thresholds of weak and strong banks are infinitely close to each other, i.e. $\lim_{\sigma \to 0} \theta^*_s = \lim_{\sigma \to 0} \theta^*_w = \lim_{\sigma \to 0} \theta^*_w$. The result that individual run thresholds are infinitely close is a typical feature of global games with heterogeneous players and infinitely precise signals (Frankel, Morris, and Pauzner, 2003). With non-negligible information friction ($\sigma > 0$), run thresholds of weak and strong banks are distinct and centered around the average threshold, $\theta^*_s(\sigma) < \theta^*(\sigma) < \theta^*_w(\sigma)$. We show in Section 5.1 how our results extend to this case.

The fact that $\theta^*_s$ and $\theta^*_w$ are infinitely close to each other implies that, from $t = 0$ point of view, fragilities of both weak and strong banks are characterized by just one threshold
θ∗. Therefore, θ∗ measures ex-ante fragility of all banks in this economy. Importantly, although θ∗s and θ∗w are infinitely close to each other, there still can be a nontrivial difference between them from investors’ perspective at the moment of runs. This is because investors receive infinitely precise signals about the aggregate fundamentals at \(t = 1\) and thus can distinguish between θ∗s and θ∗w even if they are infinitely close. As a result, run asynchronicity ∆, which by (11) is the distance between θ∗w and θ∗s in units of \(\sigma\), can still be finite. Crucially, as ∆ determines the strength of cross-bank strategic interactions, it has a first-order effect on the common run threshold θ∗.

Equation (12) shows how θ∗ is affected by run asynchronicity ∆ and model primitives, whereas (13) implicitly defines run asynchronicity ∆ as the function of model primitives. Naturally, such a formulation allows us to decompose the effect of any regulatory policy on financial stability into two parts. Holding ∆ fixed, a change in any policy-related parameter \(v\) can have a direct effect on θ∗. This channel captures policy impact holding cross-bank interactions fixed. Furthermore, a policy can affect θ∗ indirectly through reshaping cross-bank interactions and thus run asynchronicity ∆. That is,

\[
\frac{d\theta^*}{dv} = \frac{\partial \theta^*}{\partial v} + \frac{\partial \theta^*}{\partial \Delta} \frac{d\Delta}{dv}.
\]

The focus of our paper is on the indirect channel that has received much less attention in the existing literature compared to the direct one. The sign of the indirect effect depends on two components. First, it depends on the relation between run asynchronicity and financial stability \(\frac{\partial \theta^*}{\partial \Delta}\). Proposition 1 below uses Equation (12) to establish that asynchronous systems are more stable, namely, \(\frac{\partial \theta^*}{\partial \Delta} < 0\). Section 3.2 discusses economic forces behind this result. Second, the indirect effect depends on how the policy variable affects run asynchronicity, that is, on \(\frac{d\Delta}{dv}\). Equation (13) reveals various determinants of run asynchronicity, such as asset heterogeneity \(\eta\) and liquidity conditions in the asset market that shape fire-sale pressure faced by strong and weak banks. In Section 4, we examine three specific regulatory policies and discuss how they affect stability both directly and indirectly.

**Proposition 1.** When \(\Delta < \bar{\Delta} = \bar{\epsilon} - \epsilon\), run thresholds of strong and weak banks are infinitely close to the average threshold θ∗, which captures the fragility of the financial system. Moreover, \(\theta^* = \theta^*(\Delta)\) decreases in run asynchronicity ∆.

**Proof.** See Appendix A.2.
Equation (12) reveals that the common run threshold $\theta^*$ is determined by the average fire-sale pressure on strong and weak banks, $\frac{1}{2} I_s(\Delta) + \frac{1}{2} I_w(\Delta)$. Recall that, by definition of run asynchronicity (11), $\Delta$ is the normalized distance between run thresholds of weak and strong banks. From the perspective of strong-bank investors, weak banks become more fragile as $\Delta$ goes up. As a result, strong-bank investors expect a larger fire-sale externality imposed by weak banks. Mathematically, it is captured by the fact that the fire-sale pressure on strong banks $I_s(\Delta)$ increases in $\Delta$. At the same time, in the view of weak-bank investors, strong banks become more stable, and so these investors expect lower fire-sale pressure $I_w(\Delta)$. Proposition 1 shows that the latter effect is more substantial than the former, that is, the average fire-sale pressure and the run threshold $\theta^*$ decrease with $\Delta$. Intuitively, this is because weak banks suffer from more severe runs in any state of the world and are therefore more sensitive to changes in the fire-sale pressure than strong banks. Section 3.2 illustrates in detail that the key force behind this result is mutually reinforcing within- and cross-bank complementarities.

It is worth emphasizing that both weak and strong banks become more stable as the difference between their fragilities $\Delta$ enlarges. This can be clearly seen in the case with negligible information friction as weak and strong banks have infinitely close run thresholds. In Section 5.1, we analyze the non-limiting case where the run thresholds of weak and strong banks decouple. In that case, the average fragility still declines in $\Delta$, and both weak and strong banks tend to benefit from a higher $\Delta$ unless the signal noise is too large.

**Trivial cross-bank strategic interactions** When the financial system features large asynchronicity, $\Delta \geq \bar{\Delta}$, there are no strategic uncertainties across investors of weak and strong banks. That is, there are no meaningful cross-bank strategic interactions. In particular, threshold investors of strong banks are certain that all “non-sleepy” investors of weak banks will run; similarly, threshold investors of weak banks are certain that no investor in strong banks will run. Therefore, when investors make their run decisions, they only need to make inference about behaviors of investors in the same type of banks. In this case, fire-sale pressure on both strong and weak banks, $I_s(\Delta)$ and $I_w(\Delta)$, no longer depend on $\Delta$, and runs in strong and weak banks can be analyzed separately.

Formally, Equations (7) and (8) for the run thresholds of strong and weak banks
become
\[
\theta^*_s = \frac{1}{\int_0^1 \frac{1}{1-mx} dx} \left( 1 + \int_0^1 \lambda \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} \frac{\bar{m}x}{1-\bar{m}x} dx \right) - \eta, \right.
\]
\[
\theta^*_w = \frac{1}{\int_0^1 \frac{1}{1-mx} dx} \left( 1 + \int_0^1 \lambda \left( \frac{\bar{m}}{2} x \right) \frac{\bar{m}x}{1-\bar{m}x} dx \right) + \eta.
\]

In this case, run thresholds of strong and weak banks decouple, and neither varies with \( \Delta \). As a result, the average fragility of banks \( \theta^* \) does not vary with \( \Delta \) either.

### 3.2 Role of reinforcing complementarities

The key force that lies behind Proposition 1 is that within- and cross-bank complementarities are mutually reinforcing, which in the model is captured by (4). Weak banks are more sensitive to changes in the fire-sale discount because they experience more runs and need to liquidate more assets. As we discussed in the previous section, when \( \Delta < \bar{\Delta} \), an increase in \( \Delta \) alleviates cross-bank fire-sale pressure on weak banks and worsens that on strong banks. However, since weak banks are more sensitive to the change, the benefit for weak banks outweighs the loss for strong banks, and the overall stability increases. In this section, we illustrate this point formally by comparing marginal impacts of \( \Delta \) on \( I_w(\Delta) \) and \( I_s(\Delta) \).

First, consider the fire-sale pressure on strong banks. Equation (9) can be written as
\[
I_s(\Delta) = \int_{\bar{\epsilon}}^{\epsilon-\Delta} \lambda \left( \frac{\bar{m}}{2} F_\epsilon(\epsilon_s) + \frac{\bar{m}}{2} F_\epsilon(\Delta + \epsilon_s) \right) \frac{mF_\epsilon(\epsilon_s)}{1-mF_\epsilon(\epsilon_s)} dF_\epsilon(\epsilon_s) + \int_{\bar{\epsilon}}^{\epsilon-\Delta} \lambda \left( \frac{\bar{m}}{2} F_\epsilon(\epsilon_s) + \frac{\bar{m}}{2} \right) \frac{mF_\epsilon(\epsilon_s)}{1-mF_\epsilon(\epsilon_s)} dF_\epsilon(\epsilon_s),
\]
where we change the variable of integration \( x = F_\epsilon(\epsilon_s) \). Here, \( \epsilon_s \) represents the realization of signal noise for a threshold investor of a strong bank. From (1), \( \epsilon_s = \frac{\theta - \theta^*}{\sigma} \) for a strong-bank investor receiving a threshold signal \( \theta^*_s \). Because investors receiving worse signals withdraw their funds prematurely, the mass of runners on each strong bank is \( mF_\epsilon(\epsilon_s) \). Similarly, the mass of runners on each weak bank is \( mF_\epsilon(\Delta + \epsilon_s) \).
An increase in run asynchronicity bolsters the fire-sale pressure on strong banks:

\[
\frac{\partial I_s}{\partial \Delta} = 
\int_{E^-}^{E^+} f_s(\epsilon_s) \times \frac{\bar{m}}{2} f_s(\Delta + \epsilon_s) \times \lambda' \left( \frac{\bar{m}}{2} F_s(\epsilon_s) + \frac{\bar{m}}{2} F_s(\Delta + \epsilon_s) \right) \frac{\bar{m} F_s(\epsilon_s)}{1 - \bar{m} F_s(\epsilon_s)} d\epsilon_s > 0. \tag{15}
\]

The integrand can be decomposed into three parts. First, \(A_{s,1}\) represents the probability of a state in which a threshold investor receives a signal with a noise realization \(\epsilon_s\). As \(\Delta\) goes up, this investor expects a larger mass of runners on weak banks, which is captured by the second term \(A_{s,2}\). More runs on weak banks increase the fire-sale discount on the asset market and reduces the payoff to the threshold investor. The third term \(A_{s,3}\) captures the marginal decrease in the payoff. Integrating over all possible states yields the total change in the fire-sale pressure on a strong bank in response to a marginal increase in \(\Delta\).

At the same time, an increase in run asynchronicity alleviates the fire-sale pressure on weak banks:

\[
\frac{\partial I_w}{\partial \Delta} = 
- \int_{E^-+\Delta}^{E^+} f_w(\epsilon_w) \times \frac{\bar{m}}{2} f_w(-\Delta + \epsilon_w) \times \lambda' \left( \frac{\bar{m}}{2} F_w(-\Delta + \epsilon_w) + \frac{\bar{m}}{2} F_w(\epsilon_w) \right) \frac{\bar{m} F_w(\epsilon_w)}{1 - \bar{m} F_w(\epsilon_w)} d\epsilon_w = 
- \int_{E^-}^{E^+} f_s(\Delta + \epsilon_s) \times \frac{\bar{m}}{2} f_s(\epsilon_s) \times \lambda' \left( \frac{\bar{m}}{2} F_s(\epsilon_s) + \frac{\bar{m}}{2} F_s(\Delta + \epsilon_s) \right) \frac{\bar{m} F_s(\epsilon_s)}{1 - \bar{m} F_s(\epsilon_s)} d\epsilon_s < 0, \tag{16}
\]

where the second equality is obtained by changing the variable of integration, \(\epsilon_s = \epsilon_w - \Delta\). Same as (15) for the strong banks, the integrand can be decomposed into three parts.

Compare the magnitudes of marginal impact of \(\Delta\) on strong and weak banks, i.e. the absolute values of (15) and (16). For a given realization of the aggregate fundamental \(\theta\), if a threshold investor of a strong bank has a signal noise realization \(\epsilon_s = \frac{\theta - \theta^*_s}{\sigma}\), a threshold investor of a weak bank must have a noise realization \(\epsilon_w = \frac{\theta - \theta^*_w}{\sigma} = \Delta + \epsilon_s\). These two investors hold the same beliefs about the run situations in the economy: The mass of runs in all strong and all weak banks are \(\bar{m} F_s(\epsilon_s)\) and \(\bar{m} F_w(\epsilon_s + \Delta)\), respectively. From the perspectives of these investors, changes in the masses of runners in response to an increase in \(\Delta\) are symmetric, such that \(A_{s,1} \times A_{s,2} = A_{w,1} \times A_{w,2}\). Moreover, since they
hold the same view on the aggregate runs in the economy, they expect the same marginal impact on the fire-sale discount, captured by $\lambda \left( \frac{m}{2} F_e(\epsilon_s) + \frac{m}{2} F_e(\Delta + \epsilon_s) \right)$. However, the resulting changes in their payoffs are different. Specifically, weak banks experience more runs than strong banks, i.e. $F_e(\Delta + \epsilon_s) > F_e(\epsilon_s)$. Due to reinforcing within- and cross-bank complementarities, the same change in the fire-sale cost has a more profound effect on weak-bank investors than on strong-bank investors, which implies that $A_{w,3} > A_{s,3}$. Because this is true for any realization of the aggregate fundamental, we have

$$ \frac{\partial I_w}{\partial \Delta} > \frac{\partial I_s}{\partial \Delta}. $$

From Equation (12), it then follows that $\frac{\partial \theta^*}{\partial \Delta} < 0$, that is, financial systems in which runs are more asynchronous across banks are more stable. This result has important policy implications (indirect effect in Equation (14)). The fact that the two types of complementarities are mutually reinforcing plays the decisive role for this result.

In our model, mutually reinforcing complementarities is a natural implication of the existence of asset fire sales and a standard Diamond and Dybvig (1983) payoff structure. The following proposition formally establishes, with a more general payoff function, the importance of mutually reinforcing complementarities for our result in Proposition 1.

**Proposition 2.** Consider an investor of bank $i$ whose incremental payoff from staying is $\pi(z_i, m, m) = z_i \pi_1(m_i) + \pi_2(m_i, m)$, where $\pi_1(m_i)$ is positive and $\pi_2(m_i, m)$ decreases in $m$. In any threshold equilibrium, if $\Delta < \bar{\Delta}$, run thresholds of strong and weak banks are infinitely close to the average threshold $\theta^*$. Moreover, if $\frac{\partial^2 \pi}{\partial m \partial m} < 0$ then $\frac{\partial \theta^*(\Delta)}{\partial \Delta} \geq 0$.

**Proof.** See Appendix A.2.

In the baseline setting, $\pi_1(m_i) = \frac{1}{1-m_i}$ and $\pi_2(m_i, m) = -\frac{\lambda(m)m_i}{1-m_i}$, so we have two complementarities reinforcing each other, i.e., $\frac{\partial^2 \pi}{\partial m \partial m} < 0$. Appendix B.2 further generalizes Proposition 2 by imposing even milder restrictions on the incremental payoff function.

Note that if $\frac{\partial^2 \pi}{\partial m \partial m} = 0$, then $\Delta$ does not affect the common threshold $\theta^*$. This result echoes Sákovics and Steiner (2012). They show that in global games with heterogeneous agents, the weighted average belief about the aggregate action—economy-wide amount of runs in our model—is uniformly distributed. Moreover, in the absence of the interaction between the within- and cross-bank complementarities, only this weighted average belief matters for the common run threshold. Run asynchronicity therefore does not affect the run threshold.
When the two complementarities do interact, the run threshold depends on the interaction terms between the amounts of runs in the whole economy and within a particular bank. Therefore, the powerful result of Sákovics and Steiner (2012) does not hold in our setting, making the analyses much more cumbersome. By comparing (15) and (16), we can see that the interaction terms are not symmetric across weak and strong banks: a reduction in the fire-sale discount benefits weak banks more than an increase in the fire-sale discount of the same size hurts strong banks. A resulting sizable reduction in fragility of weak banks has a positive effect on strong banks, thus pushing the whole financial system to an equilibrium marked by a higher financial stability.

4 Regulatory policies and run asynchronicity

The previous section shows that run asynchronicity has important effects on financial stability in the presence of reinforcing complementarities. Notably, moving from stability to welfare is straightforward in our model. Since all agents are risk-neutral, welfare at $t = 0$ can be measured as the expected output from banks’ long-term projects net of the losses due to inefficient fire sales. Specifically,

$$W = \int \theta dF_\theta(\theta) - F_\theta(\theta^*_s) (g(\bar{m}) - \bar{m}) - (F_\theta(\theta^*_w) - F_\theta(\theta^*_s)) \left( g\left( \frac{1}{2} \bar{m} \right) - \frac{1}{2} \bar{m} \right),$$

where $F_\theta(\cdot)$ is the cumulative distribution function of the aggregate fundamental $\theta$ from the $t = 0$ perspective. Moreover, if there is a nontrivial strategic interaction between investors of weak and strong banks and the information friction is negligible, $\theta^*_s$ and $\theta^*_w$ are infinitely close (Proposition 1). Then (17) simplifies to

$$W = \int \theta dF_\theta(\theta) - F_\theta(\theta^*_s) (g(\bar{m}) - \bar{m}).$$

Equations (17) and (18) imply that reducing banks’ fragilities—that is, reducing run thresholds—is welfare-improving. In this section, we look into policies that are widely adopted by regulators in order to make the financial system more resilient. In particular, we emphasize that many policies, by affecting run synchronicity, have an indirect effect on stability (see Equation (14)) that has been previously overlooked. Interestingly, as we will show in this section, even though some policies are generally considered broad-based, they can impact certain banks more than others, thus changing run asynchronicity.
In Section 4.1, we study ring-fencing that affects asset heterogeneity across banks. In Section 4.2, we extend the model by allowing for imperfect information about bank-specific fundamentals and study how disclosing bank-specific fundamentals influences perceived heterogeneity and financial stability. In Section 4.3, we study liquidity injections that affect fire-sale discounts. Section 4.4 discusses general policy implications of our paper.

4.1 Ring-fencing

In the aftermath of the 2007–2008 financial crisis, the idea of ring-fencing has received a lot of attention. Ring-fencing refers to separating large banks’ balance sheets and restricting fund reallocations across ring-fenced subsidiaries. It is typically conducted along the following two dimensions.

First, separations can take place according to the service divisions. In the United States, the Volcker Rule restricts proprietary trading by commercial banks, essentially spinning off their investment banking activities. Similarly, starting from January 1, 2019, largest U.K. banks are required to separate core businesses in retail banking from investment banking. Second, separations can be carried out according to geographic locations. For example, the Fed requires foreign banking organizations with more than $50 billion in U.S. subsidiary assets to put all their U.S. subsidiaries under an intermediate holding company (Kreicher and McCauley, 2018). Geographic ring-fencing has also been pursued by the European regulator through increased capital and liquidity requirements on foreign-owned subsidiaries, legal restrictions on intragroup cross-border asset transfers, and limitations on the distribution of profits by foreign-owned subsidiaries (Enria, 2018).

As a regulator separates a large bank into subsidiaries with different business or geographic focuses, it effectively increases asset heterogeneity $\eta$. For instance, consider splitting a large bank issuing mortgages in New York and San Francisco into two separate banks, one of which operating only in New York while the other one only in San Francisco. As long as regional shocks of New York and San Francisco are not perfectly correlated, the two separate banks are exposed to larger bank-specific shocks than the conglomerate.

The comparative statics of the run thresholds with respect to $\eta$ are established in Proposition 3 and illustrated in Figure 2. For the rest of this section, we express run

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17Ring-fencing was first introduced through the Financial Services (Banking Reform) Act 2013, followed by adjustments in further legislation. See a summary at https://www.gov.uk/government/publications/ring-fencing-information/ring-fencing-information.
asynchronicity and run thresholds as functions of $\eta$.

**Proposition 3.** Define

$$\bar{\eta} \equiv \frac{1}{2} \int_0^1 \frac{dx}{1-mx} \int_0^1 \left[ \lambda \left( \frac{m}{2} x + \frac{\bar{m}}{2} \right) - \lambda \left( \frac{\bar{m}}{2} x \right) \right] \frac{\bar{m} x}{1-mx} dx > 0. \quad (19)$$

If $\eta \in (0, \bar{\eta})$, then cross-bank interactions are nontrivial, $\Delta (\eta) < \bar{\Delta}$, and run thresholds of strong and weak banks are infinitely close to $\theta^* (\eta)$. Moreover, $\Delta (\eta)$ increases in $\eta$ and $\theta^* (\eta)$ decreases in $\eta$.

**Proof.** See Appendix A.3.

When asset performances of weak and strong banks are not too diverse, $\eta < \bar{\eta}$, there exist nontrivial cross-bank interactions, $\Delta < \bar{\Delta}$, and the run thresholds of strong and weak banks are infinitely close to each other. It is straightforward to verify from Equation (13) that an increase in $\eta$ makes runs more asynchronous, i.e. $\frac{d\Delta}{d\eta} > 0$. Moreover, by Proposition 1, $\frac{d\theta^*}{d\Delta} < 0$. Combining these results together, we can see that the effect of $\eta$ on $\theta^*$ is negative. In words, when assets become more bank-specific, all banks become more stable, including those whose asset performances end up being weaker. Therefore, by (18), increasing $\eta$ up to $\bar{\eta}$ is unambiguously welfare-improving. Notably, existing literature studying downsides of asset commonality typically argues that more asset heterogeneity is associated with fewer systemic crises but more defaults of weak banks (e.g. Wagner, 2010 and 2011; Ibragimov et al., 2011; Cabrales et al., 2017). Different from these papers that focus on fundamental defaults, we study panic-driven runs and emphasize that more diverse asset performances are associated with lower fragility even for weak banks due to alleviation of cross-bank fire-sale externalities.

When $\eta$ increases beyond $\bar{\eta}$, banks become so different that cross-bank strategic interactions become trivial, i.e. $\Delta > \bar{\Delta}$. As there are no strategic uncertainties across investors of different banks, the run thresholds of strong and weak banks decouple in equilibrium. A further increase in $\eta$ does not affect cross-bank fire-sale externalities but only further strengthens (weakens) asset performances of strong (weak) banks. As a result, the strong-bank run threshold keeps declining while the weak-bank run threshold starts to rise (see Figure 2). A marginal increase in $\eta$ is no longer unambiguously stabilizing as weak banks become more fragile. Accordingly, whether welfare measure (17) increases or decreases in $\eta$ depends on the ex-ante distribution of the aggregate fundamental.

Another issue that ring-fencing policies intend to address is a “too-big-to-fail” problem.
Run thresholds $\theta^*(\eta), \theta^*_s(\eta), \theta^*_w(\eta)$

Figure 2: Run thresholds of strong ($\theta^*_s$) and weak ($\theta^*_w$) banks, and their average ($\theta^* = \frac{1}{2} \theta^*_s + \frac{1}{2} \theta^*_w$) as functions of the size of bank-specific shock $\eta$. Parametrization: $\bar{m} = 0.55, \lambda(m) = 1 + 2m^2, F_\epsilon(\cdot)$ is truncated standard normal over $[-1, 1]$, $\eta$ varies from 0 to 0.1.

The following corollary points out that size per se is not the key to financial stability in our model and downsizing banks into identical clones has no impact on financial stability.\(^{18}\)

**Corollary 1.** If ring-fencing does not affect asset heterogeneity $\eta$, then it does not affect stability.

*Proof.* Corollary follows directly from the scale invariance of $\pi(z, m, m_i)$.

The main economic force behind the above result is that investors of ring-fenced subsidiaries are still interconnected through the asset market. The fire-sale complementarity across these subsidiaries essentially resembles the bank-run complementarity across a large group of investors in the merged bank. The objective of ring-fencing, therefore, should not be only to downsize banks' balance sheets but also to achieve an optimal level of asset heterogeneity across institutions.

### 4.2 Disclosure of bank-specific information

In this section, we extend our benchmark model by allowing for noisy information about bank-specific fundamentals. We then analyze disclosure policies that affect the quality of bank-specific information available to bank investors.

\(^{18}\)Given the focus of this paper is on cross-bank interactions, certain features that are important to study a “too-big-to-fail” issue, such as banks’ moral hazard, are absent from our model.
In particular, investor $l$ of bank $i$ receives two distinct noisy signals. The signal about the aggregate fundamental is the same as in the baseline model, namely, $s_{il} = \theta + \sigma \epsilon_{il}$. In addition, she receives a signal $d_i$ about the bank-specific component $\zeta_i$. This signal takes two values, $G$ and $B$, with a probability mass function specified below.

$$P(d_i = G|\zeta_i = \eta) = P(d_i = B|\zeta_i = -\eta) = \alpha \in \left[\frac{1}{2}, 1\right].$$

Parameter $\alpha$ captures the quality of bank-specific information. Under this signal structure, a regulator can vary $\alpha$ by changing the stringency of bank disclosure policy. Below, we discuss how $\alpha$ affects fragility and welfare.\(^\text{19}\)

Denote the posterior belief about the probability of bank $i$ being strong as $p_G$ if $d_i = G$ and $p_B$ if $d_i = B$. We have $p_G = \alpha \geq \frac{1}{2} \geq p_B = 1 - \alpha$. The equality holds if and only if $\alpha = \frac{1}{2}$, that is, signals are uninformative about bank-specific fundamentals. Another special case is perfect bank-specific information as in our baseline setting, $\alpha = 1$.

**Proposition 4.** The model with imperfect information about bank-specific fundamentals is equivalent to the benchmark model where bank-specific shocks take values $\eta^{eff}(\alpha)$ with probability $\frac{1}{2}$ and $-\eta^{eff}(\alpha)$ with probability $\frac{1}{2}$, where $\eta^{eff}(\alpha) = (2\alpha - 1)\eta$.

**Proof.** See Appendix A.4. \(\Box\)

The main takeaway from Proposition 4 is that besides reshaping asset heterogeneity directly, regulators can improve stability by manipulating investors’ beliefs about it. If signals about bank-specific fundamentals are uninformative, then bank investors perceive strong and weak banks as homogeneous, i.e. $\eta^{eff}(\frac{1}{2}) = 0$. Run decisions of investors of strong and weak banks are then completely synchronized, and the financial system is fragile. If signals are informative, $\alpha > \frac{1}{2}$, investors are able to differentiate between banks. In that case, run decisions of investors are not perfectly synchronized, i.e. $\Delta \neq 0$. A more stringent disclosure policy improves the quality of bank-specific information $\alpha$ and enlarges the perceived difference between bank asset performances $\eta^{eff}(\alpha)$. In the presence of nontrivial cross-bank strategic interactions, this increases run asynchronicity, enhances stability of the financial system by Proposition 3, and improves welfare given by (18). In particular, the planner should seek to set $\alpha$ such that $\eta^{eff}(\alpha)$ is not smaller

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\(^{19}\)It would be interesting to endogenize the structure of signals about bank-specific fundamentals, i.e. analyze a general information design problem of the regulator. Such an extension is outside the scope of this paper.
than $\bar{\eta}$, where $\bar{\eta}$ is given by (19)."}20

Existing literature on disclosure policies, reviewed by Goldstein and Sapra (2014), highlights various costs and benefits of disclosing bank-specific information. A common benefit mentioned in the literature is that it allows investors to better monitor and discipline banks. By simultaneously considering within- and cross-bank strategic interactions, we highlight a novel benefit of disclosing bank-specific information: It enhances financial stability by boosting run asynchronicity. Without considering cross-bank strategic interactions, some papers argue that disclosure policy should be state-dependent so that bank information is opaque in good times and transparent in bad times (e.g. Bouvard et al., 2015; Goldstein and Leitner, 2018). Specifically, disclosing bank-specific information in bad times induces weak banks to fail but, at the same time, saves strong banks. In our model, disclosing bank-specific information makes weak-bank investors aware of the weak fundamentals of their banks but, at the same time, less worried about spillovers from strong banks. The latter effect dominates in the presence of reinforcing complementarities, and weak-bank investors have fewer incentives to run. As a result, instead of inducing runs, disclosure reduces fragilities of both strong and weak banks.

4.3 Asset market interventions

One of the distress resolution approaches that regulators frequently turn to during economic downturns is liquidity injections into asset markets. Prominent examples include the U.S. government’s purchases of distressed assets during the Great Recession and corporate bonds and bond ETFs via the Secondary Market Corporate Credit Facility (SMCCF) during the recent COVID-19 pandemic. In our framework, such policies reduce fire-sale discounts. Fixing run asynchronicity, a reduction in fire-sale discounts stabilizes banks directly. Interestingly, however, we find that even when such policies are broad-based and do not target specific banks, they do impact run asynchronicity and thus affect financial stability indirectly (Equation (14) defines direct and indirect effects in our setting). Via a simple numerical exercise at the end of this section, we verify that the indirect channel can be a nontrivial contributor to the overall effect of such policies on financial fragility.

We consider an extension of our baseline model in which the regulator injects $L \geq 0$

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20Improving the quality of bank-specific information might destabilize weak banks if run asynchronicity is larger than $\Delta$ (see Figure 2). At the same time, it always makes strong banks more stable. Whether additional disclosure is desirable in this case depends on the ex-ante distribution of the aggregate fundamental, as suggested by (17).
into the asset market to alleviate fire-sale concerns. Specifically, asset prices remain at their fundamental levels if the aggregate liquidity needs do not exceed $L$. Any liquidity needs beyond this point are fulfilled by outside investors as in the baseline model. As a result, the fire-sale discount factor in the asset market is

$$
\hat{\lambda}(m, L) = \begin{cases} 
1 & \text{if } m < L, \\
\lambda(m - L) & \text{if } m \geq L,
\end{cases}
$$

where $\lambda(\cdot)$ is the fire-sale discount factor in the baseline model and $m$ is the mass of runners in the whole economy. To evaluate the impacts of liquidity injections, we establish comparative statics with respect to $L$ in Proposition 5.

**Proposition 5.** Suppose that $\lambda''(\cdot) \geq 0$ and $\eta \in (0, \bar{\eta})$, where $\bar{\eta}$ is defined in Equation (19). There exists a decreasing function $\hat{L}(\eta) > 0$ such that if $L \in (0, \hat{L}(\eta))$, then cross-bank interactions are nontrivial, $\Delta < \hat{\Delta}$, and run thresholds of strong and weak banks are infinitely close to $\theta^*$. Moreover,

$$
\frac{d\theta^*}{dL} = \frac{\partial \theta^*}{\partial L} + \frac{\partial \theta^*}{\partial \Delta} \frac{d\Delta}{dL},
$$

where

1. The direct effect is stabilizing, $\frac{\partial \theta^*}{\partial L} < 0$;
2. The indirect effect is stabilizing, $\frac{\partial \theta^*}{\partial \Delta} < 0$ and $\frac{d\Delta}{dL} > 0$.

**Proof.** See Appendix A.5.

Different from ring-fencing and disclosure policies that influence fragility only through their impacts on run asynchronicity, a liquidity injection has a direct stabilizing effect, captured by $\frac{\partial \theta^*}{\partial L} < 0$. This is intuitive because such an intervention reduces fire-sale discounts faced by all banks.

Importantly, even though liquidity injections are not targeted—that is, the regulator does not restrict the purchase to assets owned by a particular group of banks—strong and weak banks are affected differently by this regulation. The reason is that strong banks end up receiving a higher implicit subsidy in the event of a run. Recall that strong banks experience runs when the liquidity conditions are particularly dire. That is, a marginal strong-bank investor expects a large amount of runs on weak banks and thus a sizable fire-sale discount even if she does not expect many runs on her own bank. In contrast, a marginal weak-bank investor expects fewer runs on strong banks and a
nonzero fire-sale discount only if runs on her bank are widespread. Therefore, a liquidity injection reduces fire-sale discount for strong banks in more states of the world, which further strengthens strong banks relative to weak banks. Formally, the fire-sale pressure on strong banks declines more than the fire-sale pressure on weak banks, \( \frac{\partial I_s}{\partial L} < \frac{\partial I_w}{\partial L} < 0 \), which by Equation (13) implies a desynchronization of runs, \( \frac{\partial \lambda}{\partial L} > 0 \). Consequently, as the injection size \( L \) increases, runs become more asynchronous, which stabilizes the financial system indirectly (Proposition 1). Although this effect arises even if the fire-sale discount function \( \lambda(\cdot) \) is linear, it becomes more pronounced if it is convex. In that case, the fire-sale discount faced by strong banks is alleviated more substantially by the same liquidity injection.

Given that liquidity injection stabilizes the financial system both directly and indirectly, it is important to understand how the latter effect compares to the former quantitatively. To get a sense of the relative magnitudes of the two effects, we conduct a simple numerical exercise in the context of the Great Recession. In particular, we pick the size of bank-specific shocks \( \eta = 0.03 \) to match the cross-sectional dispersion in cumulative returns on assets of U.S. banks between 2007Q4 and 2009Q2.\(^{21}\) We set \( \bar{m} = 0.57 \), corresponding to the fraction of bank liabilities that is not covered by deposit insurance in 2007Q4 and is thus subject to runs. We parametrize \( \lambda(m - L) = 1 + (m - L)^2 \). Under such a choice, when a systemic bank run occurs in the absence of any government liquidity injections, that is, \( m = \bar{m} \) and \( L = 0 \), asset prices are 25% below their fundamental values. This value is in line with the estimates of James (1991) and Granja, Matvos, and Seru (2017) who document comparable discounts in prices paid for assets of failed banks during the Savings and Loan Crisis and the Great Recession, respectively.\(^{22}\) Finally, we choose the noise distribution to be uniform with a support \([-1, 1]\). Our results are not sensitive to the choice of the noise distribution.

We normalize the aggregate amount of bank assets to one and vary the liquidity injection size \( L \) from 0 to 0.01. Our parametrization implies nontrivial cross-bank interactions for all \( L \in [0, 0.01] \) such that the results of Proposition 5 apply. We investigate how the common run threshold \( \theta^* = \theta^*(L) \) changes with the size of liquidity injection in two

\(^{21}\)We use data from the FR Y-9C filings for parent-level bank holding companies. To compute cross-sectional dispersion, we winsorize the top and bottom 1% of the sample. We get very similar results if we focus on large banks only (with total assets of at least $10 billion or $100 billion before the Great Recession).

\(^{22}\)Our parametrization of the fire-sale discount function is quite conservative. For example, during the Great Recession, the government actively intervened in the financial sector, likely preventing more widespread runs. In the absence of such interventions, a collapse in asset prices might have been much more substantial.
scenarios. In the baseline case, run asynchronicity $\Delta = \Delta(L)$ adjusts as $L$ increases such that $\theta^*(L)$ changes both due to direct and indirect effects. This case is illustrated by the blue solid line in Figure 3. The red dashed line in the same figure shows how the run threshold changes with $L$ if run asynchronicity is held fixed, $\Delta = \Delta(0)$. Therefore, it reflects only the direct effect of liquidity injections on stability. The difference suggests that the indirect effect accounts for about one quarter of the total reduction in financial fragility as $L$ changes from 0 to 0.01. Thus, it can indeed be an important mechanism through which government liquidity injections stabilize the financial system.

While the run threshold is directly related to the probability of a systemic financial crisis and welfare (18), to evaluate the impact of a liquidity injection on welfare, one would need to take a stand on the prior distribution of the aggregate fundamental $F_{\theta}(\cdot)$. Furthermore, it is worth noting that injecting liquidity in the financial markets is likely to be subject to fiscal costs. If this is the case, the welfare measure (18) should be adjusted to reflect such costs. In our paper, we do not take a stand on the distribution of the aggregate fundamental and the size of the fiscal costs. Instead, we argue that for a given cost of intervention, a nontrivial part of the benefit of the injection comes through its impact on run synchronicity and, via that channel, on financial stability and welfare.
4.4 Discussions

The broad message of our policy analyses is that in a financial system featuring interconnected fragilities, various factors can affect financial stability through changing run asynchronicity. Regulators should take this indirect effect into account when monitoring bank activities and developing intervention strategies. Without doing so, policies may undermine financial stability by making runs more synchronized.

For instance, regulatory policies that narrow the dispersion in bank asset performances can be harmful for stability. One example is mergers and acquisitions in the banking system, which by nature reduce bank asset heterogeneity. The outcome of bank mergers is therefore the opposite to that of ring-fencing. Another example is direct financial support—or a belief about it shared by market participants—biased toward weak banks. Although such financial support directly enhances weak banks’ resilience to negative shocks, its effectiveness can be hindered by the resulting homogeneity in (perceived) asset performances.

Stress-testing is another widely-used policy tool that can lead to an unintended reduction in run asynchronicity. Currently, regulators test bank resilience against a common set of stress scenarios. Banks that are more likely to fail these tests—weak banks in our paper—are required by regulators to increase their liquidity buffers or equity cushions, which can raise their similarity to strong banks. Furthermore, anticipating such corrective actions, weak banks might act preemptively and strengthen their balance sheet against stress test scenarios, again leading the financial system to become more homogeneous. As a result of higher similarity across banks, run asynchronicity diminishes, which can undermine the effectiveness of stress tests. Our paper suggests that a certain difference between banks’ resistances to a common set of stress scenarios is desirable.

Relatedly, regulators should closely monitor aggregate market conditions and conduct interventions in a timely manner. Recall that when strong banks are forced to liquidate their assets, the asset market is already highly stressed due to massive liquidations by weak banks. If regulators commit to intervene in situations when fire-sale discounts are especially high, then this can be particularly helpful to strong banks. As our results in Section 4.3 suggest, this can have a sizeable stabilizing effect through enlarging run asynchronicity and alleviating the vicious loop between bank runs and deteriorating liquidity conditions.

Overall, various bank activities and regulatory policies might affect run asynchronicity and, thus, impact financial stability. Regulators should take a holistic approach and
combine different regulatory tools to keep runs asynchronous in the financial system.

5 Model extensions

In this section, we consider two extensions of our baseline model. Section 5.1 allows for finitely precise signals about the aggregate fundamental. Section 5.2 generalizes the binary structure of bank-specific shocks.

5.1 Finite signal precision

In this section, we allow the precision of investors’ private signals about the aggregate fundamental to be finite, that is, we allow the signal noise \( \sigma \) to be not infinitely close to zero. To keep the analysis tractable, we maintain the assumption that signals are infinitely more precise than any prior information that investors have about the aggregate fundamental \( \theta \). Formally, this implies an uninformative prior about \( \theta \). In Appendix B.3.2, we consider a numerical example to demonstrate that the results hold even if the prior is informative.\(^{23}\)

When the prior is uninformative, there exists a unique threshold equilibrium in which investors of strong and weak banks withdraw prematurely if and only if their signals are below \( \theta^*_s \) and \( \theta^*_w \), respectively. With a non-negligible \( \sigma \), the equations defining the run thresholds (7) and (8) become

\[
\begin{align*}
\theta^*_s + \eta &= \frac{1}{\int_0^1 \frac{1}{1-mx} dx} \left( 1 + I_s(\Delta) + \sigma \int_0^1 \frac{F^{-1}_c(x)}{1-mx} dx \right), \\
\theta^*_w - \eta &= \frac{1}{\int_0^1 \frac{1}{1-mx} dx} \left( 1 + I_w(\Delta) + \sigma \int_0^1 \frac{F^{-1}_c(x)}{1-mx} dx \right),
\end{align*}
\]

where the term \( \sigma \int_0^1 \frac{F^{-1}_c(x)}{1-mx} dx \) arises because signals are not infinitely close to the aggregate fundamental when \( \sigma \) is non-negligible; the fire-sale pressure terms \( I_s(\Delta) \) and \( I_w(\Delta) \) are defined by (9) and (10), respectively; and \( \Delta \equiv \frac{\theta^*_w - \theta^*_s}{\sigma} \) is run asynchronicity.

Recall that in the main model with negligible \( \sigma \), weak and strong banks’ run thresholds are infinitely close to the average run threshold when run asynchronicity is not too large (\( \Delta < \bar{\Delta} \)) and cross-bank strategic interactions are nontrivial. In contrast, if \( \sigma \) is non-

\(^{23}\)As discussed in detail by Morris and Shin (2003), for equilibrium to be unique in global games settings, private signals should be sufficiently more precise than the prior information.
negligible, the run thresholds $\theta_s^*$ and $\theta_w^*$ are different as long as bank assets are not entirely the same, i.e. $\eta \neq 0$. In what follows, we first characterize how the average fragility of the financial system, $\theta^* = \frac{1}{2}\theta_s^* + \frac{1}{2}\theta_w^*$, depends on $\Delta$ and then consider fragilities of strong and weak banks separately.

The average fragility of the financial system is

\[
\theta^* = \frac{1}{2}\theta_s^* + \frac{1}{2}\theta_w^* = \int_0^1 \frac{1}{1-mx} \left( 1 + \frac{1}{2}I_s(\Delta) + \frac{1}{2}I_w(\Delta) + \sigma \int_0^1 \frac{F^{-1}(x)}{1-mx} dx \right),
\]

where $\Delta$ is implicitly defined by

\[
\eta = \frac{\sigma \Delta}{2} + \frac{1}{2} \int_0^1 \frac{dx}{1-mx} (I_s(\Delta) - I_w(\Delta)).
\]

These two equations generalize (12) and (13) to the case of finite signal precision. Analogous to our benchmark analysis, average fragility declines in run asynchronicity as long as there are nontrivial cross-bank strategic interactions.

**Proposition 6.** In the model with finitely precise signals about the aggregate fundamental $\theta$, the average run threshold $\theta^*(\Delta)$ is a decreasing function of $\Delta$ when $\Delta < \tilde{\Delta}$. When $\Delta \geq \tilde{\Delta}$, $\theta^*(\Delta)$ is constant.

**Proof.** See Appendix B.3.1. \(\square\)

Figure 4 illustrates Proposition 6. When cross-bank interactions are nontrivial, $\Delta < \tilde{\Delta}$, the average run threshold (the solid blue line) declines in run asynchronicity. In the region where cross-bank strategic uncertainty disappears, $\Delta \geq \tilde{\Delta}$, the threshold does not depend on run asynchronicity.

The run thresholds of strong and weak banks can be written as

\[
\theta_s^* = \theta^*(\Delta) - \frac{1}{2}\sigma \Delta,
\]

\[
\theta_w^* = \theta^*(\Delta) + \frac{1}{2}\sigma \Delta.
\]

When signals have finite precision, individual and average thresholds are no longer infinitely close. Clearly, strong banks become less fragile as $\Delta$ increases. However, this does not always hold for weak banks. Nonetheless, as $\Delta$ increases, the force that alleviates fire-sale pressure on weak banks and stabilizes them is still at work. If $\sigma$ is finite but sufficiently small, an increase in run asynchronicity stabilizes all banks, including the
Run thresholds \( \theta^*(\Delta) \), \( \theta_s^*(\Delta) \), \( \theta_w^*(\Delta) \)

Figure 4: Run thresholds of strong (\( \theta_s^* \)) and weak (\( \theta_w^* \)) banks, and their average (\( \theta^* = \frac{1}{2}\theta_s^* + \frac{1}{2}\theta_w^* \)) as functions of run synchronicity \( \Delta \). Parametrization: \( \bar{m} = 0.55, \lambda(m) = 1 + 2m^2, F_{\epsilon}(\cdot) \) is truncated standard normal over \([-1, 1]\), \( \sigma = 0.015 \), \( \eta \) varies from 0 to 0.1.

weak ones. The reason is that, unless \( \sigma \) is too large, \( \theta_w^* \) stays close to \( \theta^* \) and thus tends to decline with run asynchronicity. To illustrate that, Figure 4 shows the run thresholds \( \theta_s^* \) and \( \theta_w^* \) as functions of run asynchronicity. As is clear from the graph, if run asynchronicity is neither too small or too large, an increase in it drives down both run thresholds. Therefore, our key result that run synchronicity undermines stability carries through.

5.2 Many bank types

Our baseline model assumes that bank-specific shocks take values \( \eta \) or \(-\eta\) with equal probabilities. In this section, we show how our analyses can be extended to the case in which bank-specific shocks take \( N \geq 2 \) values.

The structure of the economy stays the same as in Section 2. The only difference is that bank-specific shock \( \zeta \) can take \( N \geq 2 \) values, \( \eta_1 \geq \eta_2 \geq \cdots \geq \eta_N \), \( \eta = \{\eta_n\}_{n=1}^{N} \), with probabilities \( \omega_1, \omega_2, \ldots, \omega_N \), respectively, where \( \omega_n \in (0, 1) \) \( \forall n \in \{1, \ldots, N\} \) and \( \sum_{n=1}^{N} \omega_n = 1 \). Without loss of generality, we assume that bank-specific shock is zero on average, \( \sum_{n=1}^{N} \omega_n \eta_n = 0 \).

Bank investors follow threshold strategies, that is, investors of a bank receiving a shock \( \eta_n \) withdraw early if their signals are below \( \theta_n^* \) and do not do so otherwise. An indifference
condition for an investor receiving a threshold signal $\theta^*_n$ is

$$\theta^*_n + \eta_n = \frac{1}{\int_0^1 \frac{1}{1-\bar{m}x} \, dx} \left( 1 + \int_0^1 \lambda \left( \tilde{m} \sum_{\tau=1}^N \omega_{\tau} F_\epsilon \left( \Delta_{n\tau} + F_\epsilon^{-1}(x) \right) \right) \frac{\bar{m}x}{1-\bar{m}x} \, dx \right), \quad (21)$$

where, as in the baseline model, signal noise is negligible, $\sigma \to 0$. $\Delta_{n\tau} = \lim_{\sigma \to 0} \frac{\theta^*_n - \theta^*_\tau}{\sigma}$ is run asynchronicity between banks receiving shocks $\eta_n$ and $\eta_\tau$, and $\Delta_{n\tau} \geq 0 \ \forall n < \tau$. A system of $N$ equations (21) is a generalized version of Equations (7)-(8). In what follows, we focus on an interesting case of nontrivial cross-bank strategic interactions. That is, $\Delta_{n\tau} < \bar{\Delta} \ \forall n < \tau$. In this case, run thresholds are infinitely close to each other with the average run threshold given by

$$\theta^* \equiv \sum_{n=1}^N \omega_n \theta^*_n = \frac{1}{\int_0^1 \frac{1}{1-\bar{m}x} \, dx} \left( 1 + \int_0^1 \sum_{n=1}^N \omega_n \lambda \left( \tilde{m} \sum_{\tau=1}^N \omega_{\tau} F_\epsilon \left( \Delta_{n\tau} + F_\epsilon^{-1}(x) \right) \right) \frac{\bar{m}x}{1-\bar{m}x} \, dx \right). \quad (22)$$

The following proposition generalizes Proposition 1.

Proposition 7. Suppose that $\Delta_{n\tau} < \bar{\Delta} \ \forall n < \tau$. In the limit of negligible information friction ($\sigma \to 0$), run thresholds of all banks are infinitely close to $\theta^*$. Moreover, any change in run asynchronicity that (i) weakly increases $\Delta_{n\tau} \ \forall n < \tau$ and (ii) strictly increases $\Delta_{n'\tau'}$ for some $n' < \tau'$ leads to a decline in $\theta^*$.

Proof. See Appendix B.4.1.

Run asynchronicity, described by pairwise distances between run thresholds $\Delta_{n\tau}$, is endogenous. As discussed in the previous sections, it depends on various primitives of the model, such as asset heterogeneity, information structure, and liquidity conditions. Given that this section generalizes the structure of bank-specific shocks, we focus on characterizing how differences in asset heterogeneity affect run asynchronicity and hence the overall financial stability. The following proposition is a version of Proposition 3 in this more general model.

Proposition 8. There exists an $\eta > 0$ such that if $|\eta_n| < \eta \ \forall n$, then in the limit of negligible information friction ($\sigma \to 0$), run thresholds of all banks are infinitely close to $\theta^*(\eta)$, which reaches a (local) maximum at $\eta = 0$.

Proof. See Appendix B.4.2.
If asset performances of all banks are identical, i.e. $\eta = 0$, then all run thresholds are exactly the same and the system is homogeneous. By continuity, all run thresholds stay infinitely close to each other for any small change in $\eta$. Pairwise asynchronicity terms $\Delta_{n\tau}$, however, adjust. In particular, if $\eta_n > \eta_\tau$, then $\Delta_{n\tau} > 0$. According to Proposition 8, such diversity in bank asset performances enhances stability of all banks.

As in the main model with two types of banks, there is a limit to which increasing diversity in bank-specific productivities is unequivocally stabilizing. In particular, if bank asset performance becomes sufficiently divergent—so that banks’ behaviors in the asset market are fully decoupled and there are no strategic interactions across investors of different banks—further divergence hurts relatively weaker banks.

6 Conclusion

This paper analyzes interactions between fragile banks through asset fire sales. Such interactions can lead to a spread of panics across institutions and thus a systemic crisis. The key feature of our model is the reinforcement between within- and cross-bank complementarities, that is, bank runs and fire sales. We highlight an important factor—run asynchronicity—that governs cross-bank interactions and plays an important role in determining fragility of the entire financial system. We show how various regulatory policies can enhance stability through increasing run asynchronicity.

Our model can be extended along various directions. First, it would be interesting to study how our results are affected by the presence of features mitigating banks runs and fire-sale spillovers, such as banks’ cash buffers, inter-bank lending, etc. Importantly, we expect that our main results will hold unless such mitigating factors completely eliminate panic runs and fire-sale externalities. Second, to focus on interactions between bank runs and fire sales, we have left ex-ante banks’ portfolio choices out of consideration. Finally, our setting can be incorporated into a richer dynamic model featuring nontrivial asset and leverage choices.

References


Shen, L. and J. Zou (2020): “Intervention with Screening in Panic-Based Runs,” Available at SSRN 3137172.


A Proofs

A.1 Liquidation price

This appendix proves Lemma 1.

Proof. Outside investors solve the problem

$$\max_{\{k_i\}_{i \in [0,1]}} \int z_i k_i \, di - g \left( \int p_i k_i \, di \right).$$

The first-order condition implies that the liquidation price of bank $i$’s assets $p_i$ is proportional to its fundamental $z_i$,

$$p_i = \frac{z_i}{g'(L)} \quad \forall i \in [0,1],$$

(23)

where $L \equiv \int p_i k_i \, di$. Aggregation of the market clearing conditions for individual banks implies that the total liquidity demand $m$ equals to the liquidity supply $L$:

$$m_i = p_i k_i \Rightarrow m = \int p_i k_i \, di = L.$$

Therefore, we obtain the equilibrium asset prices

$$p_i(z_i, m) = \frac{z_i}{\lambda(m)},$$

where $\lambda(m) \equiv g'(m)$. Since $\lambda'(m) = g''(m) > 0$, the liquidation price $p_i(z_i, m)$ is a decreasing function of the total mass of runners $m$ for any $i \in [0,1]$. \hfill \Box

A.2 Role of two complementarities

We prove Proposition 2. Note that Proposition 1 is a special case of Proposition 2 with $\pi_1(m_i) = \frac{1}{1-m_i}$ and $\pi_2(m_i, m) = -\frac{\lambda(m)m_i}{1-m_i}$. In Appendix B.2, we explore a general net payoff function $\pi(z_i, m_i, m)$ without imposing a specific functional form.

Proof. If $\Delta \equiv \lim_{\sigma \to 0} \frac{\mu - \mu'}{\sigma} < \tilde{\Delta}$, the model features nontrivial cross-bank strategic interac-
tions. The analogues of Equations (7) and (8) from the main text are
\[
\int_0^1 \left[ (\theta^*_s + \eta)\pi_1(\bar{m}x) + \pi_2\left(\bar{m}x, \frac{\bar{m}}{2}x + \frac{\bar{m}}{2} F_\epsilon(\Delta + F_\epsilon^{-1}(x))\right) \right] dx = 0, \tag{24}
\]
\[
\int_0^1 \left[ (\theta^*_w - \eta)\pi_1(\bar{m}x) + \pi_2\left(\bar{m}x, \frac{\bar{m}}{2}x + \frac{\bar{m}}{2} F_\epsilon(-\Delta + F_\epsilon^{-1}(x))\right) \right] dx = 0. \tag{25}
\]
Since we focus on the limiting case with vanishing information friction, the two run thresholds are infinitely close to the average threshold
\[
\theta^* \equiv \frac{1}{2} \theta^*_s + \frac{1}{2} \theta^*_w.
\]
Differentiating (24) and (25) with respect to \(\Delta\), we can derive
\[
\frac{\partial \theta^*_s}{\partial \Delta} = -\frac{\bar{m}}{2} \int_0^1 \frac{F_\epsilon(\xi)}{\pi_1(\bar{m}x)} \left\{ \pi_2\left(\bar{m}x, \frac{\bar{m}}{2}x + \frac{\bar{m}}{2} F_\epsilon(\Delta + F_\epsilon^{-1}(x))\right) \right\} f_\epsilon(\Delta + F_\epsilon^{-1}(x)) dx,
\]
\[
\frac{\partial \theta^*_w}{\partial \Delta} = \frac{\bar{m}}{2} \int_0^1 \frac{F_\epsilon(\xi + \Delta)}{\pi_1(\bar{m}x)} \left\{ \pi_2\left(\bar{m}x, \frac{\bar{m}}{2}x + \frac{\bar{m}}{2} F_\epsilon(-\Delta + F_\epsilon^{-1}(x))\right) \right\} f_\epsilon(-\Delta + F_\epsilon^{-1}(x)) dx
\]
\[
= \frac{\bar{m}}{2} \int_0^1 \frac{F_\epsilon(\xi)}{\pi_1(\bar{m}x)} \left\{ \pi_2\left(\bar{m}F_\epsilon(\Delta + F_\epsilon^{-1}(y)), \frac{\bar{m}}{2} F_\epsilon(\Delta + F_\epsilon^{-1}(y)) + \frac{\bar{m}}{2} y \right) \right\} f_\epsilon(\Delta + F_\epsilon^{-1}(y)) dy, \tag{27}
\]
where (27) is obtained by changing the variable of integration \(x \rightarrow F_\epsilon(\Delta + F_\epsilon^{-1}(y))\), which implies, in particular, \(dx = dF_\epsilon(\Delta + F_\epsilon^{-1}(y)) = \frac{F_\epsilon'(F_\epsilon^{-1}(y))}{f_\epsilon(F_\epsilon^{-1}(y))} dy\).

The impact of asynchronicity \(\Delta\) on \(\theta^*\) is
\[
\frac{\partial \theta^*}{\partial \Delta} = \frac{1}{2} \left( \frac{\partial \theta^*_s}{\partial \Delta} + \frac{\partial \theta^*_w}{\partial \Delta} \right).
\]
Comparing (26) and (27), we can see that
\[
\frac{\partial^2 \pi}{\partial m \partial m_i} = \frac{\partial^2 \pi_2}{\partial m \partial m_i} \leq 0 \Rightarrow -\frac{\partial \theta^*_w}{\partial \Delta} = \frac{\partial \theta^*_s}{\partial \Delta} \Rightarrow \frac{\partial \theta^*_w}{\partial \Delta} > 0.
\]

\[\boxed{}\]

### A.3 Ring-fencing

This appendix proves Proposition 3.

**Proof.** Recall that when cross-bank strategic interactions are nontrivial \((\Delta < \bar{\Delta})\), run
asynchronicity $\Delta$ is implicitly defined by Equation (13), which we repeat below,

$$
\eta = \frac{1}{2} \int_0^1 \frac{dx}{1 - \bar{m}x} \left( I_s(\Delta) - I_w(\Delta) \right).
$$

From Equations (9) and (10), it is clear that $I_s(\Delta)$ increases in $\Delta$ and $I_w(\Delta)$ decreases in $\Delta$. Hence, $\Delta(\eta)$ is an increasing function. Moreover, $\Delta(0) = 0$ and $\Delta(\bar{\eta}) = \bar{\Delta}$.

Therefore, when $\eta \in (0, \bar{\eta})$, we have $\Delta \in (0, \bar{\Delta})$, and cross-bank strategic interactions are nontrivial. From Equation (12), $\theta^*(\eta) = \theta^*(\Delta(\eta))$. Since $\Delta(\eta)$ increases in $\eta$ on $(0, \bar{\eta})$, by Proposition 1, run thresholds of strong and week banks are infinitely close to $\theta^*(\eta)$ which decreases in $\eta$ on $(0, \bar{\eta})$.

\[\square\]

### A.4 Noisy information about bank-specific fundamentals

We prove Proposition 4 in this appendix.

**Proof.** Fraction $\alpha$ of strong-bank investors and fraction $1 - \alpha$ of weak-bank investors receive signal $G$. Therefore, fraction $\frac{1}{2}\alpha + \frac{1}{2}(1 - \alpha) = \frac{1}{2}$ of all investors receive this signal. These investors run if their signals about the aggregate fundamental are below $\theta^*_G$. Similarly, fraction $\frac{1}{2}(1 - \alpha) + \frac{1}{2}\alpha = \frac{1}{2}$ of all investors receive signal $B$. These investors run if their signals about the aggregate fundamental are below $\theta^*_B$. The run thresholds for investors receiving signals $G$ and $B$ are determined by

$$
\theta^*_G + p_G \eta - (1 - p_G) \eta = \frac{1}{2} \int_0^1 \frac{dx}{1 - \bar{m}x} \left( 1 + \int_0^1 \frac{\lambda(m_G(x, \Delta))\bar{m}x}{1 - \bar{m}x} dx \right),
$$

$$
\theta^*_B + p_B \eta - (1 - p_B) \eta = \frac{1}{2} \int_0^1 \frac{dx}{1 - \bar{m}x} \left( 1 + \int_0^1 \frac{\lambda(m_B(x, \Delta))\bar{m}x}{1 - \bar{m}x} dx \right),
$$

where $\Delta = \lim_{\sigma \to 0} \frac{\theta^*_B - \theta^*_G}{\sigma}$ and

$$
m_G(x, \Delta) = \frac{1}{2} \bar{m}x + \frac{1}{2} \bar{m}F_\epsilon(\Delta + F_\epsilon^{-1}(x)),
$$

$$
m_B(x, \Delta) = \frac{1}{2} \bar{m}F_\epsilon(-\Delta + F_\epsilon^{-1}(x)) + \frac{1}{2} \bar{m}x.
$$

Define

$$
\eta^fJ(\alpha) = p_G \eta - (1 - p_G) \eta = (2\alpha - 1) \eta.
$$
It straightforward to see that the model described in Section 4.2 boils down to our baseline setting with redefined $\eta$. Therefore, the results of Section 3.1 generalize to the case of noisy bank-specific signals.

A.5 Asset market interventions

Throughout this appendix, we make the following assumption.

**Assumption 1.** $\lambda(\cdot)$ is a weakly convex function, that is, $\lambda''(\cdot) \geq 0$.

With the modified fire-sale discount function $\hat{\lambda}(m, L)$, we can express the fire-sale pressure terms as

\[
\hat{I}_s(\Delta, L) = \int_{x_s(\Delta, L)}^{1} \lambda \left( \frac{\hat{m} x}{2} + \frac{\hat{m} F_{\epsilon}(\Delta + F^{-1}_{\epsilon}(x)) - L}{1 - \hat{m} x} \right) \frac{\hat{m} x}{1 - \hat{m} x} dx + \int_0^{x_s(\Delta, L)} \frac{\hat{m} x}{1 - \hat{m} x} dx, \tag{28}
\]

\[
\hat{I}_w(\Delta, L) = \int_{x_w(\Delta, L)}^{1} \lambda \left( \frac{\hat{m} x}{2} + \frac{\hat{m} F_{\epsilon}(-\Delta + F^{-1}_{\epsilon}(x)) - L}{1 - \hat{m} x} \right) \frac{\hat{m} x}{1 - \hat{m} x} dx + \int_0^{x_w(\Delta, L)} \frac{\hat{m} x}{1 - \hat{m} x} dx, \tag{29}
\]

where $x_s(\Delta, L)$ and $x_w(\Delta, L)$ are the critical fractions of strong- and weak-bank runners below which perceived fire-sale discount is zero, i.e.

\[
x_s(\Delta, L) = \begin{cases} 0 & \text{if } L \leq \frac{\hat{m} F_{\epsilon}(\Delta + \epsilon)}{2}, \\ \hat{x}_s : \frac{\hat{m} \hat{x}_s + \frac{\hat{m}}{2} F_{\epsilon}(\Delta + F^{-1}_{\epsilon}(\hat{x}_s))}{1 - \hat{m} x} = L & \text{if } L > \frac{\hat{m} F_{\epsilon}(\Delta + \epsilon)}{2}, \end{cases} \tag{30}
\]

\[
x_w(\Delta, L) = \begin{cases} \hat{x}_w : \frac{\hat{m} \hat{x}_w + \frac{\hat{m}}{2} F_{\epsilon}(-\Delta + F^{-1}_{\epsilon}(\hat{x}_w))}{1 - \hat{m} x} = L & \text{if } L < \frac{\hat{m} F_{\epsilon}(\Delta + \epsilon)}{2}, \\ 1 & \text{if } L \geq \frac{\hat{m} F_{\epsilon}(\Delta + \epsilon)}{2}. \end{cases} \tag{31}
\]

As in the benchmark model, if cross-bank strategic interactions are nontrivial, run thresholds of strong and weak banks are infinitely close to the average threshold

\[
\theta^* = \frac{1}{\int_0^1 \frac{dx}{1 - \hat{m} x}} \left[ 1 + \frac{1}{2} \hat{I}_s(\Delta, L) + \frac{1}{2} \hat{I}_w(\Delta, L) \right], \tag{32}
\]

where run asynchronicity $\Delta = \Delta(L, \eta)$ is implicitly defined by

\[
\eta = \frac{1}{2} \int_0^1 \frac{dx}{1 - \hat{m} x} \left[ \hat{I}_s(\Delta, L) - \hat{I}_w(\Delta, L) \right]. \tag{33}
\]

In equilibrium, there exist nontrivial cross-bank interactions if (33) has a solution $\Delta = \Delta(L, \eta) < \tilde{\Delta}$.
Below, we prove several lemmas that together imply Proposition 5.

**Lemma 2.** Suppose that there exist nontrivial cross-bank strategic interactions and \( \Delta = \Delta (L, \eta) \) solves (33). Then it must be that \( L < \frac{\bar{m}}{2} + \frac{\bar{m}}{2} F_\epsilon (\Delta (L, \eta) + \bar{\epsilon}). \) Moreover, \( \frac{d \Delta}{d L} \geq 0, \) with the inequality being strict if \( L > 0 \) or \( \lambda''(\cdot) > 0. \)

**Proof.** Suppose that \( L \) and \( \Delta \) are such that \( L \geq \frac{\bar{m}}{2} + \frac{\bar{m}}{2} F_\epsilon (\Delta + \bar{\epsilon}). \) Then, using the definitions (30) and (31), we get

\[ x_s (\Delta, L) = \frac{2L}{\bar{m}} - 1 \geq F_\epsilon (\Delta + \bar{\epsilon}) \]

and \( x_w (\Delta, L) = 1. \) Then the fire-sale pressure terms (28) and (29) become

\[
\hat{I}_s (\Delta, L) = \int_{\min \left\{ \frac{2L}{\bar{m}} - 1, 1 \right\}}^{1} \lambda \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} - L \right) \frac{\bar{m} x}{1 - \bar{m} x} dx + \int_{0}^{\min \left\{ \frac{2L}{\bar{m}} - 1, 1 \right\}} \frac{\bar{m} x}{1 - \bar{m} x} dx,
\]

\[
\hat{I}_w (\Delta, L) = \int_{0}^{1} \frac{\bar{m} x}{1 - \bar{m} x} dx.
\]

In this case, the derivative of the right-hand side of (33) with respect to \( \Delta \) is 0, that is, the right-hand side of (33) does not depend on \( \Delta. \) In other words, the fire-sale pressure terms do not depend on run asynchronicity \( \Delta, \) i.e. there are no cross-bank strategic uncertainties, which is a contradiction.

Therefore, it must be that \( L < \frac{\bar{m}}{2} + \frac{\bar{m}}{2} F_\epsilon (\Delta + \bar{\epsilon}). \) Suppose that \( \Delta = \Delta (L, \eta) \) solves (33). Then \( L < \frac{\bar{m}}{2} + \frac{\bar{m}}{2} F_\epsilon (\Delta + \bar{\epsilon}), \) which implies \( 0 \leq x_s (\Delta, L) < F_\epsilon (\Delta + \bar{\epsilon}) \) and \( x_s (\Delta, L) \leq x_w (\Delta, L) < 1. \) Then, using the expressions for the fire-sale pressure terms (28) and (29), we can straightforwardly establish that

\[
\frac{\partial \hat{I}_s}{\partial \Delta} > 0 \quad \text{and} \quad \frac{\partial \hat{I}_w}{\partial \Delta} < 0.
\]

Moreover,

\[
\frac{\partial \hat{I}_s}{\partial L} - \frac{\partial \hat{I}_w}{\partial L} = - \int_{x_s (\Delta, L)}^{1} \lambda' \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F_\epsilon (\Delta + F_\epsilon^{-1}(x)) - L \right) \frac{\bar{m} x}{1 - \bar{m} x} dx + \int_{x_w (\Delta, L)}^{1} \lambda' \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F_\epsilon (\Delta + F_\epsilon^{-1}(x)) - L \right) \frac{\bar{m} x}{1 - \bar{m} x} dx \leq 0.
\]

The latter inequality holds because \( x_s (\Delta, L) \leq x_w (\Delta, L) \) and \( \lambda(\cdot) \) is weakly convex. Moreover, this inequality is strict if \( L > 0 \) (because then \( x_s (\Delta, L) < x_w (\Delta, L) \)) or if
\( \lambda'(\cdot) > 0 \).

By the implicit function theorem, it then follows that
\[
\frac{d\Delta}{dL} = \frac{\partial I_s}{\partial \Delta} - \frac{\partial I_w}{\partial \Delta} > 0.
\]

Lemma 2 implies, in particular, that if Equation (33) has a solution \( \Delta(\tilde{L}, \eta) \) for some \( L \), then it also has a solution \( \Delta(\tilde{L}, \eta) < \Delta(L, \eta) \) for any \( \tilde{L} \in [0, L) \).

Recall that \( \bar{\eta} \), defined in (19), can be written as
\[
\bar{\eta} = 1 - \frac{1}{2\int_0^1 \frac{1}{1-mx}dx} \int_0^1 \left[ \lambda \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} \right) - \lambda \left( \frac{\bar{m}}{2} x \right) \right] \frac{\bar{m}x}{1-mx} dx.
\]

Lemma 3. For each \( \eta \in (0, \bar{\eta}) \), there exists \( \bar{L} = \bar{L}(\eta) \) such that if \( L < \bar{L} \), then cross-bank interactions are nontrivial, \( \Delta = \Delta(L, \eta) < \bar{\Delta} \). Moreover, \( \bar{L}(\eta) \) is a decreasing function such that \( \sup_{\eta \in (0, \bar{\eta})} \bar{L}(\eta) = \bar{m} \) and \( \inf_{\eta \in (0, \bar{\eta})} \bar{L}(\eta) = 0 \).

Proof. Lemma 2 implies that if (33) has a solution \( \Delta = \Delta(L, \eta) \), then \( \Delta = \Delta(L, \eta) \) increases in \( L \). Define the supremum value of \( \Delta(L, \eta) \) by \( \Delta^*(\eta) \). Two cases are possible.

**Case 1:** \( \Delta^*(\eta) = \bar{\Delta} \).

Suppose that \( L = \bar{L}(\eta) \) such that \( \Delta = \Delta^*(\eta) = \bar{\Delta} \). By Lemma 2, it must be that
\[
\bar{L} \leq \frac{\bar{m}}{2} + \frac{\bar{m}F_c}{2\bar{L} + \bar{\Delta} + \bar{\epsilon}} = \frac{\bar{m}}{2}.
\]

Using the definitions (30) and (31), we find \( x_s(\bar{\Delta}, \bar{L}) = 0 \) and \( x_w(\bar{\Delta}, \bar{L}) = \frac{2\bar{L}}{\bar{m}} \). Plugging these into (33), we obtain
\[
\eta = \frac{1}{2\int_0^1 \frac{1}{1-mx}dx} \times 
\left[ \int_0^1 \lambda \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} - \bar{L} \right) \frac{\bar{m}x}{1-mx} dx - \int_{\frac{\bar{m}}{2}}^{\frac{2\bar{L}}{\bar{m}}} \lambda \left( \frac{\bar{m}}{2} x - \bar{\Delta} \right) \frac{\bar{m}x}{1-mx} dx - \int_{\frac{\bar{m}}{2}}^{\frac{2\bar{L}}{\bar{m}}} \frac{\bar{m}x}{1-mx} dx \right].
\]

This equation defines \( \bar{L} = \bar{L}(\eta) \) implicitly. It is easy to verify that the right-hand side
decreases in $\bar{L}$ if $\lambda(\cdot)$ is weakly convex. Therefore, $\bar{L}(\eta)$ is a decreasing function. Define

$$\hat{\eta} = \frac{1}{2} \int_0^1 \frac{1}{1 - \bar{m}x} dx \int_0^1 \left[ \lambda \left( \frac{\bar{m}}{2} x \right) - 1 \right] \frac{\bar{m}x}{1 - \bar{m}x} dx.$$  \hfill (35)

Note that if $\lambda(\cdot)$ is weakly convex, then $\bar{\eta} > \hat{\eta}$.

It is easy to see that $\bar{L}(\hat{\eta}) = \bar{m}$ and $\bar{L}(\bar{\eta}) = 0$. Furthermore, if $\eta < \hat{\eta}$, then $\bar{L}(\eta) > \frac{\bar{m}}{2}$, which contradicts (34).

**Case 2:** $\Delta^*(\eta) < \bar{\Delta}$.

Suppose that $L = \bar{L}(\eta)$ such that $\Delta = \Delta^*(\eta) < \bar{\Delta}$. By Lemma 2, it must be that

$$\bar{L} \leq \frac{\bar{m}}{2} + \frac{\bar{m}}{2} F_\epsilon (-\Delta^* + \bar{\epsilon}) \hfill \hfill (36)$$

Suppose that this inequality holds strictly. Then, using the definitions (30) and (31), we get $x_s(\Delta^*, \bar{L}) < F_\epsilon (-\Delta^* + \bar{\epsilon})$ and $x_w(\Delta^*, \bar{L}) < 1$. But then the right-hand side of (33) strictly increases in $\Delta$ and strictly decreases in $L$ in the neighborhood of $(\Delta^*, \bar{L})$. Therefore, for a given $\eta$, a marginal increase in $\bar{L}$ implies a marginal increase in $\Delta^*$. Moreover, such changes do not violate (36). Therefore, $\Delta^*$ is not the supremum value of $\Delta(L, \eta)$ for a given $\eta$. Therefore, it must be that (36) holds as equality, i.e.,

$$\bar{L} = \frac{\bar{m}}{2} + \frac{\bar{m}}{2} F_\epsilon (-\Delta^* + \bar{\epsilon}). \hfill \hfill (37)$$

Under (37), we get $x_s(\Delta^*, \bar{L}) = F_\epsilon (-\Delta^* + \bar{\epsilon})$ and $x_w(\Delta^*, \bar{L}) = 1$ from (30) and (31). Plugging these into (33), we obtain

$$\eta = \frac{1}{2} \int_0^1 \frac{1}{1 - \bar{m}x} dx \int_{F_\epsilon (-\Delta^* + \bar{\epsilon})}^1 \left[ \lambda \left( \frac{\bar{m}}{2} (x - F_\epsilon (-\Delta^* + \bar{\epsilon})) \right) - 1 \right] \frac{\bar{m}x}{1 - \bar{m}x} dx.$$  \hfill (38)

This equation defines $\Delta^* = \Delta^*(\eta)$ implicitly. Clearly, $\Delta^*(\eta)$ is an increasing function. From (37) it then follows that $\bar{L}(\eta)$ is a decreasing function. Note that $\Delta^* < \bar{\Delta} = \bar{\epsilon} - \epsilon \iff \eta < \hat{\eta}$, where $\hat{\eta}$ is defined by (35). Finally, $\Delta^*(0) = 0$ and $\Delta^*(\hat{\eta}) = \bar{\Delta}$, and so $\bar{L}(0) = \bar{m}$ and $L(\hat{\eta}) = \frac{\bar{m}}{2}$.

The analyses of cases 1 and 2 imply that if $L = \bar{L}(\eta)$ then (33) has a solution $\Delta = \Delta^*(\eta)$. Therefore, by Lemma 2, (33) has a solution $\Delta(L, \eta) < \Delta^*(\eta)$ for any $L \in [0, \bar{L}(\eta))$.

Finally, we establish
Lemma 4. Suppose that \( \eta \in (0, \bar{\eta}) \) and \( L < \bar{L}(\eta) \). Then \( \frac{\partial \theta^{*}}{\partial L} < 0 \) and \( \frac{\partial \theta^{*}}{\partial \Delta} < 0 \).

Proof. From Lemmas 2 and 3 it follows that if \( L < \bar{L}(\eta) \) then \( 0 \leq x_{s}(\Delta, L) < F_{\epsilon}(-\Delta + \bar{\epsilon}) \) and \( x_{s}(\Delta, L) \leq x_{w}(\Delta, L) < 1 \). Therefore, using the definitions of the fire-sale pressure terms (28) and (29), we get

\[
\frac{\partial \hat{I}_{s}}{\partial L} < 0 \quad \text{and} \quad \frac{\partial \hat{I}_{w}}{\partial L} < 0,
\]

which together imply that \( \frac{\partial \theta^{*}}{\partial L} < 0 \), where \( \theta^{*} \) is defined by (32).

In the absence of liquidity injections, \( \frac{\partial \theta^{*}}{\partial \Delta} < 0 \) by Proposition 1. Below we show that this result holds if \( L \in (0, \bar{L}(\eta)) \).

\[
\frac{\partial \theta^{*}}{\partial \Delta} \propto \int_{x_{s}(\Delta, L)}^{F_{\epsilon}(-\Delta + \bar{\epsilon})} \left( \frac{m}{2}x + \frac{\bar{m}}{2}F_{\epsilon}(\Delta + F_{\epsilon}^{-1}(x)) - L \right) f_{\epsilon}(\Delta + F_{\epsilon}^{-1}(x)) \frac{m}{1 - \bar{m}} dx - \int_{\max\{x_{w}(\Delta, L), F_{\epsilon}(\Delta + \bar{\epsilon})\}}^{1} \left( \frac{m}{2}x + \frac{\bar{m}}{2}F_{\epsilon}(-\Delta + F_{\epsilon}^{-1}(x)) - L \right) f_{\epsilon}(-\Delta + F_{\epsilon}^{-1}(x)) \frac{m}{1 - \bar{m}} dx,
\]

where \( \propto \) denotes proportionality up to a positive multiplicative term. By changing the variable of integration in the second integral, \( y = F_{\epsilon}(-\Delta + F_{\epsilon}^{-1}(x)) \), we can rewrite the expression as

\[
\frac{\partial \theta^{*}}{\partial \Delta} \propto \int_{x_{s}(\Delta, L)}^{F_{\epsilon}(-\Delta + \bar{\epsilon})} \lambda' \left( \frac{m}{2}x + \frac{\bar{m}}{2}F_{\epsilon}(\Delta + F_{\epsilon}^{-1}(x)) - L \right) f_{\epsilon}(\Delta + F_{\epsilon}^{-1}(x)) \times \left( \frac{m}{1 - \bar{m}} - \frac{mF_{\epsilon}(\Delta + F_{\epsilon}^{-1}(x))}{1 - \bar{m}F_{\epsilon}(\Delta + F_{\epsilon}^{-1}(x))} \right) dx < 0.
\]

\( \Box \)

Proposition 5 follows from Lemmas 2, 3 and 4.

B Robustness and model extensions

B.1 Inefficient asset management

In Section 2.2, the fire-sale discount is a result of liquidity shortage in the asset market. In this section, we consider an alternative setup of the asset market to illustrate that the fire-sale discount can arise when liquidity is abundant but outside investors are less efficient in managing assets than banks.
Assume that outside investors have abundant liquidity, i.e., \( g(L) = L \). However, they are less efficient in managing assets. In particular, under banks’ management, in the absence of premature liquidations, a portfolio \( \{ k_i \}_{i \in [0,1]} \) generates \( y \equiv \int z_i k_i di \) at \( t = 2 \). In contrast, if the same portfolio is managed by outside investors, the return is subject to a discount: instead of receiving \( y \), outside investors only get \( f(y) \), where \( f(y) < y \) for all \( y > 0 \) and \( f(0) = 0 \). In addition, we assume that \( f'(\cdot) > 0 \) and \( f''(\cdot) < 0 \) so that outsiders’ inefficiency in production increases in the amount of assets they absorb. Furthermore, we assume that \( y f'(y) \) is increasing in \( y \) to guarantee equilibrium uniqueness in the asset market at \( t = 1 \). These assumptions on \( f(\cdot) \) are typical in the literature on fire sales (e.g., Lorenzoni, 2008).

The outside investors’ problem therefore becomes

\[
\max_{\{k_i\}_{i \in [0,1]}} f \left( \int z_i k_i di \right) - \int p_i k_i di.
\]

The first-order conditions of the outside investors’ problem imply that

\[
p_i = \frac{\partial f(y)}{\partial y} z_i \quad \forall i \in [0,1],
\]

where \( y \equiv \int z_i k_i di \). After imposing the market clearing conditions, \( k_i = \frac{m_i}{p_i} \forall i \in [0,1] \), we obtain

\[
m_i = z_i k_i \frac{\partial f(y)}{\partial y} \Rightarrow m = \int m_i di = y \frac{\partial f(y)}{\partial y}.
\]

Since by assumption \( y f'(y) \) is an increasing function of \( y \), there is a unique solution \( y = h(m) \) to the equation above. Moreover, \( h'(\cdot) > 0 \). Plugging this into (38), we obtain the equilibrium prices of the same form as in Lemma 1,

\[
p_i(z_i, m) = z_i \frac{m}{h(m)} = \frac{z_i}{\lambda(m)},
\]

where \( \lambda(m) \equiv h(m) m \).

Moreover, the liquidation price for any asset \( i \) is a decreasing function of the total mass of early withdrawers \( m \). Indeed, using (38), we can write

\[
p_i(z_i, m) = z_i \frac{\partial f(y)}{\partial y} \bigg|_{y=h(m)} \Rightarrow \frac{\partial p_i}{\partial m} = z_i \frac{\partial^2 f(y)}{\partial y^2} \bigg|_{y=h(m)} \times \frac{\partial h(m)}{\partial m} < 0.
\]
B.2 General payoff function

Consider a general incremental payoff function \( \pi(z_i, m_i, m) \) of an investor of bank \( i \) that chooses not to withdraw her funds early. It depends on her bank’s productivity \( z_i \), mass of runners on her bank \( m_i \), and overall mass of runners in the whole economy \( m \). In the main model, \( \pi(z_i, m_i, m) = \frac{z_i - \lambda(m)m_i}{1 - m_i} - 1 \). In Section 3.2, we consider \( \pi(z_i, m_i, m) = z_i \pi_1(m_i) + \pi_2(m_i, m) \) to illustrate the role of two complementarities. In this appendix, we do not assume a specific functional form for \( \pi(\cdot, \cdot, \cdot) \). We denote partial derivatives of the \( \pi(\cdot, \cdot, \cdot) \) function by subscripts. We assume that the payoff function is smooth and that it satisfies the following monotonicity properties: \( \pi_z \equiv \partial \pi/\partial z > 0, \pi_m \equiv \partial \pi/\partial m < 0 \).

Below, we show that if \( \pi_m m \leq 0, \pi_z m \geq 0, \pi_{zz} \geq 0 \), our main result in Proposition 1 holds, i.e. \( \partial \theta^*_s/\partial \Delta \leq 0 \). We focus on the case of nontrivial cross-bank strategic interactions, that is, \( \Delta \equiv \lim_{\sigma \to 0} \theta^*_w - \theta^*_s \sigma \in (0, \tilde{\Delta}) \). The analogues of Equations (7) and (8) from the main text are

\[
\int_0^1 \pi \left( \theta^*_s + \eta, \bar{m} x, \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F_\epsilon (\Delta + F_\epsilon^{-1}(x)) \right) dx = 0,
\]

\[
\int_0^1 \pi \left( \theta^*_w - \eta, \bar{m} x, \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F_\epsilon (-\Delta + F_\epsilon^{-1}(x)) \right) dx = 0.
\]

Define

\[
m_{\text{tot}}(x, \Delta) = \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F_\epsilon (\Delta + F_\epsilon^{-1}(x)).
\]

Note that

\[
m_{\text{tot}}(x, \Delta) > \bar{m} x > m_{\text{tot}}(x, -\Delta).
\]

By the implicit function theorem,

\[
\frac{\partial \theta^*_s}{\partial \Delta} = -\frac{\frac{\bar{m}}{2} \int_0^1 F_\epsilon(x - \Delta) \pi_m (\theta^*_s + \eta, \bar{m} x, m_{\text{tot}}(x, \Delta)) f_\epsilon (\Delta + F_\epsilon^{-1}(x)) dx}{\int_0^1 \pi_z (\theta^*_s + \eta, \bar{m} x, m_{\text{tot}}(x, \Delta)) dx} > 0,
\]

\[
\frac{\partial \theta^*_w}{\partial \Delta} = -\frac{-\frac{\bar{m}}{2} \int_0^1 F_\epsilon(x + \Delta) \pi_m (\theta^*_w - \eta, \bar{m} x, m_{\text{tot}}(x, -\Delta)) f_\epsilon (-\Delta + F_\epsilon^{-1}(x)) dx}{\int_0^1 \pi_z (\theta^*_w - \eta, \bar{m} x, m_{\text{tot}}(x, -\Delta)) dx} < 0.
\]
Then the impact of $\Delta$ on the average threshold $\theta^* \equiv \frac{1}{2} \theta^*_s + \frac{1}{2} \theta^*_w$ is

$$\frac{\partial \theta^*}{\partial \Delta} = \frac{1}{2} \frac{\partial \theta^*_s}{\partial \Delta} + \frac{1}{2} \frac{\partial \theta^*_w}{\partial \Delta} = \frac{1}{2} \frac{\partial \theta^*_s}{\partial \Delta} \left( \frac{\partial \theta^*_w}{\partial \Delta} + 1 \right).$$

When $\sigma \to 0$ and $\Delta < \bar{\Delta}$, $\theta^*_s \to \theta^*$ and $\theta^*_w \to \theta^*$, so that

$$\frac{\partial \theta^*}{\partial \Delta} \propto \left. \frac{\partial \theta^*_m}{\partial \Delta} \right|_{\theta^*_m=\theta^*} + 1 = A_1 \times A_2 + 1,$$

where $\propto$ denotes proportionality up to a positive multiplicative term,

$$A_1 = \frac{\int_0^1 \pi_z (\theta^* + \eta, \bar{m}x, m_{tot}(x, \Delta)) \, dx \quad \pi_{xm} \geq 0}{\int_0^1 \pi_z (\theta^* - \eta, \bar{m}x, m_{tot}(x, -\Delta)) \, dx} \geq 1,$$

and

$$A_2 = -\frac{\int_{F_1(\xi+\Delta)} \pi_m (\theta^* - \eta, \bar{m}x, m_{tot}(x, -\Delta)) \, f_x (-\Delta + F^{-1}_e(x)) \, dx}{\int_{F_1(\xi-\Delta)} \pi_m (\theta^* + \eta, \bar{m}x, m_{tot}(x, \Delta)) \, f_x (\Delta + F^{-1}_e(x)) \, dx} \leq \frac{\int_{F_1(\xi+\Delta)} \pi_m (\theta^* + \eta, \bar{m}x, m_{tot}(x, -\Delta)) \, f_x (-\Delta + F^{-1}_e(x)) \, dx}{\int_{F_1(\xi+\Delta)} \pi_m (\theta^* - \eta, \bar{m}x, m_{tot}(x, -\Delta)) \, f_x (-\Delta + F^{-1}_e(x)) \, dx} \leq \frac{\int_{F_1(\xi+\Delta)} \pi_m (\theta^* - \eta, \bar{m}x, m_{tot}(x, -\Delta)) \, f_x (\Delta + F^{-1}_e(x)) \, dx}{\int_{F_1(\xi+\Delta)} \pi_m (\theta^* - \eta, \bar{m}x, m_{tot}(x, -\Delta)) \, f_x (-\Delta + F^{-1}_e(x)) \, dx} \leq \frac{\int_{F_1(\xi+\Delta)} \pi_m (\theta^* + \eta, \bar{m}x, m_{tot}(x, -\Delta)) \, f_x (-\Delta + F^{-1}_e(x)) \, dx}{\int_{F_1(\xi+\Delta)} \pi_m (\theta^* + \eta, \bar{m}x, m_{tot}(x, -\Delta)) \, f_x (-\Delta + F^{-1}_e(x)) \, dx} = -1.$$

Therefore, $\frac{\partial \theta^*}{\partial \Delta} \leq 0$. Moreover, if one of the inequalities $\{\pi_{xm} \leq 0, \pi_{zm} \geq 0, \pi_{zz} \geq 0\}$ holds strictly, then $\frac{\partial \theta^*}{\partial \Delta} < 0$.

In the micro-founded case considered in the main text $\pi_{zz} = 0, \pi_{zm} = 0$ and $\pi_{mx} < 0$. Therefore, the crucial underlying economic mechanism behind Propositions 1 and 2 is mutually reinforcing within- and cross-bank complementarities.
B.3 Nonzero signal precision

B.3.1 Proof of Proposition 6

Proof. The average fragility is given by Equation (20):

$$\theta^* = \frac{1}{2}\theta^*_s + \frac{1}{2}\theta^*_w = \frac{1}{2} \int_1^{1 - \bar{m}x} \left( 1 + \frac{1}{2} I_s(\Delta) + \frac{1}{2} I_w(\Delta) + \sigma \int_0^{1} \frac{F^{-1}_e(x)}{1 - \bar{m}x} dx \right).$$

Therefore,

$$\frac{\partial \theta^*}{\partial \Delta} \propto \frac{\partial (I_s(\Delta) + I_w(\Delta))}{\partial \Delta},$$

where $\propto$ denotes proportionality up to a positive multiplicative term.

From Equations (9) and (10) defining $I_s(\Delta)$ and $I_w(\Delta)$, it is clear that these terms do not depend on $\Delta$ when $\Delta \geq \bar{\Delta}$. If $\Delta < \bar{\Delta}$, we have

$$\frac{\partial (I_s(\Delta) + I_w(\Delta))}{\partial \Delta} \propto \int_0^{F_e(\bar{\Delta})} \lambda' \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F_e (\Delta + F_e^{-1}(x)) \right) f_e (\Delta + F_e^{-1}(x)) \frac{\bar{m}x}{1 - \bar{m}x} dx -$$

$$\int_{F_e(\bar{\Delta})}^{1} \lambda' \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F_e (-\Delta + F_e^{-1}(x)) \right) f_e (-\Delta + F_e^{-1}(x)) \frac{\bar{m}x}{1 - \bar{m}x} dx =$$

$$\int_{F_e(\bar{\Delta})}^{1} \lambda' \left( \frac{\bar{m}}{2} F_e (-\Delta + F_e^{-1}(x)) + \frac{\bar{m}}{2} x \right) \frac{\bar{m}F_e (-\Delta + F_e^{-1}(x))}{1 - \bar{m}F_e (-\Delta + F_e^{-1}(x))} f_e (-\Delta + F_e^{-1}(x)) dx -$$

$$\int_{F_e(\bar{\Delta})}^{1} \lambda' \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F_e (-\Delta + F_e^{-1}(x)) \right) f_e (-\Delta + F_e^{-1}(x)) \frac{\bar{m}x}{1 - \bar{m}x} dx < 0,$$

where we change the variable of integration $x \rightarrow F_e (-\Delta + F_e^{-1}(x))$ in the first integral.

\[\square\]

B.3.2 Informative prior

In this appendix, we explore the case in which standard deviation of signal noise $\sigma$ is finite and the prior about the aggregate fundamental $\theta$ is informative. Morris and Shin (2003) show that the results obtained under the improper prior assumption can be continuously extended to the case in which signals are sufficiently, but not necessarily infinitely, more precise than the prior. In what follows, we assume that signals are sufficiently precise, such that the model features unique threshold equilibrium. In particular,
Run thresholds $\theta^*(\Delta)$, $\theta^s_*(\Delta)$, $\theta^w_*(\Delta)$

Figure 5: Run thresholds of strong ($\theta^*_s$) and weak ($\theta^*_w$) banks, and their average ($\theta^* = \frac{1}{2}\theta^*_s + \frac{1}{2}\theta^*_w$) as functions of run asynchronicity $\Delta$. The prior is informative. See text for the numerical values used to get these graphs.

The run thresholds of strong- and weak-bank investors solve

$$\int \pi \left( \theta + \eta, \bar{m} F_{\epsilon} \left( \frac{\theta - \theta^*_s}{\sigma} \right) + \frac{1}{2} \bar{m} F_{\epsilon} \left( \frac{\theta^*_w - \theta}{\sigma} \right) \right) f_{\theta} (\theta) f_\epsilon \left( \frac{\theta - \theta^*_s}{\sigma} \right) d\theta = 0,$$

$$\int f_{\theta} (\theta) f_\epsilon \left( \frac{\theta - \theta^*_w}{\sigma} \right) d\theta = 0,$$

where investors of weak and strong banks run if their signals are below $\theta^*_w$ and $\theta^*_s$, respectively; $\pi (z_i, m_i, m)$ is the net benefit of not running defined in (2); $f_\theta (\cdot)$ and $f_\epsilon (\cdot)$ are probability density functions of the prior and signal noise, respectively.

We consider the following parameterization: $\bar{m} = 0.55$; $\lambda(m) = 1 + 2m^2$; noise distribution is standard normal; $\sigma = 0.005$; the prior is normal with a mean of 2 and a standard deviation of 0.15. The distribution is truncated, with a lower bound of 0.7 and an upper bound of 2.75. Figure 5 shows the run thresholds as functions of run asynchronicity $\Delta$. As in the main text, heterogeneity is beneficial for stability of both weak and strong banks as long as it is not too large or small.

Figure 6 compares the cases in which the prior is informative and uninformative. In the former case, the parameterization is the same as above. In the latter case, the only difference is the prior distribution. In these two cases bank heterogeneities are not directly comparable, in the sense that the same values of the model primitives correspond...
to different $\Delta$’s. Because of that, we analyze how the run thresholds depend on the ex-post difference between asset performances of weak and strong banks $\eta$. In both models, an increase in $\eta$ corresponds to an increase in run asynchronicity $\Delta$ (panel (a) of Figure 6). As $\Delta$ increases, the average fragility declines. Moreover, as long as $\Delta$ is not too large and there exist meaningful cross-bank strategic interactions, an increase in $\Delta$ tends to benefit both weak and strong banks. These results hold in both versions of the model (panels (b) and (c) of the same figure). One difference between the two models is that the run thresholds are overall lower when the prior is informative. This is because, for our choice of parameters, investors’ prior information about the aggregate fundamental is plausible, and worse signals are required to trigger runs.

(b) Run thresholds as functions of $\eta$, uninformative prior (c) Run thresholds as functions of $\eta$, informative prior

Figure 6: Panel (a): Run asynchronicity $\Delta(\eta)$ for the cases in which prior about the aggregate fundamental is uninformative and informative. Panels (b) and (c): Run thresholds of strong ($\theta^*_s$) and weak ($\theta^*_w$) banks, and their average ($\theta^* = \frac{1}{2}\theta^*_s + \frac{1}{2}\theta^*_w$) as functions of the size of bank-specific shock $\eta$ for the cases of uninformative and informative priors, respectively. In panels (b) and (c), the Y-axes are aligned. See text for the numerical values used to get these graphs.

B.4 Many types of banks

B.4.1 Proof of Proposition 7

Proof. From (21), the run threshold for a bank receiving a shock $\eta_n$ is

$$\theta^* = -\eta_n + \frac{1}{\int_0^1 \frac{1}{1-mx} dx} \left( 1 + \int_0^1 \lambda \left( \bar{m} \sum_{\tau=1}^N \omega_{\tau} F_\epsilon \left( \Delta_{1\tau} - \Delta_{1n} + F_\epsilon^{-1}(x) \right) \right) \frac{\bar{m}x}{1-\bar{m}x} dx \right).$$
where we impose that the run threshold $\theta^*_n$ is infinitely close to the average threshold $\theta^*$. By definition, $\Delta_{n\tau} = \Delta_{1\tau} - \Delta_{1n}$. Notice that

$$\eta_n > \eta_\tau \iff \Delta_{1n} < \Delta_{1\tau} \text{ and } \eta_n = \eta_\tau \iff \Delta_{n\tau} = 0.$$  

Recall that $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_N$. Asynchronicity in the financial system is fully described by $N - 1$ variables $0 \leq \Delta_{12} \leq \Delta_{13} \leq \cdots \leq \Delta_{1N}$. A marginal increase in asynchronicity that (weakly) increases $\Delta_{n\tau}$ for $n < \tau$ corresponds to a change $0 \leq d\Delta_{12} \leq d\Delta_{13} \leq \cdots \leq d\Delta_{1N}$. Recall that the average fragility $\theta^*$ is given by Equation (22), which can be rewritten as

$$\theta^* = \frac{1}{\int_0^1 \frac{1}{1-\overline{m}x} dx} \left( 1 + \int_0^1 \sum_n \omega_n \lambda \left( \overline{m} \sum_\tau \omega_\tau F_\epsilon \left( \Delta_{1\tau} - \Delta_{1n} + F_{\epsilon}^{-1}(x) \right) \right) \frac{\overline{m}x}{1-\overline{m}x} dx \right).$$

Denote

$$I(\Delta_{12}, \ldots, \Delta_{1N}) = \int_0^1 f(x) \sum_n \omega_n \lambda \left( \overline{m} \sum_\tau \omega_\tau F_\epsilon \left( \Delta_{1\tau} - \Delta_{1n} + F_{\epsilon}^{-1}(x) \right) \right) dx,$$

where for brevity we denote $f(x) = \frac{\overline{m}x}{1-\overline{m}x}$. Differentiating $I(\Delta_{12}, \ldots, \Delta_{1N})$ with respect to $\Delta_{1k}$, we get

$$\frac{\partial I}{\partial \Delta_{1k}} = \overline{m} \omega_k \sum_{n=1}^N \omega_n I_{nk},$$

where

$$I_{nk} = \int_0^1 \left[ f(x) - f \left( F_\epsilon \left( \Delta_{1k} - \Delta_{1n} + F_{\epsilon}^{-1}(x) \right) \right) \right] f_\epsilon \left( \Delta_{1k} - \Delta_{1n} + F_{\epsilon}^{-1}(x) \right) \times$$

$$\lambda' \left( \overline{m} \sum_\tau \omega_\tau F_\epsilon \left( \Delta_{1\tau} - \Delta_{1n} + F_{\epsilon}^{-1}(x) \right) \right) dx.$$
is
\[
dI = \sum_{k=2}^{N} \frac{\partial I}{\partial \Delta_{1k}} d\Delta_{1k} = \bar{m} \sum_{k=2}^{N} \sum_{n=1}^{N} \omega_k \omega_n I_{nk} d\Delta_{1k} = \]
\[
\bar{m} \sum_{k=2}^{N} \omega_k \omega_1 I_{1k} d\Delta_{1k} + \bar{m} \sum_{k=2}^{N} \sum_{n=2}^{N} \omega_k \omega_n I_{nk} d\Delta_{1k} \leq 0.
\]
The inequality holds because \( I_{1k} \leq 0 \) \( \forall k \in \{1, \ldots, N\} \) and \( I_{nk} d\Delta_{1k} + I_{kn} d\Delta_{1n} = I_{nk} (d\Delta_{1k} - d\Delta_{1n}) \leq 0 \). The latter is true because if \( k > n \), \( I_{nk} \leq 0 \) and \( d\Delta_{1k} \geq d\Delta_{1n} \), and if \( k < n \), \( I_{nk} \geq 0 \) and \( d\Delta_{1k} \leq d\Delta_{1n} \).

Therefore, a marginal change in asynchronicity that does not reduce pairwise asynchronicities \( \Delta_{n\tau} \) \( \forall n < \tau \) and increases \( \Delta_{n'\tau'} \) for some \( n' < \tau' \) leads to a lower \( I(\Delta_{12}, \ldots, \Delta_{1N}) \) and hence a lower \( \theta^* \).

\( \square \)

**B.4.2 Proof of Proposition 8**

*Proof.* If all bank-specific productivities are identical, i.e. \( \eta = 0 \), the banking system is homogeneous. Therefore, \( \Delta_{n\tau} = 0 \) \( \forall n, \tau \), and all investors run if their signals are below \( \theta^*(0) \). By continuity of \( \lambda(\cdot) \) and \( F_\epsilon(\cdot) \), there must exist an \( \eta > 0 \) such that if \( |\eta_n| < \eta \) \( \forall n \), then investors of all banks share the same run threshold \( \theta^*(\eta) \).

Consider an \( \eta \neq 0 \) with \( |\eta_n| < \eta \) \( \forall n \). For such \( \eta \), without loss of generality we can write \( \eta_1 \geq \eta_2 \geq \cdots \geq \eta_N \), with at least one inequality being strict. Therefore, \( 0 \leq \Delta_{12} \leq \Delta_{13} \leq \cdots \leq \Delta_{1N} \), with at least one inequality being strict. By Proposition 7, \( \theta^*(\eta) < \theta^*(0) \).

\( \square \)

**C Global games proofs**

In this appendix, we prove that our baseline model features a unique threshold equilibrium. Our proof is relatively standard and based on Morris and Shin (2003).

For investors of bank \( i \), the net benefit of not withdrawing funds early is
\[
\pi \left( \theta + \zeta_i, m_i, m \right) = \frac{\theta + \zeta_i - \lambda(m)m_i}{1 - m_i} - 1 = \frac{\theta + \zeta_i - 1 - (\lambda(m) - 1)m_i}{1 - m_i}.
\]

(39)

Here \( m_i \) is the mass of runners on bank \( i \), and \( m = \int m_i di \) is the total mass of runners in the whole economy. Idiosyncratic productivity \( \zeta_i \) takes values \( \eta \) with probability \( \frac{1}{2} \) and \( -\eta \) with probability \( \frac{1}{2} \).
Investor $l$ of bank $i$ receives a signal about the aggregate fundamental $\theta$, 
\[ s_{il} = \theta + \sigma \epsilon_{il}. \]

Noise terms $\epsilon_{il}$ are independent across investors and have identical cumulative distribution function $F_{\epsilon_{il}}(\cdot)$ that is increasing on the support $[\underline{\epsilon}, \overline{\epsilon}]$, $-\infty \leq \underline{\epsilon} \leq \overline{\epsilon} \leq \infty$. As in the main text, $\mathbb{E}\epsilon_{il} = 0$ and $\mathbb{V}\epsilon_{il} = 1$. In this appendix, we assume that noise is small, that is, $\sigma \rightarrow 0$. Using continuity arguments as in Morris and Shin (2003), one can easily show that our results can be extended to the case in which signals are sufficiently, but not necessarily infinitely, more precise than the prior.

Notice that the payoff function (39) is not always decreasing in $m_i$, that is, it does not always imply strategic complementarities across investors of the same bank. However, as we show in what follows, it features a single-crossing property. As a result, under a set of standard assumptions outlined below, the game features a unique threshold equilibrium. However, as in Morris and Shin (2003), we cannot rule out existence of non-threshold equilibria.

Since we focus on threshold equilibria, we can rewrite the payoff function as
\[
\Pi(z_i, m_i, \hat{\Delta}_i) \equiv \pi \left( z_i, m_i, \frac{m_i}{2} + \frac{\hat{\Delta}_i}{\overline{m}} F_{\epsilon_{il}} \left( \hat{\Delta}_i + \frac{1}{F_{\epsilon_{il}}^{-1} \left( \frac{m_i}{\overline{m}} \right)} \right) \right).
\]

Here $\hat{\Delta}_i$ represents a signed distance between run thresholds normalized by $\sigma$. Specifically, for strong banks $\hat{\Delta}_s = \lim_{\sigma \rightarrow 0} \theta^*_{w} - \theta^*_{s} \sigma$, and for weak banks $\hat{\Delta}_w = -\hat{\Delta}_s$. With this alternative payoff expression, we can define single-crossing property as follows.

**Definition 3.** A payoff function $\pi(z_i, m_i, m)$ satisfies a single-crossing property if for any $z_i \in [\theta - \eta, \theta + \eta]$ and any $\hat{\Delta}_i \in [\underline{\epsilon}, \overline{\epsilon}, \overline{\epsilon} - \epsilon]$, there exists at most one $m^{*}_{i} \in (0, \overline{m})$ such that $\Pi(z_i, m_i, \hat{\Delta}_i)$ switches sign, i.e., $\Pi \gtrless 0$ if $m_i \gtrless m^{*}_{i}$.

It is straightforward to verify that the payoff function in our main model, given by (39), satisfies the single-crossing property.

As we show in the proof of Proposition 9 below, the unique threshold equilibrium result can be generalized to any payoff function that satisfies the single-crossing property. We make the following standard assumptions.

**Assumption 2.** (Dominance regions) There exist $\theta^{LDR}$ and $\theta^{UDR}$ such that $\pi(\theta^{LDR} + \eta, 0, 0) < 0$ and $\pi(\theta^{UDR} - \eta, \overline{m}, \overline{m}) > 0$. 

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Assumption 3. (Monotone likelihood ratio property) Probability density function of noise \( f_\epsilon(\cdot) \) is such that \( \frac{f_\epsilon(\xi^H - \cdot)}{f_\epsilon(\xi^L - \cdot)} \) increases in \( \epsilon \) for any \( \xi^H > \xi^L \).

Proposition 9. Given any payoff function \( \pi(z, m_i, m) \) that increases in \( z_i \), decreases in \( m \) and satisfies the single-crossing property, there exists a unique threshold equilibrium in which investors of strong and weak banks withdraw early if their signals are below \( \theta_s^* \) and \( \theta_w^* \), respectively, and do not withdraw early otherwise.

Proof. Define \( \pi^s(s, k_s, k_w) \) as the net benefit of not running on her bank for a strong-bank investor that observes signal \( s \) and believes that investors of strong and weak banks run if their signals are below \( k_s \) and \( k_w \), respectively. Define \( \pi^w(s, k_w, k_s) \) in an analogous way but for a weak-bank investor.

\[
\pi^s(s, k_s, k_w) = \frac{\int_{s-\sigma \epsilon}^{s-\sigma \epsilon} \pi (\theta + \eta, \bar{m} F_e \left( \frac{k_s - \theta}{\sigma} \right), \frac{\eta}{2} F_e \left( \frac{k_s - \theta}{\sigma} \right)) f_\epsilon \left( \frac{s - \theta}{\sigma} \right) f_\theta(\theta) d\theta}{\int_{s-\sigma \epsilon}^{s-\sigma \epsilon} f_\epsilon \left( \frac{s - \theta}{\sigma} \right) f_\theta(\theta) d\theta},
\]

\[
\pi^w(s, k_w, k_s) = \frac{\int_{s-\sigma \epsilon}^{s-\sigma \epsilon} \pi (\theta - \eta, \bar{m} F_e \left( \frac{k_w - \theta}{\sigma} \right), \frac{\eta}{2} F_e \left( \frac{k_w - \theta}{\sigma} \right)) f_\epsilon \left( \frac{s - \theta}{\sigma} \right) f_\theta(\theta) d\theta}{\int_{s-\sigma \epsilon}^{s-\sigma \epsilon} f_\epsilon \left( \frac{s - \theta}{\sigma} \right) f_\theta(\theta) d\theta},
\]

where \( f_\theta(\cdot) \) is the probability density function of the prior distribution of the aggregate fundamental.

In what follows, we focus on strong banks. All derivations for weak banks are analogous. Changing the variable of integration, \( z = \frac{s - k_s}{\sigma} \), we get

\[
\pi^s(s, k_s, k_w) = \frac{\int_{s-k_s}^{s-k_s} \pi (k_s + \sigma z + \eta, \bar{m} F_e (-z), \frac{\eta}{2} F_e (-z) + \frac{m}{2} F_e \left( \frac{k_w - k_s}{\sigma} - z \right)) f_\epsilon \left( \frac{s - k_s}{\sigma} - z \right) f_\theta(k_s + \sigma z) dz}{\int_{s-k_s}^{s-k_s} f_\epsilon \left( \frac{s - k_s}{\sigma} - z \right) f_\theta(k_s + \sigma z) dz}.
\]

With infinitely small noise, we have

\[
\pi^s(s, k_s, k_w) \xrightarrow{\sigma \to 0} \bar{\pi}^s(s, k_s, k_w) = \frac{\int_{s-k_s}^{s-k_s} \pi (k_s + \eta, \bar{m} F_e (-z), \frac{\eta}{2} F_e (-z) + \frac{m}{2} F_e \left( \frac{k_w - k_s}{\sigma} - z \right)) f_\epsilon \left( \frac{s - k_s}{\sigma} - z \right) dz}{\int_{s-k_s}^{s-k_s} f_\epsilon \left( \frac{s - k_s}{\sigma} - z \right) dz}.
\]
Denote

\[ h(s, s', k_s, k_w) = \int_{\frac{z - k_s}{\sigma} - \tilde{\epsilon}}^{\frac{z - k_s}{\sigma} - \epsilon} \tilde{f}(z, s, k_s) \tilde{\pi}(z, s', k_s, k_w) \, dz, \]

where

\[ \tilde{f}(z, s, k_s) = f_\epsilon \left( \frac{s - k_s}{\sigma} - z \right), \]
\[ \tilde{\pi}(z, s', k_s, k_w) = \pi \left( s' + \eta, \bar{m} F_\epsilon(-z), \frac{\bar{m}}{2} F_\epsilon(-z) + \frac{\bar{m}}{2} F_\epsilon \left( \frac{k_w - k_s}{\sigma} - z \right) \right). \]

First, \( \tilde{f}(z, s, k_s) \) satisfies the monotone likelihood ratio property by Assumption 3, namely, \( \tilde{f}(z, s, H, k_s) \) increases in \( z \) for \( s_H > s_L \). Second, \( \tilde{\pi}(z, s', k_s, k_w) \) satisfies the single-crossing property. That is, for a given \( s' \) there exists at most one \( z \) in which \( \tilde{\pi}(z, s', k_s, k_w) \) switches sign. Therefore, by Lemma 5 in Athey (2002), \( h(s, s', k_s, k_w) \) also satisfies single-crossing, that is, if \( s < s^*(s', k_s, k_w) \) then \( h(s, s', k_s, k_w) < 0 \) and if \( s > s^*(s', k_s, k_w) \) then \( h(s, s', k_s, k_w) > 0 \).

Suppose that \( h(s, s, k_s, k_w) = 0 \). Such \( s \) exists by Assumption 2. For any \( s' < s \) we have

\[ h(s', s', k_s, k_w) < h(s', s, k_s, k_w) < h(s, s, k_s, k_w) = 0, \]

where the first inequality holds because \( h(s, s', k_s, k_w) \) increases in \( s' \) and the second inequality holds because \( h(s, s', k_s, k_w) \) satisfies single-crossing. Analogously, for any \( s' > s \) we have

\[ h(s', s', k_s, k_w) > h(s', s, k_s, k_w) > h(s, s, k_s, k_w) = 0. \]

Then there exists a cutoff \( \beta_s(k_s, k_w) \) such that \( \tilde{\pi}^s(s, k_s, k_w) = h(s, s, k_s, k_w) \) is negative (positive) if \( s \) is below (above) the cutoff and zero at the cutoff. Similarly, \( \beta_w(k_w, k_s) \) can be defined. In equilibrium, it must be that \( k_s = \beta_s(k_s, k_w) \) and \( k_w = \beta_w(k_w, k_s) \). Therefore, the run thresholds are implicitly given by

\[ \tilde{\pi}^s(\theta_s^*, \theta_s^*, \theta_w^*) = \int_{-\tilde{\epsilon}}^{-\epsilon} \pi \left( \theta_s^* + \eta, \bar{m} F_\epsilon(-z), \frac{\bar{m}}{2} F_\epsilon(-z) + \frac{\bar{m}}{2} F_\epsilon \left( \frac{\theta_w^* - \theta_s^*}{\sigma} - z \right) \right) f_\epsilon(-z) \, dz = 0. \]
Changing the variable of integration, $x = F(x) (-z)$, we get

$$\int_0^1 \pi \left( \theta_s^* + \eta, \bar{m} x, \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F \left( \frac{\theta_w^* - \theta_s^*}{\sigma} + F^{-1}(x) \right) \right) dx = 0. \quad (40)$$

Similarly, we can derive the indifference condition for marginal investors of weak banks:

$$\int_0^1 \pi \left( \theta_w^* - \eta, \bar{m} x, \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F \left( -\frac{\theta_w^* - \theta_s^*}{\sigma} + F^{-1}(x) \right) \right) dx = 0. \quad (41)$$

There exists a unique solution $(\theta_s^*, \theta_w^*)$ to equations (40) and (41). The existence follows from Assumption 2. Below we prove the uniqueness by contradiction.

Suppose there exist two equilibria with distinct thresholds $(\hat{\theta}_s^*, \hat{\theta}_w^*)$ and $(\hat{\theta}_s^*, \hat{\theta}_w^*)$. Without loss of generality, suppose $\hat{\theta}_s^* > \theta_s^*$. Since $\pi(z_i, m_i, m)$ increases in $z_i$ and decreases in $m$, Equation (40) implies that $\frac{\hat{\theta}_w^* - \theta_s^*}{\sigma} > \frac{\theta_w^* - \theta_s^*}{\sigma}$. Similarly, according to Equation (41), this implies $\hat{\theta}_w^* < \theta_w^*$. Since the difference between two thresholds $\hat{\theta}_w^* - \hat{\theta}_s^* > \theta_w^* - \theta_s^*$, it then must be $\hat{\theta}_s^* < \theta_s^*$, which contradicts the premise that $\hat{\theta}_s^* > \theta_s^*$. Therefore, the equilibrium is unique.

Substituting in the payoff function (39), thresholds $\theta_s^*$ and $\theta_w^*$ are determined by

$$\int_0^1 \frac{\theta_s^* + \eta - \lambda \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F \left( \frac{\theta_w^* - \theta_s^*}{\sigma} + F^{-1}(x) \right) \right)}{1 - \bar{m} x} \bar{m} x dx = 1,$$

$$\int_0^1 \frac{\theta_w^* - \eta - \lambda \left( \frac{\bar{m}}{2} x + \frac{\bar{m}}{2} F \left( -\frac{\theta_w^* - \theta_s^*}{\sigma} + F^{-1}(x) \right) \right)}{1 - \bar{m} x} \bar{m} x dx = 1.$$

These are Equations (7) and (8).

Note that this proof can be straightforwardly extended to the case in which bank-specific productivity shock takes $N \geq 2$ values.