A Production-Based Model for the Term Structure

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    - "Complete" production-based pricing (Cochrane 1988, 1993, Belo 2010, Jermann 2010)

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- Examine implied term structure quantitatively and analytically

Findings

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- Depreciation rates are important for term premium

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$$\text{s.t.} \quad K_j\left(s^t\right) = K_j\left(s^{t-1}\right)\left(1-\delta_j\right) + I_j\left(s^t\right),\ \forall s^t, j,$$

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- $F(...) = \sum_{j=1}^{2} A_j\left(s^t\right) K_j\left(s^{t-1}\right)$
- $H_j(...) = \left\{\frac{b_j}{\nu_j}\left(I_j\left(s^t\right)/K_j\left(s^{t-1}\right)\right)^{\nu_j} + c_j\right\} K_j\left(s^{t-1}\right)$

Real Model, 2

First-order conditions

$$1 = \sum_{s_{t+1}} P\left(s_{t+1} | s^t\right) R_j^I\left(s^t, s_{t+1}\right) \text{ for } j = 1, 2$$

with

$$R_j^I\left(s^t, s_{t+1}\right) \equiv \left(\frac{F_{K_j}(s^t, s_{t+1}) - H_{j,1}(s^t, s_{t+1}) + \left(1 - \delta_j\right) q_j(s^t, s_{t+1})}{q_j(s^t)} \right)$$

and

$$q_j\left(s^t\right) = H_{j,2}\left(...\right) = b_j \left(\frac{I_j\left(s^t\right)}{K_j\left(s^{t-1}\right)} \right)^{\nu_j - 1}$$

Real Model, 3

- Recovering state prices

$$\begin{bmatrix} R_1^l\left(s^t,\mathfrak{s}_1\right) & R_1^l\left(s^t,\mathfrak{s}_2\right) \\ R_2^l\left(s^t,\mathfrak{s}_1\right) & R_2^l\left(s^t,\mathfrak{s}_2\right) \end{bmatrix} \begin{bmatrix} P\left(\mathfrak{s}_1 | s^t\right) \\ P\left(\mathfrak{s}_2 | s^t\right) \end{bmatrix} = \mathbf{1}$$

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- so that state prices depend on

$$\left(\frac{I_1\left(s^t\right)}{K_1\left(s^{t-1}\right)}, \frac{I_2\left(s^t\right)}{K_2\left(s^{t-1}\right)}, \lambda_1^I\left(s^{t+1}\right), \lambda_2^I\left(s^{t+1}\right), A_j\left(s^{t+1}\right) \right)$$

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- Assume investment and technology not contingent on inflation. For instance,

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- Inflation not directly priced. For instance,
$$P\left(\mathfrak{s}_1, \mathfrak{z}_1 | s^t, z_t\right) = \left(\frac{\Pr(\mathfrak{s}_1, \mathfrak{z}_1 | s^t, z_t)}{\Pr(\mathfrak{s}_1, \mathfrak{z}_1 | s^t, z_t) + \Pr(\mathfrak{s}_1, \mathfrak{z}_2 | s^t, z_t)}\right) P\left(\mathfrak{s}_1 | s^t\right), \text{ and}$$
$$P\left(\mathfrak{s}_1, \mathfrak{z}_2 | s^t, z_t\right) = \left(1 - \frac{\Pr(\mathfrak{s}_1, \mathfrak{z}_1 | s^t, z_t)}{\Pr(\mathfrak{s}_1, \mathfrak{z}_1 | s^t, z_t) + \Pr(\mathfrak{s}_1, \mathfrak{z}_2 | s^t, z_t)}\right) P\left(\mathfrak{s}_1 | s^t\right)$$

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$P(\mathfrak{s}_1, \mathfrak{z}_2|s^t, z_t) = \left(1 - \frac{\Pr(\mathfrak{s}_1, \mathfrak{z}_1|s^t, z_t)}{\Pr(\mathfrak{s}_1, \mathfrak{z}_1|s^t, z_t) + \Pr(\mathfrak{s}_1, \mathfrak{z}_2|s^t, z_t)}\right) P(\mathfrak{s}_1|s^t)$
- If inflation and investment independent

$$V_t^{\$(1)}(s^t, z_t) = \left\{P(\mathfrak{s}_1'|s^t) + P(\mathfrak{s}_2'|s^t)\right\} E\left(\frac{1}{\lambda^P}|s^t, z_t\right)$$

Table 1: Parameter values

Parameter	Symbol	Value
Investment rates	$\lambda^{I}(\mathfrak{s}_1), \lambda^{I}(\mathfrak{s}_2)$	$0.9497, 1.1109$
Serial correlation		0.2
Relative freq. of low		0.8
Inflation rates	$\lambda^{P}(\mathfrak{z}_1), \lambda^{P}(\mathfrak{z}_2)$	$1.0169, 1.0763$
Serial correlation		0.8
Relative freq. of low		1.9
Depreciation rates	δ_E, δ_S	$0.112, 0.031$
Relative value of cap.	$\overline{K_E / K_S}$	0.6
Adjustment cost par.	b_E, b_S, c_E, c_S so that $\overline{q}$	1.5
Adjustment cost curv.	ν_E, ν_S	$2.2385, 4.1080$
Marginal prod. of cap.	A_E, A_S so that $\bar{R}_E, \bar{R}_S$	$1.04515, 1.05773$

Table 2: Equity returns and short term yields		
	Model	Data
$E\left(r_M - y^{(1)}\right)$ %	4.64	4.64
$\sigma\left(r_{M,r}\right)$ %	17.13	17.13
$E\left(y^{(1)}\right)$ %	5.29	5.29
$\sigma\left(y^{(1)}\right)$ %	2.98	2.98

Yields, y, are from Fama and Bliss, defined as $-\ln\left(\text{price}\right)/\text{maturity}$, stock returns are the logs of value-weighted returns from CRSP, $r_{M,r}$ is the stock return deflated by the CPI-U. All data is 1952-2010.

Table 3: Term structure

	Maturity (years)				
	1	2	3	4	5
Nominal yields					
Mean - Model %	5.29	5.44	5.58	5.72	5.86
Mean - Data %	5.29	5.49	5.67	5.81	5.90
Std - Model %	2.98	2.73	2.51	2.33	2.17
Std - Data %	2.98	2.93	2.85	2.80	2.75
Real yields					
Mean - Model %	1.68	1.84	2.00	2.15	2.31
Std - Model %	2.06	1.92	1.81	1.71	1.62

Table 4: Fama-Bliss excess return regressions				
$rx_{t+1}^{(n)} = \alpha + \beta \left(f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+1}^{(n)}$				
	Maturity (years)			
	2	3	4	5
Model - β	.3050	.3906	.5144	.6135
Data - β	.7606	1.0007	1.2723	.9952

Yields are from Fama and Bliss 1952-2010, $rx_{t+1}^{(n)}$ is the excess return of a n-period discount bond, $f_t^{(n)}$is the forward rate, $(p_t^{(n-1)} - p_t^{(n)})$, $p_t^{(n)}$ the log of the price discount bond, and $y_t^{(1)}$ is the 1 period yield.

Table 5: Fama-Bliss excess return regressions

No inflation risk

$rx_{t+1}^{(n)} = \alpha + \beta \left(f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+1}^{(n)}$

		Maturity (years)			
		2	3	4	5
Model - β	no inflation risk	.4656	.6101	.7881	.9465
Model - β	real forward premium	.4667	.6039	.7866	.9473
Model - β	benchmark	.3050	.3906	.5144	.6135
Data - β		.7606	1.0007	1.2723	.9952

Continuous-time

- Assume univariate dz with discount factor process

$$\frac{d\Lambda}{\Lambda} = -r\left(.\right)dt - \sigma\left(.\right)dz$$

with given returns for the two types of capital

$$\frac{dR_j}{R_j} = \mu_j\left(.\right)dt + \sigma_j\left(.\right)dz, \text{for } j = 1, 2$$

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- The absence of arbitrage implies that

$$0 = -r + \mu_j - \sigma_j \sigma, \text{ for } j = 1, 2$$

so that

$$\begin{aligned} r &= \frac{\sigma_2}{\sigma_2 - \sigma_1}\mu_1 - \frac{\sigma_1}{\sigma_2 - \sigma_1}\mu_2 \\ \sigma &= \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \end{aligned}$$

Capital return

The return to a given capital stock equals

$$\underbrace{\left\{ \begin{array}{c} \frac{A_j - c_j}{b_j \left(\frac{I_{j,t}}{K_{j,t}} \right)^{\nu_j - 1}} - (\nu_j - 1) \left(1 - \frac{1}{\nu_j} \right) I_{j,t} / K_{j,t} - \delta_j \\ + (\nu_j - 1) \left[\left(\lambda^{I,j} - 1 \right) + \delta_j + \frac{1}{2} (\nu_j - 2) \sigma_{I,j}^2 \right] \end{array} \right\} dt}_{\mu_j(.)}$$

$$+ \underbrace{(\nu_j - 1) \sigma_{I,j}}_{\sigma_j(.)} dz$$

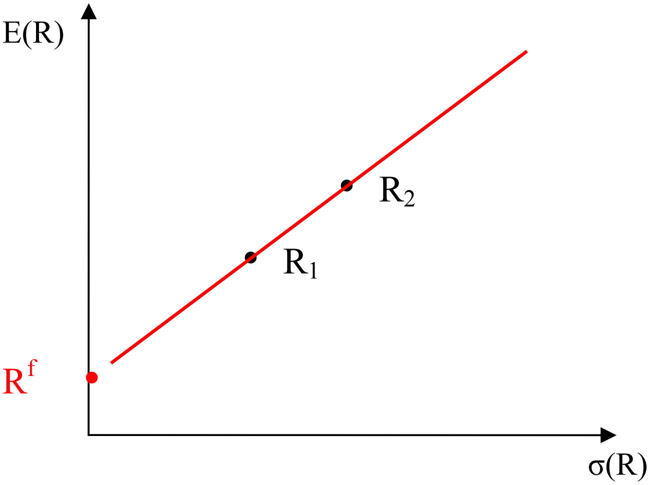
Sharpe ratio

At steady state, $I/K = \lambda^I - 1 + \delta$, and with $\sigma_{I,j} = \sigma_I$, the Sharpe ratio is given by

$$\sigma|_{ss} = \frac{\mu_j - r}{\sigma_j} = \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} = \frac{\bar{R}_2 - \bar{R}_1}{(\nu_2 - \nu_1)\,\sigma_I} + \frac{\nu_1 + \nu_2 - 3}{2}\sigma_I$$

with

$$\bar{R} = \frac{A - c}{b\left(\lambda^I - (1 - \delta)\right)^{\nu - 1}} + \left(1 - \frac{1}{\nu}\right)\lambda^I + \frac{1}{\nu}(1 - \delta)$$


E(R)
R₂
R₁
Rᶠ
σ(R)

Dynamics of the short rate

- The short rate equals

$$r = \frac{\sigma_2}{\sigma_2 - \sigma_1}\mu_1 - \frac{\sigma_1}{\sigma_2 - \sigma_1}\mu_2$$

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- $dr = \mu_r(.)\, dt + \sigma_r(.)\, dz :$ at steady state, for $\sigma_{I,j} = \sigma_I$, and $\lambda^{I,j}$ and σ_I constant,

$$\sigma_r|_{ss} = \frac{(\nu_2 - 1)(\nu_1 - 1)}{\nu_2 - \nu_1}\left[\bar{R}_2 - \bar{R}_1 + \delta_2 - \delta_1\right]\sigma_I$$

Table 6: Term premium: continuous-time versus discrete-time model

	Cont.-time	Discrete-time
	$-\sigma_r\sigma$	$E_t\left(r_{t+1}^{(2)} - y_t^{(1)}\right)$
Benchmark	.0024	.0022
$\delta_1 = \delta_2, \bar{R}_1 = \bar{R}_2,$	0	0.00001
$\delta_1 = \delta_2$	-0.00044	$-.00036$
$\bar{R}_1 = \bar{R}_2, \delta_1 = .112 > \delta_2 = .0313$	.0017	.0015
$\bar{R}_1 = \bar{R}_2, \delta_1 = .0313 < \delta_2 = .112$	$-.0017$	$-.0018$

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- Time-varying term premiums are evidenced through Fama-Bliss regressions
- Even with homoscedastic investment and inflation, the market price of risk and the volatility of the short rate are naturally time-varying, driven by time-varying investment to capital ratios