Handout 3: Proof of Mean and Variance Formulas (Optional)

Corporate Finance, Sections 001 and 002

In this handout we prove the following very useful relations:

$$E(R_p) = X_1 E(R_1) + X_2 E(R_2) \tag{1}$$

$$\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\text{Cov}(R_1, R_2) \tag{2}$$

where $E(R_p)$ represents the mean of a portfolio, and σ_p^2 represents the variance. We sometimes write the second formula in terms of correlation:

$$\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_1\sigma_2\rho.$$

First recall the rules of mean and covariance. Here, X, Y, and Z are random variables and a is a constant. Then:

1. $E(X+Y) = E(X) + E(Y)$
2. $E(aX) = aE(X)$
3. $\text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$
4. $\text{Cov}(aX, Y) = a\text{Cov}(X, Y)$

Our first goal is to prove the formula for the mean (1). From $R_p = X_1R_1 + X_2R_2$, it follows from the first rule that

$$E(R_p) = E(X_1R_1) + E(X_2R_2).$$

By the second rule:

$$E(R_p) = X_1E(R_1) + X_2E(R_2).$$

This proves the formula for the mean.

Now for the formula for the variance. First, recall that

$$\sigma_p^2 = \mathrm{Cov}(R_p, R_p).$$

By repeated applications of the third rule:

$$\begin{aligned}
\mathrm{Cov}(R_p, R_p) &= \mathrm{Cov}(X_1R_1 + X_2R_2, X_1R_1 + X_2R_2) \\
&= \mathrm{Cov}(X_1R_1, X_1R_1) + \mathrm{Cov}(X_2R_2, X_2R_2) + 2\mathrm{Cov}(X_1R_1, X_2R_2).
\end{aligned}$$

Finally, applying the fourth rule:

$$\begin{aligned}
\mathrm{Cov}(R_p, R_p) &= X_1^2\mathrm{Cov}(R_1, R_1) + X_2^2\mathrm{Cov}(R_2, R_2) + 2X_1X_2\mathrm{Cov}(R_1, R_2) \\
&= X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\mathrm{Cov}(R_1, R_2).
\end{aligned}$$

This completes the proof of the variance formula.