Handout 4: Gains from Diversification for 2 Risky Assets

Corporate Finance, Sections 001 and 002

Suppose you are deciding how to allocate your wealth between two risky assets. Recall that the expected return of a two-asset portfolio is:

$$\bar{R}_p = X_1\bar{R}_1 + X_2\bar{R}_2$$

where $\bar{R}_1$ and $\bar{R}_2$ are the expected returns on Asset 1 and 2, and X_1 and X_2 are the weights invested in each. The standard deviation (σ) on the portfolio is:

$$\sigma_p = \left[X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_1\sigma_2\rho\right]^{1/2}$$

where ρ is the correlation between the two assets.

Suppose we have the following information:

$$\bar{R}_1 = .17 \quad \sigma_1 = .25$$
$$\bar{R}_2 = .10 \quad \sigma_2 = .12$$

We also know that the correlation between these assets is $\rho = .2$. Notice that Asset 2 has a lower mean and standard deviation than Asset 1. Using the two formulas, we can compute the mean and standard deviation for a range of weights in Assets 1 and 2:

X_1	X_2	$\bar{R}_p$	σ_p
0	1	.100	.120
.2	.8	.114	.117
.4	.6	.128	.134
.6	.4	.142	.166
.8	.2	.156	.206
1	0	.170	.250

For example, consider the portfolio with 20% in Asset 1 and 80% in Asset 2. From the formula for the mean:

$$\bar{R}_p = .2(.17) + .8(.10) = .114.$$

From the formula for the standard deviation:

$$\sigma_p = \left[(.2)^2(.25)^2 + (.8)^2(.12)^2 + 2(.2)(.8)(.25)(.12)(.2)\right]^{1/2} = .117$$

The table shows that as we go from 100% in the second asset ($X_2 = 1$) to 100% in the first the mean increases. Not so for the standard deviation. Adding a little bit of Asset 1 to Asset 2 causes a *decrease* in standard deviation. This is a gain from diversification.

How is it that we can increase the mean and decrease the standard deviation at the same time? The best way to understand this gain from diversification is to consider three special cases that are particularly intuitive.

Case 1. Perfect correlation: $\rho = 1$

When $\rho = 1$, the formula for variance becomes:

$$\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_1\sigma_2$$

It turns out that this expression is a perfect square, and

$$\sigma_p^2 = (X_1\sigma_1 + X_2\sigma_2)^2.$$

That means:

$$\sigma_p = X_1\sigma_1 + X_2\sigma_2$$

In this very special case, the standard deviation is a weighted average, just like the mean. We have two linear equations, so every point on our mean-variance diagram (shown below) is a straight line. As we move along the line, for each bit of expected return, we have to pay in terms of standard deviation. In the case of $\rho = 1$ (and only in this case) we have no gains from diversification.

Case 2. Perfect negative correlation: ρ = -1

When $\rho = -1$, the formula for variance becomes:

$$\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 - 2X_1X_2\sigma_1\sigma_2.$$

This is also a perfect square:

$$\sigma_p^2 = (X_1\sigma_1 - X_2\sigma_2)^2$$

Because σ_p must be a positive number:

$$\sigma_p = |X_1\sigma_1 - X_2\sigma_2|.$$

The amazing thing about $\rho = -1$ is that we can reduce the standard deviation all the way to zero. Remember that $X_1 = 1 - X_2$. Setting σ_p to zero in the formula above implies:

$$(1 - X_2)\sigma_1 - X_2\sigma_2 = 0$$

Collecting terms:

$$\sigma_1 - X_2(\sigma_1 + \sigma_2) = 0.$$

Therefore:

$$X_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

Using $\sigma_1 = .25$ and $\sigma_2 = .12$:

$$X_2 = \frac{.25}{.25 + .12} = .68$$

Which implies $X_1 = .32$

$$\bar{R}_p = .32(.17) + .68(.10) = .12$$

This tells us that for $\rho = -1$, the investment opportunity set must touch the y-axis at .12. In the case of $\rho = -1$ we have huge gains in diversification. Perfect negative correlation

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means that when Asset 1 is high relative to its mean, Asset 2 is low. A portfolio that combines Asset 1 and 2 will not have as wide swings as Asset 1 or Asset 2 by itself. What is surprising is that we can reduce the risk all the way to zero.

The cases of $\rho = 1$ and $\rho = -1$ are depicted in the mean-variance diagram shown below.
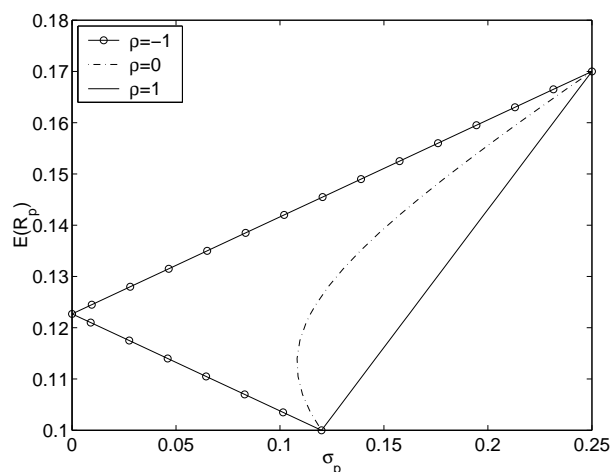
Case 3. Independence: $\rho = 0$

It may seem like in this case that we don't have gains from diversification. But we do. Consider what happens to the formula when we put in $\rho = 0$:

$$\sigma_p^2 = X_1^2 \sigma_1^2 + X_2^2 \sigma_2^2$$

This formula is *not* a weighted average, despite appearances. The squares of weights sum to less than one. In fact,

$$\sigma_p = (X_1^2 \sigma_1^2 + X_2^2 \sigma_2^2)^{1/2} < X_1 \sigma_1 + X_2 \sigma_2$$

The standard deviation is much less than it would be if $\rho = 1$. The best way to prove it to yourself is to calculate the standard deviations of the portfolios:

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X_1	X_2	$\bar{R}_p$	σ_p
0	1	.100	.120
.2	.8	.114	.108
.4	.6	.128	.123
.6	.4	.142	.158
.8	.2	.156	.201
1	0	.170	.250

Note that the means are identical to the case where $\rho = .2$. This is not surprising - the correlation does not enter into the formula for the mean.

These points are plotted in the figure on the previous page. Even in this case we can simultaneously increase our mean and decrease our standard deviation.

Real Life. $\rho > 0$

In real life, the most common case we encounter is that $\rho > 0$. For example, in the case above, $\rho = .2$. The intuition for the positive correlation case is exactly the same as in the zero-correlation case, though the gains from diversification are less powerful.

Below, $\rho = .2$ and $\rho = .5$ are added into the plot. Notice that for $\rho = .5$, there is no backward bend to the diagram. The standard deviation is steadily increasing. Even here, however, there are gains from diversification relative to the case of $\rho = 1$. Standard deviation increases, but it does so less than proportionally.

For all values of $\rho < 1$, the standard deviation of a portfolio is less than its standard deviation when $\rho = 1$. Because the standard deviation when $\rho = 1$ is the weighted average, the standard deviation when $\rho < 1$ must be less than the weighted average.

Note also that diversification can never create a "monster portfolio". That is, the standard deviation never "explodes" because it never exceeds a weighted average of the two underlying standard deviations. The reason is that ρ can never be greater than one.
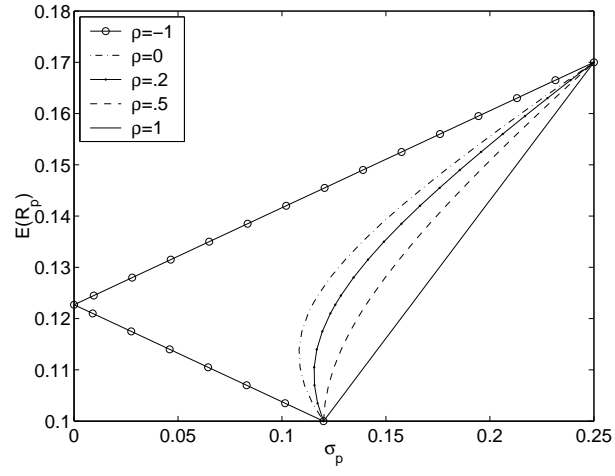
Thus the largest possible standard deviation is a weighted average of σ_1 and σ_2.