Handout 6: Portfolio Variance with Many Risky Assets

Corporate Finance, Sections 001 and 002

Case 1: Unsystematic risk only.

Recall that when the correlation ρ between two securities equals zero, the portfolio variance is given by:

$$\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2$$

A simple generalization of this formula holds for many securities *provided that* $\rho = 0$ *between all pairs of securities*:

$$\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + \cdots + X_N^2\sigma_N^2. \tag{1}$$

We will prove the following result. As $N \to \infty$, the portfolio standard deviation $\sigma_p \to 0$.

To make the notation simpler, assume that $\sigma_1 = \sigma_2 = \cdots = \sigma_N = \sigma$. This means that each asset is equally risky. Under those circumstances, we try the simple diversification strategy of dividing our wealth equally among each asset such that $w_i = \frac{1}{N}$.

These assumptions allow us to rewrite expression (1) as

$$\sigma_p^2 = \left(\frac{1}{N}\right)^2\sigma^2 + \left(\frac{1}{N}\right)^2\sigma^2 + \cdots + \left(\frac{1}{N}\right)^2\sigma^2 \tag{2}$$

There are N identical terms in expression (2), which means:

$$\begin{aligned}\sigma_p^2 &= N\left(\frac{1}{N}\right)^2\sigma^2 \\ \sigma_p^2 &= \frac{1}{N}\sigma^2\end{aligned}$$

The expression in (3) shows that as N grows larger and larger, the variance the portfolio declines. As $N \to \infty$, the variance goes to zero.

Case 2: Systematic and unsystematic risk

In fact, U.S. stocks do not have zero correlation with one another. We can capture the positive correlation of U.S. stocks with a factor model. Let R_i denote the return on an individual stock, and R_M the return on a broad market index like the S&P 500. One way to capture the common source of variation is to run a regression of the values of R_i on R_M:

$$R_i = \alpha_i + \beta_i R_M + \epsilon_i \tag{3}$$

This is equivalent to fitting a line through a scatter plot of pairs of returns (R_M, R_i). The slope of the line equals β_i. You may have heard in your statistics class that

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\sigma_M^2}. \tag{4}$$

The coefficient β_i is a measure of how much the stock moves together with the market index R_M. The error term, ϵ_i, measures the variability in R_i that is independent of all other securities in R_M.

Using (3), we can decompose the variance of a stock into its systematic and unsystematic components:

$$\begin{aligned} \sigma_i^2 = &\quad \beta_i^2 \sigma_M^2 &\quad + &\quad \sigma_\epsilon^2 \\ \text{Total risk} = &\quad \text{Systematic risk} &\quad + &\quad \text{Idiosyncratic risk} \end{aligned} \tag{5}$$

Here, σ_M^2 is the variance of R_M. Equation (5) follows from the fact that ϵ and R_M are independent random variables.

Equation (5) shows that U.S. stocks have both systematic risk $(\beta_i^2 \sigma_M^2)$ and unsystematic risk (σ_ϵ^2) . The argument from Case 1 demonstrates that the unsystematic component of stock risk goes away in a well-diversified portfolio (i.e. a portfolio with a large number of securities N). Only the systematic component remains.