Handout 7: Optimal portfolios when there is a riskfree asset

Corporate Finance, Sections 001 and 002

How does the set of possible portfolios change when you have access to a riskfree asset? We will conside the problem in two steps.

1. One riskfree asset and one risky asset

2. One riskfree asset and multiple risky assets

As we learned in our section on bond valuation, the riskfree asset is the zero-coupon bond whose maturity equals the length of your holding period. For concreteness, we will assume a one-year holding period, so the riskfree asset will be a one-year Treasury Bill.

1. One riskfree asset and one risky asset

Suppose the risky asset is a mutual fund. We will call this mutual fund "M", and the riskfree asset "f". The return on the mutual fund will therefore be called R_M, while the return on the riskfree asset will be called R_f. Using our formulas for the mean and the variance, the mean of the portfolio that puts weight X_f in the riskfree asset and weight X_M in the risky asset equals

$$\bar{R}_p = X_f R_f + X_M \bar{R}_M$$

Note that $R_f = \bar{R}_f$ because the riskfree asset is, by definition, without risk. Thus its return is always the same and equal to the mean. We also know that

$$X_f = 1 - X_M$$

Substituting in,

$$\bar{R}_p = (1 - X_M) R_f + X_M \bar{R}_M$$

$$\sigma_p^2 = X_f^2\sigma_f^2 + X_M^2\sigma_M^2 + 2\rho X_M X_f \sigma_M \sigma_f$$

But $\sigma_f = 0$. Therefore:

$$\begin{aligned} \sigma_p^2 &= X_M^2\sigma_M^2 \\ \sigma_p &= X_M\sigma_M \end{aligned}$$

Using this formula, we can show that the set of portfolios now form a straight line. Rearranging:

$$X_M = \frac{\sigma_p}{\sigma_M}$$

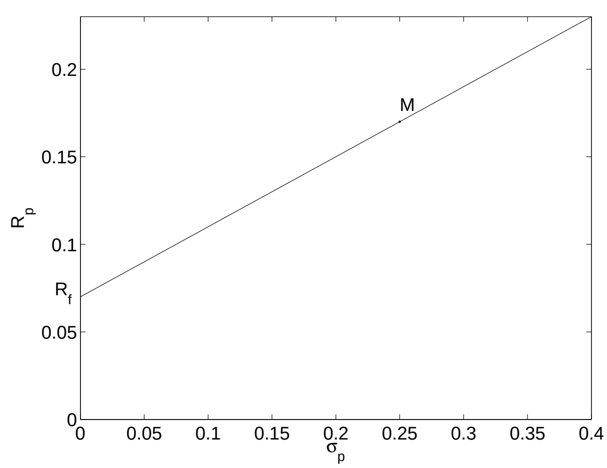
Thus

$$\bar{R}_p = \left(1 - \frac{\sigma_p}{\sigma_M}\right) R_f + \frac{\sigma_p}{\sigma_M}\bar{R}_M$$

And finally

$$\bar{R}_p = R_f + \sigma_p \frac{\bar{R}_M - R_f}{\sigma_M}$$

The slope of the line connecting the mutual fund to the riskfree asset equals $\frac{\bar{R}_M - R_f}{\sigma_M}$ and is called the *Sharpe ratio*.



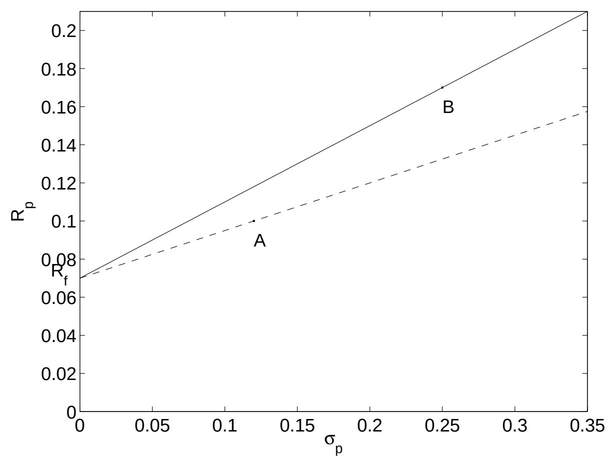
Can we achieve a higher mean than $\bar{R}_M$? We can, by setting $X_f < 0$. Economically, this corresponds to borrowing at the riskfree rate and investing the proceeds in the mutual fund.

2. One riskfree asset and multiple risky assets

Suppose you have a choice between investing in the riskfree rate and in asset A or aset B. Suppose for now that you cannot form a portfolio between A and B. Assume

$$\bar{R}_A = .10 \quad \sigma_A = .12$$
$$\bar{R}_B = .17 \quad \sigma_B = .25$$

and $R_f = .07$. Portfolios that combine R_f with B lie on the solid line. Portfolios that combine R_f with A lie on the dashed line.



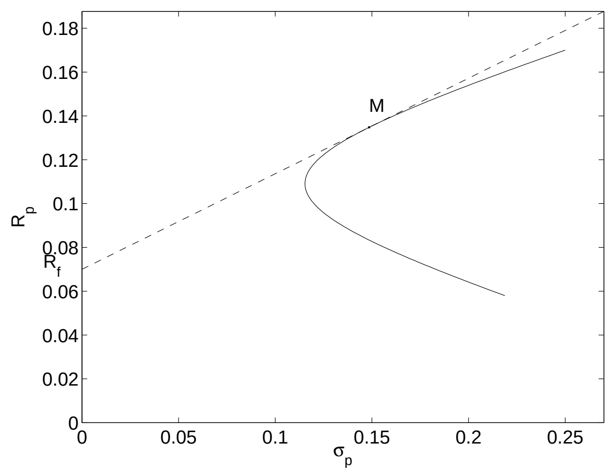
Its clear from the diagram that you would rather combine the riskfree asset with asset B, because the portfolios of B combined with R_f have a lower standard deviation for a given mean than the portfolios of A combined with R_f. In general, it is optimal to combine the riskfree asset with the risky asset for which the slope is highest:

$$\text{Sharpe Ratio for A} \qquad \frac{.10 - .07}{.12} = .25$$

$$\text{Sharpe Ratio for B} \qquad \frac{.17 - .07}{.25} = .40$$

Because the slope (Sharpe ratio) is highest for B, we want to combine the riskfree asset with B.

Now consider the most realistic case. You have access to a large universe of risky assets. Consider the minimum variance frontier for the risky assets (made up of portfolios of risky assets that give you the lowest variance for a given mean). You can combine the riskfree rate with any portfolio on the frontier you wish. Clearly the best portfolio to hold in combination with the riskfree rate is the *tangency portfolio* because it has the highest slope. Any other portfolio would be inefficient.



The line between the riskfree rate and the tangency portfolio is called the *capital allocation line*. Any investor who likes mean and dislikes variance will want to hold a portfolio along this line. This means we can divide the investment allocation problem into two steps.

1. Determine the tangency portfolio (the optimal combination of riskfree assets)

2. Determine where you want to be on the capital allocation line

Assuming that two investors have the same beliefs about means, standard deviations, and correlations, they should hold all risky assets in the same proportions. This statement is known as the *Mutual Fund Theorem*. It states that as long as investors have the same beliefs, they will all invest in an identical mutual fund, in combination with the riskfree asset.