Handout 20: Arbitrage Proofs for Put-Call Parity and Minimum Value (Optional)

Corporate Finance, Sections 001 and 002

I. Put-Call Parity

Put-call parity states that

$$C = S - Ee^{-rT} + P \tag{1}$$

To prove this statement, assume that it doesn't hold and show that it is possible to make riskless profits. We will use numbers for concreteness. Assume $S = \$110$, $E = \$100$, $t = 1$, $r = 0$. Also assume $C = \$12$ and $P = \$5$. Thus the call is undervalued and/or the put is overvalued, given $S = \$110$ and $E = \$100$.

Construct an arbitrage by buying the "cheap" call at \$12 and selling the "expensive" put at \$5. Recall that long a call and short a put both profit when S rises. Therefore, we complete the arbitrage by selling S short at \$110. Our portfolio today:

	Cash flow
Buy 1 call	- \$ 12
Short (write) 1 put	+ \$ 5
Sell short 1 share	+ \$110
Invest proceeds	-\$103
Net	0

Examine what happens on expiration (1 year from now) if $S > E$ and if $S < E$.

<u>If $S \geq E$:</u>

Exercise call	-\$100
Deliver against short	-
Put expires worthless	0
Receive proceeds from investment	\$103
	\$3

If $S < E$:

Call expires worthless	0
Put is exercised against you	-\$100
Take stock just received from put and return against short	-
Receive proceeds from investment	\$103
	\$3

There is a profit of \$3 is guaranteed no matter what happens. This profit equals the "mispricing" of $C - P$ versus $S - E$

$$\begin{aligned} C - P &= \$12 - \$5 &= \$7 \\ S - E &= \$110 - \$100 &= \$10 \end{aligned}$$

Another way of thinking about this is: Buying the call for \$12 and selling the put for \$5 allows you to be long the underlying stock (including exercise cost of \$100) for \$107. Since you sold the stock for \$110, you are entitled to \$3.

Since our position in part 4 is profitable and riskless, we do it as often as we can, driving up the value of C and driving down the value of P until they differ by exactly \$10. Thus C is driven up from \$12 and P is driven down from \$5. Whether C is \$14 and P is \$4 or C is \$15 and P is \$5 or whether C is \$12 and P is \$2 depends on the valuation of the call (or the put) from a model such as Black-Scholes. Put call parity just gives the relative value of C versus P given S and E.

II. Minimum Value

The minimum value of a call option on a non-dividend paying stock equals:

$$C \geq \max(0, S - Ee^{-rt}).$$

This relationship holds at *or before* expiration. The minimum value is greater than the intrinsic value $\max(0, S - E)$. Suppose that $S = \$101$, $E = \$100$, $r = .06$, $t = 1$. We prove the statement by contradiction, as before.

Assume $C = \$1$, i.e. calls are selling at their intrinsic value and below their minimum value (they do not reflect the interest saved by the delayed payment of E). Then there is an arbitrage opportunity. Construct the following portfolio today:

	Cash flow
Buy 1 call	-$1
Sell short one share	$101
Invest proceeds at .06	-$100
Net	0

At the end of the year, you have $\$100e^{.06} = \$106.18$ from your investment. You own a call, and you are short one share. Evaluate what your payoff is if $S \geq E$ or if $S < E$.

<u>If $S \geq E$</u>:

	Cash flow
Exercise call	-$100.00
Deliver against short S	-
Receive proceeds of investment	$106.18
Net	$6.18

If $S < E$ (say $S = \$99$):

	Cash flow
Leave call unexercised	-
Buy S in the market	-\$99.00
Deliver against short S	-
Receive proceeds of investment	\$106.18
Net	\$7.18

Conclusions:

1. This is an arbitrage strategy: the portfolio established "today" is zero-cost and earns a profit next year whether $S \geq E$ or $S < E$.

2. As you exploit this arbitrage, you will drive up the price of C, or down the price of S until C is above its minimum value.

3. The minimum value of a call reflects the interest that you save by not having to pay E until expiration. The minimum arbitrage profit is the difference between having to pay $E = \$100$ now, or E at expiration: $100e^{.06} - 100 = 6.18$. If the time value of money is not embedded into the call, the arbitrageur can capture it by selling short one share and investing the proceeds.