Internet Appendix for
"Do Rare Events Explain CDX Tranche Spreads"*

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Sections I–III of this appendix solve for utility, aggregate market prices, and firm values in closed form up to a system of ODEs. Ratios of prices to payouts are functions of the state variables λ_t and ξ_t. Section IV solves for CDX/CDX tranche spreads. In Section V, we describe how we simulate the model. Section VI solves for zero-coupon bond prices, and Section VII solves for option prices. Figures not included in the main text follow.

I. State-Price Density

Duffie and Skiadas (1994) show that the state-price density π_t equals

$$\pi_t = \exp\left\{\int_0^t \frac{\partial}{\partial V} f\left(C_s, V_s\right) ds\right\} \frac{\partial}{\partial C} f\left(C_t, V_t\right). \tag{IA.1}$$

Our goal is to obtain an expression for the state-price density in terms of C_t, λ_t, and ξ_t.

We conjecture that, in equilibrium, the continuation utility V_t equals a function J of consumption and the state variables λ_t and ξ_t such that

$$J(C_t, \lambda_t, \xi_t) = \frac{C_t^{1-\gamma}}{1-\gamma} e^{a+b_\lambda \lambda_t + b_\xi \xi_t}. \tag{IA.2}$$

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For future reference, we list the derivates of J with respect to its arguments:

$$
\begin{aligned}
\frac{\partial J}{\partial C} &= (1-\gamma)\frac{J}{C}, & \frac{\partial^2 J}{\partial C^2} &= -\gamma(1-\gamma)\frac{J}{C^2}, \\
\frac{\partial J}{\partial \lambda} &= b_\lambda J, & \frac{\partial^2 J}{\partial \lambda^2} &= b_\lambda^2 J, \\
\frac{\partial J}{\partial \xi} &= b_\xi J, & \frac{\partial^2 J}{\partial \xi^2} &= b_\xi^2 J.
\end{aligned} \tag{IA.3}
$$

Applying Ito's Lemma to $J(C_t, \lambda_t, \xi_t)$ with conjecture (IA.2) and derivatives (IA.3), we obtain

$$
\begin{aligned}
\frac{dV_t}{V_{t^-}} = (1-\gamma)(\mu_c dt + \sigma_c dB_t) - \frac{1}{2}\gamma(1-\gamma)\sigma_c^2 dt \\
+ b_\lambda\left(\kappa_\lambda(\xi_t - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t}\right) + \frac{1}{2}b_\lambda^2\sigma_\lambda^2\lambda_t dt \\
+ b_\xi\left(\kappa_\xi(\bar{\xi} - \xi_t)dt + \sigma_\xi\sqrt{\xi_t}dB_{\xi,t}\right) + \frac{1}{2}b_\xi^2\sigma_\xi^2\xi_t dt + (e^{(1-\gamma)Z_{c,t}} - 1)dN_t.
\end{aligned}
$$

Under the optimal consumption path, it must be the case that

$$
V_t + \int_0^t f(C_s, V_s)ds = E_t\left[\int_0^\infty f(C_s, V_s)ds\right] \tag{IA.4}
$$

(see Duffie and Epstein (1992)). By definition,

$$
\begin{aligned}
f(C_t, V_t) &= \beta(1-\gamma)V_t\left(\log C_t - \frac{1}{1-\gamma}\log\left[(1-\gamma)V\right]\right) \\
&= \beta(1-\gamma)V_t\log C_t - \beta V_t\log\left[(1-\gamma)V_t\right] \\
&= \beta V_t \log\left(\frac{C_t^{1-\gamma}}{(1-\gamma)V_t}\right) \\
&= -\beta V_t(a + b_\lambda\lambda_t + b_\xi\xi_t),
\end{aligned} \tag{IA.5}
$$

where the last equation follows from (IA.2).

By the law of iterated expectations, the left-hand side of (IA.4) is a martingale. Thus, the sum of the drift and the jump compensator of $(V_t + \int_0^t f(C_s, V_s)ds)$ equals zero. That is,

$$
\begin{aligned}
0 = (1-\gamma)\mu_c - \frac{1}{2}\gamma(1-\gamma)\sigma_c^2 + b_\lambda\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_\lambda^2\sigma_\lambda^2\lambda_t + b_\xi\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_\xi^2\sigma_\xi^2\xi_t \\
+ \lambda_t E_\nu\left[e^{(1-\gamma)Z_{c,t}} - 1\right] - \beta(a + b_\lambda\lambda_t + b_\xi\xi_t).
\end{aligned} \tag{IA.6}
$$

By collecting terms in (IA.6), we obtain

$$0 = \underbrace{\left[(1-\gamma)\mu_c - \frac{1}{2}\gamma(1-\gamma)\sigma_c^2 + b_\xi\kappa_\xi\bar{\xi} - \beta a\right]}_{=0} + \lambda_t \underbrace{\left[-b_\lambda\kappa_\lambda + \frac{1}{2}b_\lambda^2\sigma_\lambda^2 + E_\nu\left[e^{(1-\gamma)Z_{c,t}} - 1\right] - \beta b_\lambda\right]}_{=0} + \xi_t \underbrace{\left[b_\lambda\kappa_\lambda - b_\xi\kappa_\xi + \frac{1}{2}b_\xi^2\sigma_\xi^2 - \beta b_\xi\right]}_{=0}. \tag{IA.7}$$

Solving these equations gives us

$$a = \frac{1-\gamma}{\beta}\left(\mu_c - \frac{1}{2}\gamma\sigma_c^2\right) + \frac{b_\xi\kappa_\xi\bar{\xi}}{\beta} \tag{IA.8}$$

$$b_\lambda = \frac{\kappa_\lambda + \beta}{\sigma_\lambda^2} - \sqrt{\left(\frac{\kappa_\lambda + \beta}{\sigma_\lambda^2}\right)^2 - 2\frac{E_\nu\left[e^{(1-\gamma)Z_{c,t}} - 1\right]}{\sigma_\lambda^2}} \tag{IA.9}$$

$$b_\xi = \frac{\kappa_\xi + \beta}{\sigma_\xi^2} - \sqrt{\left(\frac{\kappa_\xi + \beta}{\sigma_\xi^2}\right)^2 - 2\frac{b_\lambda\kappa_\lambda}{\sigma_\xi^2}}, \tag{IA.10}$$

where we have chosen the negative root based on the economic consideration that when there are no disasters, λ_t and ξ_t should not appear in the value function. That is, for $Z_{c,t} = 0$, $b_\lambda = b_\xi = 0$. Note that these results verify the conjecture (IA.2).

It follows from (IA.5) that

$$\frac{\partial}{\partial C} f\left(C_t, V_t\right) = \beta(1-\gamma)V_t C_t^{\gamma-1} C_t^{-\gamma}$$

$$\frac{\partial}{\partial V} f(C_t, V_t) = \beta(1-\gamma)\left(\log C_t - \frac{1}{1-\gamma}\log\left((1-\gamma)V_t\right)\right) + \beta.$$

By (IA.2), in equilibrium, we have that

$$\frac{\partial}{\partial C} f\left(C_t, V_t\right) = \beta C_t^{-\gamma} e^{a + b_\lambda\lambda_t + b_\xi\xi_t}$$

$$\frac{\partial}{\partial V} f(C_t, V_t) = -\beta a - \beta - \beta b_\lambda\lambda_t - \beta b_\xi\xi_t.$$

Therefore, from (IA.1), it follows that the state-price density can be written as

$$\pi_t = \exp\left\{-\beta(a+1)t - \beta b_\lambda \int_0^t \lambda_s ds - \beta b_\xi \int_0^t \xi_s ds\right\} \beta C_t^{-\gamma} e^{a+b_\lambda \lambda_t + b_\xi \xi_t}. \qquad \text{(IA.11)}$$

II. Dynamics of the Aggregate Market

Let $F(D_t, \lambda_t, \xi_t)$ denote the price of the dividend claim. The pricing relation implies

$$\begin{aligned} F(D_t, \lambda_t, \xi_t) &= E_t\left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds\right] \\ &= \int_t^\infty E_t\left[\frac{\pi_s}{\pi_t} D_s\right] ds. \end{aligned}$$

Let $H(D_t, \lambda_t, \xi_t, s-t)$ denote the price of the asset that pays the aggregate dividend at time s:

$$H(D_t, \lambda_t, \xi_t, s-t) = E_t\left[\frac{\pi_s}{\pi_t} D_s\right].$$

By the law of iterated expectations, it follows that $\pi_t H_t$ is a martingale:

$$\pi_t H(D_t, \lambda_t, \xi_t, s-t) = E_t[\pi_s D_s].$$

Conjecture that

$$H(D_t, \lambda_t, \xi_t, \tau) = D_t \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda_t + b_{\phi\xi}(\tau)\xi_t\right). \qquad \text{(IA.12)}$$

Applying Ito's Lemma to conjecture (IA.12) implies

$$\begin{aligned} \frac{dH_t}{H_{t-}} = \Big\{ &\mu_d + b_{\phi\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2} b_{\phi\lambda}(\tau)^2 \sigma_\lambda^2 \lambda_t + b_{\phi\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2} b_{\phi\xi}(\tau)^2 \sigma_\xi^2 \xi_t \\ &- a_\phi'(\tau) - b_{\phi\lambda}'(\tau)\lambda_t - b_{\phi\xi}'(\tau)\xi_t \Big\} dt \\ &+ \phi\sigma_c dB_t + b_{\phi\lambda}(\tau)\sigma_\lambda\sqrt{\lambda_t} dB_{\lambda,t} + b_{\phi\xi}(\tau)\sigma_\xi\sqrt{\xi_t} dB_{\xi,t} + (e^{\phi Z_{c,t}} - 1) dN_t. \qquad \text{(IA.13)} \end{aligned}$$

It follows from (IA.13), (3), and the product rule for stochastic processes that

$$\begin{aligned}\frac{d(\pi_t H_t)}{\pi_{t^-} H_{t^-}} = \Big\{ & -\beta - \mu_c + \gamma\sigma_c^2 - \lambda_t E_\nu\left[e^{(1-\gamma)Z_{c,t}} - 1\right] \\ & + \mu_d + b_{\phi\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{\phi\lambda}(\tau)^2\sigma_\lambda^2\lambda_t \\ & + b_{\phi\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{\phi\xi}(\tau)^2\sigma_\xi^2\xi_t \\ & - a'_\phi(\tau) - b'_{\phi\lambda}(\tau)\lambda_t - b'_{\phi\xi}(\tau)\xi_t \\ & - \gamma\phi\sigma_c^2 + b_\lambda b_{\phi\lambda}(\tau)\sigma_\lambda^2\lambda_t + b_\xi b_{\phi\xi}(\tau)\sigma_\xi^2\xi_t \Big\} dt \\ & + (\phi - \gamma)\sigma_c dB_t + (b_\lambda + b_{\phi\lambda}(\tau))\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + (b_\xi + b_{\phi\xi}(\tau))\sigma_\xi\sqrt{\xi_t}dB_{\xi,t} \\ & + (e^{(\phi-\gamma)Z_{c,t}} - 1)dN_t.\end{aligned}$$

Since $\pi_t H_t$ is a martingale, the sum of the drift and the jump compensator of $\pi_t H_t$ equals zero. Thus,

$$\begin{aligned}0 = & -\beta - \mu_c + \gamma\sigma_c^2 - \lambda_t E_\nu\left[e^{(1-\gamma)Z_{c,t}} - 1\right] \\ & + \mu_d + b_{\phi\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{\phi\lambda}(\tau)^2\sigma_\lambda^2\lambda_t \\ & + b_{\phi\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{\phi\xi}(\tau)^2\sigma_\xi^2\xi_t \\ & - a'_\phi(\tau) - b'_{\phi\lambda}(\tau)\lambda_t - b'_{\phi\xi}(\tau)\xi_t \\ & - \gamma\phi\sigma_c^2 + b_\lambda b_{\phi\lambda}(\tau)\sigma_\lambda^2\lambda_t + b_\xi b_{\phi\xi}(\tau)\sigma_\xi^2\xi_t + \lambda_t E_\nu\left[e^{(\phi-\gamma)Z_{c,t}} - 1\right]. \quad \text{(IA.14)}\end{aligned}$$

Collecting terms of (IA.14) results in the following equation:

$$
\begin{aligned}
0 = &\underbrace{\left[-\beta - \mu_c + \gamma\sigma_c^2 + \mu_d + b_{\phi\xi}(\tau)\kappa_\xi\bar{\xi} - \gamma\phi\sigma_c^2 - a_\phi'(\tau)\right]}_{=0} \\
&+ \lambda_t \underbrace{\left[-b_{\phi\lambda}(\tau)\kappa_\lambda + \frac{1}{2}b_{\phi\lambda}(\tau)^2\sigma_\lambda^2 + b_\lambda b_{\phi\lambda}(\tau)\sigma_\lambda^2 + E_\nu\left[e^{(\phi-\gamma)Z_{c,t}} - e^{(1-\gamma)Z_{c,t}}\right] - b_{\phi\lambda}'(\tau)\right]}_{=0} \\
&+ \xi_t \underbrace{\left[b_{\phi\lambda}(\tau)\kappa_\lambda - b_{\phi\xi}(\tau)\kappa_\xi + \frac{1}{2}b_{\phi\xi}(\tau)^2\sigma_\xi^2 + b_\xi b_{\phi\xi}(\tau)\sigma_\xi^2 - b_{\phi\xi}'(\tau)\right]}_{=0}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
a_\phi'(\tau) &= \mu_d - \mu_c - \beta + \gamma\sigma_c^2(1-\phi) + \kappa_\xi\bar{\xi}b_{\phi\xi}(\tau) \\
b_{\phi\lambda}'(\tau) &= \frac{1}{2}\sigma_\lambda^2 b_{\phi\lambda}(\tau)^2 + (b_\lambda\sigma_\lambda^2 - \kappa_\lambda)b_{\phi\lambda}(\tau) + E_\nu\left[e^{(\phi-\gamma)Z_{c,t}} - e^{(1-\gamma)Z_{c,t}}\right] \qquad \text{(IA.15)} \\
b_{\phi\xi}'(\tau) &= \frac{1}{2}\sigma_\xi^2 b_{\phi\xi}(\tau)^2 + (b_\xi\sigma_\xi^2 - \kappa_\xi)b_{\phi\xi}(\tau) + \kappa_\lambda b_{\phi\lambda}(\tau).
\end{aligned}
$$

This establishes that H satisfies the conjecture (IA.12). We note that by no-arbitrage,

$$H(D_t, \lambda_t, \xi_t, 0) = D_t.$$

This condition provides the boundary conditions for the system of ODEs (IA.15):

$$a_\phi(0) = b_{\phi\lambda}(0) = b_{\phi\xi}(0) = 0.$$

Finally,

$$
\begin{aligned}
F(D_t, \lambda_t, \xi_t) &= \int_t^\infty E_t\left[\frac{\pi_s}{\pi_t}D_s\right]ds \\
&= \int_t^\infty H(D_t, \lambda_t, \xi_t, s-t)ds \\
&= D_t\int_t^\infty \exp\left(a_\phi(s-t) + b_{\phi\lambda}(s-t)\lambda_t + b_{\phi\xi}(s-t)\xi_t\right)ds \\
&= D_t\int_0^\infty \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda_t + b_{\phi\xi}(\tau)\xi_t\right)d\tau.
\end{aligned}
$$

The price-dividend ratio can be written as

$$G(\lambda_t, \xi_t) = \int_0^{\infty} \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda_t + b_{\phi\xi}(\tau)\xi_t\right) d\tau.$$

III. Individual Firm Value Dynamics

Let $H_i(D_{i,t}, \lambda_t, \xi_t, s-t)$ denote the time-t value of firm i's payoff at time s. That is,

$$H_i(D_{i,t}, \lambda_t, \xi_t, s-t) = E_t\left[\frac{\pi_s}{\pi_t} D_{i,s}\right],$$

where $D_{i,t}$ is determined by (12).[1]

We conjecture that $H_i(\cdot)$ has the following functional form:

$$H_i(D_{i,t}, \lambda_t, \xi_t, \tau) = D_{i,t} \exp\left(a_i(\tau) + b_{i\lambda}(\tau)\lambda_t + b_{i\xi}(\tau)\xi_t\right). \tag{IA.16}$$

To verify this conjecture, we apply Ito's Lemma to the process $\pi_t H_i(D_{i,t}, \lambda_t, \xi_t, s-t)$ and derive the conditional expectation of its instantaneous change. This conditional expectation must equal zero because of (IA.16), which implies that $\pi_t H_i(D_{i,t}, \lambda_t, \xi_t, s-t)$ is a martingale.

By applying Ito's Lemma to equation (IA.16), it follows that

$$\begin{aligned}\frac{dH_{i,t}}{H_{i,t-}} = \Big\{&\mu_i + b_{i\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{i\lambda}(\tau)^2\sigma_\lambda^2\lambda_t + b_{i\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{i\xi}(\tau)^2\sigma_\xi^2\xi_t - a_i'(\tau) \\ &- b_{i\lambda}'(\tau)\lambda_t - b_{i\xi}'(\tau)\xi_t\Big\} dt + \phi_i\sigma_c dB_{c,t} + b_{i\lambda}(\tau)\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + b_{i\xi}(\tau)\sigma_\xi\sqrt{\xi_t}dB_{\xi,t} \\ &+ (e^{\phi_i Z_{c,t}} - 1)dN_{c,t} + I_{i,t}(e^{Z_{i,t}} - 1)dN_{i,t} + (e^{Z_{S_i,t}} - 1)dN_{S_i,t}.\end{aligned}$$

Note that firm i is hit by the idiosyncratic shock $dN_{i,t}$ and the sector shock $dN_{S_i,t}$. The SDE for π_t is given in (3). By applying Ito's Lemma for the product of two stochastic processes,

[1] In the equations that follow, we allow Z_i and Z_{S_i} to be random variables (with independent and time-invariant distributions) rather than constants, and denote them as $Z_{i,t}$ and $Z_{S_i,t}$ respectively. We assume $Z_{i,t}$ to be independent and identically distributed across firms, while $Z_{S_i,t}$ is independent and identically distributed across sectors.

we obtain the SDE for $\pi_t H_{i,t}$:

$$
\begin{aligned}
\frac{d(\pi_t H_{i,t})}{\pi_{t^-} H_{i,t^-}} = \Bigg\{ & -\beta - \mu_c + \gamma\sigma_c^2 - \lambda_t E\left[e^{(1-\gamma)Z_{c,t}} - 1\right] + \mu_i + b_{i\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{i\lambda}(\tau)^2\sigma_\lambda^2\lambda_t \\
& + b_{i\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{i\xi}(\tau)^2\sigma_\xi^2\xi_t - a_i'(\tau) - b_{i\lambda}'(\tau)\lambda_t - b_{i\xi}'(\tau)\xi_t - \gamma\phi_i\sigma_c^2 \\
& + b_\lambda b_{i\lambda}(\tau)\sigma_\lambda^2\lambda_t + b_\xi b_{i\xi}(\tau)\sigma_\xi^2\xi_t \Bigg\} dt + (\phi_i - \gamma)\sigma_c dB_{c,t} \\
& + (b_\lambda + b_{i\lambda}(\tau))\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + (b_\xi + b_{i\xi}(\tau))\sigma_\xi\sqrt{\xi_t}dB_{\xi,t} \\
& + (e^{(\phi_i-\gamma)Z_{c,t}} - 1)dN_{c,t} + I_{i,t}(e^{Z_{i,t}} - 1)dN_{i,t} + (e^{Z_{S_{i,t}}} - 1)dN_{S_{i,t}}.
\end{aligned}
$$

Since $\pi_t H_t$ is a martingale, the sum of the drift and the jump compensator of $\pi_t H_t$ equals zero. This zero mean condition provides the system of ODEs for $a_i(\tau)$, $b_{i\lambda}(\tau)$, and $b_{i\xi}(\tau)$:

$$
\begin{aligned}
a_i'(\tau) &= \mu_i - \mu_c - \beta + \gamma\sigma_c^2(1 - \phi_i) + \lambda_i E\left[e^{Z_{i,t}} - 1\right] + p_i w_0 E\left[e^{Z_{S_{i,t}}} - 1\right] + \kappa_\xi\bar{\xi}b_{i\xi}(\tau) \\
b_{i\lambda}'(\tau) &= \frac{1}{2}\sigma_\lambda^2 b_{i\lambda}(\tau)^2 + (b_\lambda\sigma_\lambda^2 - \kappa_\lambda)b_{i\lambda}(\tau) + E\left[e^{(\phi_i-\gamma)Z_{c,t}} - e^{(1-\gamma)Z_{c,t}}\right] + p_i w_\lambda E\left[e^{Z_{S_{i,t}}} - 1\right] \\
b_{i\xi}'(\tau) &= \frac{1}{2}b_{i\xi}(\tau)^2\sigma_\xi^2 + (b_\xi\sigma_\xi^2 - \kappa_\xi)b_{i\xi}(\tau) + \kappa_\lambda b_{i\lambda}(\tau) + +p_i w_\xi E\left[e^{Z_{S_{i,t}}} - 1\right].
\end{aligned}
$$

This shows that H_i satisfies the conjecture (IA.16). Furthermore, since $H_i(D_{i,t}, \lambda_t, \xi_t, 0) = D_{i,t}$, we obtain the following boundary conditions:

$$a_i(0) = b_{i\lambda}(0) = b_{i\xi}(0) = 0.$$

With the solution for the ODEs, equation (13) can be written as

$$
\begin{aligned}
A_i(D_{i,t}, \lambda_t, \xi_t) &= \int_t^\infty H_i(D_{i,t}, \lambda_t, \xi_t, s - t)\, ds \\
&= D_{i,t}\int_0^\infty \exp\left(a_i(\tau) + b_{i\lambda}(\tau)\lambda_t + b_{i\xi}(\tau)\xi_t\right)\, d\tau.
\end{aligned}
$$

IV. Computing CDX and Tranche Prices

Given the closed-form expressions for asset prices, the prices for the CDX and its tranches must be computed by simulation. That is, for each pair of state variables (λ_t, ξ_t), we compute the expectations that determine the protection legs (22) and (18) and the premium legs (19) and (25) by simulating 100,000 sample paths for the 125 firms (see Internet Appendix Section V for more details on how we simulate the state variables and firm values).

To reduce computation time, we compute $n_{t,s}$, $L_{t,s}$, $T^L_{j,t,s}$, and $T^R_{j,t,s}$ at quarterly intervals, which correspond to the timing of payment premium dates. Given these series, we compute the value of cash flows paid by the protection seller by assuming that default occurs at the midpoint between two premium payment dates. This follows standard practice (Mortensen 2006), and is more accurate than simply assuming that default occurs on the premium payment date itself. We compute

$$\begin{aligned}\mathbf{Prot}_{\text{CDX}}(\lambda_t, \xi_t; T-t) &= E_t^Q\left[\int_t^T e^{-\int_t^s r_u du} dL_{t,s}\right] \\ &\simeq \sum_{m=1}^{4T} E_t^Q\left[e^{-\int_t^{t+\vartheta(m-\frac{1}{2})} r_u\, du}(L_{t,t+\vartheta m} - L_{t,t+\vartheta(m-1)})\right]. \quad \text{(IA.17)}\end{aligned}$$

Because default occurs at the midpoint between two payment periods, $L_{t,t+\vartheta m} = L_{t,t+\vartheta m-\vartheta/2}$.

It will be computationally useful to write (IA.17) in terms of the risk-neutral, discounted expectation of $L_{t,t+\vartheta m}$ and $L_{t,t+\vartheta(m-1)}$. Note that the risk-free rate has continuous sample paths, and ϑ is small. We therefore approximate

$$\int_{t+\vartheta(m-\frac{1}{2})}^{t+\vartheta m} r_u\, du \simeq \frac{\vartheta}{2} r_{t+\vartheta m}. \quad \text{(IA.18)}$$

Combining (IA.17) and (IA.18), we have

$$\mathbf{Prot}_{\text{CDX}}(\lambda_t, \xi_t; T-t) \simeq$$
$$\frac{1}{4}\sum_{m=1}^{4T}\left(E_t^Q\left[e^{\frac{1}{2}\vartheta r_{t+\vartheta m}} e^{-\int_t^{t+\vartheta m} r_u du} L_{t,t+\vartheta m}\right] - E_t^Q\left[e^{-\frac{1}{2}\vartheta r_{t+\vartheta(m-1)}} e^{-\int_t^{t+\vartheta(m-1)} r_u du} L_{t,t+\vartheta(m-1)}\right]\right).$$

Recall that the premium leg equals

$$\begin{aligned}
&\mathbf{Prem}_{\mathrm{CDX}}(\lambda_t, \xi_t; T-t, S) = \\
&\qquad SE_t^Q\left[\frac{1}{4}\sum_{m=1}^{4T} e^{-\int_t^{t+\vartheta m} r_u du}(1-n_{t,t+\vartheta m}) + \int_{t+\vartheta(m-1)}^{t+\vartheta m} e^{-\int_t^s r_u du}(s-t-\vartheta(m-1))dn_{t,s}\right] \\
&\qquad \simeq S\sum_{m=1}^{4T}\frac{1}{4}E_t^Q\left[e^{-\int_t^{t+\vartheta m} r_u du}(1-n_{t,t+\vartheta m}) + e^{-\int_t^{t+(\vartheta-\frac{1}{2})m} r_u du}\left(\frac{n_{t,t+\vartheta m}-n_{t,t+\vartheta(m-1)}}{2}\right)\right],
\end{aligned}$$

where the approximation in the second equation holds if default is close to the midpoint between two premium payment dates.

Our goal is to compute the risk-neutral, discounted expectation of $n_{t,t+\vartheta m}$. Using the approximation $e^{-\int_t^{t+\vartheta(m-\frac{1}{2})} r_u du} \simeq e^{-\int_t^{t+\vartheta m} r_u du}$,

$$\begin{aligned}
&\mathbf{Prem}_{\mathrm{CDX}}(\lambda_t, \xi_t; T-t, S) \\
&\quad \simeq \frac{S}{4}\sum_{m=1}^{4T} E_t^Q\left[e^{-\int_t^{t+\vartheta m} r_u du}\left(1-\frac{1}{2}n_{t,t+\vartheta m}-\frac{1}{2}n_{t,t+\vartheta(m-1)}\right)\right] \\
&\quad = \frac{S}{4}\sum_{m=1}^{4T}\left(H_0(\lambda_t,\xi_t,\vartheta m) - \frac{1}{2}E_t^Q\left[e^{-\int_t^{t+\vartheta m} r_u du} n_{t,t+\vartheta m}\right] - \frac{1}{2}E_t^Q\left[e^{-\int_t^{t+\vartheta m} r_u du} n_{t,t+\vartheta(m-1)}\right]\right) \\
&\quad \simeq \frac{S}{4}\sum_{m=1}^{4T}\left(H_0(\lambda_t,\xi_t,\vartheta m) - \frac{1}{2}E_t^Q\left[e^{-\int_t^{t+\vartheta m} r_u du} n_{t,t+\vartheta m}\right]\right. \\
&\qquad\qquad \left. - \frac{1}{2}E_t^Q\left[e^{-\vartheta r_{t+\vartheta(m-1)}} e^{-\int_t^{t+\vartheta(m-1)} r_u du} n_{t,t+\vartheta(m-1)}\right]\right).
\end{aligned}$$

where $H_0(\lambda_s, \xi_s, \tau)$ is the price of the default-free zero-coupon bond with maturity τ, which we derive in Internet Appendix Section VI.[2]

Like the computation for the protection leg on the CDX, our computation for the protection leg for tranche j assumes that the default occurs at the midpoint between payment

[2]In economic terms, this interest rate approximation implies that the accrued interest payment comes at the same time as the premium payment, rather than upon default. Based on the argument in Section I.E of the main text, this implies an equivalence between the premium leg for the CDX and the premium leg for the tranches.

periods, and uses approximation (IA.18):

$$\begin{aligned}
\mathbf{Prot}_{\mathrm{Tran},j}(\lambda_t, \xi_t, T-t) &\simeq \sum_{m=1}^{4T} E_t^Q \left[e^{-\int_t^{t+(\vartheta-\frac{1}{2})m} r_u du} (T^L_{j,t,t+\vartheta m} - T^L_{j,t,t+\vartheta(m-1)}) \right] \\
&\simeq \sum_{m=1}^{4T} \Big(E_t^Q \left[e^{\frac{\vartheta}{2} r_{t+\vartheta m}} e^{-\int_t^{t+\vartheta m} r_u du} T^L_{j,t,t+\vartheta m} \right] - \\
&\qquad E_t^Q \left[e^{-\frac{\vartheta}{2} r_{t+\vartheta(m-1)}} e^{-\int_t^{t+\vartheta(m-1)} r_u du} T^L_{j,t,t+\vartheta(m-1)} \right] \Big).
\end{aligned}$$

Recall that

$$\mathbf{Prem}_{\mathrm{Tran},j}(\lambda_t, \xi_t; T-t, U, S) = \\
U + S E_t^Q \left[\sum_{m=1}^{4T} \left(e^{-\int_t^{t+\vartheta m} r_s ds} \int_{t+\vartheta(m-1)}^{t+\vartheta m} \left(1 - T^L_{j,t,s} - T^R_{j,t,s}\right) ds \right) \right].$$

Under the assumption that any default occurs at the midpoint between the two payment periods, the integral above is an average:

$$\mathbf{Prem}_{\mathrm{Tran},j}(\lambda_t, \xi_t; T-t, U, S) \simeq \\
U + \frac{S}{4} \sum_{m=1}^{4T} E_t^Q \left[e^{-\int_t^{t+\vartheta m} r_s ds} \frac{(1 - T^L_{j,t,t+\vartheta m} - T^R_{j,t,t+\vartheta m}) + (1 - T^L_{j,t,t+\vartheta(m-1)} - T^R_{j,t,t+\vartheta(m-1)})}{2} \right].$$

(recall $\vartheta = 1/4$). Because we want to write the premium leg in terms of risk-neutral expectations of discounted variables, we approximate

$$\begin{aligned}
e^{-\vartheta r_{t+\vartheta(m-1)}} e^{-\int_t^{t+\vartheta m} r_s ds} &= e^{-\int_{t+\vartheta(m-1)}^{t+\vartheta m} r_s\, ds} e^{-\int_t^{t+\vartheta(m-1)} r_s ds} \\
&\simeq e^{-\vartheta r_{t+\vartheta(m-1)}} e^{-\int_t^{t+\vartheta(m-1)} r_s ds}
\end{aligned}$$

so that

$$\mathbf{Prem}_{\mathrm{Tran},j}(\lambda_t, \xi_t; T-t, U, S) \simeq U + \\
\frac{S}{4} \sum_{m=1}^{4T} \Big(H_0(\lambda_t, \xi_t, \vartheta m) - \frac{1}{2} E_t^Q \left[e^{-\int_t^{t+\vartheta m} r_s ds} T^L_{j,t,t+\vartheta m} \right] - \frac{1}{2} E_t^Q \left[e^{-\int_t^{t+\vartheta m} r_s ds} T^R_{j,t,t+\vartheta m} \right] \\
- \frac{1}{2} E_t^Q \left[e^{-\vartheta r_{t+\vartheta(m-1)}} e^{-\int_t^{t+\vartheta(m-1)} r_s ds} T^L_{j,t,t+\vartheta(m-1)}) \right] - \frac{1}{2} E_t^Q \left[e^{-\Delta r_{t+\vartheta(m-1)}} e^{-\int_t^{t+\vartheta(m-1)} r_s ds} T^R_{j,t,t+\vartheta(m-1)} \right] \Big).$$

Next, for any $u \in \mathbb{R}$, we define the following four expectations:

$$\begin{aligned}
\text{EDR}(u, \tau, \lambda_t, \xi_t) &= E_t^Q \left[e^{ur_{t+\tau}} e^{-\int_t^{t+\tau} r_s ds} n_{t,t+\tau} \right] \\
\text{ELR}(u, \tau, \lambda_t, \xi_t) &= E_t^Q \left[e^{ur_{t+\tau}} e^{-\int_t^{t+\tau} r_s ds} L_{t,t+\tau} \right] \\
\text{ETLR}_j(u, \tau, \lambda_t, \xi_t) &= E_t^Q \left[e^{ur_{t+\tau}} e^{-\int_t^{t+\tau} r_s ds} T_{j,t,t+\tau}^L \right] \\
\text{ETRR}_j(u, \tau, \lambda_t, \xi_t) &= E_t^Q \left[e^{ur_{t+\tau}} e^{-\int_t^{t+\tau} r_s ds} T_{j,t,t+\tau}^R \right]
\end{aligned} \tag{IA.19}$$

We rewrite the pricing formulas for the CDX index and its tranches as follows:

$$\begin{aligned}
\mathbf{Prot}_{\text{CDX}}(\lambda_t, \xi_t; T-t) &= \sum_{m=1}^{4T} \left(\text{ELR}\left(\frac{\vartheta}{2}, \vartheta m, \lambda_t, \xi_t\right) - \text{ELR}\left(-\frac{\vartheta}{2}, \vartheta(m-1), \lambda_t, \xi_t\right) \right) \\
\mathbf{Prem}_{\text{CDX}}(\lambda_t, \xi_t; T-t, S) &= \frac{S}{4} \sum_{m=1}^{4T} \Bigg(H_0(\lambda_t, \xi_t, \vartheta m) - \frac{1}{2}\text{EDR}\left(0, \vartheta m, \lambda_t, \xi_t\right) \\
&\qquad - \frac{1}{2}\text{EDR}\left(\vartheta, \vartheta(m-1), \lambda_t, \xi_t\right) \Bigg) \\
\mathbf{Prot}_{\text{Tran},j}(\lambda_t, \xi_t; T-t) &= \sum_{m=1}^{4T} \left(\text{ETLR}_j\left(\frac{\vartheta}{2}, \vartheta m, \lambda_t, \xi_t\right) - \text{ETLR}_j\left(-\frac{\vartheta}{2}, \vartheta(m-1), \lambda_t, \xi_t\right) \right) \\
\mathbf{Prem}_{\text{Tran},j}(\lambda_t, \xi_t; T-t, U, S) &= U + \frac{S}{4} \sum_{m=1}^{4T} \Bigg(H_0(\lambda_t, \xi_t, \vartheta m) - \frac{[\text{ETLR}_j + \text{ETRR}_j]\left(0, \vartheta m, \lambda_t, \xi_t\right)}{2} \\
&\qquad - \frac{[\text{ETLR}_j + \text{ETRR}_j]\left(\vartheta, \vartheta(m-1), \lambda_t, \xi_t\right)}{2} \Bigg).
\end{aligned}$$

To price the CDX index and its tranches, it suffices to calculate the four expectations above. Note that

$$\text{EDR}(u, \tau, \lambda_t, \xi_t) = E_t^Q \left[e^{ur_{t+\tau}} e^{-\int_t^{t+\tau} r_s ds} n_{t,t+\tau} \right] = E_t^Q \left[e^{-\int_t^{t+\tau} r_s ds} e^{ur_{t+\tau}} n_{t,t+\tau} \right] = E_t \left[\frac{\pi_{t+\tau}}{\pi_t} e^{u \cdot r_{t+\tau}} n_{t,t+\tau} \right]$$

and similarly for the other expectations in (IA.19). It therefore suffices to calculate the physical processes for $n_{t,s}$, $L_{t,s}$, $T_{j,t,s}^R$, and $T_{j,t}^L$ at a quarterly frequency. For each value of the state variables, we do this 100,000 times to obtain the expectation. This requires 100,000 5-year simulations of the 125 firms in the index.

V. Model Simulation

As discussed in Internet Appendix Section IV, we must simulate from the model in order to price the CDX and its tranches.

The first step is to simulate a series of state variables (λ_t, ξ_t). The variable ξ_t follows the square-root process of Cox, Ingersoll, and Ross (1985), and thus $\xi_{t+\Delta t}|\xi_t$ has a noncentral Chi-squared distribution with $\left(\frac{4\kappa_\xi \bar{\xi}}{\sigma_\xi^2}\right)$ degrees of freedom and noncentrality parameter $\left(\frac{4\xi_t \kappa_\xi e^{-\kappa_\xi \Delta t}}{(1-e^{-\kappa_\xi \Delta t})\sigma_\xi^2}\right)$.

Over a short time interval, λ_t will be well approximated by a CIR process. That is, we approximate the conditional distribution $\lambda_{t+\Delta t}|\lambda_t$ with a noncentral Chi-squared distribution with $\left(\frac{4\kappa_\lambda \xi_t}{\sigma_\lambda^2}\right)$ degrees of freedom and noncentrality parameter $\left(\frac{4\lambda_t \kappa_\lambda e^{-\kappa_\lambda \Delta t}}{(1-e^{-\kappa_\lambda \Delta t})\sigma_\lambda^2}\right)$.[3]

Given λ_t, log consumption growth $(\log(C_{t+\Delta t}/C_t))$, and each firm's log payout growth $(\log(D_{i,t+\Delta t}/D_{i,t}))$ can be drawn by discretizing the following SDEs, which follow from Ito's Lemma, applied to (1) and (12):

$$\begin{aligned} d\log C_t &= \left(\mu_c - \frac{1}{2}\sigma_c^2\right) dt + \sigma_c dB_{c,t} + Z_{c,t} N_{c,t} \\ d\log D_{i,t} &= \left(\mu_i - \frac{1}{2}\phi_i^2\sigma_c^2\right) dt + \phi_i \sigma_c dB_{c,t} + \phi_i Z_{c,t} dN_{c,t} + I_{i,t} Z_{S_i,t} dN_{S_i,t} + Z_{i,t} dN_{i,t}. \end{aligned}$$

Firm value can then be computed as

$$\begin{aligned} \frac{A_{i,t+\Delta t}}{A_{i,t}} &= \frac{D_{i,t+\Delta t}}{D_{i,t}} \frac{G_i(\lambda_{t+\Delta t}, \xi_{t+\Delta t})}{G_i(\lambda_t, \xi_t)} \\ &= \exp\left[\log\left(\frac{D_{i,t+\Delta t}}{D_{i,t}}\right)\right] \frac{G_i(\lambda_{t+\Delta t}, \xi_{t+\Delta t})}{G_i(\lambda_t, \xi_t)}, \end{aligned} \qquad \text{(IA.20)}$$

[3] The advantage of this approach over an Euler approximation of (2a) using a conditional normal process is that, due to the presence of ξ_t, λ_t can spend long periods of time close to zero. Thus, the Euler method can lead to negative values of λ_t (which, strictly speaking, are impossible under the model), which reduces its accuracy.

while the pricing kernel can be computed as

$$\frac{\pi_{t+\Delta t}}{\pi_t} \simeq \exp\left[\eta\Delta t - \beta b_\lambda \lambda_{t+\Delta t}\Delta t - \beta b_\xi \xi_{t+\Delta t}\Delta t \right.$$
$$\left. -\gamma \log\left(\frac{C_{t+\Delta t}}{C_t}\right) + b_\lambda(\lambda_{t+\Delta t} - \lambda_t) + b_\xi(\xi_{t+\Delta t} - \xi_t)\right]. \quad \text{(IA.21)}$$

Using (IA.20), we can obtain a series of n_t, $L_{t,s}$, $T^L_{j,t,s}$, and $T^R_{j,t,s}$ for all j. From these series, (IA.21) and the equation for r_t, (6), we compute CDX and tranche pricing using simulations as described in Internet Appendix Section IV.

VI. Zero-Coupon Bond Price

Let $H_g(L^g_t, \lambda_t, \xi_t, s-t)$ denote the time-t price of the zero-coupon government bond maturing at time $s > t$. Barro (2006) assumes that short-term government debt experiences a partial default with probability q during disasters; that is, its face value declines by the same percentage as consumption. We extend this to long-term debt.

Let L^g_t be the face value of the government bond. Assume that

$$\frac{dL^g_t}{L^g_{t^-}} = (e^{Z_{g,t}} - 1)dN_{c,t},$$

where

$$Z_{g,t} = \begin{cases} Z_{c,t} & \text{with probability } q \\ 0 & \text{otherwise.} \end{cases}$$

The pricing relation implies that

$$H_g(L^g_t, \lambda_t, \xi_t, s-t) = E_t\left[\frac{\pi_s}{\pi_t} L^g_s\right]. \quad \text{(IA.22)}$$

By multiplying π_t on both sides of (IA.22), we obtain a martingale:

$$\pi_t H_g(L^g_t, \lambda_t, \xi_t, s-t) = E_t\left[\pi_s L^g_s\right].$$

Conjecture that

$$H_g(L_t^g, \lambda_t, \xi_t, \tau) = L_t^g \exp\left(a_g(\tau) + b_{g\lambda}(\tau)\lambda_t + b_{g\xi}(\tau)\xi_t\right). \tag{IA.23}$$

By Ito's Lemma,

$$\begin{aligned}\frac{dH_{g,t}}{H_{g,t^-}} = &\bigg(b_{g\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{g\lambda}(\tau)^2\sigma_\lambda^2\lambda_t + b_{g\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{g\xi}(\tau)^2\sigma_\xi^2\xi_t \\ &- a_g'(\tau) - b_{g\lambda}'(\tau)\lambda_t - b_{g\xi}'(\tau)\xi_t\bigg) dt + b_{g\lambda}(\tau)\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + b_{g\xi}(\tau)\sigma_\xi\sqrt{\xi_t}dB_{\xi,t} \\ &+ (e^{Z_{L,t}} - 1)dN_{c,t}.\end{aligned} \tag{IA.24}$$

Next, derive the SDE for $\pi_t H_{g,t}$ by combining equation (IA.24) and (3) using Ito's Lemma:

$$\begin{aligned}\frac{d(\pi_t H_{g,t})}{\pi_{t^-} H_{g,t^-}} = &\bigg(- \beta - \mu_c + \gamma\sigma_c^2 - \lambda_t E\left[e^{(1-\gamma)Z_{c,t}} - 1\right] \\ &+ b_{g\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{g\lambda}(\tau)^2\sigma_\lambda^2\lambda_t \\ &+ b_{g\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{g\xi}(\tau)^2\sigma_\xi^2\xi_t \\ &- a_g'(\tau) - b_{g\lambda}'(\tau)\lambda_t - b_{g\xi}'(\tau)\xi_t \\ &+ b_\lambda b_{g\lambda}(\tau)\sigma_\lambda^2\lambda_t + b_\xi b_{g\xi}(\tau)\sigma_\xi^2\xi_t\bigg) dt - \gamma\sigma_c dB_{c,t} \\ &+ (b_\lambda + b_{g\lambda}(\tau))\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + (b_\xi + b_{g\xi}(\tau))\sigma_\xi\sqrt{\xi_t}dB_{\xi,t} + (e^{Z_{L,t}-\gamma Z_{c,t}} - 1)dN_{c,t}.\end{aligned}$$

Since $\pi_t H_{g,t}$ is a martingale, the sum of the drift and the jump compensator of $\pi_t H_{g,t}$ equals zero. That is,

$$\begin{aligned}0 = &\underbrace{\left[-\beta - \mu_c + \gamma\sigma_c^2 + b_{g\xi}(\tau)\kappa_\xi\bar{\xi} - a_g'(\tau)\right]}_{=0} \\ &+ \lambda_t \underbrace{\left[-b_{g\lambda}(\tau)\kappa_\lambda + \frac{1}{2}b_{g\lambda}(\tau)^2\sigma_\lambda^2 + b_\lambda b_{g\lambda}(\tau)\sigma_\lambda^2 + (1-q)E\left[e^{Z_{L,t}-\gamma Z_{c,t}} - e^{(1-\gamma)Z_{c,t}}\right] - b_{g\lambda}'(\tau)\right]}_{=0} \\ &+ \xi_t \underbrace{\left[b_{g\lambda}(\tau)\kappa_\lambda - b_{g\xi}(\tau)\kappa_\xi + \frac{1}{2}b_{g\xi}(\tau)^2\sigma_\xi^2 + b_\xi b_{g\xi}(\tau)\sigma_\xi^2 - b_{g\xi}'(\tau)\right]}_{=0}.\end{aligned}$$

These conditions provide a system of ODEs:

$$\begin{aligned}
a_g'(\tau) &= -\beta - \mu_c + \gamma\sigma_c^2 + b_{g\xi}(\tau)\kappa_\xi\bar{\xi} \\
b_{g\lambda}'(\tau) &= -b_{g\lambda}(\tau)\kappa_\lambda + \frac{1}{2}b_{g\lambda}(\tau)^2\sigma_\lambda^2 + b_\lambda b_{g\lambda}(\tau)\sigma_\lambda^2 + (1-q)E\left[e^{-\gamma Z_{c,t}} - e^{(1-\gamma)Z_{c,t}}\right] \\
b_{g\xi}'(\tau) &= b_{g\lambda}(\tau)\kappa_\lambda - b_{g\xi}(\tau)\kappa_\xi + \frac{1}{2}b_{g\xi}(\tau)^2\sigma_\xi^2 + b_\xi b_{g\xi}(\tau)\sigma_\xi^2. \qquad \text{(IA.25)}
\end{aligned}$$

This shows that H_g satisfies the conjecture (IA.23). We can obtain the boundary conditions for (IA.25) because $H_g(L_t^g, \lambda_t, \xi_t, 0) = L_t^g$, which is equivalent to

$$a_g(0) = b_{g\lambda}(0) = b_{g\xi}(0) = 0.$$

Note that the time-t price of the default-free zero-coupon bond maturing at time $s > t$,

$$H_0(\lambda_t, \xi_t, s-t) = E_t\left[\frac{\pi_s}{\pi_t}\right],$$

is obtained as a special case with $q = 0$ and $L_t^g = 1$.

VII. Option Pricing

A. A Log-Linear Approximation for the Price-Dividend Ratio

The transform analysis we use to price options requires that the log of the price-dividend ratio be linear. Fortunately, the exact price-dividend ratio we derive can be closely approximated by a log-linear function.

Let $g(\lambda, \xi) = \log G(\lambda, \xi)$. For given λ^* and ξ^*, the two-dimensional Taylor approximation implies

$$g(\lambda, \xi) \simeq g(\lambda^*, \xi^*) + \left.\frac{\partial g}{\partial \lambda}\right|_{\lambda^*,\xi^*}(\lambda - \lambda^*) + \left.\frac{\partial g}{\partial \xi}\right|_{\lambda^*,\xi^*}(\xi - \xi^*). \qquad \text{(IA.26)}$$

We note that

$$
\begin{aligned}
\left.\frac{\partial g}{\partial \lambda}\right|_{\lambda^*,\xi^*} &= \frac{1}{G(\lambda^*,\xi^*)} \left.\frac{\partial G}{\partial \lambda}\right|_{\lambda^*,\xi^*} \\
&= \frac{1}{G(\lambda^*,\xi^*)} \int_0^\infty b_{\phi\lambda}(\tau) \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda^* + b_{\phi\xi}(\tau)\xi^*\right) d\tau \qquad \text{(IA.27)}
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\left.\frac{\partial g}{\partial \xi}\right|_{\lambda^*,\xi^*} &= \frac{1}{G(\lambda^*,\xi^*)} \left.\frac{\partial G}{\partial \xi}\right|_{\lambda^*,\xi^*} \\
&= \frac{1}{G(\lambda^*,\xi^*)} \int_0^\infty b_{\phi\xi}(\tau) \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda^* + b_{\phi\xi}(\tau)\xi^*\right) d\tau. \qquad \text{(IA.28)}
\end{aligned}
$$

Expression (IA.27) and (IA.28) can be interpreted as weighted averages of the coefficients $b_{\phi\lambda}(\tau)$ and $b_{\phi\xi}(\tau)$, respectively. The average is over τ, and the weights are proportional to $\exp\left\{a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda^* + b_{\phi\xi}(\tau)\xi^*\right\}$. With this in mind, we define the notation

$$
b^*_{\phi\lambda} = \frac{1}{G(\lambda^*,\xi^*)} \int_0^\infty b_{\phi\lambda}(\tau) \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda^* + b_{\phi\xi}(\tau)\xi^*\right) d\tau \qquad \text{(IA.29)}
$$

$$
b^*_{\phi\xi} = \frac{1}{G(\lambda^*,\xi^*)} \int_0^\infty b_{\phi\xi}(\tau) \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda^* + b_{\phi\xi}(\tau)\xi^*\right) d\tau, \qquad \text{(IA.30)}
$$

and the log-linear function

$$
\hat{G}(\lambda_t,\xi_t) = G(\lambda^*,\xi^*) \exp\left\{b^*_{\phi\lambda}(\lambda_t - \lambda^*) + b^*_{\phi\xi}(\xi_t - \xi^*)\right\}. \qquad \text{(IA.31)}
$$

It follows from exponentiating both sides of (IA.26) that

$$
G(\lambda_t,\xi_t) \simeq \hat{G}(\lambda_t,\xi_t).
$$

In our analysis, we choose λ^* and ξ^* to be $\bar{\xi}$, the stationary mean of both processes.

This log-linearization method differs from the more widely used method of Campbell (2003), applied in continuous time by Chacko and Viceira (2005). However, in this application it is more accurate. This is not surprising, since we are able to exploit the fact that the true solution for the price-dividend ratio is known. In dynamic models with the EIS not equal to one, the solution is typically unknown.

B. Transform Analysis

The normalized put option price is given as

$$P^n(\lambda_t, \xi_t, T-t; K^n) = E_t\left[\frac{\pi_T}{\pi_t}\left(K^n - \frac{F(D_T, \lambda_T, \xi_T)}{F(D_t, \lambda_t, \xi_t}\right)^+\right]. \tag{IA.32}$$

It follows from (IA.11) that

$$\frac{\pi_T}{\pi_t} = \exp\left\{\int_t^T -\beta(1 + a + b_\lambda \lambda_s + b_\xi \xi_s)\, ds - \gamma \log\left(\frac{C_T}{C_t}\right) + b_\lambda(\lambda_T - \lambda_t) + b_\xi(\xi_T - \xi_t)\right\},$$

where b_λ and b_ξ are defined by (4) and (5), respectively. It follows from $F(D_t, \lambda_t, \xi_t) = D_t G(\lambda_t, \xi_t)$, $D_t = C_t^\phi$, and (IA.31) that

$$\frac{F_T}{F_t} \quad = \quad \exp\left\{\phi \log\left(\frac{C_T}{C_t}\right) + b^*_{\phi\lambda}(\lambda_T - \lambda_t) + b^*_{\phi\xi}(\xi_T - \xi_t)\right\},$$

where $b^*_{\phi\lambda}$ and $b^*_{\phi\xi}$ are constants defined by (IA.29) and (IA.30), respectively.

To use the method of Duffie, Pan, and Singleton (2000), it is helpful to write down the following stochastic process, which, under our assumptions, is well defined for given λ_t and ξ_t:

$$X_\tau = \begin{bmatrix} \log C_{t+\tau} - \log C_t \\ \lambda_{t+\tau} \\ \xi_{t+\tau} \end{bmatrix}.$$

Note that the $\{X_\tau\}$ process is defined purely for mathematical convenience. We further define

$$d_1 = \begin{bmatrix} 0 \\ b_\lambda \\ b_\xi \end{bmatrix}, \quad d_2 = \begin{bmatrix} -\gamma \\ b_\lambda \\ b_\xi \end{bmatrix}, \quad d_3 = \begin{bmatrix} 0 \\ b^*_{\phi\lambda} \\ b^*_{\phi\xi} \end{bmatrix}, \quad d_4 = \begin{bmatrix} \phi \\ b^*_{\phi\lambda} \\ b^*_{\phi\xi} \end{bmatrix}.$$

Using this notation, (IA.32) can be rewritten as

$$\begin{aligned} P^n(\lambda_t, \xi_t, T-t; K^n) = K^n E_t & \left[e^{-\int_0^{T-t} R(X_\tau)\, d\tau + d_2^\top X_{T-t} - d_1^\top X_0}\, 1_{\left\{\frac{F_T}{F_t} \le K^n\right\}} \right] \\ & - E_t \left[e^{-\int_0^{T-t} R(X_\tau)\, d\tau + (d_2+d_4)^\top X_{T-t} - (d_1+d_3)^\top X_0}\, 1_{\left\{\frac{F_T}{F_t} \le K^n\right\}} \right], \quad \text{(IA.33)} \end{aligned}$$

where

$$\begin{aligned} R(X_\tau) &= \beta d_1^\top X_\tau + \beta(1+a) \\ 1_{\left\{\frac{F_T}{F_t} \le K^n\right\}} &= 1_{\left\{d_4^\top X_{T-t} \le \log K^n + d_3^\top X_0\right\}}. \end{aligned}$$

Since $\{X_\tau\}$ is an affine process in the sense defined by Duffie, Pan, and Singleton (2000), (IA.33) characterizes the put option price in terms of expectations that can be computed using their transform analysis. Specifically, if we define

$$\mathcal{G}_{p,q}(y; X_0, T-t) \equiv E\left[e^{-\int_0^{T-t} R(X_\tau) d\tau} e^{p^\top X_{T-t}} 1_{\{q^\top X_{T-t} \le y\}} \right], \qquad \text{(IA.34)}$$

then the normalized put price can be expressed as

$$\begin{aligned} P^n(\lambda_t, \xi_t, T-t; K^n) = e^{-d_1^\top X_0} K^n \mathcal{G}_{d_2,d_4} & \left(\log K^n + d_3^\top X_0; X_0, T-t\right) \\ & - e^{-(d_1+d_3)^\top X_0} K^n \mathcal{G}_{d_2+d_4,d_4} \left(\log K^n + d_3^\top X_0; X_0, T-t\right), \end{aligned}$$

where $X_0 = [0, \lambda_t, \xi_t]$. The terms written using the function $\mathcal{G}$ can then be computed tractably using the transform analysis of Duffie, Pan, and Singleton (2000): this analysis requires only the solution of a system of ODEs and a one-dimensional numerical integration.

C. *Implied Volatilities as Functions of State Variables*

Figure IA.1 shows that increases in ξ_t have a larger effect on stock prices than increases in λ_t. However, increases in λ_t raise the immediate probability of disaster more than increases in ξ_t. We would thus expect ξ_t to have a larger effect on ATM options, and λ_t to have a larger effect on OTM options. Figure IA.2 shows that this is indeed the case. This figure plots three-month implied volatilities as a function of moneyness (the strike price of the option divided by the index price) for the state variables at their median levels and at the 20th and 80th percentiles. The level of implied volatilities is increasing in both λ_t and ξ_t. However, ξ_t mainly affects ATM implied volatilities while λ_t has a slightly greater effect for OTM implied volatilities. Increases in ξ_t that are not accompanied by increases in λ_t bring the stock price distribution closer to log-normality, flattening the implied volatility curve.

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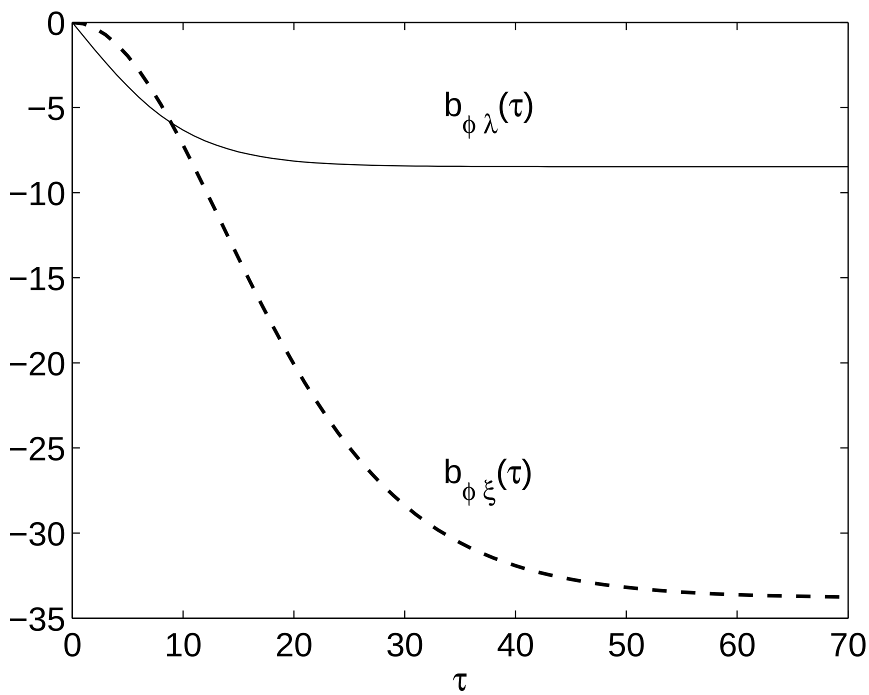
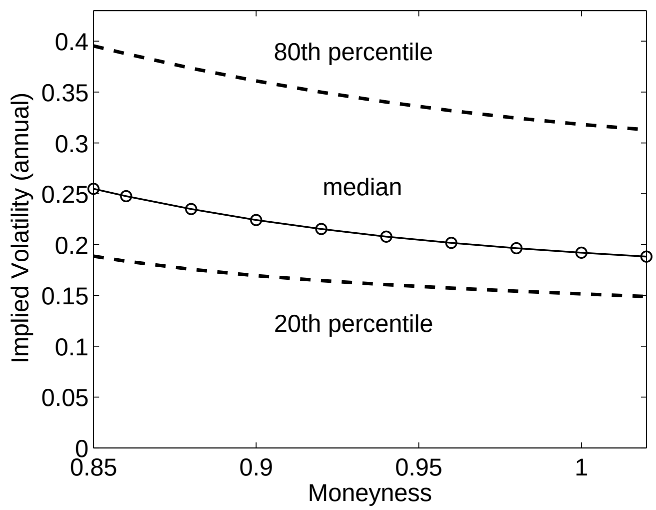


Figure IA.1. Solution for the price-dividend ratio. This figure plots the functions $b_{\phi\lambda}(\tau)$ and $b_{\phi\xi}(\tau)$, which determine the sensitivity of the aggregate market to changes in the disaster probability λ_t and to its time-varying mean ξ_t. That is, the price-dividend ratio on the aggregate market is given by $G(\lambda_t, \xi_t) = \int_0^\infty \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda_t + b_{\phi\xi}(\tau)\xi_t\right)\, d\tau$. The horizon τ is in years.

Panel A: Varying λ_t



Panel B: Varying ξ_t

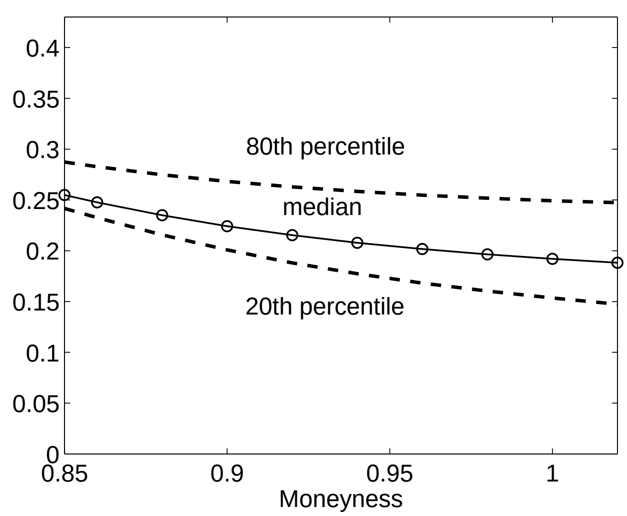


Figure IA.2. Implied volatilities at functions of moneyness. This figure plots implied volatilities for three-month put options on the equity index, shown as a function of moneyness (the strike price divided by the index price), as calculated in the model. The panels show the effects of varying the state variables λ_t (the disaster probability) and ξ_t (the value to which λ_t reverts). Panel A sets ξ_t equal to its median value and varies λ_t, while Panel B sets λ_t equal to its median value and varies ξ_t.

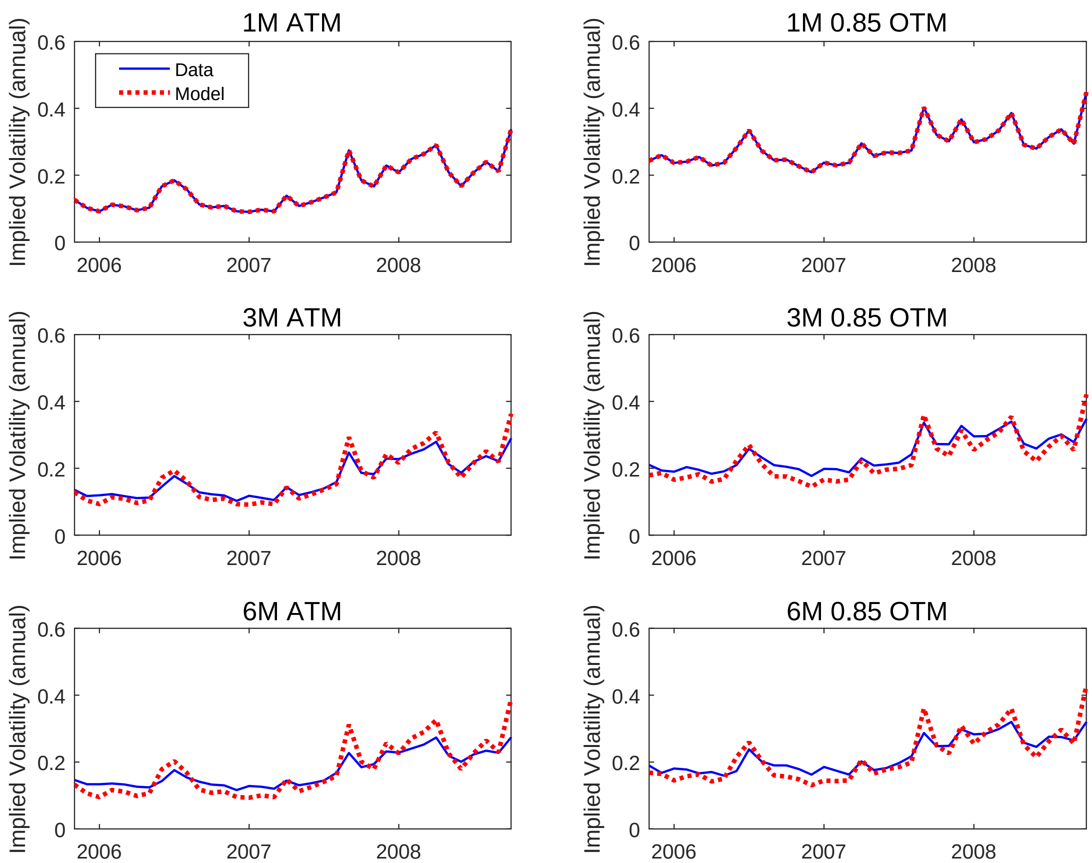


Figure IA.3. Time series of option-implied volatilities. This figure plots monthly time series of option-implied volatilities in the data (blue solid lines) and in the model (red dotted lines). Results are shown for one-, three-, and six-month options. State variables are computed to match the one-month ATM and 0.85 OTM implied volatilities exactly.

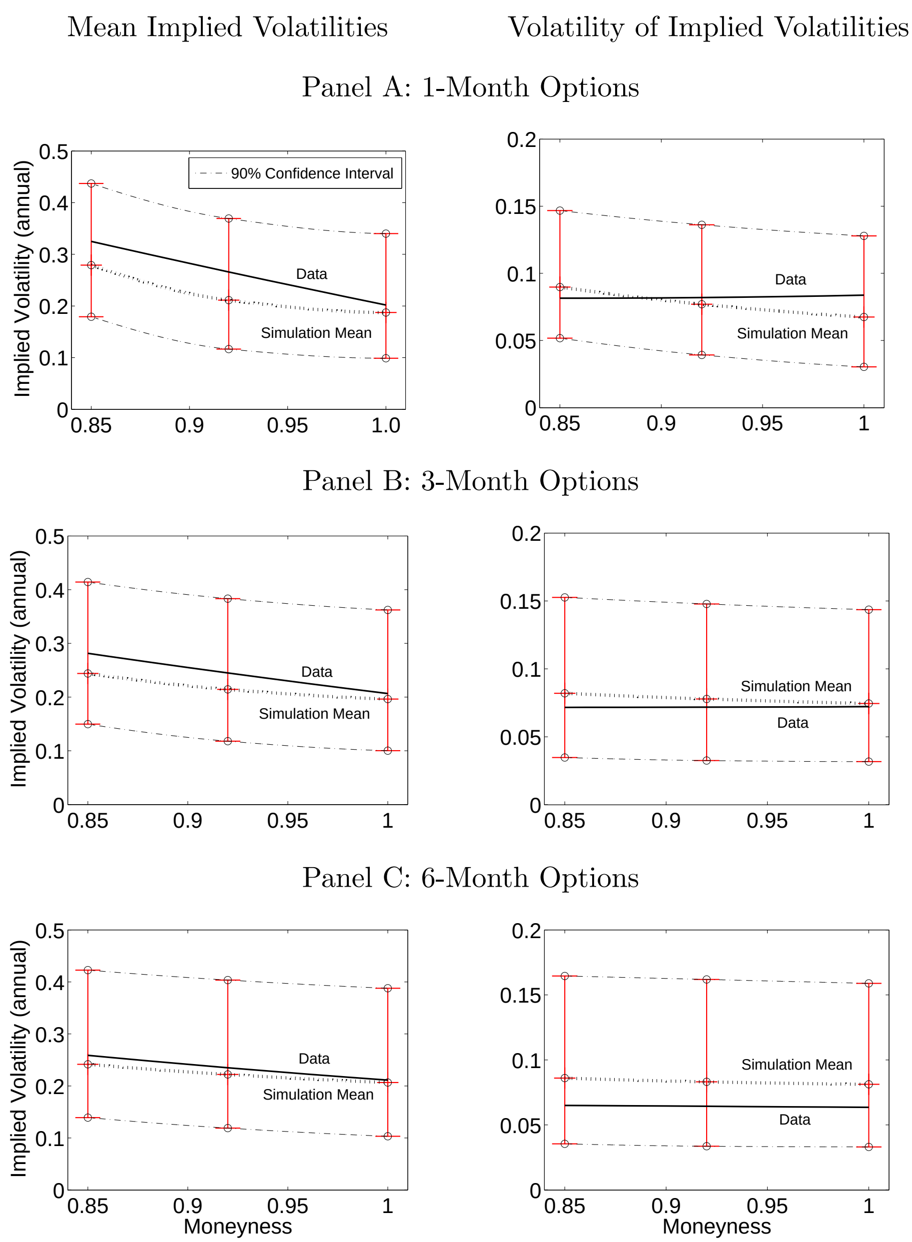


Figure IA.4. Means and standard deviations of implied volatilities in simulated data. This figure plots the mean and standard deviations of implied volatilities after simulating 1000 monthly sample paths of length 17 years (to match the options sample). We consider only sample paths that do not contain disasters. For each sample path, we compute the mean and the standard deviation of implied volatilities from equity index put options for different levels of moneyness (strike price divided by index price) and for three maturities. The dotted lines show the means of these statistics across sample paths, while the dashed-dotted lines show 95th and 5th percentiles. The solid lines show the means and standard deviation of implied volatilities on S&P 500 put options calculated over the 1996 to 2012 period.