Disaster risk and its implications for asset pricing – Online appendix*

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A The iid model

This section derives some useful results for the iid model. The utility is given by

$$V_t = E_t \int_t^{\infty} f(C_s, V_s)\, ds, \tag{A.1}$$

where

$$f(C, V) = \frac{\beta}{1-\frac{1}{\psi}} \frac{C^{1-\frac{1}{\psi}} - ((1-\gamma)V)^{\frac{1}{\theta}}}{((1-\gamma)V)^{\frac{1}{\theta}-1}}, \tag{A.2}$$

and $\theta = (1-\gamma)/(1-\frac{1}{\psi})$. For $\psi = 1$, we assume

$$f(C, V) = \beta(1-\gamma)V\left(\log C - \frac{1}{1-\gamma}\log((1-\gamma)V)\right). \tag{A.3}$$

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Proposition 1. *When consumption growth is iid, along the optimal consumption path, utility satisfies*

$$V_t \equiv J(C_t) = j\frac{C_t^{1-\gamma}}{1-\gamma}.$$

where $j > 0$ and given by

$$j = \left(1 + \frac{1}{\beta}\left(-\left(1-\frac{1}{\psi}\right)\mu + \frac{1}{2}\gamma\left(1-\frac{1}{\psi}\right)\sigma^2 - \frac{1}{\theta}\lambda\left(E_\nu\left[e^{(1-\gamma)Z_t} - 1\right]\right)\right)\right)^{-\theta}. \tag{A.4}$$

Proof of Proposition 1 Conjecture:

$$J(C_t) = j\frac{C_t^{1-\gamma}}{1-\gamma}, \tag{A.5}$$

where $\theta = (1-\gamma)/(1-\frac{1}{\psi})$.

For convenience, let $J_t = J(C_t)$. The HJB equation is:

$$\mathcal{D}J_t + f(C_t, J_t) = 0.$$

Plug (A.5) into (A.2) we get:

$$f(C_t, J_t) = \beta\theta J_t\left[j^{-\frac{1}{\theta}} - 1\right], \tag{A.6}$$

and by Ito's Lemma:

$$\frac{\mathcal{D}J}{J} = \frac{1}{J}\left(\frac{\partial J}{\partial C}C\mu + \frac{1}{2}\frac{\partial^2 J}{\partial C^2}C^2\sigma^2 + \lambda E_\nu\left[J(C\,e^{Z_t}) - J(C)\right]\right). \tag{A.7}$$

The HJB can be rewritten as:

$$0 = \left(1-\frac{1}{\psi}\right)\mu - \frac{1}{2}\gamma\left(1-\frac{1}{\psi}\right)\sigma^2 + \frac{1}{\theta}\lambda\left(E_\nu\left[e^{(1-\gamma)Z_t} - 1\right]\right) + \beta j^{-\frac{1}{\theta}} - \beta. \tag{A.8}$$

Solving this, we get

$$j = \left(1 + \frac{1}{\beta}\left(-\left(1 - \frac{1}{\psi}\right)\mu + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma^2 - \frac{1}{\theta}\lambda\left(E_\nu\left[e^{(1-\gamma)Z_t} - 1\right]\right)\right)\right)^{-\theta}.$$

□

Proposition 2. *When consumption growth is iid,*

$$\pi_t \propto C_t^{-\gamma}.$$

That is, the state-price density is proportional to $C_t^{-\gamma}$ with a positive constant of proportionality.

Proof. It follows from the form of $f(C_t, V_t)$ that

$$\begin{aligned} \frac{\partial f}{\partial C} &= \frac{\beta}{1 - \frac{1}{\psi}} \frac{C_t^{-\frac{1}{\psi}}}{((1-\gamma)V_t)^{\frac{1}{\theta}-1}} \\ &\propto \frac{C_t^{-\frac{1}{\psi}}}{C_t^{(1-\gamma)(\frac{1}{\theta}-1)}} \\ &\propto C_t^{-\gamma}. \end{aligned}$$

where the second line follows from Proposition 1. □

Proposition 3. *When consumption growth is iid, the riskfree rate is equal to*

$$r = \beta + \frac{1}{\psi}\mu - \frac{1}{2}\left(\gamma + \frac{\gamma}{\psi}\right)\sigma^2 + \lambda E_\nu\left[-(e^{-\gamma Z_t} - 1) + \left(1 - \frac{1}{\theta}\right)(e^{(1-\gamma)Z_t} - 1)\right]. \tag{A.9}$$

Proof of Proposition 3 In the i.i.d case:

$$\frac{\partial f}{\partial C}(C_t, V_t) = \beta C_t^{-\gamma} j^{1-\frac{1}{\theta}}$$

and

$$\frac{\partial f}{\partial V}(C_t, V_t) = \beta\left(\frac{\theta - 1}{j^{\frac{1}{\theta}}} - \theta\right).$$

Therefore,

$$\pi_t = \exp\left(-\int_0^t \beta\left(\theta + \frac{1-\theta}{j^{\frac{1}{\theta}}}\right) ds\right) \beta^\theta C_t^{-\gamma} k^{\theta-1} \tag{A.10}$$

By Ito's Lemma:

$$\mu_\pi = -\beta\left(\theta + \frac{1-\theta}{j^{\frac{1}{\theta}}}\right) - \frac{\gamma}{C_t} C_t \mu + \frac{1}{2}\frac{\gamma(\gamma+1)}{C_t^2} C_t^2 \sigma^2 \tag{A.11}$$

Substituting j using (A.4) in the main text and rearrange:

$$\mu_\pi = -\beta - \frac{1}{\psi}\mu + \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)\sigma^2 - \left(1 - \frac{1}{\theta}\right)\lambda E_\nu\left[e^{(1-\gamma)Z_t} - 1\right]. \tag{A.12}$$

It also follows from no-arbitrage that

$$\begin{aligned} r &= -\mu_\pi - \lambda E_\nu[e^{-\gamma Z_t} - 1] \\ &= \beta + \frac{1}{\psi}\mu - \frac{1}{2}\left(\gamma + \frac{\gamma}{\psi}\right)\sigma^2 + \lambda E_\nu\left[-(e^{-\gamma Z_t} - 1) + \left(1 - \frac{1}{\theta}\right)\left(e^{(1-\gamma)Z_t} - 1\right)\right]. \end{aligned}$$

□

Proposition 4. *When consumption growth is iid, and dividend is $D_t = C_t^\phi$*

$$\frac{dD_t}{D_{t^-}} = \mu_D\, dt + \phi\sigma\, dB_t + (e^{\phi Z_t} - 1)\, dN_t.$$

Let S_t denote the price of the dividend claim,then the price-dividend ratio is given by

$$\begin{aligned} \frac{S_t}{D_t} &= E_t \int_t^\infty \frac{\pi_s}{\pi_t}\frac{D_s}{D_t}\, ds \\ &= \Bigg(\beta - \mu_D + \frac{1}{\psi}\mu - \frac{1}{2}\left(\gamma + \frac{\gamma}{\psi} - 2\phi\gamma\right)\sigma^2 \\ &\qquad + \lambda E_\nu\left[\left(1 - \frac{1}{\theta}\right)\left(e^{(1-\gamma)Z_t} - 1\right) - \left(e^{(\phi-\gamma)Z_t} - 1\right)\right]\Bigg)^{-1}. \end{aligned}$$

Proof of Proposition 4 Section A in the Appendix in the main text gives the general no-arbitrage condition:

$$\mu_{\pi,t} + \mu_{S,t} + \frac{D_t}{S_t} + \sigma_{\pi,t}\sigma_{S,t}^\top + \lambda_t^\top E_\nu \left[e^{Z_\pi + Z_S} - 1\right] = 0. \tag{A.13}$$

Conjecture:

$$\frac{S_t}{D_t} = G,$$

therefore $S_t = GD_t$ and by Ito's Lemma:

$$\frac{dS_t}{S_t} = \mu_D \, dt + \phi\sigma \, dB_t + (e^{\phi Z_t} - 1) \, dN_t. \tag{A.14}$$

In the iid case

$$\pi_t \propto C_t^{-\gamma},$$

therefore

$$\sigma_\pi = -\gamma\sigma, \tag{A.15}$$

$$Z_\pi = -\gamma Z_t. \tag{A.16}$$

Substituting (A.12) and (A.14)-(A.16) into (A.13) implies

$$\begin{aligned} &-\beta - \frac{1}{\psi}\mu + \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)\sigma^2 + \mu_D + G^{-1} - \gamma\phi\sigma^2 \\ &\qquad - \left(1 - \frac{1}{\theta}\right)\lambda E_\nu\left[e^{(1-\gamma)Z_t} - 1\right] + \lambda E_\nu\left[e^{(\phi-\gamma)Z_t} - 1\right] = 0. \end{aligned}$$

Rearranging this verifies the conjecture and the dividend-price ratio is:

$$\begin{aligned} G^{-1} = \beta - \mu_D + \frac{1}{\psi}\mu - \frac{1}{2}\left(\gamma + \frac{\gamma}{\psi} - 2\phi\gamma\right)\sigma^2 \\ + \lambda E_\nu\left[\left(1 - \frac{1}{\theta}\right)\left(e^{(1-\gamma)Z_t} - 1\right) - \left(e^{(\phi-\gamma)Z_t} - 1\right)\right]. \end{aligned}$$

□

B Data and results for time-varying disaster probabilities model

We use annual data from 1948 to 2013. The aggregate market data come from CRSP. The market return is the gross return on the NYSE/AMEX/NASDAQ value-weighted index. Dividend growth is computed from the dividends on this index. The price-dividend ratio is price divided by the previous 12 months of dividends to remove the effect of seasonality in dividend payments (in computing this dividend stream, we assume that dividends on the market are not reinvested). For the government bill rate, we use real returns on the 3-month Treasury Bill. We compute market returns, dividend growth, and government bill returns in real terms by adjusting for inflation using changes in the consumer price index (also available from CRSP). We also compute consumption growth using real, per capital expenditures on non-durables and services for the U.S., available from the Bureau of Economic Analysis.

Table A.1-A.6 reports results for the time varying disaster probability case. Parameters are as follows: using the 15% cutoff for disasters we set the average disaster probability $\bar{\lambda} = 0.0218$, and assume that there is a 40% probability of default on government bills in the case of a disaster. Discount rate $\beta = 0.01$ to match the low risk-free rate. Normal time consumption growth and volatility are set to match the postwar data, $\mu = 0.0195$, and $\sigma = 0.0125$. We set dividend growth $\mu_D = 0.04$ to get realistic price-dividend ratios and idiosyncratic volatility $\sigma_i = 0.05$ to match the postwar dividend growth volatility. We set risk aversion $\gamma = 3$ and leverage $\phi = 3$ to match the equity premium and return volatility. The mean-reversion $\kappa_\lambda = 0.12$ to match persistence for price-dividend ratio. We look at six cases – time additive utility (EIS = 1/3), EIS = 1 and EIS = 2, for each of the three cases, we compare results with time-varying disaster probability, $\sigma_\lambda = 0.08$[1], to ones with constant disaster probability, $\sigma_\lambda = 0.00$[2].

[1] Given other parameters and disaster distribution, the existence of the value function imposes a constraint on κ_λ and σ_λ. In particular, the disaster probability process cannot be highly persistent and "too" highly volatile at the same time. We choose κ_λ and σ_λ to best to match both the volatility of stock returns, the persistence of the price-dividend ratio, and the volatility of Treasury Bill returns

[2] In order to solve the model using the general case, ψ can not be exactly one and σ_λ can not be exactly

We simulate monthly data for 600,000 years to obtain population moments. We also simulate 100,000 65-year samples. In the tables, we report population values for each statistic, percentile values from the small-sample simulations, and percentile values for the subset of small-sample simulations that do not contain jumps. It is this subset of simulations that is the most interesting comparison for postwar data.

C Production economy

This section solves the two models with endogenous consumption choice are considered by Barro (2009). The first one allows for labor-leisure choice, and productivity is subject to disaster shocks, we will show that this case is equivalent to the iid endowment case. Then we will consider another case with capital accumulation.

C.1 Recursive utility in discrete time

Assume that continuation utility takes the following form:

$$V_t = \left[(1 - e^{-\beta}) C_t^{1-\frac{1}{\psi}} + e^{-\beta} \left(E_t [V_{t+1}^{1-\gamma}] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}, \tag{C.1}$$

with intertemporal marginal rate of substitution (IMRS):

$$M_{t+1} = e^{-\beta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{V_{t+1}}{E_t [V_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}} \right)^{\frac{1}{\psi} - \gamma}.$$

For these preferences, Epstein and Zin (1989) show that the intertemporal marginal rate of substitution (IMRS) can be expressed as:

$$M_{t+1} = e^{-\beta\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{W,t+1}^{\theta-1}, \tag{C.2}$$

zero, they are numbers very close to one and zero ($1 + 10^{-8}$ and 10^{-8}, respectively). Since σ_λ is not exactly zero, the autocorrelation of price-dividend ratios in Table A.4 and A.6 are not all equals one.

where $\theta = (1-\gamma)/(1-\frac{1}{\psi})$, and $R_{W,t+1}$ is the gross return on an asset that delivers aggregate consumption as its dividend.

C.2 Leisure-labor choice

This is the first case in Barro (2009). In this model, output is given by

$$Y_t = A_t L_t^{\alpha}, \tag{C.3}$$

where A_t is productivity and L_t is the quantity of labor employed. Here, assumptions on the endowment process are replaced by assumptions on productivity, so that A_t is given by:

$$\log \frac{A_{t+1}}{A_t} = \mu + \epsilon_{t+1} + \begin{cases} 0 & \text{if there is no disaster at } t+1 \\ Z_{t+1} & \text{if there is a disaster at } t+1 \end{cases} \tag{C.4}$$

where ϵ_{t+1} is an iid standard normal shock and disaster occurs with probability $1-e^{-\lambda}$. We modify (C.1) by replacing C_t with period utility that allows for preferences over leisure:

$$U_t = C_t(1-L_t)^{\chi}. \tag{C.5}$$

As we show in Section 6.1, a constant share (α) of the consumption comes from labor income, and the rest $(1-\alpha)$ comes from dividend. Therefore, consumption/output growth and dividend growth both equal to technology growth:

$$\log \frac{C_{t+1}}{C_t} = \log \frac{D_{t+1}}{D_t} = \log \frac{Y_{t+1}}{Y_t} = \log \frac{A_{t+1}}{A_t}.$$

That is, log consumption growth process is the same as log productivity process[3]:

$$\log C_{t+1} = \log C_t + \mu + \epsilon_{t+1} + \begin{cases} 0 & \text{with prob. } e^{-\lambda} \\ Z_{t+1} & \text{with prob. } 1 - e^{-\lambda}. \end{cases} \tag{C.6}$$

Period utility depends on both consumption and leisure in this model, as stated by (C.5). In the iid case, however, labor share L_t, is constant over time, it follows that the IMRS only depends on consumption growth and return on consumption wealth.

Conjecture the consumption-to-wealth ratio is constant, that is, $C_t/W_t = cw$ for all t. The budget constraint for the representative agent:

$$W_{t+1} = R_{W,t+1}(W_t - C_t),$$

can be written as

$$R_{W,t+1} = \frac{1}{1 - cw}\frac{C_{t+1}}{C_t}. \tag{C.7}$$

And we can rewrite the IMRS by plugging (C.7) into (C.2):

$$M_{t+1} = e^{-\beta\theta}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{1}{1 - cw}\right)^{\theta - 1}. \tag{C.8}$$

The Euler equation implies that the consumption-wealth ratio is determined by

$$E_t\left[M_{t+1}\frac{W_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right] = \frac{W_t}{C_t} - 1,$$

or,

$$E_t\left[M_{t+1}\frac{C_{t+1}}{C_t}\frac{1}{cw}\right] = \frac{1}{cw} - 1,$$

[3]Notice that this specification is almost the same as the one in the continuous time endowment economy. The only difference is the interpretation of μ. In normal times, log consumption growth is normally distributed with mean μ in this discrete time model, and it is $(\mu - \frac{1}{2}\sigma^2)\Delta t$ in the continuous time model.

Substituting (C.8) into the IMRS and rearrange:

$$
\begin{aligned}
(1-cw)^{\theta} &= E_t\left[e^{-\beta\theta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right] \\
&= \exp\left\{-\theta\beta + (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}\left(e^{-\lambda} + (1-e^{-\lambda})E_{\nu}\left[e^{(1-\gamma)Z_t}\right]\right).
\end{aligned}
$$

Therefore,

$$
\log(1-cw) = -\beta + \left(1-\frac{1}{\psi}\right)\mu + \frac{1}{2}\frac{1}{\theta}(1-\gamma)^2\sigma^2 + \frac{1}{\theta}\log\left(e^{-\lambda} + (1-e^{-\lambda})E_{\nu}\left[e^{(1-\gamma)Z_t}\right]\right), \tag{C.9}
$$

which verifies the conjecture. We are interested in the value when the time interval goes to zero. Since λ scale with the time interval, in the limit, λ is close to zero and:

$$
1-e^{-\lambda} \approx \lambda,
$$

moreover, for x close to zero,

$$
\log(1+x) \approx \frac{1}{1+x}|_{x=0}x = x.
$$

Therefore the last term in (C.9),

$$
\log\left(e^{-\lambda} + (1-e^{-\lambda})E_{\nu}\left[e^{(1-\gamma)Z_t}\right]\right) \approx \log\left(1-\lambda+\lambda E_{\nu}\left[e^{(1-\gamma)Z_t}\right]\right) \approx \lambda E_{\nu}\left[e^{(1-\gamma)Z_t}-1\right].
$$

In the limit, Equation (C.9) can be written as:

$$
\log(1-cw) \approx -\beta + \left(1-\frac{1}{\psi}\right)\mu + \frac{1}{2}\frac{1}{\theta}(1-\gamma)^2\sigma^2 + \frac{1}{\theta}\lambda E_{\nu}\left[e^{(1-\gamma)Z_t}-1\right]. \tag{C.10}
$$

Next we can solve for the risk-free rates and returns on consumption claim in the economy, and we will focus on the limit when time interval approaches zero:

$$\begin{aligned}
\log R^f_{t+1} &= -\log E_t[M_{t+1}] \\
&= \theta\beta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 - \log\left(e^{-\lambda} + (1-e^{-\lambda})E_\nu\left[e^{-\gamma Z_t}\right]\right) + (\theta-1)\log(1-cw) \\
&= \beta + \frac{1}{\psi}\mu - \frac{1}{2}\sigma^2\left(\gamma^2 - \frac{\theta-1}{\theta}(1-\gamma)^2\right) \\
&\quad - \log\left(e^{-\lambda} + (1-e^{-\lambda})E_\nu\left[e^{-\gamma Z_t}\right]\right) + \frac{\theta-1}{\theta}\log\left(e^{-\lambda} + (1-e^{-\lambda})E_\nu\left[e^{(1-\gamma)Z_t}\right]\right) \\
&\approx \beta + \frac{1}{\psi}\mu - \frac{1}{2}\sigma^2\left(\gamma^2 - \frac{\theta-1}{\theta}(1-\gamma)^2\right) \\
&\qquad - \lambda E_\nu\left[e^{-\gamma Z_t} - 1\right] + \frac{\theta-1}{\theta}\lambda E_\nu\left[e^{(1-\gamma)Z_t} - 1\right] \\
&= \beta + \frac{1}{\psi}\mu - \frac{1}{2}\sigma^2\left(\gamma + \frac{\gamma}{\psi} - \frac{1}{\psi}\right) + \lambda E_\nu\left[-\left(e^{-\gamma Z_t} - 1\right) + \frac{\theta-1}{\theta}\left(e^{(1-\gamma)Z_t} - 1\right)\right].
\end{aligned} \tag{C.11}$$

Comparing with the continuous time version, there is an extra $\frac{1}{2}\frac{1}{\psi}\sigma^2$ in the risk-free rate, and this is because the mean of log consumption growth is μ instead of $\mu - \frac{1}{2}\sigma^2$.

The return on consumption claim is given by (C.7):

$$\begin{aligned}
\log E_t[R_{W,t+1}] &= \mu - \frac{1}{2}\sigma^2 + \log\left(e^{-\lambda} + (1-e^{-\lambda})E_\nu\left[e^{Z_t}\right]\right) - \log(1-cw) \\
&= \beta + \frac{1}{\psi}\mu + \frac{1}{2}\sigma^2\left(1 - \frac{1}{\theta}(1-\gamma^2)\right) \\
&\quad + \log\left(e^{-\lambda} + (1-e^{-\lambda})E_\nu\left[e^{Z_t}\right]\right) - \frac{1}{\theta}\log\left(e^{-\lambda} + (1-e^{-\lambda})E_\nu\left[e^{(1-\gamma)Z_t}\right]\right) \\
&\approx \beta + \frac{1}{\psi}\mu + \frac{1}{2}\sigma^2\left(1 - \frac{1}{\theta}(1-\gamma^2)\right) + \lambda E_\nu\left[\left(e^{Z_t} - 1\right) - \frac{1}{\theta}\left(e^{(1-\gamma)Z_t} - 1\right)\right].
\end{aligned} \tag{C.12}$$

Next we can calculate the risk premium for the consumption claim. Using (C.11) and (C.12):

$$\log E_t[R_W] - \log R^f = \gamma\sigma^2 - \lambda E_\nu\left[\left(e^{-\gamma Z_t} - 1\right)\left(e^{Z_t} - 1\right)\right]. \tag{C.13}$$

Notice that the equity premium on consumption claim in the continuous time limit is the same as the one we obtain in the iid endowment economy (Section 3.1).

C.3 Capital accumulation

In the second case of Barro (2009), output is given by:

$$Y_t = AK_t,$$

where K_t is the capital stock. Productivity A is constant in this model while capital evolves according to:

$$K_{t+1} = K_t + I_t - \delta_{t+1} K_t,$$

where δ_{t+1} is depreciation. Depreciation has a normal time and disaster component:

$$\delta_{t+1} = \delta + \begin{cases} 0 & \text{if there is no disaster at t} \\ 1 - e^{Z_t} & \text{if there is a disaster at t.} \end{cases}$$

First, conjecture a constant investment-to-capital ratio $\zeta = I_t/K_t$, consumption growth equals capital growth, thus investment growth:

$$\frac{C_{t+1}}{C_t} = \frac{(A-\zeta)K_{t+1}}{(A-\zeta)K_t} = \frac{K_{t+1}}{K_t} = 1 + \zeta - \delta_{t+1}. \tag{C.14}$$

Similar to the previous case, we can conjecture and verify that the consumption-wealth ratio is constant:

$$(1-cw)^{\theta} = E_t\left[e^{-\beta\theta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right],$$

with (C.14), the right hand side is a constant.

In this case, return on capital equals return on wealth:

$$R_{W,t+1} = R^S_{t+1} = 1 + A - \delta_{t+1},$$

and the Euler equation can be written as:

$$e^{-\beta\theta} E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} \left(R^S_{t+1} \right)^{\theta} \right] = 1. \tag{C.15}$$

We can rewrite (C.15) as:

$$e^{-\beta\theta} E_\nu \left[(1+\zeta-\delta_{t+1})^{-\frac{\theta}{\psi}} (1+A-\delta_{t+1})^{\theta} \right] = 1,$$

or

$$-\beta\theta + \log\left(E_\nu \left[(1+\zeta-\delta_{t+1})^{-\frac{\theta}{\psi}} (1+A-\delta_{t+1})^{\theta} \right] \right) = 0. \tag{C.16}$$

We are interested in the limit when time interval goes to zero:

$$\begin{aligned}
& E_\nu \left[(1+\zeta-\delta_{t+1})^{-\frac{\theta}{\psi}} (1+A-\delta_{t+1})^{\theta} \right] \\
&= e^{-\lambda} (1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta} + (1-e^{-\lambda}) E_\nu \left[(1+\zeta-\delta+e^{Z_t}-1)^{-\frac{\theta}{\psi}} (1+A-\delta+e^{Z_t}-1)^{\theta} \right] \\
&= (1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta} \left(e^{-\lambda} + (1-e^{-\lambda}) E_\nu \left[\frac{(e^{Z_t}+\zeta-\delta)^{-\frac{\theta}{\psi}} (e^{Z_t}+A-\delta)^{\theta}}{(1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta}} \right] \right) \\
&\approx (1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta} \left(1-\lambda + \lambda E_\nu \left[\frac{(e^{Z_t}+\zeta-\delta)^{-\frac{\theta}{\psi}} (e^{Z_t}+A-\delta)^{\theta}}{(1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta}} \right] \right) \\
&= (1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta} \left(1 + \lambda E_\nu \left[\frac{(e^{Z_t}+\zeta-\delta)^{-\frac{\theta}{\psi}} (e^{Z_t}+A-\delta)^{\theta}}{(1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta}} - 1 \right] \right).
\end{aligned}$$

Therefore:

$$\begin{aligned}
& \log\left((1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta} \left(1 + \lambda E_\nu \left[\frac{(e^{Z_t}+\zeta-\delta)^{-\frac{\theta}{\psi}} (e^{Z_t}+A-\delta)^{\theta}}{(1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta}} - 1 \right] \right) \right) \\
&\approx -\frac{\theta}{\psi} (\zeta-\delta) + \theta (A-\delta) + \lambda E_\nu \left[\frac{(e^{Z_t}+\zeta-\delta)^{-\frac{\theta}{\psi}} (e^{Z_t}+A-\delta)^{\theta}}{(1+\zeta-\delta)^{-\frac{\theta}{\psi}} (1+A-\delta)^{\theta}} - 1 \right] \\
&\approx -\frac{\theta}{\psi} (\zeta-\delta) + \theta (A-\delta) + \lambda E_\nu \left[e^{(1-\gamma)Z_t} - 1 \right].
\end{aligned}$$

We can then solve for the investment-capital ratio ζ using (C.16):

$$-\beta\theta - \frac{\theta}{\psi}(\zeta - \delta) + \theta (A - \delta) + \lambda E_\nu \left[e^{(1-\gamma)Z_t} - 1 \right] = 0,$$

and

$$\zeta = \psi \left(\frac{\frac{1}{\psi} - 1}{\gamma - 1} \right) \lambda E_\nu \left[e^{(1-\gamma)Z_t} - 1 \right] + \delta + \psi (A - \delta - \beta). \tag{C.17}$$

Next we can also solve for the risk-free rate using the Euler equation:

$$\begin{aligned} 1 &= e^{-\beta\theta} E_\nu \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} (R^S_{t+1})^{\theta - 1} R_f \right] \\ &= e^{-\beta\theta} R^f E_\nu \left[(1 + \zeta - \delta_{t+1})^{-\frac{\theta}{\psi}} (1 + A - \delta_{t+1})^{\theta - 1} \right]. \end{aligned}$$

Similar to the above calculation, the Euler condition becomes:

$$-\beta\theta + r^f - \frac{\theta}{\psi}(\zeta - \delta) + (\theta - 1)(A - \delta) + \lambda E_\nu \left[e^{-\gamma Z_t} - 1 \right] = 0,$$

where $\log R^f = \log(1 + r^f) \approx r^f$. Plugging in ζ using (C.17):

$$r^f = A - \delta + \lambda E_\nu \left[e^{-\gamma Z_t}(e^{Z_t} - 1) \right],$$

or

$$R^f = 1 + A - \delta + \lambda E_\nu \left[e^{-\gamma Z_t}(e^{Z_t} - 1) \right].$$

Recall that the expected return on equity is a constant:

$$E[R^S] = 1 + A - \delta + \lambda E_\nu \left[e^{Z_t} - 1 \right],$$

hence the equity premium for consumption claim is:

$$E[R^S] - R_f = -\lambda E_\nu \left[(e^{-\gamma Z_t} - 1)(e^{Z_t} - 1) \right].$$

The equity premium here again is the same as the one in the iid endowment economy model, except for not having the volatility term.

References

Barro, Robert J., 2009, Rare disasters, asset prices, and welfare costs, *American Economic Review* 99, 243–264.

Epstein, Larry, and Stan Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Table A.1: Power utility and time-varying disaster probability

		No-Disaster Simulations			All Simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
$E[R^b]$	1.24	0.08	3.55	5.26	-3.17	2.36	4.89	1.79
$\sigma(R^b)$	2.57	1.51	3.03	6.00	1.86	4.49	9.57	6.00
$E[R^e - R^b]$	7.69	1.51	3.79	7.40	0.99	3.80	8.27	4.15
$\sigma(R^e)$	17.72	5.71	6.68	7.69	6.12	10.43	18.12	11.58
Sharpe ratio	0.44	0.23	0.57	1.13	0.08	0.38	0.91	0.36
$\exp(E[p-d])$	33.33	34.52	34.52	34.52	34.52	34.52	34.52	34.52
$\sigma(p-d)$	0.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AR1$(p-d)$	0.91	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$E[\Delta c]$	1.93	1.69	1.94	2.20	-0.49	1.42	2.09	1.21
$\sigma(\Delta c)$	1.25	1.07	1.24	1.43	1.14	3.74	12.97	6.08
$E[\Delta d]$	2.06	2.66	3.92	5.20	-3.44	2.31	4.69	1.73
$\sigma(\Delta d)$	6.95	5.33	6.22	7.14	5.71	12.33	39.34	18.90

Notes: Parameters are as follows: average disaster probability $\bar{\lambda} = 0.0218$, discount rate $\beta = 0.01$, risk aversion $\gamma = 3$, normal time consumption growth $\mu_c = 0.0195$, consumption growth volatility $\sigma_c = 0.0125$, dividend growth $\mu_d = 0.04$, idiosyncratic volatility $\sigma_i = 0.05$, leverage $\phi = 3$, and mean-reversion $\kappa_\lambda = 0.12$, and volatility $\sigma_\lambda = 0.08$.

Table A.2: Power utility and constant disaster probability

		No-Disaster Simulations			All Simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
$E[R^b]$	1.24	1.93	1.93	1.93	0.93	1.93	1.93	1.69
$\sigma(R^b)$	2.57	0.00	0.00	0.00	0.00	0.00	6.69	2.89
$E[R^e - R^b]$	7.69	3.98	5.35	6.73	1.91	4.31	6.19	4.25
$\sigma(R^e)$	17.72	5.72	6.68	7.68	6.20	10.79	17.52	11.65
Sharpe ratio	0.44	0.59	0.80	1.04	0.12	0.40	0.92	0.36
$\exp(E[p-d])$	33.33	34.52	34.52	34.52	34.52	34.52	34.52	34.52
$\sigma(p-d)$	0.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AR1$(p-d)$	0.91	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$E[\Delta c]$	1.93	1.69	1.94	2.20	-0.30	1.37	2.07	1.20
$\sigma(\Delta c)$	1.25	1.07	1.24	1.43	1.15	3.97	12.80	6.90
$E[\Delta d]$	2.06	2.65	3.94	5.22	-2.90	2.15	4.63	1.72
$\sigma(\Delta d)$	6.95	5.33	6.23	7.14	5.78	12.99	38.73	18.95

Notes: Parameters are as follows: average disaster probability $\bar{\lambda} = 0.0218$, discount rate $\beta = 0.01$, risk aversion $\gamma = 3$, normal time consumption growth $\mu_c = 0.0195$, consumption growth volatility $\sigma_c = 0.0125$, dividend growth $\mu_d = 0.04$, idiosyncratic volatility $\sigma_i = 0.05$, leverage $\phi = 3$, and mean-reversion $\kappa_\lambda = 0.12$, and volatility $\sigma_\lambda = 0.00$.

Table A.3: EIS=1 and time-varying disaster probability

		No-Disaster Simulations			All Simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
$E[R^b]$	1.24	-0.04	1.48	2.22	-1.84	0.84	2.05	0.58
$\sigma(R^b)$	2.57	0.65	1.31	2.65	0.80	2.34	7.36	3.78
$E[R^e - R^b]$	7.69	4.74	7.04	10.65	4.02	6.90	11.33	7.17
$\sigma(R^e)$	17.72	10.75	15.05	21.34	12.16	18.83	28.00	19.73
Sharpe ratio	0.44	0.35	0.47	0.61	0.21	0.38	0.56	0.36
$\exp(E[p-d])$	33.33	25.79	31.42	34.67	22.09	29.68	34.06	28.87
$\sigma(p-d)$	0.42	0.09	0.18	0.34	0.11	0.22	0.44	0.30
AR1$(p-d)$	0.91	0.55	0.77	0.90	0.59	0.81	0.93	0.89
$E[\Delta c]$	1.93	1.69	1.94	2.20	-0.52	1.42	2.09	1.18
$\sigma(\Delta c)$	1.25	1.07	1.24	1.43	1.14	3.75	13.06	6.19
$E[\Delta d]$	2.06	2.65	3.93	5.22	-3.54	2.30	4.70	1.66
$\sigma(\Delta d)$	6.95	5.33	6.22	7.15	5.70	12.35	39.56	19.23

Notes: Parameters are as follows: average disaster probability $\bar{\lambda} = 0.0218$, discount rate $\beta = 0.01$, risk aversion $\gamma = 3$, normal time consumption growth $\mu_c = 0.0195$, consumption growth volatility $\sigma_c = 0.0125$, dividend growth $\mu_d = 0.04$, idiosyncratic volatility $\sigma_i = 0.05$, leverage $\phi = 3$, and mean-reversion $\kappa_\lambda = 0.12$, and volatility $\sigma_\lambda = 0.08$.

Table A.4: The equity premium – EIS=1 and constant disaster probability

		No-Disaster Simulations			All Simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
$E[R^b]$	1.24	0.81	0.81	0.81	-0.18	0.81	0.81	0.56
$\sigma(R^b)$	2.57	0.00	0.00	0.00	0.00	0.00	6.62	2.90
$E[R^e - R^b]$	7.69	3.93	5.28	6.63	1.91	4.27	6.10	4.13
$\sigma(R^e)$	17.72	5.65	6.60	7.61	6.13	10.66	17.29	11.58
Sharpe ratio	0.44	0.59	0.80	1.04	0.12	0.40	0.91	0.36
$\exp(E[p-d])$	33.33	55.98	55.98	55.98	55.98	55.98	55.98	55.98
$\sigma(p-d)$	0.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AR1$(p-d)$	0.91	0.68	0.84	0.92	0.68	0.84	0.92	0.89
$E[\Delta c]$	1.93	1.68	1.94	2.20	-0.30	1.37	2.07	1.18
$\sigma(\Delta c)$	1.25	1.07	1.24	1.43	1.15	3.97	12.78	6.17
$E[\Delta d]$	2.06	2.65	3.93	5.20	-2.90	2.15	4.60	1.62
$\sigma(\Delta d)$	6.95	5.33	6.22	7.15	5.78	12.95	38.69	19.18

Notes: Parameters are as follows: average disaster probability $\bar{\lambda} = 0.0218$, discount rate $\beta = 0.01$, risk aversion $\gamma = 3$, normal time consumption growth $\mu_c = 0.0195$, consumption growth volatility $\sigma_c = 0.0125$, dividend growth $\mu_d = 0.04$, idiosyncratic volatility $\sigma_i = 0.05$, leverage $\phi = 3$, and mean-reversion $\kappa_\lambda = 0.12$, and volatility $\sigma_\lambda = 0.00$.

Table A.5: EIS=2 and time-varying disaster probability.

		No-Disaster Simulations			All Simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
$E[R^b]$	1.24	-0.80	0.61	1.29	-2.56	0.01	1.13	-0.27
$\sigma(R^b)$	2.57	0.60	1.20	2.45	0.74	2.17	7.20	3.68
$E[R^e - R^b]$	7.69	5.50	7.91	11.83	4.86	7.86	12.68	8.19
$\sigma(R^e)$	17.72	12.04	17.13	24.60	13.66	20.94	30.86	22.07
Sharpe ratio	0.44	0.36	0.47	0.59	0.23	0.39	0.55	0.37
$\exp(E[p-d])$	33.33	27.24	34.21	38.39	22.70	32.01	37.58	30.92
$\sigma(p-d)$	0.42	0.11	0.21	0.40	0.13	0.26	0.50	0.34
AR1$(p-d)$	0.91	0.54	0.77	0.90	0.59	0.81	0.93	0.88
$E[\Delta c]$	1.93	1.69	1.94	2.20	-0.53	1.43	2.09	1.16
$\sigma(\Delta c)$	1.25	1.07	1.24	1.43	1.14	3.72	13.06	6.32
$E[\Delta d]$	2.06	2.64	3.93	5.20	-3.55	2.32	4.70	1.60
$\sigma(\Delta d)$	6.95	5.34	6.21	7.15	5.69	12.26	39.57	19.61

Notes: Parameters are as follows: average disaster probability $\bar{\lambda} = 0.0218$, discount rate $\beta = 0.01$, risk aversion $\gamma = 3$, normal time consumption growth $\mu_c = 0.0195$, consumption growth volatility $\sigma_c = 0.0125$, dividend growth $\mu_d = 0.04$, idiosyncratic volatility $\sigma_i = 0.05$, leverage $\phi = 3$, and mean-reversion $\kappa_\lambda = 0.12$, and volatility $\sigma_\lambda = 0.08$.

Table A.6: EIS=2 and constant disaster probability

		No-Disaster Simulations			All Simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
$E[R^b]$	1.24	0.53	0.53	0.53	-0.46	0.53	0.53	0.28
$\sigma(R^b)$	2.57	0.00	0.00	0.00	0.00	0.00	6.61	2.88
$E[R^e - R^b]$	7.69	3.91	5.26	6.64	1.89	4.26	6.10	4.16
$\sigma(R^e)$	17.72	5.64	6.58	7.57	6.11	10.62	17.29	11.48
Sharpe ratio	0.44	0.59	0.80	1.04	0.12	0.40	0.91	0.36
$\exp(E[p-d])$	33.33	66.27	66.27	66.27	66.27	66.27	66.27	66.27
$\sigma(p-d)$	0.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AR1$(p-d)$	0.91	0.68	0.84	0.92	0.68	0.84	0.92	0.89
$E[\Delta c]$	1.93	1.68	1.94	2.20	-0.30	1.37	2.07	1.19
$\sigma(\Delta c)$	1.25	1.07	1.25	1.43	1.15	3.96	12.78	6.11
$E[\Delta d]$	2.06	2.65	3.92	5.22	-2.89	2.15	4.61	1.68
$\sigma(\Delta d)$	6.95	5.33	6.22	7.13	5.78	12.93	38.67	18.99

Notes: Parameters are as follows: average disaster probability $\bar{\lambda} = 0.0218$, discount rate $\beta = 0.01$, risk aversion $\gamma = 3$, normal time consumption growth $\mu_c = 0.0195$, consumption growth volatility $\sigma_c = 0.0125$, dividend growth $\mu_d = 0.04$, idiosyncratic volatility $\sigma_i = 0.05$, leverage $\phi = 3$, and mean-reversion $\kappa_\lambda = 0.12$, and volatility $\sigma_\lambda = 0.00$.