Core-Periphery Trading Networks

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Abstract

Core-periphery trading networks arise endogenously in over-the-counter markets as an equilibrium balance between trade competition and inventory efficiency. A small number of firms emerge as core dealers to intermediate trades among a large number of peripheral firms. The equilibrium number of dealers depends on two countervailing forces: *(i)* competition among dealers in their pricing of immediacy to peripheral firms, and *(ii)* the benefit of concentrated intermediation in balancing dealer inventory through dealers' ability to quickly net purchases against sales. For an asset with a lower frequency of trade demand, intermediation is concentrated among fewer dealers, and interdealer trades account for a greater fraction of total trade volume. These two predictions are strongly supported by evidence from the Bund and U.S. corporate bond markets. From a welfare viewpoint, I show that there are too few dealers for assets with frequent trade demands, and too many for assets with infrequent trade demands.

JEL Classifications: C73, D43, D85, L13, L14, G14

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1 Introduction

Using a continuous-time model of network formation and trading in over-the-counter (OTC) markets, I show how a highly concentrated core-periphery network arises endogenously as an equilibrium balance between trade competition and inventory efficiency. Even when agents are all ex-ante identical, a small number of them emerge as core agents, known as "dealers," to provide trade intermediation for a large number of peripheral "buyside firms." The equilibrium number of dealers is determined by a key trade-off between two countervailing forces: *(i)* competition among dealers in their pricing of immediacy to buyside firms, and *(ii)* the benefits of concentrated intermediation in balancing dealer inventory through dealers' ability to quickly offset purchases against sales. The equilibrium balance between these two forces need not be efficient. In addition to predicting the number of dealers providing intermediation, my results point to under-provision of dealer intermediation for actively traded assets, and over-provision for infrequently traded assets.

Most OTC markets, such as those for bonds, swaps and inter-bank lending, exhibit a clear and stable core-periphery trading network.[1] Roughly the same 10 to 15 dealers, all affiliated with large banks, form the core. The vast majority of trades have one of these dealers on at least one side. For example: The largest sixteen derivatives dealers, known as the "G16,"[2] intermediate 53% of the total notional amount of interest rate swaps, 62% of credit default swaps, and 40% of foreign exchange forwards.[3] Figure 1 illustrates the trading networks of some OTC markets, all of which exhibit a clear core-periphery pattern.

The model works roughly as follows. A finite number of ex-ante identical agents form bilateral trading relationships in a continuous-time game. It is costly for agents to hold asset inventory beyond their immediate needs. Dealers arise endogenously to serve requests for quote from peripheral agents, exploiting their central position to balance inventory by quickly

[1] Li and Schürhoff (2019) provides evidence on municipal bonds, Di Maggio, Kermani, and Song (2017) on corporate bonds, Bech and Atalay (2010), Afonso, Kovner, and Schoar (2014) on federal funds, Craig and von Peter (2014), in't Veld and van Lelyveld (2014) on foreign interbank lending, Peltonen, Scheicher, and Vuillemey (2014) on credit default swaps, Hollifield, Neklyudov, and Spatt (2017) on asset-backed securities and King, Osler, and Rime (2012) on currencies.

[2] The G16 dealers are BoA, Barclays, BNP Paribas, Citi, Crédit Agricole, Credit Suisse, Deutsche Bank, Goldman Sachs, HSBC, JPMorgan, Morgan Stanley, Nomura, RBS, Société Générale, UBS and Wells Fargo.

[3] These statistics are computed by Abad, Aldasoro, Aymanns, D'errico, Fache, Hoffmann, Langfield, Neychev, and Roukny (2016) using EMIR data as of November 2015.

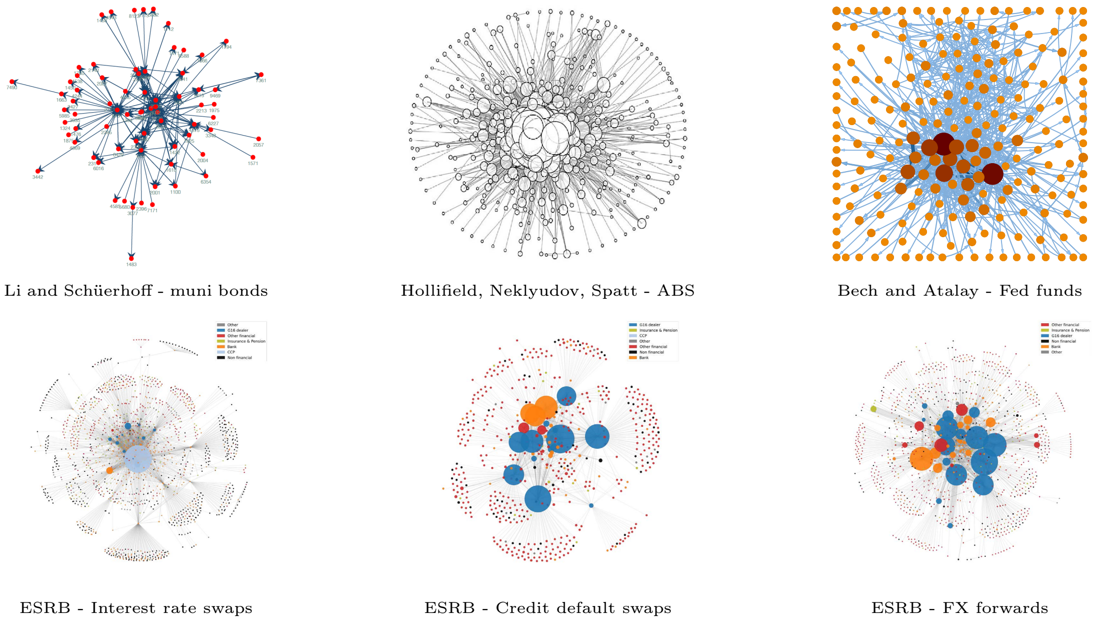


Figure 1: Core-periphery trading networks in OTC markets

netting many purchases against many sales. Dealers compete in their pricing of immediacy to retain peripheral buyside customers. As more dealers compete for trades, each dealer must offer a narrower bid-ask spread, while requiring a higher intermediation compensation given its reduced ability to balance inventory. The equilibrium number of dealers is such that the equilibrium spread, disciplined by trade competition, is just enough to cover the minimum spread sustainable by dealer inventory balancing capacity. Figure 2 depicts an example equilibrium network with 23 agents, 3 of whom emerge as dealers.

Each dealer induces a negative externality on other dealers' inventory efficiency by taking away some of their customer order flow. This externality pushes toward over-provision of dealer intermediation, and is especially pronounced for assets with infrequent trade demands which limits the scope for netting. For actively traded assets, however, this externality is inconsequential relative to the distortion caused by dealers' market power over their customers. The bilateral nature of OTC trading gives dealers a temporary monopolistic position in each contact with buyside firms, causing a "holdup" distortion by which dealer rent extraction deters buyside firms from forming some beneficial trading relationships. For actively traded assets, the holdup distortion dominates the inventory-efficiency externality, leading overall to under-provision of dealer intermediation.

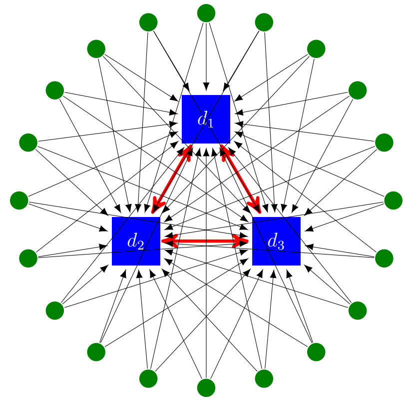


Figure 2: An example of a core-periphery network with 3 dealers and 20 buyside firms

Many studies[4] have argued that recent illiquidity in bond markets has been worsened by crisis-induced regulations (such as the Volcker Rule) and higher bank capital requirements, which have increased dealers' cost of accessing their balance sheets. My results suggest that, aside from financial stability benefits (which I do not model), weighting capital requirements by asset liquidity can foster more efficient provision of dealer intermediation.

Partly in response to post-crisis regulation, the basic core-periphery network of some OTC markets includes additional structure in the form of trading platforms on which multiple dealers provide quotes. Multilateral trading platforms have appeared in OTC markets for foreign exchange, treasuries, some corporate bonds, and (especially through the force of recent regulation) standardized swaps. Examples of such platforms include MarketAxess and Neptune for bonds, 360T and Hotspot for currencies, and Bloomberg for swaps. This paper restricts its focus, however, to the more "classical" case of purely bilateral OTC trade.

There is a rising interest in providing theoretical foundations for the endogenous core-periphery structure of OTC markets. In many prior work on this topic, the agents who form the core have some ex-ante special advantages in serving this role. For example, Hugonnier, Lester, and Weill (2018) derive the "coreness" of investors from their valuations of the asset. Those with average valuations act as intermediaries between high and low-value investors; The models of Neklyudov (2019) and Üslü (2019) are based on exogenous heterogeneity in investors' search technologies. On the other hand, Farboodi, Jarosch, Menzio, and Wiriadinata (2019), Farboodi, Jarosch, and Shimer (2020) allow agents to actively acquire special

[4] Adrian, Fleming, Goldberg, Lewis, Natalucci, and Wu (2013) provide a recent discussion. Prior studies on the relationship between dealer inventories and market liquidity include Ho and Stoll (1983), Grossman and Miller (1988), Comerton-Forde, Hendershott, Jones, Moulton, and Seasholes (2010), Weill (2007).

advantages in serving as intermediaries, such as superior search technology and negotiation skill. In these models, inevitably, ex-ante identical agents also have identical equilibrium payoff, despite some end up in the core and some on the periphery. Moreover, models based on random matching lead to a continuum of "core" agents, thus missing realistic predictions related to the number of dealers intermediating a given market, and more importantly, ruling out strategic behavior arising from repeated interactions in a dealer-client trading relationship. Chang and Zhang (2019) offers a different mechanism using directed search, showing a multitiered trading pattern in which each agent trades away other agents' misallocation until she is no longer misallocated and is publicly revealed to be so. Also using directed search, Sambalaibat (2019) shows that agents cluster according to their different frequencies of preference shocks, resulting in differentially active clusters. Both papers establish positive assortative matching based on agents' need to trade, Chang and Zhang (2019) based on whether agents need to trade, Sambalaibat (2019) on how often they need to trade. In Farboodi (2017), the endogenous network structure is generated by counterparty default risk management, and not (as in my model) by trade competition and inventory risk management.

My results contribute to this literature in three ways: First, I provide a non-cooperative game-theoretic foundation for the formation of core-periphery networks in OTC markets that is motivated by inventory balancing and trade competition. Even when agents are all ex-ante identical, an ex-post separation of core from peripheral agents is determined solely by endogenous forces that tend to concentrate the provision of intermediation. The endogenous set of dealers has significantly a higher equilibrium payoff than peripheral buyside firms. Second, I explicitly calculate the equilibrium number of dealers as a function of market characteristics. Finally, my model characterizes the endogenous relationships among welfare, trade concentration and asset trade frequency, pointing to under-provision of intermediation for actively traded assets, and over-provision for infrequently traded assets.

The paper is organized as follows. Section 2 presents the setup of the symmetric-agent model. Section 3 shows that a core-periphery network emerges in equilibrium, solving for the endogenous number of dealers as a function of market characteristics. Section 4 provides comparative statics and welfare analysis, and discusses policy implications. Section 5 extends the symmetric-agent model by introducing an inter-dealer market. Section 6 concludes.

2 The Benchmark Model

Asset and preferences. I fix a probability space and the time domain $[0, \infty)$. A non-divisible asset generates independent payoffs whose expected value is normalized to be 0 without loss of generality. A finite number n of ex-ante identical risk-neutral agents start with 0 initial endowment of the asset. Every agent incurs a quadratic cost βx^2 per unit of time when holding an asset inventory[5] of size x. That is, the agent experiences an instantaneous disutility when her inventory position deviates from a bliss point, which is normalized to 0. All agents are infinitely-lived with time preferences determined by a constant discount rate r, and can borrow and lend in a frictionless money market at the risk-free rate r.

Network formation, search and trade protocols. Each agent i is shocked by exogenously determined needs to buy or sell (equally likely) one unit of the asset at Poisson arrival times with a mean rate λ, independently of asset payoffs and across agents. If agent i can immediately execute such a trade, then she immediately consumes the traded asset and captures an immediacy benefit π. If she cannot, then she would need to wait until the next trade opportunity arrives. These demand shocks create motives for trading and can be viewed as outside customer orders, arbitrage opportunities or private hedging needs.

At any time $t \geq 0$, agent i can open a trading account with any other agent j, allowing i to request quotes from j. Setting up a trading account is costless, but maintaining an account incurs an ongoing cost of c per unit time to agent i, which can be viewed as a monitoring or operational cost. Agent i is permitted to terminate any of her accounts at any time to save the associated maintenance costs. On the equilibrium path, these trading accounts, once set up, will be maintained forever. The option to close an account, however, plays an important role in supporting competition in equilibrium as a credible threat that discourages quote providers such as agent j from extracting excessive trading rents.

At any time $t > 0$, agent i may search, at no additional cost, among the quote providers with whom she currently has an active account. A search among her m current quote

[5] Broker-dealers and asset-management firms have extra costs for holding inventory of illiquid risky assets. These costs may be related to regulatory capital requirements, collateral requirements, financing costs, and the expected cost of being forced to raise liquidity by quickly disposing of inventory in an illiquid market. The quadratic-holding-cost assumption is common in both static and dynamic trading models, including those of Vives (2011), Rostek and Weretka (2012).

providers is successful with probability θ_m, upon which one of the m quote providers offers a quote with equal probability $1/m$. These search outcomes are independent of asset payoffs, demand shocks, and across searches. The probability $\theta_m \in (0,1)$ of a successful search is increasing and concave in $m \geq 0$, and $\theta_0 = 0$.[6] A feature of the model is that for each trade request, only one quote provider is available. This gives that quote provider a temporary monopolistic position in a given trade. Competition is indirect, however, in that a quote provider j would lose i to other quote providers as a future customer should j "gouges" i.

At the point of a successful search contact with some quote provider j, agent i submits a request for quote (RFQ) indicating a desired trade direction (buy or sell). Agent j then posts an executable quote (ask or bid), a binding take-it-or-leave-it offer to trade one unit of the asset in the desired direction. The quote is observed and executable only by agent i, and is good only when offered. The restriction to a trade size of one unit is not realistic for inter-dealer trading, and will be lifted in Section 5.

From this point, if agent j has been selected as a quote provider by one or more other agents, then I will refer to agent j as a "dealer." Agents that are not dealers are called "buyside firms." For clarity, I assume that dealers simply ignore their individual exogenous trade opportunities. This eases the exposition of an equilibrium. Anticipating the equilibrium, it will be the case, with reasonable parameters, that the expected payoff that could be generated by individual trade opportunities is only a small fraction of the aggregate payoff from serving RFQs from buyside customers. This assumption reflects dealers' specialized role in providing intermediation to buyside firms. In practice, a holding company usually imposes a "China Wall" between its market making arm and its asset management arm, even before the Volcker Rule (which bans dealers from proprietary trading) was implemented.

Figure 3 illustrates the sequence of events that *could* happen at a given time $t > 0$.

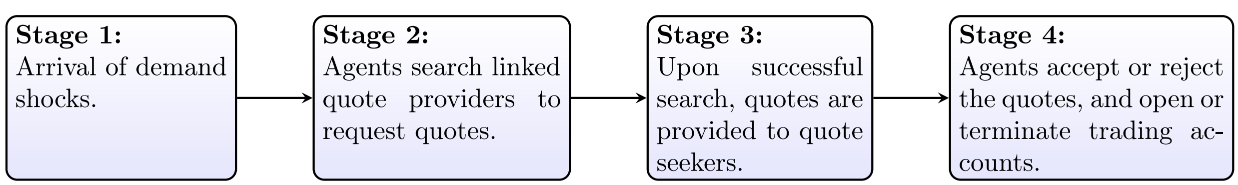


Figure 3: Sequence of events that could occur at a given time $t > 0$

[6] Appendix A microfounds this search technology, and discusses its connection to Diamond Paradox.

Information structure and solution concept. In a nutshell, each agent only observes events—demand shocks, trading activities and trading relationships—pertaining to herself. In particular, each agent does not observe the inventories or trade relationships of other agents. Technically, if each agent were to observe other agents' inventories, then that would set up a complicated problem because all the inventories would become state variables. Making inventories private information simplifies, but does not entirely bypass this issue. This is because at some point, the inventory of a given dealer j would be so large in magnitude that j would want to reject more inventory-expanding trade requests by quoting a price that is prohibitively expensive. Then its buyside customer i could infer the dealer's extreme inventory position from that price. This inference complicates the strategy of agent i in an intractable way. (Agent i would learn that keeping an account with j is relatively more likely to be a waste of time.) To avoid this informational complexity, I allow a quote provider the option, whenever a quote is accepted, of not taking the trade on her own account, instead allowing the transaction with agent i to be diverted, without cost or benefit, to a neutral third-party account called a "deep pocket." Agent i does not learn whether or not agent j invokes this deep pocket. In equilibrium, it turns out that the deep pocket is invoked only when the inventory of agent j hits an exogenous boundary, to prevent the extreme inventory position from being inferred through pricing. Based on this, I will show in Section 5 that this inference problem vanishes in a large market. This will allow me to eliminate the deep-pocket assumption (Proposition 11). I will also show that the deep-pocket account, when assumed, is technically feasible, in that all trades diverted to it generate positive equilibrium profit.

To formally define agents' strategies, I let $\mathcal{F}_{it^-}$ represent the information available to a given agent i up to but excluding time t, consisting of the agent's past demand shocks (arrival times and directions), her past trading activities (arrival times and directions of RFQ, quotes, the identities of quote providers or quote seekers, transaction prices and inventories) and her past trading relationships (incoming or outgoing links). I let $\mathcal{F}_{it}^k$ represent the information available to agent i right after Stage k at time t, where Stage k is described in Figure 3 ($k = 1, 2$ and 3). That is, $\mathcal{F}_{it}^k$ represents the information of $\mathcal{F}_{it^-}$ combined with the outcomes of stages $1, \ldots, k$. A strategy for agent i consists of

(i) A search strategy s_i specifying a search decision $s_{it} \in \{\text{Search, Do Not Search}\}$ for every

time $t > 0$. When making this decision, agent i possesses her prior information $\mathcal{F}_{it}^1$. Thus, s_{it} must be measurable with respect to $\mathcal{F}_{it}^1$. In addition, it is assumed that agent i can search only a finite number of times during any bounded time interval to be consistent with real-life search behavior.[7]

(ii) A pricing strategy p_i specifying, for every time $t > 0$, a price p_{it}, measurable with respect to $\mathcal{F}_{jt}^2$, that i would quote upon receiving a request for quote.

(iii) A quote acceptance strategy ρ_i specifying, for every time $t > 0$, a trade decision $\rho_{it} \in \{\text{Accept, Reject}\}$ that is measurable with respect to $\mathcal{F}_{it}^3$.

(iv) An account maintenance strategy N_i specifying, for every time $t > 0$, a set N_{it}, measurable with respect to $\mathcal{F}_{it}^3$, of agents with whom i maintains an account. The process $(N_{it})_{t\geq 0}$ is taken to be RCLL.[8]

An agent's continuation utility at time t right after Stage k is

$$U_{it}^k = \mathrm{E}\left(\int_t^\infty e^{-r(s-t)} \left(-\beta x_{is}^2 - |N_{is}|c\right) \, ds + \sum_{\tau_\ell \geq t} e^{-r(\tau_{i\ell}-t)} \, \kappa_{i\ell} \;\middle|\; \mathcal{F}_{it}^k \right), \tag{1}$$

where x_{is} is the agent's inventory size at time s, $|N_{is}|$ is her number active accounts, $\kappa_{i\ell}$ is her ℓ'th lump-sum benefit, which could be either an immediacy benefit π for fulfilling a need to trade or a price transferred at a trade, and time $\tau_{i\ell}$ is when she receives the benefit $\kappa_{i\ell}$.

In a perfect Bayesian equilibrium (PBE), each agent maximizes her continuation utility U_{it}^k after each Stage k at each time t, given the strategies of other agents. Appendix C provides a basic definition of PBE for continuous-time games.[9] I focus on PBE with Markovian and stationary strategies, where agents' strategies depend only on payoff-relevant histories in a time-homogenous manner. Specifically, the payoff-relevant history of agent i at time t consists of her inventory x_{it-} and her trading links (incoming and outgoing) connected right

[7]This also ensures that an agent's inventory process $(x_{it})_{t\geq 0}$ is right continuous with left limits (RCLL).

[8]This ensures that the process N_i is progressively measurable (Karatzas and Shreve (1998) 1.13 Proposition p5), and that the set N_{it-} of quote providers connected right before time t is well defined.

[9]One issue with defining PBE in a continuous-time game is that *all* information sets, even those on the equilibrium path, are reached with probability 0 due to the continuous-time nature and the randomness of event times. Appendix C deals with this issue by providing a basic definition of agents' beliefs at information sets on the equilibrium path using regular conditional probabilities.

before time t, the directions of a potential demand shock or an RFQ at time t, the identity of the quote seeker, or a potential quote received at time t and the identity of the quote provider. Letting Y_{it} be the Markov state variable encoding the payoff-relevant history of agent i at time t, a PBE is a stationary equilibrium if the strategies of every agent i can be written as $[f_i(Y_{it})]_{t>0}$ for some measurable function f_i not depending on t.

Agent i having a trading account with j is represented by a directed link $i \to j$. A network G is an *equilibrium trading network* if some stationary equilibrium σ induces G at all time $t \geq 0$ on any equilibrium path. Then σ is said to be a *supporting equilibrium* for G.

3 Core-Periphery Network

It is impossible to solve for all equilibria of the model. The focus of this paper, however, is the structure of equilibrium networks. Given a set of model parameters $(n, \beta, \lambda, \pi, c, \theta, r)$, I show that a large class of equilibrium networks all exhibit the "core-periphery" trading pattern. I also provide selection criteria that select a unique equilibrium core-periphery network.

A family of concentrated core-periphery networks

There exists a family of equilibrium core-periphery networks of the form depicted in Figure 4. In each such network, agents endogenously partition themselves into two types, $I \cup J = N$, with $|J| = m$ "dealers" only providing quotes and $|I| = n - m$ "buyside firms" only requesting quotes. This network, denoted by $G(m)$, is called a *concentrated core-periphery network*, in that each buyside firm has trading accounts with all the m dealers. The dealers do not trade with each other in equilibrium due to the lack of an inter-dealer market. A connected core will emerge in Section 5, which introduces an interdealer trade protocol that does not restrict a trade size to one unit.

Theorem 1. *The concentrated core-periphery network $G(m)$ with m dealers is supported by a buyside-symmetric equilibrium*[10] *if and only if $m \leq m^*$ for some maximum core size m^*.*

Proofs of results in this section are in Appendices B and D. I now show how the maximum number m^* of dealers, equilibrium pricing and dealer inventory size are jointly determined as functions of model parameters, stating when buyside symmetry is required.

[10] In a buyside-symmetric supporting equilibrium, buyside firms follow the same strategy.

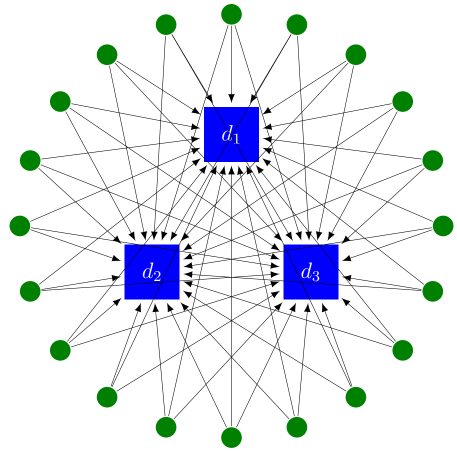


Figure 4

There are two sides of the market—the buyside and the dealer side. I first consider the buyside problem, which will uniquely pin down equilibrium pricing without requiring symmetry across buyside firms. On the equilibrium path of a stationary equilibrium, a buyside firm i cannot be strictly worse off by terminating any one of its m dealer accounts. Otherwise, every dealer would extract all trading rents from i knowing that i never has an incentive to terminate any account. On the other hand, i cannot be strictly better off by closing accounts, because by the definition of an equilibrium, m is the optimal number of dealer accounts for i. Therefore, a buyside firm must be indifferent between maintaining m or $m-1$ dealer accounts. This indifference condition implies that all the m dealers must post the same constant spread in equilibrium, because otherwise a dealer posting a wide spread would be too "expensive" to be maintained relative to a dealer posting a narrow spread. This is shown more formally in Appendix D. Then a buyside firm's instantaneous rate of benefit when it maintains accounts with all the m dealers is

$$\Phi_{m,p} = \lambda\theta_m(\pi - p) - mc, \tag{2}$$

where p is the mid-to-bid spread charged by the dealers, thus the expected profit given to a dealer for completing a trade. The mean rate of benefit $\Phi_{m,p}$ is the product of the arrival rate λ of exogenous trade opportunities, the probability θ_m of successfully executing a trade, and the expected profit $\pi-p$ of a successful trade, net of the flow cost mc of account maintenance.

If buyside firm i discontinues its account with a given dealer j at some time t, only

i and j would observe this off-the-equilibrium-path deviation. Therefore, the remaining dealers would continue to post the mid-to-bid spread p to i who now only has $m-1$ dealer accounts left. The instantaneous rate of benefit to i thus becomes $\Phi_{m-1,p}$ in this continuation game. The buyside indifference condition $\Phi(m,p) = \Phi(m-1,p)$ uniquely determines the equilibrium mid-to-bid spread $p^*(m)$:

$$\Phi(m,p) = \Phi(m-1,p) \quad \Longleftrightarrow \quad p = p^*(m) = \pi - \frac{c}{\lambda(\theta_m - \theta_{m-1})}. \tag{3}$$

Figure 5 illustrates $\Phi_{j,p}$ as a function of d for $p = p^*(m)$ and $p = p^*(m+1)$.

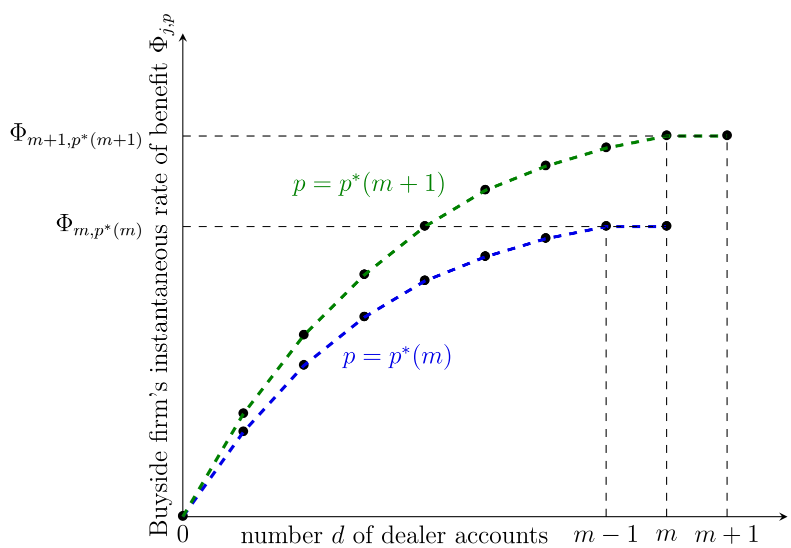


Figure 5: Indifference condition for buyside firms—If the mid-to-bid spread is $p = p^*(m)$, then a buyside firm reaches its highest instantaneous rate of benefit $\Phi_{j,p}$ with $j = m-1$ or m dealer accounts, while being indifferent between these two choices. If $p = p^*(m+1)$, the optimal rate of benefit improves, in that $\Phi_{m+1,p^*(m+1)} > \Phi_{m,p^*(m)}$.

The indifference condition (3), reflected as the flat plateau in Figure 5 between having $j = m-1$ and m dealer accounts, gives every buyside firm the ability to costlessly terminate any given dealer account. No account termination occurs on the equilibrium path. However, the ability of buyside firms to discontinue a trading relationship constitutes a credible threat to all the m dealers should they extract excessive rents. This off-the-equilibrium-path threat disciplines the dealers' quotes, and is the key source of competition among dealers. The next proposition formally summarizes the results from analyzing the buyside problem and further shows how dealer competition compresses the equilibrium spread $p^*(m)$.

Proposition 1. *On the equilibrium path of any supporting equilibrium for $G(m)$, each dealer j offers each buyside firm i some constant ask a^*_{ji} and bid b^*_{ji}, with a spread $a^*_{ji} - b^*_{ji} = 2p^*(m)$. The equilibrium mid-to-bid spread $p^*(m)$ is strictly decreasing in the number m of dealers.*

I now let buyside firms use "grim trigger" to most effectively implements the threat of account termination. That is, a buyside firm discontinues its trading account with a given dealer when *(a)* the buyside firm has m dealer accounts in total, and *(b)* the dealer gouges the buyside firm by offering an ask $a > p^*(m)$ or a bid $b < -p^*(m)$. When these two conditions are both met, the buyside firm, after accepting the current quote if the trade is still profitable (that is, if $a \leq \pi$ or $b \geq -\pi$), immediately closes its account with this dealer. Since grim trigger implements the most strict punishment, it gives dealers the strongest disincentive to gouge buyside customers. Thus, it supports the widest range of equilibrium networks.[11]

As the number of dealers increases, dealers compete for trades more intensely by offering tighter bid-ask quotes (Proposition 1). Hence, profit on each trade declines. On the other hand, each dealer receives a thinner order flow from buyside firms, thus becomes less efficient in balancing its inventory. The benefits of acting as a dealer thus decrease. At some point, each dealer starts to feel the pressure of its incentive to gouge buyside firms thus has difficulty convincing buyside firms to open accounts with it at the first place. This limits the equilibrium scope for dealer competition. Next, I demonstrate this intuition by analyzing the dealer's problem and calculate the maximum number m^* of dealers.

Each dealer j solves the following dynamic programing problem to determine its optimal pricing strategy: Each buyside firms submits RFQ to j at some Poisson rate η, requesting to buy or sell (equally likely) one unit of the asset. Upon receiving a request to buy, if j quotes an ask price a such that $a \leq p$ for some cutoff price p, then the buyside firm accepts the quote and continues to come back to j at the same Poisson rate η. If $p < a \leq \pi$, then the buyside firm accepts the quote but never comes back to j again. If $p > \pi$, then the buyside firm rejects the quote and never comes back to j again. Whenever a quote is accepted, j can divert the transaction to its deep pocket. The case of a request to sell is symmetric. The dealer optimizes its pricing strategy to maximize its continuation utility U_{jt} given by (1).

[11] In a grim trigger, the respecitve cutoffs for the bid and ask need not be symmetric around 0, whereas the spread between the two cutoffs must be $2p^*(m)$ (Proposition 1). However, such an asymmetric grim trigger cannot expand the range of equilibrium networks supported by a symmetric grim trigger (Appendix D).

I let $V_k(x)$ be the continuation value of j when it has some number k of buyside customers and some asset inventory of size $x \in \mathbb{Z}$. When $k = 0$, j has no buyside customers, thus $V_0(x) = \int_0^\infty e^{-rs}(-\beta x^2)ds = -\beta x^2/r$. The optimal pricing strategy is characterized by the following Hamilton-Bellman-Jacob (HJB) equations: for $k = 1, 2, \ldots$ and every $x \in \mathbb{Z}$:

$$
\begin{aligned}
rV_k(x) = -\beta x^2 + k\,\frac{\eta}{2}\,\max\{V_k(x-1) - V_k(x) + p,\ 0,\ V_{k-1}(x-1) - V_k(x) + \pi\} \\
+ k\,\frac{\eta}{2}\,\max\{V_k(x+1) - V_k(x) + p,\ 0,\ V_{k-1}(x+1) - V_k(x) + \pi\}.
\end{aligned}
\tag{4}
$$

The first term $-\beta x^2$ is the inventory flow cost for holding x units of the asset. The second and third terms are the dealer's expected rates of profit associated with serving requests to buy and sell, respectively. Upon receiving a request to buy, j has three options: on the equilibrium path, j sets its ask price $a = p$ just low enough to retain its buyside customer for future trades, earning a trade profit $V_k(x-1) + p - V_k(x)$; in addition, if j invokes the deep pocket, then j earns a trade profit of 0; off the equilibrium path, however, j could "gouge" the buyside customer by posting the maximum ask price $a = \pi$ that would be accepted, at the expense of losing the buyside customer for future trades. Gouging earns j a trade profit of $V_{k-1}(x-1) + \pi - V_k(x)$. Dealer j maximizes its net trade profit, taking into account its continuation value after the trade. The case of a request to sell is symmetric.

On the equilibrium path, j has no incentive to gouge any buyside firm by raising its ask or lowering its bid. I let $\underline{V}$ be the value function of j if j was restricted from gouging buyside firms. Then $\underline{V}$ satisfies the HJB equations

$$
\begin{aligned}
r\underline{V}_k(x) = -\beta x^2 + \frac{k\eta}{2}\,\max\{\underline{V}_k(x-1) - \underline{V}_k(x) + p,\ 0\} \\
+ \frac{k\eta}{2}\,\max\{\underline{V}_k(x+1) - \underline{V}_k(x) + p,\ 0\}.
\end{aligned}
\tag{5}
$$

The HJB equation (5) for $\underline{V}$ is obtained from the HJB equation (4) for V by eliminating the option of gouging. On the equilibrium path, the HJB equation for $\underline{V}$ determines a unique optimal pricing strategy for j, characterized by some inventory boundary $\bar{x}$:

Proposition 2. *On the equilibrium path, each dealer has a unique optimal pricing strategy* $[a^*(\cdot), b^*(\cdot)]$, *determined by the HJB equation* (5) *and characterized by an inventory boundary*

$\bar{x}_{k\eta} \in \mathbb{Z}$, *such that:*

- *If the dealer's inventory x is within $-\bar{x}_{k\eta} < x < \bar{x}_{k\eta}$, then $a^*(x) = p$ and $b^*(x) = -p$.*
- *If $x \leq -\bar{x}_{k\eta}$, then $a^*(x) = p_{DP}$ and $b^*(x) = -p$.*
- *If $x \geq \bar{x}_{k\eta}$, then $a^*(x) = p$ and $b^*(x) = -p_{DP}$.*

Here, the subscript "DP" indicates the use of the deep pocket. If the dealer's inventory is within the range $(-\bar{x}_{k\eta}, \bar{x}_{k\eta})$, the spread p gives it enough profit incentive to warehouse additional inventory. Whenever the dealer's inventory drops below $-\bar{x}_{k\eta}$, it is not willing to sell more assets on its own account at the price p, which no longer covers its indirect marginal inventory cost. Using the deep pocket allows the dealer to divert a sell trade. Similarly, when the dealer's inventory exceeds $\bar{x}_{k\eta}$, it uses the deep pocket to divert a buy trade. In equilibrium, the dealer optimally controls its inventory within the interval $[-\bar{x}_{k\eta}, \bar{x}_{k\eta}]$, and uses the deep pocket only when its inventory hits the boundaries $\pm\bar{x}_{k\eta}$.

Now, I examine the dealer's incentive to "gouge" buyside customers. Upon gouging, the dealer earns a one-shot benefit of $\pi - p$ for the current trade at the expense of losing a buyside customer and the associated future profit stream. The future profits forgone from losing a buyside customer lowers the dealer's continuation value by

$$L_{k,\eta,p}(x) = \underline{V}_k(x) - V_{k-1}(x)$$

The One-Shot Deviation Principle implies that the dealer has no incentive to gouge if and only if the one-shot benefit of gouging does not exceed its expected cost, in that

$$\pi - p \leq \mathcal{L}(k, \eta, p) = \min_{x \in \mathbb{Z}} \; L(k, \eta, p)(x), \tag{6}$$

Lemma 1. *The expected cost $\mathcal{L}(k, \eta, p)$ of losing a buyside customer is continuous in p and, when $\bar{x}_p \geq 1$, strictly increasing in p. Thus, the no-gouging condition (6) is satisfied if and only if $p \geq \underline{p}(k, \eta)$, where $\underline{p}(k, \eta)$ is uniquely determined by*

$$\pi - \underline{p}(k, \eta) = \mathcal{L}\left(k, \eta, \underline{p}(k, \eta)\right).$$

Intuitively, when the dealer is receiving a higher spread p from buyside customers, losing a customer is more costly. Lemma 1 implies that $\underline{p}(k, \eta)$ is the tightest mid-to-bid spread that the dealer can sustainably offer without having incentive to gouge. On the equilibrium path, a given dealer receives requests to trade from $k = n - m$ buyside customers at the rate $\eta_m = \lambda\theta_m/m$ each. Then $\underline{p}(m) = \underline{p}(n - m, \lambda\theta_m/m)$ is the dealer's tightest sustainable spread. Next is the key result that illustrates the effect of inventory balancing.

Proposition 3. *The tightest sustainable spread $\underline{p}(k, \eta)$ is strictly decreasing in a dealer's number k of buyside customers and the Poisson arrival rate η of RFQ from each customer. Hence, the tightest sustainable spread $\underline{p}(m)$ is strictly increasing in the number m of dealers.*

When a dealer has more buyside customers or receives more frequent RFQ from each buyside customer, it can offer a tighter spread thanks to its ability to more efficiently balance inventory by more quickly netting purchases against sales. A well connected dealer is in this sense a liquidity hub. When there are more dealers in the market, however, each dealer receives a thinner order flow from each buyside firm and becomes less efficient in balancing its inventory. Thus every dealer has less incentive to sustain a tight spread. Figure 6 provides a graphical illustration of Lemma 1 and Proposition 3.

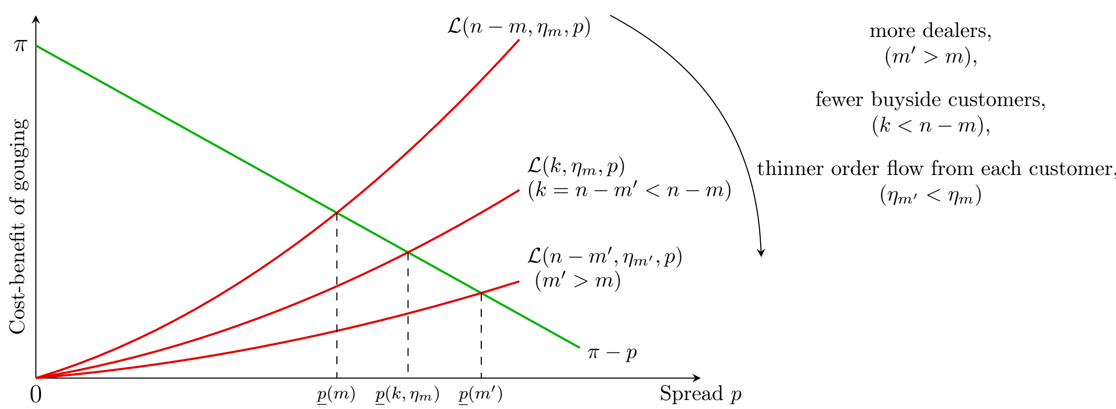


Figure 6: The tradeoff between the one-shot benefit $\pi - p$ and the expected cost $\mathcal{L}(k, \eta, p)$ of gouging. The cost $\mathcal{L}(k, \eta, p)$ associated with forgone future profits is increasing in k, η and p. Hence, the dealer-sustainable spread $\underline{p}(m)$ is increasing in m.

The equilibrium spread $p^*(m)$, given in (3), must be dealer-sustainable in that $p^*(m) \geq \underline{p}(m)$. Since the equilibrium spread $p^*(m)$ is strictly decreasing in the number m of dealers thanks to better competition, while the tightest sustainable spread $\underline{p}(m)$ is strictly increasing in m due to worse inventory efficiency, then $p^*(m) \geq \underline{p}(m)$ is equivalent to $m \leq m^*$, where m^* is the largest integer such that

$$p^*(m^*) \geq \underline{p}(m^*). \tag{7}$$

The number m^* is the *maximally sustainable core size*. Figure 7 plots both spread curves.

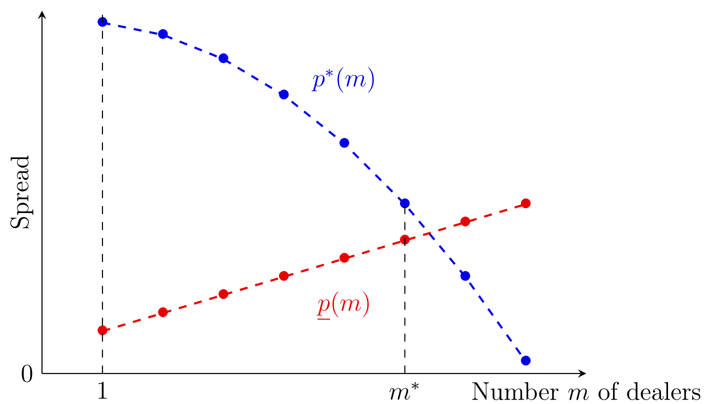


Figure 7: The equilibrium spread $p^*(m)$, dealer's sustainable spread $\underline{p}(m)$ and the maximum core size m^*

In a supporting equilibrium of $G(m)$ ($m = 1, \ldots, m^*$), dealers are deterred from gouging by the fear of losing buyside customers. Buyside firms do not terminate any account on the equilibrium path, but their ability to do so constitutes a credible threat that discourages dealers from gouging, sustaining the equilibrium. Appendix D completes the construction of the supporting equilibrium by filling in off-the-equilibrium-path actions and a belief system.

Proposition 1 implies that on the equilibrium path of any supporting equilibrium, each dealer j must offer each buyside customer i a constant mid quote $\text{mid}^*_{ji} = (a^*_{ji} + b^*_{ji})/2$. However, dealers need not offer a mid quote of 0 as they do in the candidate equilibrium above. If buyside firms impose, on dealer j, a grim trigger with bid and ask cutoffs that are not symmetric around 0, j would respond by offering them a non-zero mid quote and optimally shifts its inventory boundaries from $[-\bar{x}, \bar{x}]$ to some $[\underline{x}', \bar{x}']$ accordingly. Appendix D shows that by offering a non-zero mid-quote, j has a stronger incentive to gouge, making it more difficult to support an equilibrium network, as long as j offers the same dealer-specific non-zero mid^*_j to every buyside customer. In a buyside-symmetric equilibria, buyside firms employ the same account maintenance strategy, thus a dealer must offer, in response, the same mid^*_j to every buyside customer. Hence, dealers offering dealer-specific non-zero mid quotes cannot expand the range of equilibrium networks supported by dealers offering a zero mid quote. When a dealer offers different mid quotes to different buyside customers, it would have a different inventory boundary $[\underline{x}_i, \bar{x}_i]$ for each different buyside customer i. The dealer's problem then becomes excessively complicated.

Considering buyside-symmetric equilibria is tractable, but too restrictive because it rules out equilibrium networks in which buyside firms are asymmetrically positioned. Without imposing the full buyside symmetry, I next characterize alternative networks supported by a *one-dealer-one-mid* equilibrium, where each dealer j offers a dealer-specific mid_j^* to its every buyside customer on the equilibrium path, without being restricted to do so off the path.[12]

Core-periphery pattern of all other equilibrium networks

In practice, core-periphery structures in OTC markets are less "concentrated," in that a typical buyside firm is connected to some but not all dealers. Figure 8 illustrates an example of such core-periphery network. I show that other equilibrium networks all have a flavor of "core-periphery" structure, but are less efficient, in a sense to be specified, than some concentrated core-periphery equilibrium network. Roughly speaking, a network is an equilibrium network only if *(i)* the number of dealers is relatively small (Corollary 1), and *(ii)* every dealer has enough buyside customers to efficiently balance its inventory (Theorem 2).

For any given network G, agents can be partitioned into $I \cup J = N$ as follows: Agents in I, representing "buyside firms," have no incoming links; Agents in J, representing "dealers," have at least one incoming link. I consider a network G with *uniform outdegree* m, in which every buyside firm has the same number m of dealer accounts. A dealer with k buyside customers can only sustain a spread of $\underline{p}(k, \eta_m)$. Since $\underline{p}(k, \eta_m)$ is strictly decreasing in k (Proposition 3), then $\underline{p}(k, \eta_m) \leq p^*(m)$ if and only if $k \geq k^*(m)$ for some $k^*(m)$. That is,

[12] An alternative way (other than buyside symmetry) to motivate the consideration of "same-mid-quote" equilibria is by assuming that the quote seeker's identity is not revealed to the quote provider at the time of an RFQ. This trade protocol, known as "anonymous RFQ," is being implemented and gaining popularity as OTC markets move toward electronic trading. On Swap Execution Facilities in 2014, 31% of investors prefer trading via anonymous RFQ, and 52% prefer name give-up RFQ, according to McPartland (2014) based on survey responses. With anonymous RFQ, a dealer must offer the same mid-quote to its every buyside customer, and buyside firms need not follow the same equilibrium strategy. All results in this paper continue to hold assuming anonymous RFQ. However, the anonymous-RFQ assumption is still too big a hammer from a modeling perspective because it restricts a dealer's strategy set thus possible deviations. It weakens the generality of the results thus is avoided.

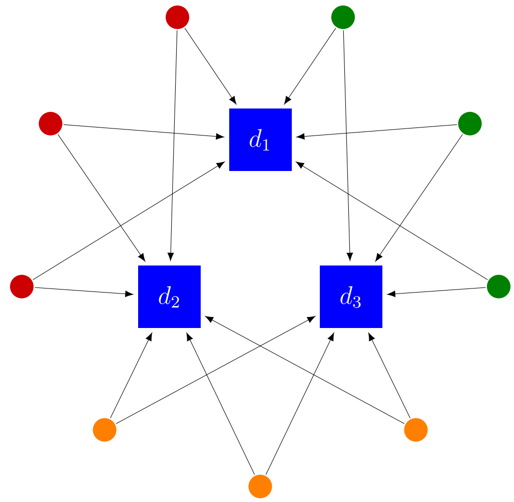


Figure 8: Example: every buyside firm is connected to 2 of the 3 dealers

the dealer needs more than $k \geq k^*(m)$ buyside customers to sustain the equilibrium spread $p^*(m)$ without having an incentive to gouge. This result is summarized by the next Theorem.

Theorem 2. *A network G with uniform outdegree m is supported by a one-dealer-one-mid equilibrium if and only if (i) $m \leq m^*$, and (ii) every dealer has more than $k^*(m)$ buyside customers. In any supporting equilibrium, every dealer offers the equilibrium mid-to-bid spread $p^*(m)$, and every buyside firm's equilibrium utility is $\Phi(m, p^*(m))/r$.*

Later, I will provide an equilibrium selection criterion, based on trade competition, that selects $m = m^*$. Focusing on equilibrium networks with $m = m^*$, I derive, from Theorem 2, an explicit upper bound on the equilibrium number of dealers.

Corollary 1. *If network G with uniform outdegree m^* is supported by a one-dealer-one-mid equilibrium, then the total number $|J|$ of dealers is bounded by*

$$|J| < \frac{m^* n}{k^*(m^*) + m^*}.$$

Intuitively, because each buyside firm can have at most m^* dealer accounts, and each dealer needs at least $k^*(m^*)$ buyside customers, then there cannot be too many dealers. As a numerical example, I consider a market with $n = 1000$ agents, $\beta = 0.1$, $\pi = 1$, $\lambda = 3, \theta_m = 1 - 0.8^m$, $c = 0.09$ and $r = 0.1$. The upper bound on the number of dealers is $|J| \leq 17$.

I made the deep-pocket assumption to avoid an inventory inference problem. Admittedly, this is not the most natural assumption. It is at least desired that a deep pocket is used in a non-biased manner. In a one-dealer-one-mid equilibrium, the expected asset position of each dealer's deep pocket is 0 at all time (that is, the deep pocket is neither a net buyer nor a net seller in expectation) if and only if the dealer offers a mid quote of 0 on the equilibrium path. In this case, for every trade diverted to it on the equilibrium path, the deep pocket receives the positive payment $p^*(m)$ from a buyside customer and does not subsidize the dealer.

However, this does not rule out any equilibrium network, because dealers offering a zero mid quote actually makes them less tempted to gouge, thus can support a wider range of equilibrium networks. Next, I propose two equilibrium selection criteria, based on inventory balancing efficiency and dealer trade competition respectively, that jointly select the concentrated core-periphery network $G(m^*)$ with m^* dealers.

Equilibrium selection

The next result, based on inventory balancing efficiency, shows that the concentrated core-periphery networks induce higher welfare than other equilibrium networks. Given a strategy profile σ, I define welfare $U(\sigma)$ as the sum of all agents' utilities.

Proposition 4. *Given a network G with uniform outdegree m that is not a concentrated core-periphery network, if G is supported by some zero-mid equilibrium σ where dealers offer mid quote of 0 to every buyside firms on the equilibrium path, then $U(\sigma') < U(\sigma^*_m)$ where σ^*_m is the supporting equilibrium for the concentrated core-periphery network $G(m)$.*

A more concentrated network allows more efficient netting of trade. Specifically, from an equilibrium network G in which every buyside firm has accounts with some but not all dealers, $G(m)$ is obtained by concentrating buyside firms' accounts toward the same, smaller set of dealers. The networks G and $G(m)$ have the same total number of trading lines, thus the same total trading volume and account maintenance cost. The same volume of trade is intermediated, however, by a smaller set of dealers in $G(m)$ relative to G. Trade concentration leads to more efficient netting of trades and thus a lower aggregate dealer inventory cost, which results in higher welfare in $G(m)$.

The second criterion, based on trade competition, uses the fact that the equilibria are Pareto-ranked for buyside firms.

Corollary 2. *Given two equilibrium networks G and G' with uniform outdegrees m and m', the equilibrium utility of buyside firms is strictly lower in G than in G' if $m < m'$.*

Corollary 2 follows from Theorem 2 and $\Phi(m, p^*(m)) < \Phi(m', p^*(m'))$ if $m < m'$. All buyside firms prefer an equilibrium where they are connected to more competing dealers, since the benefit associated with a tighter equilibrium spread and better trade execution outweighs additional account maintenance costs. Figure 5 provides an illustration of Corollary 2.

Corollary 2 leads to a natural equilibrium selection criterion: an equilibrium network G with uniform outdegree $m < m^*$ can be overturned if agents can actively coordinate the selection of dealers. For example, in a concentrated core-periphery network $G(m)$ where $m < m^*$, the $n-m$ buyside firms can jointly deviate to get rid of the m incumbent dealers and trade in their own concentrated core-periphery network $G_{n-m}(m')$ with m' ($m' > m$) dealers. Those m' buyside firms selected to become dealers in the joint deviation are strictly better off by exploiting their new network positions as dealers to earn intermediation profits. The remaining buyside firms also benefit from greater dealer competition, earning $\Phi(m', p^*(m'))/r$ instead of $\Phi(m, p^*(m))/r$ (Corollary 2). On the other hand, the concentrated core-periphery network $G(m^*)$ with m^* dealers cannot be overturned by competitive pressure in this manner.

This selection procedure closely mimics the logic of *coalition-proof Nash equilibrium* of Bernheim, Peleg, and Whinston (1987), which can be adapted to this dynamic network formation setting to obtain a formal equilibrium selection criterion.

Definition 1. (i) Given an equilibrium network G supported by some equilibrium σ, a single-player deviation is coalition-proof if it makes the player strictly better off at some information set given its belief.

(ii) For any $\ell > 1$, I assume that coalition-proofness has been defined for any joint deviation by any ℓ'-agent coalition where $\ell' < \ell$.

(a) A joint deviation (σ'_S) by an ℓ-agent coalition S is *self-enforcing* if it is stationary, if (σ'_S, σ_{-S}) induces a stable network almost surely and if the deviation's restriction $(\sigma'_{S'})$ to any proper sub-coalition $S' \subset S$ is coalition-proof with respect to (σ'_S, σ_{-S}).

(b) A joint deviation by an ℓ-agent coalition S is *coalition-proof* if it is self-enforcing, and if there is no other self-enforcing deviation by S that makes everyone in S strictly better off.

(iii) An equilibrium network is coalition-proof if it has a supporting equilibrium that cannot be blocked by any coalition-proof deviation that makes every member in the deviating coalition strictly better off.

Part (i) says that coalition-proofness imposes no additional restriction on a unilateral deviation. Part (ii) then recursively defines a coalition-proof deviation as one that cannot be blocked by any other coalition-proof deviation of any sub-coalition.

Proposition 5. *If an equilibrium network G is coalition-proof, then its maximum outdegree is m^*. The concentrated core-periphery network $G(m^*)$ with m^* dealers is coalition-proof.*

The rational of requiring deviation to be coalition-proof is that "players can freely discuss their strategies, but cannot make binding commitments...In [such] environments..., any meaningful agreement to deviate must also be self-enforcing (i.e., immune to deviations by subcoalitions)" (Bernheim, Peleg, and Whinston (1987)). This underlying assumption fits well with the non-cooperative game here.

Here, the most interesting application of coalition-proofness is that it rules out collusive price rigging by any dealer coalition. I suppose that a coalition of dealers were to collude to offer a wider mid-to-bid spread $p > p^*(m)$. Since buyside firms continue to use the grim trigger, and losing a buyside customer is costly for any given dealer, each dealer in the coalition would have an incentive to unilaterally switch away from the agreed-upon collusive spread p back to the equilibrium spread $p^*(m)$ in order to retain its own customers. That is, the dealers cannot credibly engage in such a collusion. In other words, the supporting equilibrium for $G(m)$ constructed here is robust against dealer collusion (though not against the joint deviation by buyside firms to push for more dealers described above unless $m = m^*$).

Jointly, the two selection criteria in Propositions 4 and 5, based on inventory balancing and trade competition respectively, select the concentrated core-periphery network $G(m^*)$.

4 Core Size, Welfare and Policy Implications

To develop comparative statics on the equilibrium number of dealers, I focus attention on the concentrated core-periphery network $G(m^*)$ that is selected by Proposition 4 and Corollary 2.

The equilibrium core size m^* and the equilibrium spread $p^*(m^*)$

The next result shows how the core size varies as a function of the model parameters $(n, \beta, \lambda, \pi, c, \theta, r)$, fixing all but one parameter. Proofs are available in Appendix F of the Full Appendix on my website (Wang (2021)).

Proposition 6. *(i) The core size m^* is weakly increasing in the total number n of agents, with a finite limit size m^*_∞. The limit size m^*_∞ is the largest integer m such that*

$$\frac{mr\pi}{\lambda\theta_m + mr} < p^*(m).$$

(ii) The core size m^* is increasing in the arrival rate λ of demand shocks and the total gain per trade π, and decreasing in the account maintenance cost c.

(iii) The equilibrium spread $p^*(m^*)$ is weakly decreasing in the total number n of agents.

Part *(i)* of Proposition 6 has a simple intuitive proof, as follows. As the total number n of agents increases, each dealer becomes more efficient in balancing inventory, thus can sustain a tighter spread $\underline{p}(n-m, \eta_m)$ (Proposition 3). The equilibrium spread $p^*(m)$, however, does not depend on n. The core size m^* is thus increasing in n, as shown by Figure 9.

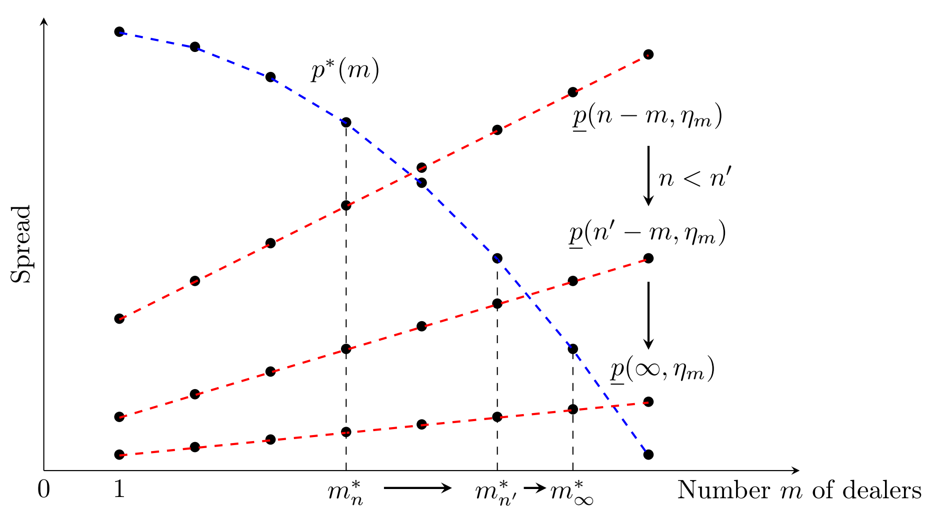


Figure 9: The core size m^* is increasing in the total number n of agents.

Part *(i)* also implies that even for an "infinite" set of investors, one should anticipate only a finite number m^*_∞ of dealers. This limiting core size m^*_∞ does not depend on the inventory cost coefficient β, because dealers are very efficient in balancing their inventories with an infinite number of buyside firms. To provide a numerical example of the number m^*_∞ of dealers in a large market, I let $\pi = 1,\ \lambda = 3,\ \theta_m = 1 - 0.8^m,\ c = 0.1$, and $r = 0.1$. Then $m^*_\infty = 7$, and the equilibrium spread in the large market is $p^*(m^*_\infty) \simeq 0.36$.

As π increases, dealers extract a higher rent per trade (reflected by a wider equilibrium spread $p^*(m)$), but also have a stronger incentive to gouge (wider sustainable spread $\underline{p}(m)$). It is shown, in Appendix F, that the equilibrium spread $p^*(m)$ increases more than the sustainable spread $\underline{p}(m)$. The equilibrium core size m^* thus increases with π.

The parameter λ measures each buyside firm's frequency of liquidity trade demand. With more frequent buyside demand, it is natural that more dealers emerge to facilitate the intermediation of the asset, leading to a lower market concentration. This prediction of a negative relationship between trade frequency and market concentration is consistent with empirical evidences from OTC markets. Using data on the German Bund market, de Roure and Wang (2016) shows that higher trade frequency leads to lower Herfindahl index of trade concentration. In the foreign exchange derivatives market, the Herfindahl index ranking is, from low to high, USD, EUR, GBP, JPY, CHF, CAD and SEK. The order of the outstanding notional amounts of these currencies is almost entirely reversed (with the exception of JPY and GBP, which are close in both measures). Across asset classes, the Herfindahl index is lowest in the interest rate derivatives market, followed by the credit derivatives market and finally the equity derivatives market. As a time-series example, Cetorelli, Hirtle, Morgan, Peristiani, and Santos (2007) document a substantial decline in the market concentration of the credit derivatives market during 2000-04, as "financial institutions have rushed to take part in this exploding market." Figures 10 to 12 and Table 1 illustrate these four examples.

	Four firms		Eight firms		HHI
	Notional	Percent	Notional	Percent	
Interest rate	173.5	40.0	272.9	62.9	629.4
Credit	10.7	40.8	18.4	69.9	738.5
Equity	2.7	43.0	4.5	70.8	747.9
Total	184.6	39.5	293.2	62.8	630.1

Table 1: Souce: ISDA Market Survey, Mid-Year 2010, by Mengle (2010)

Dealer inventory levels and turnover, aggregate dealer inventory cost.

Given the model parameters $(n, \beta, \lambda, \pi, c, \theta, r)$, the total arrival rate of demand shocks is $n\lambda$.

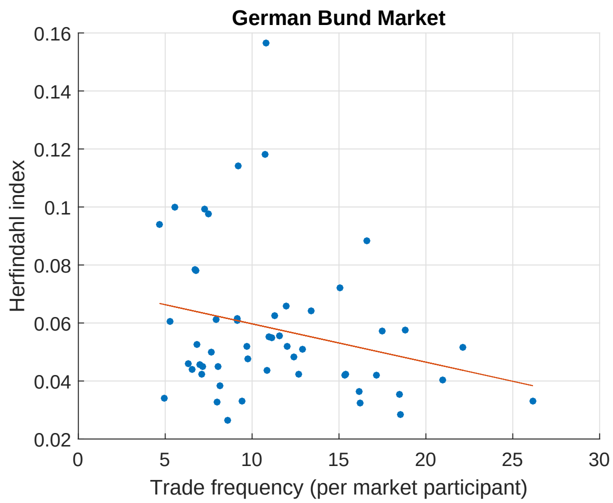


Figure 10: Source: de Roure and Wang (2016)

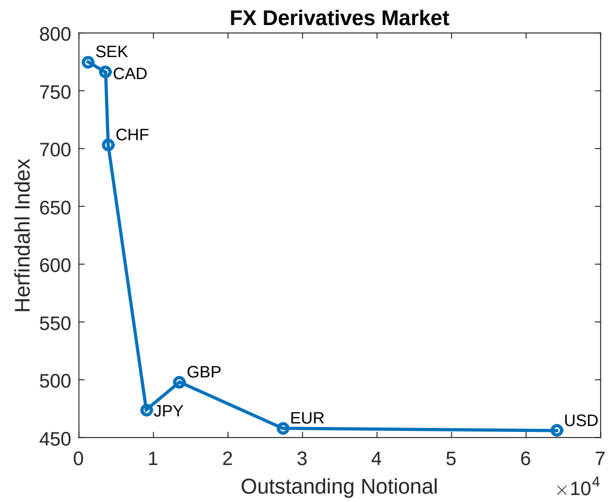


Figure 11: Source: Semiannual Statistics (BIS)

TABLE 2

Concentration Trends in Interest Rate and Foreign Exchange Over-the-Counter Derivatives Markets

Market	Average HHI	Growth in HHI (Percent)
Panel A: Global concentration: BIS surveys, 1998-2004		
U.S. interest rate derivatives		
Forward rate agreements	843	4.64
Interest rate swaps	591	8.20
Options	908	0.75
Foreign exchange derivatives		
Forwards and swaps	420	5.30
Options	544	2.31
Panel B: Federal Reserve Bank of New York Estimates of Concentration: 2000-04		
Credit derivatives	825	-14.04

Figure 12: Source: Cetorelli, Hirtle, Morgan, Peristiani, and Santos (2007)

In a concentrated core-periphery network $G(m)$ $(m \leq m^*)$, I examine how the dealer inventory dynamic depends on n and λ. I let $\bar{x}$ denote the inventory boundary $\bar{x}_{(n-m)\eta_m, p^*(m), \beta}$.

Proposition 7. *The inventory boundary $\bar{x}$ is decreasing in β and increasing in $n\lambda$. As $n\lambda$ goes to infinity, $\bar{x}$ goes to infinity at the rate $(n\lambda)^{1/3}$, and the mixing time*[13] *of the dealer inventory process goes to 0 at the rate $(n\lambda)^{-1/3}$.*

Fixing the number of dealers, as it becomes more costly to warehouse inventory (higher

[13] The mixing time of a Markov process $(X_t)_{t\geq 0}$ is defined as $t_{\text{mix}} = \inf\{t : \sup_{x_0} ||X_t, \mu||_{\text{TV}} \leq 1/4\}$, where $||X_t, \mu||_{\text{TV}}$ is the total variation distance between X_t and the stationary distribution μ of the Markov process $(X_t)_{t\geq 0}$. Levin, Peres, and Wilmer (2009) provide background on Markov chains and mixing times.

β), dealers optimally reduce their inventory size. When the asset has more frequent trade demand (because of either a larger rate λ or a larger number n of market participants), dealers expand their inventory boundary to take advantage of thicker order flow from buyside firms.

Roughly speaking, the mixing time of dealer inventory process is the expected time it takes for a dealer to rebalance its inventory. For an actively traded asset, dealer inventory has quick turnover and exhibits fast mixing. The positive relationship between asset liquidity and the speed of dealer inventory rebalancing, as predicted by the model, is consistent with prior empirical studies. Using data on the actual daily U.S.-dollar inventory held by a major dealer, Duffie (2012) estimates that the "expected half-life" of inventory imbalances is approximately 3 days for the common shares of Apple, versus two weeks for a particular investment-grade corporate bond. Figure 13 illustrates the two inventory processes of the Dealer. The data also reveal substantial cross-sectional heterogeneity across individual equities handled by the same market maker, with the expected half-life of inventory imbalances being the highest for (the least liquid) stocks with the highest-bid-ask spreads and the lowest trading volume.

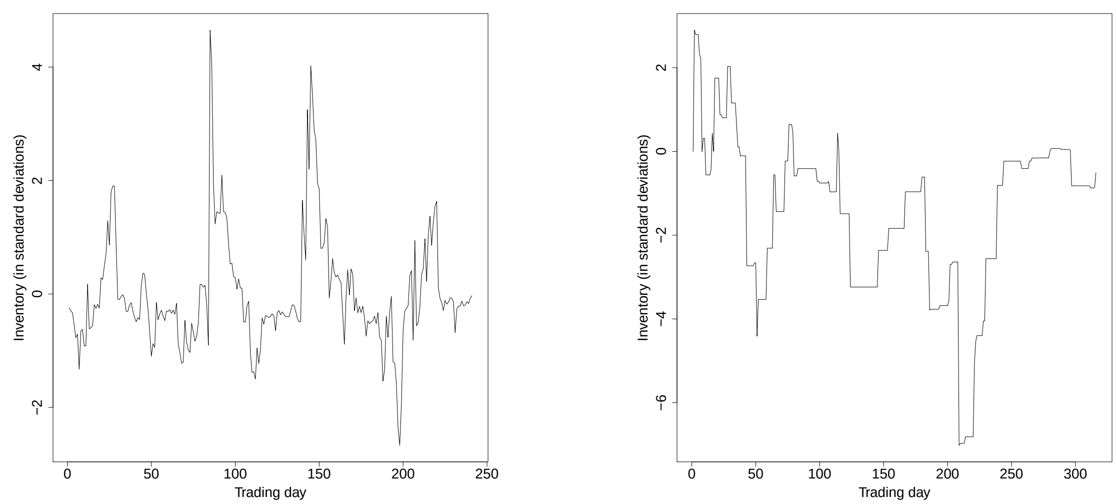


Figure 13: A major US dealer's inventory processes for Apple (left) and an investment-grade corporate bond (right) – Source: Duffie (2012)

Next, I examine properties of dealer inventory cost. The equilibrium utility of a dealer can be decomposed into its inventory cost and profits from trading with buyside firms:

$$V_{n-m,\eta_m,p^*(m)}(0) = -C(n,\lambda,m) + \lambda\,(n-m)\,\frac{\theta_m}{m}\,\frac{p^*(m)}{r}.$$

Proposition 8. *(i) The present value $C(n, \lambda, m)$ of individual dealer inventory cost is increasing and strictly concave in n and λ. As $n\lambda \to \infty$, $C(n, \lambda, m)$ goes to infinity at the rate $(n\lambda)^{2/3}$. (ii) The aggregate dealer inventory cost $mC(n, \lambda, m)$ is strictly increasing in m.*

The inventory holding cost βx^2 is quadratic in inventory size, whereas the present value $C(n, \lambda, m)$ of individual dealer inventory cost grows only sublinearly with n and λ. This captures the netting benefit: a dealer is more efficient in balancing its inventory when receiving a thicker order flow, and the associated netting benefit more than offsets the convexity of the inventory cost function.

Property *(ii)* follows from the decreasing returns to scale of the individual inventory cost function $C(n, \lambda, m)$ and Jensen's inequality. It implies that in order to minimize the aggregate dealer inventory cost, it is better to concentrate the provision of intermediation at a smaller set of dealers to maximize the netting efficiency.

Inventory-efficiency externality and holdup distortion.

In OTC markets, it is extremely rare for regulators to directly intervene in asset allocation. However, regulators may impose transaction tax, capital requirements or some pricing rule to induce a different equilibrium outcome, in which decisions related to trading and link formation are still left to market participants. In the model, the concentrated core-periphery network $G(m^*)$ emerges as the unique equilibrium network that balances market forces. Therefore, feasible regulations amount to induce a different endogenous core size. Regulators thus face a one-dimensional problem, in which they choose the optimal number of dealers intermediating a given market. From a welfare viewpoint, the next result points to under-provision of dealer intermediation for liquid assets, and over-provision for illiquid assets. I discuss the effects of three regulation policies—a "soft" stub-quote rule, a transaction tax, and capital requirements—in inducing a more efficient level of dealer intermediation.

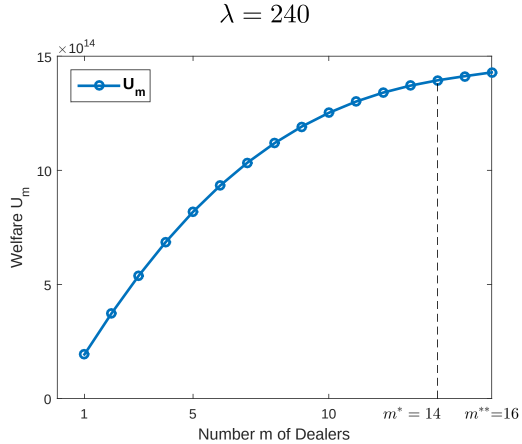
I let $\overline{m}$ be the largest integer such that $p^*(\overline{m}) > 0$. For any given $m \leq \overline{m}$, I let

$$U_m = m\underline{V}_{n-m,\eta_m,p^*(m)}(0) + (n-m)\frac{\Phi_{m,p^*(m)}}{r}, \qquad m^{**} = \underset{m\leq\overline{m}}{\operatorname{argmax}}\ U_m.$$

That is, m^{**} is the socially efficient number of dealers, if the social planner were able to to dictate the number of dealers.

Proposition 9. *(i) If $n\lambda$ is sufficiently large, one has $m^* \leq m^{**}$ and under-provision of dealer intermediation. (ii) Under certain parameter conditions that reduce $n\lambda$, one has $m^* > m^{**}$ and over-provision of dealer intermediation.*

That is, in equilibrium, there are too few dealers for actively traded assets, and too many for thinly traded assets, relative to the socially efficient number of dealers. There are two sources of inefficiency. First, each dealer induces a negative externality on other dealers' inventory efficiency by taking away some of their customer order flow (Proposition 8). This externality pushes toward over-provision of dealer intermediation, and is especially pronounced for infrequently traded assets which have limited scope for netting. For actively traded assets, however, this inventory-efficiency externality is inconsequential, relative to the distortion caused by the market power of dealers over their buyside customers. This market power comes from the bilateral nature of OTC trading, which gives dealers a temporary monopolistic position and thus a private incentive to gouge in each contact with a buyside firm. Even though dealers do not gouge on the equilibrium path, their incentive to gouge induces a holdup distortion, by which dealers extract moral hazard rents that preempts buyside firms from establishing trading links that are socially beneficial. For actively traded assets, the holdup distortion dominates the inventory-efficiency externality, leading overall to an under-provision of dealer intermediation. Figure 14 numerically illustrates the welfare U_m as a function of m for a liquid ($\lambda = 240$, or 20 trade demands per agent per month) and an illiquid asset ($\lambda = 24$, or 2/agent/month), respectively.



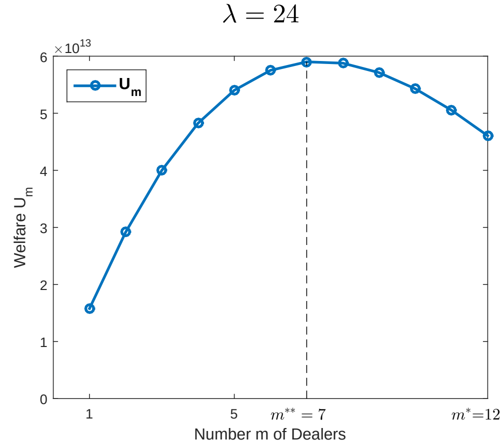


Figure 14: The welfare U_m, where the number of dealers is $m = 1, 2, \ldots, \overline{m}$

To improve market efficiency, under-intermediation can be mitigated by regulations that aim to discourage dealers from gouging. Such regulations can be, for example, a "soft" stub-quote rule that imposes a penalty should a dealer widen its spread relative to the market-prevailing level. Every dealer would internalize the penalty cost C_{penalty} into its loss $\mathcal{L}(n-m, \eta_m, p)$ from gouging and could thus sustain a tighter spread $\underline{\underline{p}}(m)$. Such a penalty cost on dealers—never triggered on the equilibrium path—allows more dealers to compete in equilibrium by strengthening dealer's commitment not to gouge buyside customers. By choosing an appropriate penalty cost C_{penalty}, regulators can achieve the socially optimal level of intermediation provision. Figure 15 illustrates the effect of such a penalty cost. Sometimes, such rules are proposed by self-regulatory organizations (SRO), such as FINRA. Broker-dealers have an incentive to join such an SRO, as the improvement of their commitment power via self regulation gives them an advantage over other non-member competitors.

To reduce dealer intermediation, regulators can impose a transaction tax on dealers, which would widen their sustainable spread, reducing the endogenous core size m^*.

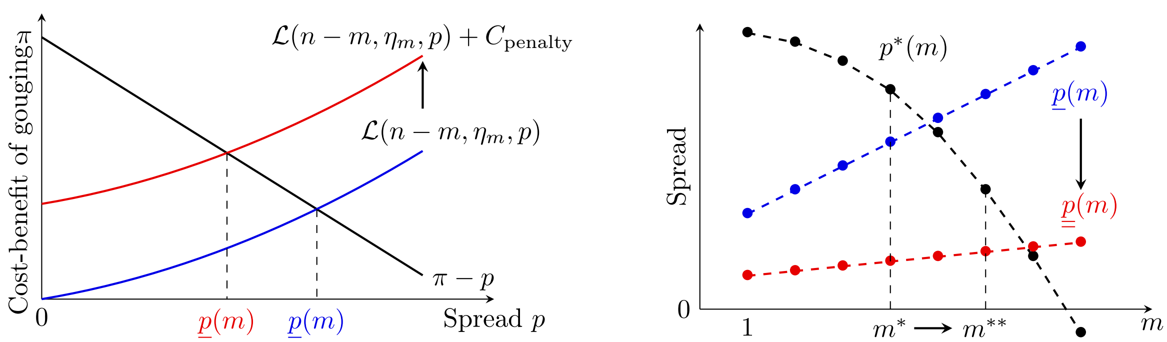


Figure 15: Introducing a penalty cost C_{penalty} on gouging increases dealer competition.

After the financial crisis, the Basel committee implemented new balance-sheet regulations for bank-affiliated dealers. Among these regulations, the capital requirements use risk-weighted assets, requiring a bank to hold more capital for holdings of risker assets. However, it does not distinguish between the liquidity of the underlying asset. Likewise, the Basel III Net Stable Funding Ratio (NSFR) and Supplementary Leverage Ratio (SLR)[14] treat high quality liquid assets (HQLA) equally as non-HQLA. My results suggest that this approach

[14]The NSFR, implemented in 2018, requires banks to maintain sufficient available stable funding (ASF) relative to the amount of required stable funding (RSF). The SLR, also implemented in 2018, requires U.S. globally systemically important bank holding companies to have capital equal to or greater than 5% of their total assets, regardless of the risk and liquidity composition of the assets.

can be improved by adding a “liquidity weight” for assets: Regulators should require dealers to have more capital and stable funding for holdings of illiquid assets in order to discourage inefficient dealer competition, while encouraging dealer intermediation for liquid assets such as the Treasuries to reduce dealers’ market power. This result is obtained based on intermediation efficiency, and is in line with regulator’s primary goal of improving financial stability, which I do not consider in this paper.

Recently, more non-bank firms such as fund managers have begun to act as liquidity providers. However, many question whether these firms can substitute for dealers by taking an effective role of market makers. This paper highlights the importance of having a large customer base for a market maker to efficiently balance inventory. Being in a central network position is essential for enabling financial institution to “lean against the wind”—that is, to provide liquidity during financial disruptions. Buyside firms are not natural liquidity hubs. Without having access to the same number of trading lines and a global customer base as traditional dealers, these firms may be unable or unwilling to absorb external selling pressure in a selloff. It is worrisome that the liquidity provided by non-bank firms may be “illusory,” in that liquidity may vanish when it is most needed. This paper does not cover this topic.

5 Inter-Dealer Trading

In the symmetric-agent model, dealers do not trade with each other due to the lack of an interdealer market. In practice, dealers form a completely connected core to trade with each other—usually in large quantities—to offset their inventory imbalances accumulated from trading with buyside firms. Introducing such an interdealer market would thus mitigate the concern of dealer inventory efficiency when there are too many dealers in the market, thus supporting a larger core in equilibirum. In this section, I add, on top of the symmetric-agent model, an artificially efficient interdealer market, in which every interdealer trade clears a dealer’s entire excess inventory. I show that even with such artificially efficient interdealer trading, the equilibrium core size remains the same when the market is sufficiently large.

Now, I add an interdealer market to the symmetric-agent model. I partition agents into $I \cup J = N$ with $|J| = m$ dealers and $|I| = n - m$ buyside firms. Encounters between pairs of connected dealers are based on independent random matching, with some pair-wise meeting intensity ξ. Upon a meeting, each dealer clears all its excess inventories, resetting its inventory back to zero. This protocol is unrealistically efficient, because two dealers usually do not have the exact opposite asset positions that could allow them to completely offset their inventory imbalances with each other. Assuming such interdealer trading keeps the model tractable, while providing a bound on the effect of interdealer trading on the equilibrium core size. The buyside firms do not have access to this interdealer market, and trades only via RFQ as in the benchmark model. This extension model no longer preserves agent symmetry, as a subset of agents (dealers) have access to an interdealer market that is not available to other agents (buyside firms). In practice, dealers resist buyside firms' participation in the interdealer segment and accuse them of taking liquidity without exposing themselves to the risks of providing liquidity. Others criticize dealers for trying to prevent competition that would compress bid-ask spreads.[15]

Other than the addition of the interdealer market, the remaining setup is identical to the symmetric-agent model. That is, every agent has 0 initial endowment of a non-divisible asset with 0 expected payoffs, and is subject to the quadratic inventory flow cost βx^2. Every buyside firm in I has an exogenously determined desire to buy of sell (equally likely) one unit of the asset at mean rate λ, and receives a fixed benefit π for each immediate execution of such trade. Dealers in J do not receive demand shocks. At any time $t \geq 0$, agents can open new and terminate existing trading accounts. Maintaining an account costs c per unit of time. Buyside firms use the same search technology θ_m to request quotes among its m connected quotes providers. The time discount rate is r. Therefore, this extension model embeds the baseline symmetric-agent model when the interdealer meeting intensity $\xi = 0$.

To take advantage of interdealer trading, dealers will now establish bilateral trading relationships with each other, forming a completely connected core. As a result, a core-periphery network $\widehat{G}(m)$ as depicted in Figure 2 emerges in equilibrium. The next result shows that, when the market is sufficiently large, introducing the interdealer market does not affect the equilibrium number of dealers.

Theorem 3. *For a given set of model parameters $(\beta, \lambda, \pi, \xi, \theta, c, r)$ except n and m, there is some constant n_0 such that if $n > n_0$, then $\widehat{G}(m)$ is an equilibrium trading network if and*

[15] Some recent electronic facilities such as SEF blur the exclusivity of an interdealer market.

only if $m \leq m^*$, where m^* is the maximum number of dealers defined by (7).

Proofs are available in Appendix G of the Full Appendix on my website (Wang (2021)). The supporting equilibrium for $\widehat{G}(m)$ in this extension model is identical to that for $G(m)$ in the symmetric-agent model, with the exception that dealers now set a larger inventory boundary $\hat{x} > \bar{x}$ because they are less concerned about temporarily holding a large inventory in the presence of the interdealer market. I let $\hat{\sigma}^*(m)$ denote this supporting equilibrium.

Behind Theorem 3 is the intuition that in a large market, dealers can efficiently balance their inventories with buyside orders and thus are less reliant on each other to layoff their inventory risk. This intuition is illustrated by the next proposition, predicting that interdealer trade accounts a small fraction of total trade volume for more actively traded assets.

Proposition 10. *As $n\lambda \to \infty$, the fraction of interdealer volume is on the order of $(n\lambda)^{-2/3}$.*

Using TRACE transaction data for U.S. corporate bonds between 2005-2014, I estimate the relationship between the fraction of interdealer volume and annual trade volume across all 61,823 bonds. Proposition 10 predicts that the logarithms of these two variables are linearly related, with a negative slope. Consistent with this prediction, the data shows that a 10% increase in total volume is associated with a 1% decrease in the fraction of interdealer volume. The t-statistic is -7, with standard errors clustered at the company level.

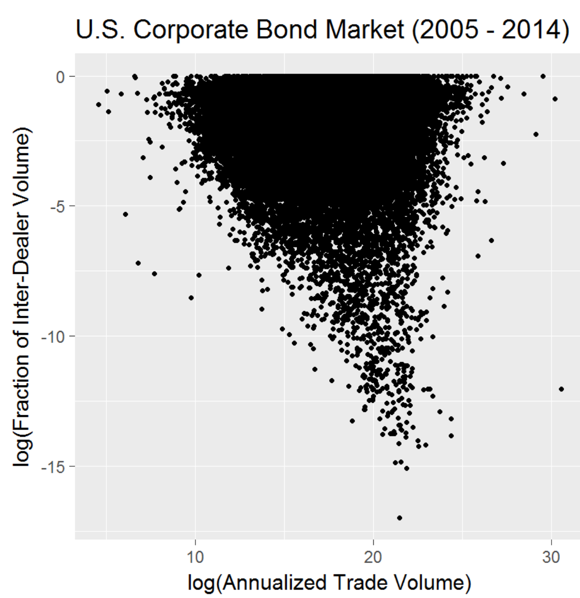


Figure 16: Interdealer Trading in the U.S. Corporate Bond Market

A side benefit of working with a large market is that the deep pocket assumption can be eliminated. Instead of invoking the deep pocket, a dealer would simply quote an ask $a = \infty$ or a bid $b = -\infty$ to reject a trade request. I let $\hat{\sigma}^*_{\text{No DP}}(m)$ be the strategy profile that is obtained from $\hat{\sigma}^*(m)$ by making this substitution. I use an approximate equilibrium concept: In a *perfect ε-equilibrium*, each agent's continuation utility is within ε of her maximum attainable continuation utility at each information set, given her belief and other agents' strategies.[16]

Proposition 11. *Given any $\varepsilon > 0$, if n is sufficiently large, the strategy profile $\hat{\sigma}^*_{No\ DP}(m)$ is a perfect ε-equilibrium when agents do not have access to deep pockets.*

Proposition 11 also holds in the symmetric-agent model (which corresponds to the case $\xi = 0$). That is, this result is orthogonal to the addition of the interdealer market.

Intuitively, a dealer's inventory process exhibits fast mixing in a large market. Even when dealer j hits its inventory boundary and thus rejects a trade request from a buyside firm i, the dealer's inventory would have almost totally remixed by the next time i requests a quote from j. Hence, a buyside firm can safely disregard its inference regarding a dealer's inventory position. The deep pocket assumption—designed to circumvent this inventory inference problem, which vanishes in a large market by itself—becomes no longer necessary.

The next result predicts that there are too few dealers relative to the socially efficient number of dealers in a large market. I let $\widehat{U}_m = U(\hat{\sigma}^*(m))$ be the welfare induced by $\hat{\sigma}^*(m)$.

Proposition 12. *When n is sufficiently large, the welfare $\widehat{U}_m$ is strictly increasing in the number m of dealers, for $m \le \overline{m}$. There is under-provision of dealer intermediation.*

This result is identical to the welfare result in the symmetric-agent model that predicts under-provision of dealer intermediation in a large market, despite the introduction of the interdealer market. Again, this is because the relative importance of the interdealer market vanishes in a large market.

6 Conclusion

Extensive empirical work has shown that core-periphery networks dominate conventional OTC markets. However, few theoretical foundations have been provided. Existing literature

[16] Mailath, Postlewaite, and Samuelson (2005) defines this solution concept in games of perfect information.

has a continuum of dealers, and exploits some ex-ante heterogeneity of agents to explain the ex-post differentiation in their "network" positions. This paper is original in its ability to *(i)* provide an endogenous separation of core from peripheral agents solely based on trade competition and inventory balancing—two countervailing forces that have opposing effects on the degree of trade concentration, and to *(ii)* explicitly determine the equilibrium number of dealers as a trade-off between these two forces. Although financial institutions are heterogeneous in real OTC markets, the core-periphery separation obtained in this paper highlights the importance of these two economic forces in determining market structure.

From a welfare viewpoint, the model identifies two sources of inefficiency: (1) the negative externality of each individual dealer on other dealers' inventory efficiency, and (2) dealers' market power over their buyside customers. The first inventory-efficiency externality results in over-provision of dealer intermediation, and is especially pronounced for an infrequently traded asset. For actively traded assets, however, this externality is dominated by the holdup distortion caused by dealers' market power over their customers, leading overall to under-provision of dealer intermediation. Regulators or an SFO can implement a soft stub-quote rule to deter dealers from gouging. Such pricing rule improves dealers' commitment power, and therefore creates room for greater dealer competition. These welfare results suggest balance-sheet regulations that treat assets differently according to their trade demand through, for example, the introduction of a "liquidity weight," in addition to the currently adopted "risk weight."

One useful direction of future research is to introduce agent heterogeneity in order to study the relationship between dealer centrality and the pricing of immediacy. Recent empirical work suggests that the price-centrality relationship changes across different markets. In the municipal bond market, central dealers earn higher markups compared with less central dealers.[17] This pattern is reversed in the market for asset-backed securities.[18]

[17] Li and Schürhoff (2019) provide evidence from the municipal bond market.

[18] Hollifield, Neklyudov, and Spatt (2017) provide evidence from the market of asset-backed securities.

Appendices

A A Microfoundation of the Search Technology

This appendix provides an example microfoundation of the search technology in Section 2. When agent i receives a demand shock at time t, the trade opportunity is assumed to be lost after an exponentially distributed time with infinitesimal[19] mean $\nu \in {}^*\mathbb{R}$. To get connected, each line of contact of i has an independent and exponentially distributed latency time with infinitesimal mean $\chi \in {}^*\mathbb{R}$. The two infinitesimal means ν and χ are "on the same order," in that neither is infinitely larger than the other. Hence, upon receiving a demand shock, agent i reaches one of her m quote providers before the trade opportunity expires with probability

$$\theta_m = \frac{m\nu}{m\nu + \chi} \in \mathbb{R},$$

which is increasing and concave in $m \geq 0$, and $\theta_0 = 0$. Each of m quote providers are equally likely to be the first to be reached.

Further, after rejecting a quote, agent i can be given a chance to receive a subsequent quote from a potentially different quote provider, while being subject to the same search latency and risk of losing the trade opportunity. This outside option, however, does not change quote providers' pricing behavior, and agent i accepts the first offer in the unique equilibrium. The equilibrium outcome of this search model, with or without access to another quote, is equivalent to the reduce-form search technology in Section 2 where upon a successful search, only one of the m quote providers offers a quote with equal probability $1/m$.

The irrelevance of search friction exhibited in this model was predicted by a simple version of Diamond (1971). However, unlike Diamond (1971), a dealer in this paper offers a non-monopolistic price, leaving some rent to a buyside customer, for the fear of losing future trades with the customer. This circumvents the monopolistic pricing problem of Diamond Paradox. Bagwell and Ramey (1992) also circumvents Diamond Paradox in a similar way. The distinction is that Bagwell and Ramey (1992) gives a range of sustainable equilibrium

[19] The hyperreals, ${}^*\mathbb{R}$, are an extension of the real numbers that contain infinite and infinitesimal numbers. An infinitesimal $\nu \in {}^*\mathbb{R}$ is a hyperreal such that $|\nu| < 1/n$, $\forall n \in \mathbb{N}$. The hyperreals are used in a branch of mathematics known as nonstandard analysis (Anderson (2000)).

prices, even when the number of price-setting firms is fixed. This paper predicts, for a given number m of dealers, a unique equilibrium price $p^*(m)$ which renders a buyside firm indifferent to whether it terminates a trade account.

B Selected Proofs for Section 3

This appendix includes proofs for results concerning the dealer's problem.

B.1 Proof of Proposition 2

I reparametrize the subscripts by letting $\underline{V}_\vartheta$ denote $\underline{V}_{k,\eta}$, where $\vartheta = k\eta$.

Lemma 2. *For every $\vartheta \geq 0$, the value function $\underline{V}_\vartheta$ is even and strictly concave, with*

$$\underline{V}_\vartheta(x) - \underline{V}_\vartheta(x+1) \geq \frac{(2x+1)\beta}{r+\vartheta}.$$

Proof. I let Γ be a set of functions from $\mathbb{Z}$ to $\mathbb{R}$ defined as

$$\Gamma \equiv \left\{ f : \mathbb{Z} \to \mathbb{R} \text{ such that } f(x) \geq -\frac{\beta}{r}x^2 \text{ for all } x, \text{ and } f \leq \bar{f} \text{ for some constant } \bar{f} \in \mathbb{R} \right\}.$$

I let $\underline{B}_{\vartheta,p,\beta}$ be an operator on Γ defined as

$$\begin{aligned} \underline{B}_{\vartheta,p,\beta}(f)(x) = \frac{1}{r+\vartheta} \bigg[&-\beta x^2 + \frac{\vartheta}{2} \max\{f(x-1)+p,\ f(x))\} \\ &+ \frac{\vartheta}{2} \max\{f(x+1)+p,\ f(x)\} \bigg], \end{aligned} \tag{8}$$

With a slight abuse of notation, I omit some or all subscripts whenever there is no ambiguity. The operator $\underline{B}$ maps Γ into itself, and satisfies

- (monotonicity) Given two functions $f, g \in \Gamma$, if $f \leq g$, then $\underline{B}(f) \leq \underline{B}(g)$.

I let the functional space Γ be equipped with the weighted sup-norm d defined as

$$\varrho(f,g) = \sup_{x\in\mathbb{Z}} \frac{|f(x)-g(x)|}{\phi(x)}, \tag{9}$$

where $\phi(x) = \beta x^2/r + a|x| + b$, and $a, b > 0$ are such that operator $\underline{B}$ satisfies

- (discounting) For every $f \in \Gamma$ and $A > 0$, $\underline{B}(f + A\phi) \leq \underline{B}(f) + \alpha A\phi$ for some $\alpha < 1$.

One can choose α to be any number between $\vartheta/(r+\vartheta)$ and 1, and a, b such that

$$\vartheta \left[\frac{\beta}{r} x^2 + a|x| + b + \frac{\beta}{r}(2|x| + 1) + a \right] \leq \alpha(r+\vartheta) \left(\frac{\beta}{r} x^2 + a|x| + b \right)$$

for every $x \in \mathbb{Z}$. For example, a and b can be chosen such that

$$a\left[\alpha(r+\vartheta) - \vartheta\right] > 2\vartheta\beta/r, \qquad \text{and} \qquad b\left[\alpha(r+\vartheta) - \vartheta\right] > \vartheta(\beta/r + a).$$

Operator $\underline{B}$ satisfies *Blackwell-Boyd sufficient conditions* as a contraction on the complete metric space (Γ, ϱ). By the Banach Fixed Point Theorem, $\underline{B}$ admits a unique fixed point.

The HJB equation (5) for $\underline{V}_\vartheta$ (with $k\eta$ replaced by ϑ) can be rewritten as $\underline{V}_\vartheta = \underline{B}(\underline{V}_\vartheta)$. Since $\underline{V}_\vartheta \in \Gamma$, then $\underline{V}_\vartheta$ is the unique fixed point of $\underline{B}$. I let $\underline{V}^0 = -\beta x^2/r$, and $\underline{V}^h = \underline{B}^h(\underline{V}^0)$ for every $h \geq 1$. As $h \to \infty$, one has

$$\varrho\left(\underline{V}^h, \underline{V}_\vartheta\right) \to 0,$$

Convergence in ϱ implies that $\underline{V}^h$ converges to $\underline{V}_\vartheta$ pointwise. Since the Bellman operator $\underline{B}$ preserves concavity (Lemma D.1), it follows by an induction that for every $h \geq 0$, $\underline{V}^h$ is even and strictly concave, with

$$\underline{V}^h(x+1) + \underline{V}^h(x-1) - 2\underline{V}^h(x) \leq -\frac{2\beta}{r+\vartheta}.$$

By letting $h \to \infty$, one obtains Lemma 2. □

Proof of Proposition 2. It follows from Lemma 2 that there exists a unique $\bar{x}_{k\eta}$ is such that

$$\underline{V}_{k\eta}(\bar{x}-1) - \underline{V}_{k\eta}(\bar{x}) \leq p, \qquad \underline{V}_{k\eta}(\bar{x}-1) - \underline{V}_{k\eta}(\bar{x}) > p. \qquad \square$$

Then the dealer's optimal pricing strategy is characterized by the inventory boundary $\bar{x}_{k\eta}$ as provided in Proposition 2.

B.2 Proof of Lemma 1

A function $f : \mathbb{Z} \to \mathbb{R}$ is said to be *W-shaped* if f is even and $f(x) < f(x+2)$, $\forall x \geq 0$.

Lemma 3. *If $\bar{x}_\vartheta \geq 1$ for some ϑ, then $\underline{V}_\vartheta - V_k$ is W-shaped for every k such that $k\eta < \vartheta$.*

The proof of Lemma 3 is long thus is deferred to the Online Appendix (Appendix D).

Proof of Lemma 1. If $\bar{x}_{k,p_1} \geq 1$ and given some $p_2 > p_1$, I let $f = \underline{V}_{k,p_2} - \underline{V}_{k,p_1}$ and $\underline{\vartheta} = \sup\left\{\vartheta \geq 0 : \underline{V}_{\vartheta,p_2}(0) = 0\right\}$. For every $k' < k$, I let $\vartheta_{k'} = k'\eta \vee \underline{\vartheta}$, $M_{k'} = V_{k',p_2} - \underline{V}_{\vartheta_{k'},p_2}$, and

$$g_{k'}(x) = \begin{cases} f(x) & \forall\,|x| \leq 1, & \text{if } M_{k'}(0) < M_{k'}(1), \\ \delta_{k\eta}^{1-|x|} f(0) & \forall\,|x| \leq 1, & \text{if } M_{k'}(0) \geq M_{k'}(1), \quad \text{where } \delta_\vartheta = \dfrac{\vartheta}{\vartheta + r}, \\ f(x-1) & \forall\,2 \leq |x| \leq \bar{x}_{k,p_2}, & \text{if } M_{k'}(\bar{x}_{k,p_2}) < M_{k'}(\bar{x}_{k,p_2}+1), \\ f(x) & \forall\,2 \leq |x| \leq \bar{x}_{k,p_2}, & \text{if } M_{k'}(\bar{x}_{k,p_2}) \geq M_{k'}(\bar{x}_{k,p_2}+1), \\ f(\bar{x}_k - 1) & \forall\,|x| > \bar{x}_{k,p_2}. & \text{if } M_{k'}(x) < M_{k'}(x+1), \\ f(\bar{x}_k) & \forall\,|x| > \bar{x}_{k,p_2}, & \text{if } M_{k'}(x) \geq M_{k'}(x+1), \end{cases}$$

$$W_{k'} = V_{k',p_2} - g_{k'}.$$

I show that $V_{k',p_1} \geq W_{k'}$ for every $k' < k$. When $k' = 0$, $V_{0,p_1} = V_{0,p_2} \geq W_0$. I suppose $V_{k'-1,p_1} \geq W_{k'-1}$ for some $k' < k$. Since V_{k',p_1} is the unique fixed point for B_{k',p_1}, where

$$B_{k',p}(V)(x) = \frac{1}{r + k'\eta}\Bigg[-\beta x^2 + \frac{k'\eta}{2}\max\left\{V_{k'-1,p}(x-1)+\pi,\ V(x-1)+p,\ V(x)\right\}$$
$$+ \frac{k'\eta}{2}\max\left\{V_{k'-1,p}(x+1)+\pi,\ V(x+1)+p,\ V(x)\right\}\Bigg],$$

to show that $V_{k',p_1} \geq W_{k'}$, it suffices to verify that $B_{k',p_1}(W_{k'}) \geq W_{k'}$. If $M_{k'}(0) < M_{k'}(1)$,

$$\begin{aligned} &\max\left\{V_{k'-1,p_1}(1)+\pi,\ W_{k'}(1)+p_1,\ W_{k'}(0)\right\} \\ \geq\ &\max\left\{V_{k'-1,p_2}(1)+\pi,\ V_{k',p_2}(1)+p_2,\ V_{k',p_2}(0)\right\} - \max\left\{f(0),\ f(1)+p_2-p_1\right\}, \\ \geq\ &\max\left\{V_{k'-1,p_2}(1)+\pi,\ V_{k',p_2}(1)+p_2,\ V_{k',p_2}(0)\right\} - [f(1)+p_2-p_1], \\ \Longrightarrow\ &B_{k',p_1}(W_{k'})(0) = V_{k',p_2}(0) - \delta_{k'\eta}[f(1)+p_2-p_1] > V_{k',p_2}(0) - f(0) = W_{k'}(0). \end{aligned}$$

The first inequality above follows from $V_{k'-1,p_1} \geq W_{k'-1}$ and the second from Lemma D.6.

Next, I show that $B_{k',p_1}(W_{k'})(1) \geq (W_{k'})(1)$. If $k'\eta < \underline{\vartheta}$, then the fact that $M_{k'}$ is inverse

W-shaped (Lemma 3) implies

$$V_{k',p_2}(1) = \underline{V}_{\underline{\vartheta},p_2}(1)+M_{k'}(1) > \underline{V}_{\underline{\vartheta},p_2}(2)+p_2+M_{k'}(0) > \underline{V}_{\underline{\vartheta},p_2}(2)+p_2+M_{k'}(2) = V_{k',p_2}(2)+p_2.$$

Hence,
$$\begin{aligned} & \max\left\{V_{k'-1,p_1}(2)+\pi,\ W_{k'}(2)+p_1,\ W_{k'}(1)\right\} \\ \geq\ & \max\left\{V_{k'-1,p_2}(2)+\pi,\ V_{k',p_2}(1)\right\} - f(1) \\ =\ & \max\left\{V_{k'-1,p_2}(2)+\pi,\ V_{k',p_2}(2)+p_2,\ V_{k',p_2}(1)\right\} - f(1). \end{aligned} \tag{10}$$

On the other hand,

$$\begin{aligned} & \max\left\{V_{k'-1,p_1}(0)+\pi,\ W_{k'}(0)+p_1,\ W_{k'}(1)\right\} \\ \geq\ & \max\left\{V_{k'-1,p_2}(0)+\pi,\ V_{k',p_2}(0)+p_2,\ V_{k',p_2}(1)\right\} - \max\left\{f(0)+p_2-p_1,\ f(1)\right\}, \\ =\ & \max\left\{V_{k'-1,p_2}(0)+\pi,\ V_{k',p_2}(0)+p_2,\ V_{k',p_2}(1)\right\} - \left[f(0)+p_2-p_1\right]. \end{aligned} \tag{11}$$

Combining (11) with (10), one obtains,

$$B_{k',p_1}(W_{k'})(1) \geq V_{k',p_2}(1) - \frac{\delta_{k'\eta}}{2}\left[f(1)+f(0)+p_2-p_1\right] > V_{k',p_2}(1) - f(1) = W_{k'}(1).$$

If $k'\eta \geq \underline{\vartheta}$, then

$$\begin{aligned} M_{k'}(1) &= \frac{\delta_{k'\eta}}{2}\left[\max\{M_{k'}(0),\ \pi-p_2-L_{k',p_2}(0)\} + \max\{M_{k'}(2),\ \pi-p_2-L_{k',p_2}(2)\}\right] \\ &\leq \delta_{k'\eta}\max\{M_{k'}(0),\ \pi-p_2-L_{k',p_2}(0)\} \qquad \text{(Lemma 3)} \\ &< \max\{M_{k'}(0),\ \pi-p_2-L_{k',p_2}(0)\}. \end{aligned}$$

Then, $M_{k'}(0) < M_{k'}(1) < \pi - p_2 - L_{k',p_2}(0)$, thus $V_{k',p_2}(0)+p_2 < V_{k'-1,p_2}(0)+\pi$. Hence,

$$\begin{aligned} & \max\left\{V_{k'-1,p_1}(0)+\pi,\ W_{k'}(0)+p_1,\ W_{k'}(1)\right\} \\ \geq\ & \max\left\{V_{k'-1,p_2}(0)+\pi,\ V_{k',p_2}(1)\right\} - \max\left\{f(0),\ f(1)\right\}, \\ =\ & \max\left\{V_{k'-1,p_2}(0)+\pi,\ V_{k',p_2}(0)+p_2,\ V_{k',p_2}(1)\right\} - f(0). \end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned} & \max\left\{V_{k'-1,p_1}(2)+\pi,\ W_{k'}(2)+p_1,\ W_{k'}(1)\right\} \\ \geq\ & \max\left\{V_{k'-1,p_2}(2)+\pi,\ V_{k',p_2}(2)+p_2,\ V_{k',p_2}(1)\right\} - [f(1)+p_2-p_1]. \end{aligned} \tag{13}$$

Combining (12) with (13), one obtains,

$$B_{k',p_1}(W_{k'})(1) \geq V_{k',p_2}(1) - \frac{\delta_{k'\eta}}{2}\left[f(0)+f(1)+p_2-p_1\right] > V_{k',p_2}(1) - f(1) = W_{k'}(1).$$

That is, if $M_{k'}(0) < M_{k'}(1)$, and $|x| \leq 1$,

$$B_{k',p_1}(W_{k'})(x) > W_{k'}(x). \tag{14}$$

Using the same technique, one can verify that (14) also holds if $M_{k'}(0) \geq M_{k'}(1)$ or $|x| > 1$. Thus, $B_{k',p_1}(W_{k'}) > W_{k'}$. Therefore, $V_{k',p_1} > W_{k'}$, or equivalently, $V_{k',p_2} - V_{k',p_1} < g_{k'}$ for every $k' < k$ and in particular, for $k' = k-1$. If $M_{k-1}(0) < M_{k-1}(1)$, then

$$\begin{aligned} &L_{k,p_1}(x) < L_{k,p_2}(x) - [f(x) - g_{k-1}(x)] = L_{k,p_2}(x), \qquad \text{for } |x| \leq 1, \\ \implies &\mathcal{L}(k,\eta,p_1) = \min_{|x|\leq 1} L_{k,p_1}(x) \leq \min_{|x|\leq 1} L_{k,p_2}(x) = \mathcal{L}(k,\eta,p_2). \end{aligned}$$

If $M_{k-1}(0) \geq M_{k-1}(1)$, then

$$\mathcal{L}(k,\eta,p_1) \leq L_{k,p_1}(0) < L_{k,p_2}(0) - [f(0) - g_{k-1}(0)] < L_{k,p_2}(0) = \mathcal{L}(k,\eta,p_2).$$

That is, $\mathcal{L}(k,\eta,p_1) < \mathcal{L}(k,\eta,p_2)$ if $p_1 < p_2$ and $\bar{x}_{k,p_1} \geq 1$, completing the proof. □

B.3 Proof of Proposition 3

A function $f : \mathbb{Z} \to \mathbb{R}$ is said to be *U-shaped* if f is even and $f(x+1) > f(x)$, $\forall x \geq 0$.

Lemma 4. *Function $\frac{\partial}{\partial\vartheta}\underline{V}$ is U-shaped. Thus, inventory boundary $\bar{x}_\vartheta$ is increasing in ϑ.*

Proof. The proof of Lemma D.6 can be adapted to show that $\frac{\partial}{\partial\vartheta}\underline{V}$ is U-shaped. Details are omitted. Then $\underline{V}_\vartheta(x) - \underline{V}_\vartheta(x+1)$ is strictly decreasing in ϑ. Proposition 2 then implies that $\bar{x}_\vartheta$ is increasing in ϑ. □

A function $f : \mathbb{Z} \to \mathbb{R}$ is said to be *weakly U-shaped* if f is even, non-constant and $f(x+1) \geq f(x)$, $\forall x \geq 0$.

Lemma 5. *The function $\psi_k = V_k + \beta x^2/r$ is convex and weakly U-shaped for every $k \geq 1$. The function $\underline{\psi}_\vartheta = \underline{V}_\vartheta + \beta x^2/r$ is convex and U-shaped for every $\vartheta > 0$.*

The proof of Lemma 5 is deferred to Appendix D

Lemma 6. *If $\bar{x}_\vartheta \geq 1$, then $\underline{V}_\vartheta(0)$ is strictly increasing and strictly convex in ϑ.*

Proof. Since $\underline{V}_\vartheta(0)$ is continuous in ϑ (Lemma D.7), it is equivalent to establish that $\frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(0)$ is strictly increasing in ϑ. I formally differentiate (5) with respect to ϑ twice to obtain $A\frac{\partial^2}{\partial\vartheta^2}\underline{V}_\vartheta = \chi$ where A is the matrix in (E.1) with $n = (2\bar{x}_\vartheta + 1)$ and $\zeta = (\vartheta + r)/\vartheta$, and

$$\chi(x) = \begin{cases} \frac{1}{\vartheta}\left[\frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x+1) + \frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x-1) - 2\frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x)\right], & |x| \leq \bar{x}_\vartheta, \\ \frac{1}{\vartheta}\left[\frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x-1) - \frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x)\right], & x = \bar{x}_\vartheta, \\ \frac{1}{\vartheta}\left[\frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x+1) - \frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x)\right], & x = -\bar{x}_\vartheta, \end{cases}$$

Since the function $\frac{\partial}{\partial\vartheta}\underline{V}_\vartheta$ is U-shaped (Lemma 4), thus the function χ is even and

$$\sum_{\tilde{x}=-x}^{x}\chi(\tilde{x}) = \begin{cases} \frac{2}{\vartheta}\left[\frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x+1) - \frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(x)\right] > 0, & \text{for } 0 \leq x < \bar{x}_\vartheta, \\ 0, & \text{for } x = \bar{x}_\vartheta. \end{cases} \tag{15}$$

Then $\frac{\partial^2}{\partial\vartheta^2}\underline{V}_\vartheta = A^{-1}\chi$. In particular,

$$\frac{\partial^2}{\partial\vartheta^2}\underline{V}_\vartheta(0) = \sum_{x=-\bar{x}_\vartheta}^{\bar{x}_\vartheta} A^{-1}_{0,x}\cdot\chi(x) = \sum_{x=0}^{\bar{x}_\vartheta-1}\left(A^{-1}_{0,x} - A^{-1}_{0,x+1}\right)\sum_{\tilde{x}=-x}^{x}\chi(\tilde{x}) + A^{-1}_{0,\bar{x}_\vartheta}\sum_{\tilde{x}=-\bar{x}_\vartheta}^{\bar{x}_\vartheta}\chi(\tilde{x}) > 0.$$

The last inequality follows from (15) and property *(ii)* of Lemma E.1 in Online Appendix E.

It remains to show that the left derivative $\frac{\partial}{\partial\vartheta}\underline{V}_{\vartheta-}(0)$ of $\underline{V}(0)$ is not greater than its right derivative $\frac{\partial}{\partial\vartheta}\underline{V}_{\vartheta+}(0)$. That is, $\frac{\partial}{\partial\vartheta}\underline{V}_\vartheta(0)$ cannot jump downward as ϑ increases. I take the right derivative in (5) to obtain $A\frac{\partial}{\partial\vartheta}\underline{V}_{\vartheta+} = r\underline{\psi}_\vartheta/\vartheta$. On the other hand, for every $\vartheta' < \vartheta$, $\underline{V}_{\vartheta'}$ is at least as high as if the dealer used $\bar{x}_\vartheta$ as its inventory boundary:

$$r\underline{V}_{\vartheta'}(x) \geq \begin{cases} -\beta x^2 + \frac{\vartheta'}{2}\left[\underline{V}_{\vartheta'}(x-1) + \underline{V}_{\vartheta'}(x+1) - 2\underline{V}_{\vartheta'}(x) + 2p\right], & |x| < \bar{x}_\vartheta, \\ -\beta x^2 + \frac{\vartheta'}{2}\left[\underline{V}_{\vartheta'}(x-1) - \underline{V}_{\vartheta'}(x) + p\right], & x = \bar{x}_\vartheta, \\ -\beta x^2 + \frac{\vartheta'}{2}\left[\underline{V}_{\vartheta'}(x+1) - \underline{V}_{\vartheta'}(x) + p\right], & |x| = -\bar{x}_\vartheta. \end{cases}$$

This implies $A \frac{\partial}{\partial \vartheta} \underline{V}_{\vartheta -} \leq r\underline{\psi}_{\vartheta}/\vartheta$. Then $\frac{\partial}{\partial \vartheta} \underline{V}_{\vartheta -}(0) \leq \frac{\partial}{\partial \vartheta} \underline{V}_{\vartheta +}(0)$. □

Lemma 7. *If $L_k(0) \geq \pi - p$ for some k, then*

$$L_k(0) < L_k(1),$$

and in particular, $L_k(0) = \min_x L_k(x)$ since L_k is W-shaped (Lemma 3).

The proof of Lemma 7 is deferred to the Online Appendix (Appendix D).

Lemma 8. *If $\mathcal{L}(k, \eta, p) \geq \pi - p$, then $\mathcal{L}(k, \eta', p) > \mathcal{L}(k, \eta, p)$ for every $\eta' > \eta$.*

Proof. I proceed with an induction over k. The case $k = 1$ is straightforward:

$$\mathcal{L}(k, \eta', p) = \min\left(\underline{V}_{k,\eta'} - V_0\right) > \min\left(\underline{V}_{k,\eta} - V_0\right) = \mathcal{L}(k, \eta, p).$$

I suppose that the statement of Lemma 8 holds for $k-1$, and consider the case for k. I let η_0 satisfy $(k-1)\eta_0 = k\eta$, I first establish Lemma 8 for $\eta' \geq \eta_0$. Since $(k-2)\eta_0 < (k-1)\eta$, then $V_{k-2,\eta_0} < V_{k-1,\eta}$. Thus,

$$\mathcal{L}(k-1, \eta_0, p) = \min\left(\underline{V}_{k-1,\eta_0} - V_{k-2,\eta_0}\right) > \min\left(\underline{V}_{k,\eta} - V_{k-1,\eta}\right) = \mathcal{L}(k, \eta, p) \geq \pi - p.$$

Then the induction hypothesis implies that $\mathcal{L}(k-1, \eta', p) > \mathcal{L}(k-1, \eta_0, p) \geq \pi - p$ for every $\eta' > \eta_0$. Then $V_{k-1,\eta'} = \underline{V}_{k-1,\eta'}$. Thus,

$$\begin{aligned}
\mathcal{L}(k, \eta', p) &= \min\left(\underline{V}_{k,\eta'} - V_{k-1,\eta'}\right) \\
&= \min\left(\underline{V}_{k,\eta'} - \underline{V}_{k-1,\eta'}\right) = \underline{V}_{k,\eta'}(0) - \underline{V}_{k-1,\eta'}(0) && \text{(Lemma 4)} \\
&\qquad > \underline{V}_{k,\eta}(0) - \underline{V}_{k-1,\eta}(0) \geq \mathcal{L}(k, \eta, p) && \text{(Lemma 6)}
\end{aligned}$$

It remains to establish Lemma 8 for $\eta' < \eta_0$. For such a η', I conjecture that

$$\psi_{k',\eta'} \leq a(\eta')\psi_{k',\eta} + b(\eta')\psi_{k'+1,\eta} \qquad \forall k' < k,$$

where $a(\eta') + b(\eta') = 1$ for some $a(\eta'), b(\eta') > 0$ to be determined. That is, $V_{k',\eta'}$ is bounded by some weighted average of $V_{k',\eta}$ and $V_{k'+1,\eta}$. This is trivially the case when $k' = 0$. If this holds for $k'-1$ for some $1 \le k' < k$, then, letting $\bar\psi_{k',\eta'} = a(\eta')\psi_{k',\eta} + b(\eta')\psi_{k'+1,\eta}$,

$$
\begin{aligned}
& T_{k',\eta'}\left(\bar\psi_{k',\eta'}\right)(x) \\
\le\ & \frac{1}{1+\frac{2r}{k'\eta'}}\Bigg(\left[\bar\psi_{k',\eta'}(x-1)+\frac{\beta(2x-1)}{r}+p\right]\vee\bar\psi_{k',\eta'}(x)\vee\left[\bar\psi_{k'-1,\eta'}(x-1)+\frac{\beta(2x-1)}{r}+\pi\right] \\
& \qquad +\left[\bar\psi_{k',\eta'}(x+1)-\frac{\beta(2x+1)}{r}+p\right]\vee\bar\psi_{k',\eta'}(x)\vee\left[\bar\psi_{k'-1,\eta'}(x+1)-\frac{\beta(2x+1)}{r}+\pi\right]\Bigg) \\
\le\ & \frac{1+\frac{2r}{k'\eta}}{1+\frac{r}{k'\eta'}}\,a(\eta')\,\psi_{k',\eta}(x)+\frac{1+\frac{2r}{(k'+1)\eta}}{1+\frac{2r}{k'\eta'}}\,b(\eta')\,\psi_{k'+1,\eta}(x)
\end{aligned}
$$

for every $x \in \mathbb{Z}$, where $T_{k,\eta}$ is defined in (D.1). To have $\psi_{k',\eta'} \le \bar\psi_{k',\eta'}$, it suffices to have $T_{k',\eta'}\left(\bar\psi_{k',\eta'}\right) \le \bar\psi_{k',\eta'}$. Since $0 \le \psi_{k',\eta} \le \psi_{k'+1,\eta}$, then it suffices to have

$$
\frac{1+\frac{2r}{k'\eta}}{1+\frac{2r}{k'\eta'}}\,a(\eta')+\frac{1+\frac{2r}{(k'+1)\eta}}{1+\frac{2r}{k'\eta'}}\,b(\eta') \le a(\eta')+b(\eta') = 1.
$$

Therefore, it suffices to set

$$
b(\eta') = k\,\frac{\eta'-\eta}{\eta'}, \qquad a(\eta') = 1-b(\eta').
$$

It then follows from an induction over k' that

$$
\begin{aligned}
\psi_{k-1,\eta'} &\le a\left(\eta'\right)\psi_{k-1,\eta}+b\left(\eta'\right)\psi_{k,\eta} \\
\Longrightarrow\ \psi_{k-1,\eta'}(0)-\psi_{k-1,\eta}(0) &\le b\left(\eta'\right)\left[\psi_{k,\eta}(0)-\psi_{k-1,\eta}(0)\right] \\
&< k(\eta'-\eta)\frac{\left[\psi_{k,\eta}(0)-\psi_{k-1,\eta}(0)\right]}{\eta} \\
&\le k(\eta'-\eta)\frac{\left[\underline{\psi}_{k,\eta}(0)-\underline{\psi}_{k-1,\eta}(0)\right]}{\eta} \\
&< \underline{\psi}_{k,\eta'}(0)-\underline{\psi}_{k,\eta}(0) \qquad\qquad (\text{Lemma 6}).
\end{aligned}
$$

Therefore, $\underline{\psi}_{k,\eta'}(0)-\psi_{k-1,\eta'}(0) > \underline{\psi}_{k,\eta}(0)-\psi_{k-1,\eta}(0) \ge \pi - p$. It then follows from Lemmas 3 and 7 that $\mathcal{L}(k,\eta',p) > \mathcal{L}(k,\eta,p)$. □

Lemma 9. *If $\underline{V}_{k+1,\eta,p} - \underline{V}_{k,\eta,p} \ge \pi - p$, then $\mathcal{L}(k,\eta,p) < \mathcal{L}(k+1,\eta,p)$.*

Proof. I show that $V_{k,\eta,p} \leq W$ where

$$W = \underline{V}_{k,\eta,p} + [\pi - p - \mathcal{L}(k,\eta,p)]^+.$$

$$\text{Then} \quad \min_{x\in\mathbb{Z}} [W(x) - V_{k-1,\eta,p}(x)] \geq \pi - p.$$

Thus, $B_{k,\eta,p}(W) = \underline{B}_{k,\eta,p}(W) \leq \overline{W}$. It follows from the monotonicity of operator $B_{k,\eta,p}$ that $V_{k,\eta,p} \leq W$. Since $\underline{V}_{k+1,\eta,p} - \underline{V}_{k,\eta,p} \geq \pi - p$, it then follows that

$$\begin{aligned} \mathcal{L}(k+1,\eta,p) \geq \min_{x\in\mathbb{Z}} \left[\underline{V}_{k+1,\eta,p} - W\right] &= \underline{V}_{k+1,\eta,p}(0) - \underline{V}_{k,\eta,p}(0) - [\pi - p - \mathcal{L}(k,\eta,p)]^+ \qquad (16) \\ &> \max\left\{\pi - p, \underline{V}_{k,\eta,p}(0) - \underline{V}_{k-1,\eta,p}(0)\right\} - [\pi - p - \mathcal{L}(k,\eta,p)]^+ \geq \mathcal{L}(k,\eta,p). \quad \square \end{aligned}$$

Proposition 3 follows immediately from Lemmas 8 and 9.

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Online Appendix

C Perfect Bayesian Equilibrium in Continuous-Time Games

I define a basic version of perfect Bayesian equilibrium (PBE) for continuous-time games with imperfect information. Players' beliefs are given in the form of *regular conditional probabilities* that satisfy *no-signaling-what-you-don't-know*.[20] A *game* consists of *(i)* a finite set N of players, *(ii)* a nice measurable space[21] $(\Omega, \mathcal{F})$ and a filtration $(\mathcal{F}_t^1)_{t\geq 0}$, where each outcome $\omega \in \Omega$ is a complete play of the game, and the σ-algebra $\mathcal{F}_t^1$ describes what has happened when players are about to take their time-t actions simultaneously, *(iii)* for each player $i \in N$, a sub-filtration $(\mathcal{F}_{it}^1)_{t\geq 0}$ where the sub-σ-algebra $\mathcal{F}_{it}^1 \subseteq \mathcal{F}_t$ represents the information available to i before i takes her time-t action, *(iv)* a measurable action space $\mathbb{A}$, *(v)* a utility function u_i for each player i that is a measurable function from the outcome space Ω to $\mathbb{R}$, and *(vi)* a probability measure P_σ on the outcome space $(\Omega, \mathcal{F})$ induced by each strategy profile σ, and a regular condition probability $\kappa_{t,\sigma} : \Omega \times \mathcal{F} \mapsto [0, 1]$ of P_σ given $\mathcal{F}_t^1$ that describes the likelihood of potential outcomes given what has happened up to time t. A strategy of player i is a process $(\sigma_{it})_{t\geq 0}$, where σ_{it} is a mapping from Ω to $\mathbb{A}$ that is $\mathcal{F}_{it}^1$-measurable. A strategy profile σ is the collection of all players' strategies.

Given a strategy profile σ, a system of consistent beliefs is a system of regular conditional probabilities[22] $\mu_{it} : \Omega \times \mathcal{F}_t^1 \mapsto [0, 1]$ given $\mathcal{F}_{it}^1$ for every player $i \in N$ and time $t \geq 0$,[23] such that for every time t and $B \in \mathcal{F}_{t^+}^1 \equiv \bigcap_{s>t} \mathcal{F}_s^1$, the mapping $s \to \mu_{is}(\omega, B)$ has at every $t \geq 0$ a right-hand limit $\lim_{s\downarrow t} \mu_{is}(\omega, B)$. It is easy to show that these limits constitute a regular conditional probability μ_{it^+} on $(\Omega, \mathcal{F}_{t^+}^1)$ given $\mathcal{F}_{it^+}^1$. The system of beliefs satisfies *no-signaling-what-you-don't-know*[24] if for every given player i, time t and $B \in \mathcal{F}_t^1$, the player's belief $\mu_{it^+}(\omega, B)$ does not depend on ω through her own action taken at or after time t. A

[20] The notion was first set forth by Fudenberg and Tirole (1991) for discrete-time games.

[21] That is, there is a 1-1 map ϕ from $(\Omega, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B})$ so that ϕ and ϕ^{-1} are both measurable. Assuming a nice measurable space guarantees the existence of regular conditional probabilities. Most spaces arising in applications are nice. Durrett (2019) provides more details.

[22] The definition of a regular condition probability embeds the Bayesian-updating requirement on the equilibrium path, without imposing any restriction on off-the-path beliefs.

[23] For every $\omega \in \Omega$, $\mu_{it}(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{F}_t^1)$ that describes the player's belief regarding what has happened so far given her available information $\mathcal{F}_{it}^1$.

[24] Watson (2016) imposes a stronger independence restriction on players' beliefs, which is unnecessary here.

PBE is a strategy profile σ and a system μ of consistent beliefs such that, for any player i, strategy σ_i' and time t,

$$\int_{\omega''}\int_{\omega'}\mu_{it}(\omega,d\omega')\,\kappa_{t,\sigma}(\omega',d\omega'')\,u_i(\omega'')\geq\int_{\omega''}\int_{\omega'}\mu_{it}(\omega,d\omega')\,\kappa_{t,(\sigma_{-i},\sigma_i')}(\omega',d\omega'')\,u_i(\omega'')$$

for every $\omega\in\Omega$. The right hand side above is the player's expected utility under belief μ_{it} if she deviates to σ_i' from time t onward. Requiring the inequality above to hold *everywhere* instead of *almost surely* is to require sequentiality both on and off the equilibrium path.

D Supporting Lemmas, Remaining Proofs for Section 3

This appendix contains supporting lemmas and the remaining proofs for Section 3.

D.1 Supporting Lemmas

The proof of Lemma 2 in Appendix B invoked the next lemma.

Lemma D.1. *I let T_1 and T_2 be two functional operators such that for every $f:\mathbb{Z}\mapsto\mathbb{R}$,*

$$\begin{aligned}T_1(f)(x)&=\max\{f(x-1)+a,f(x)\}\quad\forall x\in\mathbb{Z},\quad\text{and}\\ T_2(f)(x)&=\max\{f(x+1)-b,f(x)\}\quad\forall x\in\mathbb{Z},\end{aligned}$$

where $a,b\in\mathbb{R}$ are two constants. Then T_1 and T_2 preserve concavity.

Proof. If f is a concave function from $\mathbb{Z}$ to $\mathbb{R}$, then for every $x\in\mathbb{Z}$,

$$\begin{aligned}T_1(f)(x-1)-T_1(f)(x)&\leq\max\{f(x-2)-f(x-1),\ f(x-1)-f(x)\}\\&\leq\min\{f(x-1)-f(x),\ f(x)-f(x+1)\}\\&\leq T_1(f)(x)-T_1(f)(x+1).\end{aligned}$$

Therefore, T_1 preserves concavity. The same property holds for T_2. □

The proof of Lemma 5 is given here:

Lemma D.2. *The function $\psi_k=V_k+\beta x^2/r$ is convex and weakly U-shaped for every $k\geq 1$. The function $\underline{\psi}_\vartheta=\underline{V}_\vartheta+\beta x^2/r$ is convex and U-shaped for every $\vartheta>0$.*

Proof of Lemma 5. I show that ψ_k is convex by an induction over $k \geq 0$. When $k = 0$, function $\psi_0 = 0$ is convex. I suppose that ψ_{k-1} is convex. The HJB equation (4) for V_k can be rewritten as $\psi_k = T_k(\psi_k)$, where

$$\begin{aligned} &T_k(\psi)(x) \\ =& \frac{\delta_{k\eta}}{2}\Bigg(\left[\psi(x-1) + \frac{\beta(2x-1)}{r} + p\right] \vee \psi(x) \vee \left[\psi_{k-1}(x-1) + \frac{\beta(2x-1)}{r} + \pi\right] \\ &+ \left[\psi(x+1) - \frac{\beta(2x+1)}{r} + p\right] \vee \psi(x) \vee \left[\psi_{k-1}(x+1) - \frac{\beta(2x+1)}{r} + \pi\right]\Bigg), \end{aligned} \tag{D.1}$$

for every $x \in \mathbb{Z}$, where $\delta_{k\eta} = k\eta/(r + k\eta)$. It follows again from Blackwell-Boyd sufficiency conditions and the Contraction Mapping Theorem that operator T_k admits a unique fixed point which is ψ_k. Since operator T_{k+1} preserves convexity, then its unique fixed point ψ_k is even and convex. If function ψ_k is not weakly U-shaped, it must be that ψ_k is constant. However, no constant function solves $\psi = T_k(\psi)$. Therefore, function ψ_k is weakly U-shaped. Similarly, $\underline{\psi}_\vartheta$ is convex since it is the unique fixed point for the operator $\underline{T}_\vartheta$ defined as

$$\begin{aligned} \underline{T}_\vartheta(\psi)(x) = \frac{\delta_\vartheta}{2}\Bigg(&\left[\psi(x-1) + \frac{\beta(2x-1)}{r} + p\right] \vee \psi(x) \\ &+ \left[\psi(x+1) - \frac{\beta(2x+1)}{r} + p\right] \vee \psi(x)\Bigg), \qquad \text{where } \delta_\vartheta = \frac{\vartheta}{\vartheta + r}. \end{aligned}$$

Further, if the dealer never gouges buyside customers and never invokes the deep pocket, then its continuation utility would be $-\frac{\beta x^2}{r} + \frac{\vartheta}{r}(p - \frac{\beta}{r})$ which is a strict lower bound for $\underline{V}_\vartheta$. Then $\underline{\psi}_\vartheta > \frac{\vartheta}{r}(p - \frac{\beta}{r})$. Thus,

$$\underline{\psi}_\vartheta(0) \leq \delta_\vartheta\left[\underline{\psi}_\vartheta(1) + p - \frac{\beta}{r}\right] < \underline{\psi}_\vartheta(1).$$

Therefore, $\underline{\psi}$ must be U-shaped. $\square$

The proof of Lemma 3 in Appendix B is given here:

Lemma D.3. *If $\bar{x}_\vartheta \geq 1$ for some ϑ, then $\underline{V}_\vartheta - V_k$ is W-shaped for every k such that $k\eta < \vartheta$.*

Proof. For simplicity of notation, I write $L_{\vartheta,k}$ for $\underline{V}_\vartheta - V_k$ and $\delta_\vartheta = \vartheta/(\vartheta + r)$. First,

$$L_{\vartheta,k}(x) = \underline{\psi}_\vartheta(x) - \psi_k(x) = \frac{\delta_{k\eta}}{\delta_\vartheta}\,\underline{\psi}_\vartheta(x) - \psi_k(x) + \left(1 - \frac{\delta_{k\eta}}{\delta_\vartheta}\right)\underline{\psi}_\vartheta(x)$$

It follows from Lemma D.4 that for every $x \geq 1$,

$$
\begin{aligned}
& \frac{\delta_{k\eta}}{\delta_\vartheta}\, \underline{\psi}_\vartheta(x) - \psi_k(x) \\
= {} & \frac{\delta_{k\eta}}{\delta_\vartheta}\, \underline{T}_\vartheta(\underline{\psi}_\vartheta)(x) - T_k(\psi_k) \\
= {} & \frac{\delta_{k\eta}}{2} \big[\, (\underline{V}_\vartheta(x-1) + p - \max\{V_{k-1}(x-1) + \pi,\ V_k(x-1) + p\}) \\
& \quad + (\max\{\underline{V}_\vartheta(x+1) + p,\ \underline{V}_\vartheta(x)\} - \max\{V_{k-1}(x+1) + \pi,\ V_{k-1}(x+1) + p,\ V_k(x)\})\,\big].
\end{aligned}
$$

Since $\bar{x}_\vartheta \geq 1$, the second equality above becomes "$\leq$" when $x = 0$. Then for every $x \geq 1$ (when $x = 0$, the equality below becomes "$\leq$"),

$$
\begin{aligned}
& L_{\vartheta,k}(x) \\
= {} & \frac{\delta_{k\eta}}{2} \big[\min\{L_{\vartheta,k-1}(x-1) + p - \pi,\ L_{\vartheta,k}(x-1)\} \\
& \quad + (\max\{\underline{V}_\vartheta(x+1) + p,\ \underline{V}_\vartheta(x)\} - \max\{V_{k-1}(x+1) + \pi,\ V_{k-1}(x+1) + p,\ V_k(x)\})\,\big] \\
& + \left(1 - \frac{\delta_{k\eta}}{\delta_\vartheta}\right) \underline{\psi}_\vartheta(x)
\end{aligned}
$$

When x is sufficiently large,

$$
\begin{aligned}
& \underline{V}_\vartheta(x) > \underline{V}_\vartheta(x+1) + p, \\
& V_k(x) > \max\{V_k(x+1) + p,\ V_{k+1}(x+1) + \pi\}.
\end{aligned}
$$

Hence, for x sufficiently large,

$$
L_{\vartheta,k}(x) = \frac{\delta_{k\eta}}{2 - \delta_{k\eta}} \min\{L_{\vartheta,k-1}(x-1) + p - \pi, L_{\vartheta,k}(x-1)\} + \frac{1 - \frac{\delta_{k\eta}}{\delta_\vartheta}}{1 - \frac{\delta_{k\eta}}{2}}\, \underline{\psi}_\vartheta(x).
$$

When $k = 0$, it follows from Lemma 5 that $L_{\vartheta,0} = \underline{\psi}_\vartheta$ is U-shaped and thus W-shaped. I suppose that $L_{\vartheta,k-1}$ is W-shaped, but $L_{\vartheta,k}$ is not. For x sufficiently large, if $L_{\vartheta,k}(x-1) - L_{\vartheta,k}(x+1) \geq 0$, then

$$
\begin{aligned}
& L_{\vartheta,k}(x-1) - L_{\vartheta,k}(x+1) \\
< {} & \min\{L_{\vartheta,k-1}(x-2) + p - \pi,\ L_{\vartheta,k}(x-2)\} - \min\{L_{\vartheta,k-1}(x) + p - \pi,\ L_{\vartheta,k}(x)\} \\
\leq {} & L_{\vartheta,k}(x-2) - L_{\vartheta,k}(x).
\end{aligned}
$$

Therefore, one can find some $x' \geq 1$ such that

$$x' = \operatorname*{argmax}_{x \geq 1} \left\{ L_{\vartheta,k}(x-1) - L_{\vartheta,k}(x+1) \right\}.$$

I consider four cases and show that they all lead to a contraction. First, if

$$\underline{V}_\vartheta(x'-1) \leq \underline{V}_\vartheta(x') + P, \quad \text{and} \quad V_k(x'+1) \leq \max\{V_k(x'+2) + P,\ V_{k-1}(x'+2) + \pi\},$$

$$\begin{aligned}
\text{then} \quad & L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1) \\
< \quad & \frac{\delta_{k\eta}}{2} \Big([\min\{L_{\vartheta,k-1}(x'-2) + p - \pi,\ L_{\vartheta,k}(x'-2)\} - \min\{L_{\vartheta,k-1}(x') + p - \pi,\ L_{\vartheta,k}(x')\}] \\
& + [\min\{L_{\vartheta,k-1}(x') + p - \pi,\ L_{\vartheta,k}(x')\} - \min\{L_{\vartheta,k-1}(x'+2) + p - \pi,\ L_{\vartheta,k}(x'+2)\}] \Big) \\
\leq \quad & \frac{\delta_{k\eta}}{2} \Big([L_{\vartheta,k}(x'-2) - L_{\vartheta,k}(x')]^+ + [L_{\vartheta,k}(x') - L_{\vartheta,k}(x'+2)]^+ \Big) \\
\leq \quad & L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1).
\end{aligned}$$

This leads to a contradiction. Second, if

$$\underline{V}_\vartheta(x'-1) \leq \underline{V}_\vartheta(x') + p, \quad \text{and} \quad V_k(x'+1) > \max\{V_k(x'+2) + P,\ V_{k-1}(x'+2) + \pi\},$$

$$\begin{aligned}
\text{then} \quad & L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1) \\
< \quad & \frac{\delta_{k\eta}}{2} \Big([L_{\vartheta,k}(x'-2) - L_{\vartheta,k}(x')]^+ + [\min\{L_{\vartheta,k-1}(x') + p - \pi,\ L_{\vartheta,k}(x')\} - L_{\vartheta,k}(x'+1)] \Big) \\
< \quad & \frac{\delta_{k\eta}}{2} \Big([L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1)] + [\min\{L_{\vartheta,k-1}(x') + p - \pi,\ L_{\vartheta,k}(x')\} - L_{\vartheta,k}(x'+1)] \Big).
\end{aligned}$$

This implies that

$$0 \leq L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1) < \min\{L_{\vartheta,k-1}(x') + p - \pi,\ L_{\vartheta,k}(x')\} - L_{\vartheta,k}(x'+1) \tag{D.2}$$

$$\leq L_{\vartheta,k}(x') - L_{\vartheta,k}(x'+1). \tag{D.3}$$

However,

$$\begin{aligned}
& L_{\vartheta,k}(x') - L_{\vartheta,k}(x'+1) \\
< \ & \frac{\delta_{k\eta}}{2} \Big(L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1) - \min\{L_{\vartheta,k-1}(x') + p - \pi,\ L_{\vartheta,k}(x')\} \\
& + [\max\{\underline{V}_\vartheta(x'+1) + p,\ \underline{V}_\vartheta(x')\} - \max\{V_{k-1}(x'+1) + \pi,\ V_k(x'+1) + p,\ V_k(x')\}] \Big)
\end{aligned}$$

Since $\quad \max\left\{\underline{V}_\vartheta(x'+1)+p,\ \underline{V}_\vartheta(x')\right\} - \max\left\{V_{k-1}(x'+1)+\pi,\ V_k(x'+1)+p,\ V_k(x')\right\}$

$$\begin{aligned}
&\le \max\left\{\underline{V}_\vartheta(x'+1)+p,\ \underline{V}_\vartheta(x')\right\} - \max\left\{V_k(x'+1)+p,\ V_k(x')\right\}\\
&\le \max\left\{L_{\vartheta,k}(x'+1),\ L_{\vartheta,k}(x')\right\},
\end{aligned}$$

and $\quad \max\left\{L_{\vartheta,k}(x'+1),\ L_{\vartheta,k}(x')\right\} - \min\left\{L_{\vartheta,k-1}(x')+p-\pi,\ L_{\vartheta,k}(x')\right\}$

$$\begin{aligned}
&\le \max\left\{L_{\vartheta,k}(x'+1),\ L_{\vartheta,k}(x')\right\} - L_{\vartheta,k}(x'+1) && \text{(following from Inequality (D.2))}\\
&\le L_{\vartheta,k}(x') - L_{\vartheta,k}(x'+1) && \text{(following from Inequality (D.3))},
\end{aligned}$$

thus, $$L_{\vartheta,k}(x') - L_{\vartheta,k}(x'+1) < \delta_{k\eta}\left[L_{\vartheta,k}(x') - L_{\vartheta,k}(x'+1)\right],$$

contradicting (D.3). Third, if

$$\underline{V}_\vartheta(x'-1) > \underline{V}_\vartheta(x')+p, \quad \text{and} \quad V_k(x'+1) \le \max\{V_k(x'+2)+p,\ V_{k-1}(x'+2)+\pi\},$$

then
$$\begin{aligned}
&L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1)\\
&< \frac{\delta_{k\eta}}{2}\Big(\left[L_{\vartheta,k}(x'-2) - L_{\vartheta,k}(x')\right]^+\\
&\quad + \Big[\quad \left(\underline{V}_\vartheta(x'-1) - \max\left\{V_{k-1}(x')+\pi,\ V_k(x')+p\right\}\right)\\
&\quad\quad - \left(\underline{V}_\vartheta(x'+1) - \max\left\{V_{k-1}(x'+2)+\pi,\ V_k(x'+2)+p\right\}\right)\Big]\Big).
\end{aligned}$$

Since $$\underline{V}_\vartheta(x'-1) - \underline{V}_\vartheta(x'+1) < \underline{V}_\vartheta(x') - \underline{V}_\vartheta(x'+2),$$

and $\quad \min\left\{L_{\vartheta,k-1}(x')-\pi,\ L_{\vartheta,k}(x')-p\right\} - \min\left\{L_{\vartheta,k-1}(x'+2)-\pi,\ L_{\vartheta,k}(x'+2)-p\right\}$

$$\le \left[L_{\vartheta,k}(x') - L_{\vartheta,k}(x'+2)\right]^+,$$

it then follows that

$$\begin{aligned}
L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1) &< \frac{\delta_{k\eta}}{2}\left(\left[L_{\vartheta,k}(x'-2) - L_{\vartheta,k}(x')\right]^+ + \left[L_{\vartheta,k}(x') - L_{\vartheta,k}(x'+2)\right]^+\right)\\
&\le L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1).
\end{aligned}$$

This leads to a contradiction. Lastly, if

$$\underline{V}_\vartheta(x'-1) > \underline{V}_\vartheta(x')+p, \quad \text{and} \quad V_k(x'+1) > \max\{V_k(x'+2)+p,\ V_{k-1}(x'+2)+\pi\},$$

then

$$\begin{aligned}
L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1) &< \frac{\delta_{k\eta}}{2}\left(\left[L_{\vartheta,k}(x'-2) - L_{\vartheta,k}(x')\right]^+ + \left[L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1)\right]\right)\\
&\le L_{\vartheta,k}(x'-1) - L_{\vartheta,k}(x'+1).
\end{aligned}$$

This leads to a contradiction. Hence, $L_{\vartheta,k}$ must be W-shaped, completing the proof. $\square$

The proof of Lemma 3 invoked the next lemma.

Lemma D.4. *For every $x \geq 1$, $\max\{V_{k-1}(x-1)+\pi,\ V_k(x-1)+p\} > V_k(x)$.*

Proof. If there was some $x \geq 1$ such that $\max\{V_{k-1}(x-1)+\pi,\ V_k(x-1)+p\} \leq V_k(x)$, then

$$\begin{aligned}
&\max\left\{V_{k-1}(x+1)+\pi-V_k(x),\ V_k(x+1)+p-V_k(x),\ 0\right\} = \frac{2r\psi_k(x)}{k\eta} \\
\geq & \frac{2r\psi_k(x-1)}{k\eta} \geq \max\left\{V_{k-1}(x)+\pi-V_k(x-1),\ V_k(x)+p-V_k(x-1),\ 0\right\} \geq 2p.
\end{aligned}$$

Since $\qquad V_{k-1}(x+1)+\pi-V_k(x) < V_{k-1}(x-1)+2p+\pi-V_k(x) \leq 2p \qquad$ (Lemma D.5),

$$\begin{aligned}
\text{then} \quad & V_k(x+1)+p-V_k(x) \geq \max\left\{V_{k-1}(x)+\pi-V_k(x-1),\ V_k(x)+p-V_k(x-1),\ 0\right\} \\
\Longrightarrow \quad & V_k(x+1) \geq \max\{V_{k-1}(x)+\pi,\ V_k(x)-p\}.
\end{aligned}$$

An induction implies that $V_k(x+y) \geq V(x)+yP$ for every $y \in \mathbb{Z}^+$, which contridicts with the value function V_k being upper-bounded. Hence, Lemma D.4 holds for every $k \geq 0$. □

The proof of Lemma D.4 invoked the next lemma:

Lemma D.5. *For every $x \geq 1$, $V_k(x-1)+2p > V_k(x+1)$.*

Proof. Lemma D.5 trivially holds for $k=0$. I suppose that Lemma D.5 holds for $k-1$ but not for k. I let x be the smallest integer in $\mathbb{Z}^+$ such that $V_k(x-1)+2p \leq V_k(x+1)$. Then

$$\begin{aligned}
& \max\{V_{k-1}(x-2)-V_k(x-1)+\pi,\ V_k(x-2)-V_k(x-1)+p,\ 0\} \\
& + \max\{V_{k-1}(x)-V_k(x-1)+\pi,\ V_k(x)-V_k(x-1)+p,\ 0\} \\
= & \frac{2r\psi_k(x-1)}{k\eta} \leq \frac{2r\psi_k(x+1)}{k\eta} \\
= & \max\{V_{k-1}(x)-V_k(x+1)+\pi,\ V_k(x)-V_k(x+1)+p,\ 0\} \\
& + \max\{V_{k-1}(x+2)-V_k(x+1)+\pi,\ V_k(x+2)-V_k(x+1)+p,\ 0\}.
\end{aligned}$$

Since
$$\begin{aligned}
V_{k-1}(x-2)-V_k(x-1)+\pi &> V_{k-1}(x)-V_k(x+1)+\pi, \\
V_k(x-2)-V_k(x-1)+p &> V_k(x)-V_k(x+1)+p, \\
V_{k-1}(x)-V_k(x-1)+\pi &> V_{k-1}(x+2)-V_k(x+1)+\pi,
\end{aligned}$$

thus,
$$V_k(x)-V_k(x-1)+p \leq V_k(x+2)-V_k(x+1)+p.$$

Then $V_k(x) + 2p \leq V_k(x+2)$. An induction implies that $V_k(x+2y) \geq V(x) + 2yp$ for every $y \in \mathbb{Z}^+$, which contradicts with the value function V_k being upper-bounded. Hence, Lemma D.5 holds for every $k \geq 0$. □

The proof of Lemma 1 in Appendix B.2 invoked the next lemma:

Lemma D.6. *If $p_1 < p_2$ and $\bar{x}_{p_1} \geq 1$, then $\bar{x}_{p_1} \leq \bar{x}_{p_2}$ and for every $x > 0$,*

$$0 < \underline{V}_{p_2}(x-1) - \underline{V}_{p_1}(x-1) - \left[\underline{V}_{p_2}(x) - \underline{V}_{p_1}(x)\right] < p_2 - p_1.$$

Proof. Given any p such that $\bar{x}_p \geq 1$, for every $|x| \leq \bar{x}_p$, I formally differentiate (5) with respect to p to obtain

$$A \frac{\partial}{\partial p} \underline{V}_p = (1/2,\, 1, \ldots, 1,\, 1/2)^\top$$

where A is the matrix in (E.1) in Online Appendix E with $n = (2\bar{x}_p + 1)$ and $\zeta = (\vartheta + r)/\vartheta$. Since $(1/2,\, 1, \ldots, 1,\, 1/2)^\top$ is weakly inverse U-shaped, then $\underline{V}_p$ is inverse U-shaped and

$$\frac{\vartheta}{2r} < \frac{\partial}{\partial p} \underline{V}_p(x) < \frac{\vartheta}{r}.$$

for $-\bar{x}_p \leq x \leq \bar{x}_p$ (property *(v)* of Lemma E.1 in Online Appendix E). For every $x \geq \bar{x}_p$,

$$\frac{\vartheta}{2r} < \frac{\partial}{\partial p} \underline{V}_p(x) = \frac{\vartheta}{\vartheta + 2r} \left(\frac{\partial}{\partial p} \underline{V}_p(x-1) + 1 \right) < \frac{\vartheta}{r},$$

That is, $\vartheta/(2r) < \frac{\partial}{\partial p} \underline{V}_p(x) < \vartheta/r$ for every $x \in \mathbb{Z}$. Thus, for every $x \geq \bar{x}_p$,

$$\frac{\partial}{\partial p} \underline{V}_p(x-1) - 1 < \frac{\partial}{\partial p} \underline{V}_p(x) = \frac{\vartheta}{\vartheta + 2r} \left(\frac{\partial}{\partial p} \underline{V}_p(x-1) + 1 \right) < \frac{\partial}{\partial p} \underline{V}_p(x-1). \tag{D.4}$$

The same inequalities hold for every $0 < x < \bar{x}_p$, since

$$\begin{aligned} 2\zeta \frac{\partial}{\partial p} \underline{V}_p(x) &= \frac{\partial}{\partial p} \underline{V}_p(x+1) + \frac{\partial}{\partial p} \underline{V}_p(x-1) + 2 > \frac{\partial}{\partial p} \underline{V}_p(x) + \frac{\partial}{\partial p} \underline{V}_p(x-1) + 1 \\ \Longrightarrow \frac{\partial}{\partial p} \underline{V}_p(x) &> \frac{\vartheta}{\vartheta + 2r} \left(\frac{\partial}{\partial p} \underline{V}_p(x-1) + 1 \right) > \frac{\partial}{\partial p} \underline{V}_p(x-1) - 1. \end{aligned}$$

That is, $\underline{V}_p(x-1) - \underline{V}_p(x) - p$ is strictly decreasing in p as long as $\bar{x}_p \geq 1$. Thus, $\bar{x}_p$ is increasing in p. Integrating (D.4) over $p \in [p_1, p_2]$, one obtains the desired inequalities. □

The proof of Lemma 6 in Appendix B invoked the next lemma.

Lemma D.7. *For every $x \in \mathbb{Z}$, $\underline{V}_{\vartheta,p,\beta}(x)$ is jointly continuous in $(\vartheta, p, \beta) \in \mathbb{R}^{+3}$.*

Proof. First, given some fixed (p, β), if $0 \leq \vartheta_1 \leq \vartheta_2$, then $\underline{V}_{\vartheta_1} \leq \underline{V}_{\vartheta_2}$. This is because $\underline{B}^{\ell+1}_{\vartheta_2}(\underline{V}_{\vartheta_1}) \geq \underline{B}^{\ell}_{\vartheta_2}(\underline{V}_{\vartheta_1})$ for every $\ell \geq 0$ by induction, and $\underline{B}^{\ell}_{\vartheta_2}(\underline{V}_{\vartheta_1})$ converges to $\underline{V}_{\vartheta_2}$ pointwise. Likewise, $\underline{V}_{\vartheta,p,\beta}(x)$ is non-decreasing in p and non-increasing in β for every $x \in \mathbb{Z}$.

Given a converging sequence of triples $(\vartheta_\ell, p_\ell, \beta_\ell)_{\ell \geq 0}$ of non-negative reals with some limit $(\vartheta_\infty, p_\infty, \beta_\infty)$. The sequence $(\vartheta_\ell, p_\ell, \beta_\ell)_{\ell \geq 0}$ must be bounded. For simplicity, I write $\underline{V}_\ell$ for $\underline{V}_{\vartheta_\ell, p_\ell, \beta_\ell}$ and $\underline{B}_\ell$ for $\underline{B}_{\vartheta_\ell, p_\ell, \beta_\ell}$. For every $x \in \mathbb{Z}$, the sequence $(\underline{V}_\ell(x))_{\ell \geq 0}$ is bounded. Thus, there exists a subsequence $\left(\underline{V}_{\varphi(\ell)}\right)_{\ell \geq 0}$ that converges pointwise to some $\underline{V}$. It is easy to verify that $\underline{V} = \underline{B}_\infty(\underline{V})$. Thus, $V = V_\infty$. Likewise, every subsequence of $(V_\ell)_{\ell \geq 0}$ admits a sub-subsequence that converges to V_∞ pointwise. The next lemma implies that V_ℓ converges to V_∞ pointwise. Thus, for every $x \in \mathbb{Z}$, $V_{\vartheta,p,\beta}(x)$ is jointly continuous in $(\vartheta, p, \beta) \in \mathbb{R}^{+3}$. $\square$

Lemma D.8. *If a real sequence $(y_\ell)_{\ell \geq 0}$ is such that every subsequence of $(y_\ell)_{\ell \geq 0}$ admits a sub-subsequence that converges to the same constant $y_\infty \in \mathbb{R}$, then y_ℓ converges to y_∞.*

Proof. Otherwise, there exists some $\varepsilon > 0$ and a subsequence $\left(y_{\varphi(\ell)}\right)$ such that $\left|y_{\varphi(\ell)} - y_\infty\right| > \varepsilon$ for all ℓ. Subsequence $\left(y_{\varphi(\ell)}\right)$ would not admit a sub-subsequence that converges to y_∞. $\square$

The proof of Lemma 7 in Appendix B is given here:

Lemma D.9. *If $L_k(0) \geq \pi - p$ for some k, then*

$$L_k(0) < L_k(1),$$

and in particular, $L_k(0) = \min_x L_k(x)$ since L_k is W-shaped (Lemma 3).

Proof. If $L_k(0) = 0$, then Lemma 7 holds trivially. If $L_k(0) > 0$, then $\underline{V}_k(0) > 0$ thus $\bar{x}_k \geq 1$. In this case, since $\underline{V}_k(1) + p - \underline{V}_k(0) = r\underline{\psi}_k(0)/(k\eta)$ and $V_{k-1}(1) + p - V_{k-1}(0) \leq r\psi_{k-1}(0)/[(k-1)\eta]$, then it suffices to show that

$$\psi_{k-1}(0) < \frac{k-1}{k}\,\underline{\psi}_k(0).$$

I suppose that the inequality above does not hold. Then

$$\pi - p \leq \underline{\psi}_k(0) - \psi_{k-1}(0) \leq \frac{1}{k}\,\underline{\psi}_k(0).$$

If $\psi_{k'}(0) \geq k'\underline{\psi}_k(0)/k$ for some $0 < k' < k$, then

$$\max\{V_{k'-1}(1) + \pi,\ V_{k'}(1) + p\} - V_{k'}(0) \geq \underline{V}_k(1) + p - \underline{V}_k(0).$$

If $V_{k'}(1) + p - V_{k'}(0) \geq \underline{V}_k(1) + p - \underline{V}_k(0)$, then

$$\psi_{k'}(1) \geq \underline{\psi}_k(1) - [\underline{V}_k(0) - V_{k'}(0)] \geq \underline{\psi}_k(1) - \frac{k-k'}{k}\,\underline{\psi}_k(0) > \frac{k'}{k}\,\underline{\psi}_k(1). \tag{D.5}$$

Thus,

$$\begin{aligned} &\max\{V_{k'-1}(0) + \pi,\ V_{k'}(0) + p\} + \max\{V_{k'-1}(2) + \pi,\ V_{k'}(2) + p,\ V_{k'}(1)\} - 2V_{k'}(1) \\ &> \underline{V}_k(0) + p + \max\{\underline{V}_k(2) + p,\ \underline{V}_k(1)\} - 2\underline{V}_k(1) \end{aligned} \tag{D.6}$$

Since $L_{k,k'}$ and $L_{k,k'-1}$ are W-shaped (Lemma 3) and $L_{k,k'}(0) \geq L_{k,k'}(1)$, (D.6) implies that

$$\begin{aligned} 2L_{k\eta,k'}(1) &> \min\{L_{k,k'-1}(0) - (\pi - p), L_{k,k'}(0)\} + \min\{L_{k,k'-1}(2) - (\pi - p), L_{k,k'}(2), L_{k,k'}(1)\} \\ &\geq 2\min\{L_{k,k'-1}(0) - (\pi - p),\ L_{k,k'}(0),\ L_{k,k'}(1)\}. \end{aligned}$$

Thus,
$$L_{k,k'-1}(0) - (\pi - p) < L_{k,k'}(1) \leq L_{k,k'}(0) \leq \frac{k-k'}{k}\,\underline{\psi}_k(0).$$

Thus, $\psi_{k'-1}(0) > (k'-1)\underline{\psi}_k(0)/k$. If $V_{k'-1}(1) + \pi - V_{k'}(0) \geq \underline{V}_k(1) + p - \underline{V}_k(0)$, then

$$\psi_{k'-1}(1) \geq \underline{\psi}_k(1) - \left[\underline{\psi}_k(0) - \psi_{k'}(0)\right] - (\pi - p) \geq \underline{\psi}_k(1) - \frac{k-k'+1}{k}\,\underline{\psi}_k(0) > \frac{k'-1}{k}\,\underline{\psi}_k(1).$$

Then it must be that $k' > 1$. The same steps following (D.5) implies that $\psi_{k'-2}(0) > (k'-2)\underline{\psi}_k(0)/k$. An immediate induction over k' implies that $0 = \psi_0(0) > 0 \cdot \underline{\psi}_k(0)/k = 0$ which is a contradiction, thus completing the proof. □

D.2 Proof of Proposition 1

Step 1 (indifference condition): I first show that each individual dealer j is dispensable, in that almost surely, a buyside firm is never strictly worse off by permanently terminating its account with j.

Formally, since the state evolution of the trading game is right-continuous, then the regular condition probability $\kappa_{s,\sigma} : \Omega \times \mathcal{F} \mapsto [0,1]$ that describes the likelihood of potential outcomes under any strategy profile σ given what has happened up to Stage 3[25] at time s has at every $t \geq 0$ a right-hand limit $\kappa_{t^+,\sigma} = \lim_{s \downarrow t} \kappa_{s,\sigma}$. For every outcome ω whose Stage 4 at time t proceeds as prescribed by σ (that is, if $\omega_{t4} = \sigma_{t4}(\omega)$, where ω_{t4} is what happens in Stage 4 at time t under outcome ω, and $\sigma_{t4}(\omega)$ are agents' actions prescribed by strategy profile σ),

$$\kappa_{t^+,\sigma}(\omega, B) = \kappa_{t,\sigma}(\omega, B), \ \forall B \in \mathcal{F}. \tag{D.7}$$

That is, the likelihood of potential outcomes does not alter as long as the trading game proceeds as prescribed by strategy profile σ.

I fix a supporting equilibrium (σ, μ) for $G(m)$, and a buyside firm i and some $\mathcal{F}^3_{it}$-stopping time[26] τ. Then for almost every[27] time-τ-Stage-3 information set $h_{i\tau 3}$,

$$\mu_{i\tau}(h_{i\tau 3}, B) = \int_{\omega''} \int_{\omega'} \mu_{i\tau}(h_{i\tau 3}, d\omega')\, \kappa_{\tau,\sigma}(\omega', d\omega'')\, \mu_{i\tau^+}(\omega', B) = \mu_{i\tau^+}(h_{i\tau,J}, B), \ \forall B \in \mathcal{F}^3_\tau,$$

where $h_{it,J'} = [h_{it3}, \omega_{it4} = (\rho_{it}(h_{it3}), J')]$ is the information set consisting of h_{it3} followed by i, as its time-t-Stage-4 actions, accepting/rejecting a potential quote as prescribed by the equilibrium acceptance strategy ρ_{it} and opening accounts with some subset J' of dealers, and μ_{it^+} is the buyside firm's right-hand limiting belief which does not depend on its own account choice[28] J'. Then for almost every $h_{i\tau 3}$,

$$\mu_{i\tau}(h_{i\tau 3}, B) = \mu_{i\tau^+}(h_{i\tau,J'}, B), \ \forall B \in \mathcal{F}^3_\tau, \tag{D.8}$$

for any $J' \subseteq J$. On the other hand, almost surely, the buyside firm's belief $\mu_{i\tau^+}(h_{i\tau,J'}, \cdot)$

[25] Since agents open/terminate their accounts in Stage 4, hence the proof focuses on Stage-4 actions at Stage-3 information sets.

[26] A $\mathcal{F}^3_{it}$-stopping time τ is a (potentially) random time at which i can be asked to take its Stage-4 actions. Karatzas and Shreve (1998) provides the definition and properties for a stopping time.

[27] *Almost every* is with respect to the σ-induced unconditional probability distribution P_σ over the outcome space $(\Omega, \mathcal{F})$.

[28] This is the *no-signaling-what-you-don't-know* requirement in the continuous-time game, as provided by Appendix C.

must be consistent with its information set $h_{i\tau,J'}$ being reached:

$$\operatorname{supp}(\mu_{i\tau^+}(h_{i\tau,J'},\cdot)) \subseteq \{\omega' \,:\, h_{i\tau}(\omega') = h_{i\tau,J'}\} \quad \text{almost surely.} \tag{D.9}$$

For any strategy σ'_i such that $\rho'_{i\tau} = \rho_{i\tau}$, I let $J' = N'_{i\tau}$ be the buyside firm's account choice, $h_{i(\tau+s)3,J'}$ be the time-$(\tau+s)$-Stage-3 information set that succeeds $h_{i\tau,J'}$ after nothing happens to i between time τ and $\tau+s$ for every $s>0$, and $U_i(\sigma'_i \,|\, h_{i(\tau+s)3,J'})$ be the buyside firm's payoff conditional on reaching the information set $h_{i(\tau+s)3,J'}$ following which i deviates to σ'_i, given belief μ. Then for almost-every $h_{i\tau3}$,

$$\begin{aligned}
& U_i\left(\sigma'_i \,|\, h_{i(\tau+s)3,J'}\right) \\
&= \int_{\omega''}\int_{\omega'} \mu_{i(\tau+s)}(h_{i(\tau+s)3,J'}, d\omega')\, \kappa_{\tau+s,\sigma'_i}(\omega', d\omega'')\, u_i(\omega'') \\
&\xrightarrow{s\downarrow 0} \int_{\omega''}\int_{\omega'} \mu_{i\tau^+}(h_{i\tau,J'}, d\omega')\, \kappa_{\tau^+,\sigma'_i}(\omega', d\omega'')\, u_i(\omega'') \\
&= \int_{\omega''}\int_{\omega'} \mu_{i\tau^+}(h_{i\tau,J'}, d\omega')\, \kappa_{\tau,\sigma'_i}(\omega', d\omega'')\, u_i(\omega'') && \text{(following from (D.7) and (D.9))} \\
&= \int_{\omega''}\int_{\omega'} \mu_{i\tau}(h_{i\tau3}, d\omega')\, \kappa_{\tau,\sigma'_i}(\omega', d\omega'')\, u_i(\omega'') && \text{(following from (D.8))} \\
&= U_i\left(\sigma'_i \,|\, h_{i\tau3}\right).
\end{aligned}$$

I let $\tilde{\sigma}_i$ be the strategy obtained from σ_i by substituting the account maintenance strategy N_i with the constant strategy $\tilde{N}_i = J$ which always maintain accounts with all dealers. Then

$$\begin{aligned}
&\int_{\omega''} \kappa_{t,\tilde{\sigma}_i}(\omega', d\omega'')\, u_i(\omega'') = \int_{\omega''} \kappa_{t^+,\tilde{\sigma}_i}(\omega', d\omega'')\, u_i(\omega'') \\
&\qquad \text{for every } t \geq 0 \text{ and every } \omega' \text{ such that } \omega'_{it4} = [\rho_{it}(\omega'), J''] \text{ for any } J'' \subseteq J.
\end{aligned} \tag{D.10}$$

Since i maintains accounts with all the dealers on the equilibrium path (almost surely, $N_{it} = J,\ \forall t$), then

$$\text{For almost-every } \omega',\ \kappa_{t,\sigma_i}(\omega', B) = \kappa_{t,\tilde{\sigma}_i}(\omega', B),\ \text{for every } B \in \mathcal{F} \text{ and } t, \tag{D.11}$$

Then for almost every $h_{i\tau 3}$,

$$\begin{aligned}
&U_i(\tilde{\sigma}_i \,|\, h_{i(\tau+s)3,J'}) \\
&= \int_{\omega''} \int_{\omega'} \mu_{i(\tau+s)}(h_{i(\tau+s)3,J'}, d\omega')\, \kappa_{\tau+s,\tilde{\sigma}_i}(\omega', d\omega'')\, u_i(\omega'') \\
&\xrightarrow{s\downarrow 0} \int_{\omega''} \int_{\omega'} \mu_{i\tau^+}(h_{i\tau,J'}, d\omega')\, \kappa_{\tau^+,\tilde{\sigma}_i}(\omega', d\omega'')\, u_i(\omega'') \\
&= \int_{\omega''} \int_{\omega'} \mu_{i\tau^+}(h_{i\tau,J'}, d\omega')\, \kappa_{\tau,\tilde{\sigma}_i}(\omega', d\omega'')\, u_i(\omega'') && \text{(following from (D.10))} \\
&= \int_{\omega''} \int_{\omega'} \mu_{i\tau}(h_{i\tau 3}, d\omega')\, \kappa_{\tau,\tilde{\sigma}_i}(\omega', d\omega'')\, u_i(\omega'') && \text{(following from (D.8))} \\
&= \int_{\omega''} \int_{\omega'} \mu_{i\tau}(h_{i\tau 3}, d\omega')\, \kappa_{\tau,\sigma_i}(\omega', d\omega'')\, u_i(\omega'') && \text{(following from (D.11))} \\
&= U_i(\sigma_i \,|\, h_{i\tau 3}).
\end{aligned}$$

Now, I fix some dealer j. I first show that almost surely, i is not worse off by switching from the equilibrium strategy σ_i to some stationary strategy $\sigma_i^{(0)}$ following Stage 3 at time τ that has the same quote acceptance strategy as σ_i ($\rho'_{i\tau} = \rho_{i\tau}$) but connects i to a subset $J^{(0)}$ of dealers not including j at time τ. If this is not the case, that is, if given every stationary strategy σ'_i such that $\rho'_{i\tau} = \rho_{i\tau}$ and $j \notin J'$ everywhere, one has $U_i(\sigma'_i \,|\, h_{i\tau 3}) < U_i(\sigma_i \,|\, h_{i\tau 3})$ for $h_{i\tau 3}$ in a non-zero probability set. Then $U_i(\sigma'_i \,|\, h_{i(\tau+s)3,J'}) < U_i(\tilde{\sigma}_i \,|\, h_{i(\tau+s)3,J'})$ for s sufficiently close to 0. Then it is strictly optimal for i to open an account with j at $h_{i(\tau+s)3,J'}$ given belief μ, that is, $j \in N_{i\tau+s}(h_{i(\tau+s)3,J'})$. Since the search strategy of i is stationary, and i can search only a finite number of times during any finite time interval, thus i must not search without receiving an exogenous need to trade. Hence, i always has an inventory of size 0. In addition, since the equilibrium account maintenance strategy N_{it} is stationary, then it is strictly optimal for i to open an account with j whenever i doesn't have one yet. Then, j would extract all rent in every trade with i, knowing that i would never terminate its account with j. It is then suboptimal for i to maintain its account with j, contradicting the optimality of σ_i.

Next, I show that i can permanently terminate its account with j at and after time τ without being worse off. Since the equilibrium strategy σ_i is optimal at every information set, including $h_{i\tau 3}$, then $\sigma_i^{(0)}$ must be optimal at almost every $h_{i\tau 3}$ given belief μ. Since the account maintenance strategy $N_i^{(0)}$ of $\sigma_i^{(0)}$ is stationary, it suffices to modify $\sigma_i^{(0)}$ at times when i receives exogenous needs to trade and show that not opening an account with j at those times remains optimal.

I let $\tau^{(1)}$ be the first time that i receives an exogenous need to trade after time τ. Viewing $\mathrm{P}_{\sigma_i^{(0)}}$ as a probability measure on $\left(\Omega, \mathcal{F}^3_{i\tau^{(1)}}\right)$, I let

$$S = \operatorname{supp}\left(\mathrm{P}_{\sigma_i^{(0)}}\right) \in \mathcal{F}^3_{i\tau^{(1)}}.$$

I let ϕ be an operator on Ω that modifies an outcome $\omega'' \in \Omega$ by switching the dealer accounts of i from $J^{(0)}(\omega'')$ to J for $t \in \left[\tau(\omega''), \tau^{(1)}(\omega'')\right)$. Then ϕ is a 1-1 mapping between S and $\phi(S)$, and both ϕ and ϕ^{-1} are $\mathcal{F}^3_{i\tau^{(1)}}$-measurable. Then for every $A \in \mathcal{F}^3_{i\tau^{(1)}}$ such that $A \subseteq S$,

$$\mathrm{P}_{\sigma_i}(\phi(A)) = 0 \implies \mathrm{P}_{\sigma_i^{(0)}}(A) = 0$$

It follows from the Radon-Nikodym Theorem that there exists some $\mathcal{F}^3_{i\tau^{(1)}}$-measurable $\mathrm{P}_{\sigma_i} \circ \phi$-integrable function $f : S \mapsto \mathbb{R}^+$ such that

$$\mathrm{P}_{\sigma_i^{(0)}}(A) = \int_{\omega \in A} (P_{\sigma_i} \circ \phi)(d\omega)\, f(\omega) = \int_{\omega \in \phi(A)} \mathrm{P}_{\sigma_i}(d\omega)\, f(\phi^{-1}(\omega)) \tag{D.12}$$

I define a transition kernel $\mu^{(1)}$ from $(\Omega, \mathcal{F}^3_{i\tau^{(1)}})$ to $(\Omega, \mathcal{F}^3_{\tau^{(1)}})$ as follows:

$$\mu^{(1)}(\omega, B) = \mu_{i\tau^{(1)}}(\phi(\omega), \phi(B)) \qquad \forall \omega \text{ and } B \in \mathcal{F}^3_{\tau^{(1)}},$$

Then for every $B \in \mathcal{F}^3_{\tau^{(1)}}$ and $A \in \mathcal{F}^3_{i\tau^{(1)}}$,

$$\begin{aligned}
&\int_{\omega} \mathrm{P}_{\sigma_i^{(0)}}(d\omega)\, \mu^{(1)}(\omega, B)\, \mathbb{1}_A(\omega) \\
&= \int_{\omega \in S} \mathrm{P}_{\sigma_i^{(0)}}(d\omega)\, \mu_{i\tau^{(1)}}(\phi(\omega'), \phi(B)), \mathbb{1}_A(\omega) \\
&= \int_{\omega \in \phi(S)} \mathrm{P}_{\sigma_i}(d\omega)\, f(\phi^{-1}(\omega))\, \mu_{i\tau^{(1)}}(\omega, \phi(B))\, \mathbb{1}_{\phi(A)}(\omega) \quad \text{(following from (D.12))} \\
&= \int_{\omega \in \phi(S)} \mathrm{P}_{\sigma_i}(d\omega)\, f(\phi^{-1}(\omega))\, \mathbb{1}_{\phi(B \cap A)}(\omega) \\
&= \mathrm{P}_{\sigma_i^{(0)}}(A \cap B) \qquad\qquad\qquad\qquad\qquad\qquad \text{(following from (D.12))}.
\end{aligned}$$

Then $$\mu^{(1)}(\cdot, B) = \mathrm{P}_{\sigma_i^{(0)}}\left(B \,|\, \mathcal{F}^3_{i\tau^{(1)}}\right), \ \mathrm{P}_{\sigma_i^{(0)}}\text{-almost-surely.} \tag{D.13}$$

Now, I turn to modify $\sigma_i^{(0)}$. There exists some stationary strategy σ_i' that is optimal at almost every $h_{i\tau^{(1)}3}$ given belief μ with the same quote acceptance strategy as σ_i ($\rho'_{i\tau^{(1)}} = \rho_{i\tau^{(1)}}$ everywhere) without having an account with j ($j \notin N'_{i\tau^{(1)}}$ everywhere). I define $\sigma_i^{(1)}$ as follows: for every time-t-Stage-k information set h_{itk} ($k = 1, 2, 3$), I let

$$\sigma_{itk}^{(1)}(\omega) = \begin{cases} \sigma_{itk}^{(0)}(\omega) & t < \tau^{(1)}, \text{ or } t = \tau^{(1)}, \ k = 1, 2, \\ \sigma'_{itk}(\phi(\omega)) & t = \tau^{(1)}, \ k = 3, \\ \sigma'_{itk}(\omega) & t > \tau^{(1)}, \ k = 1, 2, 3. \end{cases}$$

That is, $\sigma_i^{(1)}$ is obtained by switching from $\sigma_i^{(0)}$ to σ_i' following Stage 3 at time $\tau^{(1)}$. Then $\rho^{(1)}_{i\tau^{(1)}} = \rho_{i\tau^{(1)}} \circ \phi$.

If σ_i' is optimal at some $h_{i\tau^{(1)}3} \subseteq \phi(S)$ given belief μ, then $\sigma_i^{(1)}$ is optimal at $\phi^{-1}\left(h_{i\tau^{(1)}3}\right)$ given belief $\mu^{(1)}$. Since σ_i' is optimal at almost every $h_{i\tau^{(1)}3}$ given belief μ, and

$$\begin{aligned} &\text{For some } A \in \mathcal{F}^3_{i\tau^{(1)}}, \ \mathrm{P}_{\sigma_i}(A) = 1, \\ \Longrightarrow\ &\mathrm{P}_{\sigma_i^{(0)}}(\phi^{-1}(A \cap \phi(S))) = \int_{\omega' \in A \cap \phi(S)} \mathrm{P}_{\sigma_i}(d\omega')\, f(\omega') \quad \text{(following from (D.12))} \\ &\qquad = \int_{\omega' \in \phi(S)} \mathrm{P}_{\sigma_i}(d\omega')\, f(\omega') = \mathrm{P}_{\sigma_i^{(0)}}(S) = 1, \end{aligned}$$

then $\sigma_i^{(1)}$ is optimal at $\mathrm{P}_{\sigma_i^{(0)}}$-almost-every $h_{i\tau^{(1)}3}$ given belief $\mu^{(1)}$. Since $\sigma_i^{(1)}$ and $\sigma_i^{(0)}$ are identical prior to Stage 3 of time $\tau^{(1)}$, thus $\mathrm{P}_{\sigma_i^{(0)}}$ and $\mathrm{P}_{\sigma_i^{(1)}}$ are identical as a probability distribution on $(\Omega, \mathcal{F}^3_{i\tau^{(1)}})$. Thus

$$\begin{aligned} &\mathrm{E}_{\sigma_i^{(1)}}\left(u_i \,|\, \mathcal{F}^3_{i\tau^{(1)}}\right) \geq \mathrm{E}_{\sigma_i^{(0)}}\left(u_i \,|\, \mathcal{F}^3_{i\tau^{(1)}}\right) \quad \mathrm{P}_{\sigma_i^{(0)}}\text{-almost-surely,} \qquad \text{(following from (D.13))} \\ \Longrightarrow\ &\mathrm{E}_{\sigma_i^{(1)}}\left(u_i \,|\, \mathcal{F}^3_{i\tau}\right) \geq \mathrm{E}_{\sigma_i^{(0)}}\left(u_i \,|\, \mathcal{F}^3_{i\tau}\right) \qquad \text{almost surely.} \end{aligned}$$

Since $\sigma_i^{(0)}$ is optimal at almost every $h_{i\tau 3}$ given belief μ, then $\sigma_i^{(1)}$ is also optimal at almost every $h_{i\tau 3}$ given belief μ, with the same equilibrium time-$\tau^{(1)}$ quote acceptance strategy as if i had maintained accounts with all dealers ($\rho^{(1)}_{i\tau^{(1)}} = \rho_{i\tau^{(1)}} \circ \phi$ everywhere) and without opening an account with j at time $\tau^{(1)}$ under any circumstance ($j \notin N^{(1)}_{i\tau^{(1)}}$ everywhere).

It follows from an induction over $k = 1, 2, \ldots$, that there exists some stationary strategy

$\sigma_i^{(k)}$ such that $\sigma_{it}^{(k)} = \sigma_{it}^{(k-1)}$ at all times $t < \tau^{(k)}$ prior to the arrival of the k'th exogenous need to trade, $\sigma_i^{(k)}$ is optimal at almost every $h_{i\tau 3}$ given belief μ with the same equilibrium time-$\tau^{(k)}$ quote acceptance strategy as if i had maintained accounts with all dealers and without opening an account with j at time $\tau^{(k)}$ under any circumstance ($j \notin N_{i\tau^{(k)}}^{(k)}$ everywhere). Then the limiting stationary strategy $\sigma_i^{(\infty)}$ is optimal at almost every $h_{i\tau 3}$ given belief μ without opening an account with j at and after time τ. That is, i can permanently terminate its account with j following almost every $h_{i\tau 3}$ without being worse off.

Step 2 (constant pricing): Now, I show that each dealer j offers each buyside firm i some constant ask a_{ji}^* and bid b_{ji}^* with a spread $a_{ji}^* - b_{ji}^* = 2p^*(m)$ almost surely, where $p^*(m)$ is the equilibrium mid-to-bid spread given by (3). Intuitively, if j were to offer non-constant prices, then i cannot always be indifferent to whether it terminates its account with j—keeping an account with j is relatively more likely to be a waste of time when i expects to receive a worse price from j. This contradicts the indifference condition established in *Step 1*.

Formally, I let a_{ji}^* and b_{ji}^* be the highest ask and the lowest bid from j that i would accept on the equilibrium path without terminating its account with j. Then, on the equilibrium path, j always offers i a_{ji}^* and b_{ji}^* whenever it executes a trade on its own account.

Hence, the optimal pricing strategy of j is characterized by the HJB equation

$$rV(x) = -\beta x^2 + \sum_{i \in I} \frac{\eta_m}{2} \left(\left[V(x-1) - V(x) + a_{ji}^* \right]^+ + \left[V(x+1) - V(x) - b_{ji}^* \right]^+ \right).$$

Adapting the proof of Proposition 2 shows that V is concave, thus j offers i the ask a_{ji}^* (or the bid b_{ji}^*) when its inventory position is above (below) certain threshold $\bar{x}_{ji}^-$ ($\bar{x}_{ji}^+$) and otherwise would not take a trade into its inventory.

Next, I show that j offers i the same a_{ji}^* and b_{ji}^* on the equilibrium path even when its inventory position is outside the range $(\bar{x}_{ji}^-, \bar{x}_{ji}^+)$. If this is not the case, say j offers i some bid $b < b^*$ at some inventory position in excess of the upper-threshold $\bar{x}_{ji}^+$, and

$$\mathrm{P}(\text{At some time, } j \text{ offers } i \text{ the bid price } b) > 0.$$

I now show that it is more likely to be a waste of time for i to keep an account with j after receiving the lower bid b relative to the case where i had received the bid b_{ji}^* instead, all else the same. Thus, i cannot be indifferent to whether it maintains its account with j in both cases. Upon receiving the bid b, since i is not worse off by immediately and permanently terminating its account with j, then its conditional payoff is the same as if i had received the bid b^*_{ji} instead of b, all else the same. If i maintains its account with j for an extra instant dt, its continuation payoff after time dt is again the same regardless of whether i had received b or b^*_{ji}, all else the same, because i is again not worse off by permanently terminating its account with j after time dt. This would imply that the instantaneous rate of benefit to i of keeping its account with j should the same regardless of whether i received the bid b^*_{ji} instead of b. However, conditional on receiving the bid b, i would reject the bid because otherwise, j would be strictly better off raising its bid to b^*_{ji}. At the same time, i would infer that the inventory position of j is in excess of the upper-threshold $\bar{x}^+_{ji}$. Thus, if i maintains its account with j for an extra instant dt, and if i were to receive another quote from j in the next instant, then i expects to be offered the ask a^*_{ji} and the same low bid b. Therefore, the expected benefit to i of having another trade with j in the next instant is lower than that if i had received b^*_{ji} instead of b at the first place. This contradiction shows that j must offer i the constant bid b^*_{ji} (and likewise, the constant ask a^*_{ji}) with probability 1.

Then the buyside firm indifference condition established in *Step 1* implies that the bid-ask spread $a^*_{ji} - b^*_{ji}$ offered by j must the same across all dealers $j \in J$. Otherwise, if dealer j offered a wider spread than another dealer j', then terminating the account with j strictly dominates terminating the account with j'. Then i cannot be indifferent to whether it terminates its account with j while also being indifferent to whether it terminates its account with j', contradicting the indifference condition.

Letting p be the mid-to-bid spread offered by the dealers, then the rate of benefit when maintaining d dealer accounts is $\Phi_{d,p}$ given by (2), which is strictly concave in d (Figure 5). Every individual dealer to be dispensable if and only if $\Phi_{m,p} = \Phi_{m-1,p}$, or $p = p^*_m$, where the equilibrium spread $p^*(m)$, given in (3), is strictly decreasing in the number m of dealers.

D.3 Proof of Theorem 1

Sufficient condition: I first show that $m \leq m^*$ is sufficient for $G(m)$ to be an equilibrium networkrk by formally completing the construction of the supporting equilibrium.

Buyside firms employ the same strategy, as follows: Each buyside firm i searches among its connected dealer counterparties upon receiving an exogenous need to trade, and accepts any ask $a \leq \pi$ or any bid $b \geq -\pi$. For account maintenance, each buyside firm employs the "grim-trigger" strategy as described in Section 3.

Next, I complete each dealer's strategy. Upon receiving an RFQ from a buyside firm, each dealer j employs the optimal pricing strategy characterized by its HJB equation Equation (4). If j receives an RFQ from another dealer j' (which is an off-the-equilibrium path event), then j quotes the highest ask (or the lowest bid) that j' would ever possibly accept. Formally, I let $\{\pi_\ell\}_{\ell=1,\ldots,h}$ be all conceivable reservation prices of j', with $\pi_1 \leq \pi_2 \leq \ldots \leq \pi_h$, and each $\pi_\ell = V_k(x+1) - V_k(x)$ for some $k \leq n - m$ and x. Then j quotes $a = \pi_h$ or $b = -\pi_h$. If j receives a quote from another dealer (which is an off-the-equilibrium path event), then j accepts the quote if and only if the quote is within its reservation price. A dealer never opens any trade accounts, on or off the equilibrium path, thus never searches. This completes the equilibrium strategy profile.

I next turn to the belief system. Since all strategies are Markovian, it suffices to specify, at each information set, an agent's belief about the game's current state consisting of the current network structure and the current inventory position of each agent. Each agent always believes, on or off the equilibrium path, that other agents are connected to each other as prescribed by the equilibrium network G_m, and holds any consistent belief about other dealers' inventory positions[29], with the following exception. If dealer j receives an RFQ from another dealer j' (which is an off-the-equilibrium-path event), then j believes that j' is in a state (k, x) such that j' has the widest conceivable reservation price $V_k(x+1) - V_k(x) = p_h$.[30] Each agent always believes, on or off the equilibrium path, that other buyside firms have an inventory of size 0. This completes the construction of the belief system.

[29]There is no need to assume any specific off-the-equilibrium-path belief about the inventory position of a given dealer j because with $n - m$ buyside customers, j quotes constant bid and ask regardless of its inventory position.

[30]This is the most natural belief given iterated elimination of dominated strategies as follows: for j, quoting any ask price $a < \pi_1$ is strictly dominated by quoting $a = \pi_1$ because any type of j' would be willing to accept $a = \pi_1$. After eliminating these dominated pricing choices of j, then for the lowest type π_1 of j', opening an account with j is strictly dominated by not opening one, because type π_1 would only have link cost to incur with no gains from trade to capture. Then for j, quoting any ask price $a < \pi_2$ is strictly dominated by quoting $a = \pi_2$, and so on.... Such choice of off-the-equilibrium-path beliefs is not necessary for ruling out non-equilibrium networks, as shown later.

It is easy to check that the belief system is consistent with the strategy profile, which in turn is sequentially rational given the belief system. Therefore, the strategy profile and the belief system constitute a PBE.

Necessary condition: Next, I show that $m \leq m^*$ is necessary. If dealer j offers $a^* = p^*(m)$ and $b^* = -p^*(m)$ with a mid-price of 0 to its buyside customers, then the no-gouging condition for j implies that that $m \leq m^*$. It remains to establish the same if j offers $a^* = p^*(m) + h$ and $b^* = -p^*(m) + h$ with a potentially non-zero mid-price h. Intuitively, offering a negative (positive) equilibrium mid price gives the dealer a larger one-shot benefit from gouging upon receiving a request to buy (sell) from i, which tightens the dealer's no-gouging condition.

The one-shot benefit from gouging a buyside firm is

$$\max\{\pi - p - h, \pi - p + h\} = \pi - p + |h|,$$

For every $h \in \mathbb{R}$, I let $V_{k,h}$ be the value function of a dealer offering mid price h and mid-to-bid spread p to its k buyside customers, $\underline{V}_{k,h}$ be the dealer's value function if the dealer was restricted from gouging buyside firms, and $\psi_{k,h} = V_{k,h} + \beta x^2/r$, $\underline{\psi}_{k,h} = \underline{V}_{k,h} + \beta x^2/r$. Then the dealer optimally controls its inventory within some range $[\underline{x}, \bar{x}]$. Thus the expected cost of losing a buyside customer is at most

$$\mathcal{L}(h) = \min_{x \in (\underline{x}, \bar{x})} [\underline{V}_{k,h}(x) - V_{k-1,h}(x)] = \min_{x \in (\underline{x}, \bar{x})} \left[\underline{\psi}_{k,h}(x) - \psi_{k-1,h}(x)\right]$$

Then a necessary condition for no-gouging is that the one-shot benefit not exceeding the expected cost of gouging: $\mathcal{L}(h) \geq \pi - p + |h|$.

The function $\psi_{k-1,h}$ solves the fixed point problem $\psi = T_{k-1,h}(\psi)$, where

$$\begin{aligned} &T_{k-1,h}(\psi)(x) \\ = &\frac{\delta_{(k-1)\eta}}{2}\Bigg(\left[\psi(x-1) + \frac{\beta(2x-1)}{r} + p + h\right] \vee \psi(x) \vee \left[\psi_{k-1}(x-1) + \frac{\beta(2x-1)}{r} + \pi\right] \\ &\qquad + \left[\psi(x+1) - \frac{\beta(2x+1)}{r} + p - h\right] \vee \psi(x) \vee \left[\psi_{k-1}(x+1) - \frac{\beta(2x+1)}{r} + \pi\right]\Bigg). \end{aligned}$$

The fixed point problem $\psi = T_{k-1,h}(\psi)$ can be viewed as the HJB equation to the following *hypothetical* dynamic programming problem: at mean rate $\eta/2$, an agent with an inventory position x, which is not subject to any inventory cost, receives an opportunity to trade one unit of the asset with customer i and receive $\beta(2x-1)/r+p+h$ when selling $(-\beta(2x+1)/r+p-h$ when buying) or a higher payment $\beta(2x-1)/r+\pi$ when selling $(-\beta(2x+1)/r+\pi$ when buying) at the expense of losing the customer.

For a given $x_0 \in \mathbb{Z}$, I let $\tilde{\psi}_{k-1,h}$ be the agent's value function if the agent acts as if it had an inventory position of $x - x_0$ while its actual inventory position is x. For every $h \in \mathbb{R}$, the agent follows the same strategy which is symmetric around x_0: for example, if the agent sells without gouging at inventory position $x - x_0$, then it would buy without gouging at inventory position $2x_0 - x$. Thus, the value $\tilde{\psi}_{k-1,h}(x_0)$ evaluated at x_0 is not affected by h: offering a positive mid-quote h means that the dealer receives more when selling but less when buying, with the net effect being 0. Hence,

$$\tilde{\psi}_{k-1,h}(x_0) = \tilde{\psi}_{k-1,0}(x_0), \text{ and likewise, } \tilde{\psi}_{k-1,0}(x_0) = \psi_{k-1,0}(0).$$

Since the optimized value function $\psi_{k-1,h}$ is at least as large as the non-optimized value function $\tilde{\psi}_{k-1,h}$, it then follows that

$$\psi_{k-1,h}(x_0) \geq \tilde{\psi}_{k-1,h}(x_0) = \tilde{\psi}_{k-1,0}(x_0) = \psi_{k-1,0}(0), \qquad \forall x_0 \in \mathbb{Z}.$$

Then the no-gouging condition $\mathcal{L}(h,k) \geq \pi - p + |h|$ implies that

$$\min_{x \in (\underline{x}, \bar{x})} \underline{\psi}_{k,h}(x) - \psi_{k-1,0}(0) \geq \pi - p + |h|,$$

If I can show that

$$\min_{x \in (\underline{x}, \bar{x})} \underline{\psi}_{k,h}(x) - \underline{\psi}_{k,0}(0) \leq |h|, \tag{D.14}$$

then the no-gouging condition would imply $\underline{\psi}_{k,0}(0) - \psi_{k,0}(0) \geq \pi - p$, which is equivalent to the desired necessary condition $m \leq m^*$ when $k = n - m$ and $p = p^*(m)$.

To establish (D.14), I remove k from subscripts to ease notation. It can be verified that

$$\underline{\psi}_h(x) = \underline{\psi}_{h+2\beta x/r}(0)$$

I can then extend domain of $\underline{\psi}_h(x)$ from $x \in \mathbb{Z}$ to $x \in \mathbb{R}$ using the above equation, and the domain of $\underline{V}_h(x)$ from $x \in \mathbb{Z}$ to $x \in \mathbb{R}$ by defining $\underline{V}_h = \underline{\psi}_h - \beta x^2/r$. The value function $\underline{V}_h$ solves the HJB equation: for every $x \in \mathbb{R}$,

$$
\begin{aligned}
r\underline{V}_h(x) = -\beta x^2 + \frac{k\eta}{2}\big(& [\underline{V}_h(x-1) - \underline{V}_h(x) + p + h)]^+ \\
& + [\underline{V}_h(x+1) - \underline{V}_h(x) + p - h]^+ \big).
\end{aligned} \tag{D.15}
$$

Since $\underline{\psi}_h(x)$ and thus $V_h(x)$ are jointly continuous in (x, h), the Intermediate Value Theorem implies the existence of some $\tilde{x}_0 \in \mathbb{R}$ such that

$$
\bar{x} = \lfloor \tilde{x} \rfloor, \qquad V_0(\tilde{x} - 1) - V_0(\tilde{x}) = p.
$$

Then the dealer optimally controls its inventory within the interval $I_h = \left[-\tilde{x} - \frac{rh}{2\beta}, \tilde{x} - \frac{rh}{2\beta}\right]$.

I formally differentiate $\underline{V}_h(x)$ with respect to h in (D.15) to obtain, for every $x \in \mathbb{Z}$,

$$
\zeta \frac{\partial}{\partial h} \underline{V}_h(x) = \begin{cases} \frac{1}{2}\left[\frac{\partial}{\partial h}\underline{V}_h(x-1) + \frac{\partial}{\partial h}\underline{V}_h(x+1)\right], & -\tilde{x} + 1 \le x + \frac{rh}{2\beta} \le \tilde{x} - 1, \\ \frac{1}{2}\left[\frac{\partial}{\partial h}\underline{V}_h(x-1) + \frac{\partial}{\partial h}\underline{V}_h(x) + 1\right], & x + \frac{rh}{2\beta} > \tilde{x} - 1, \\ \frac{1}{2}\left[\frac{\partial}{\partial h}\underline{V}_h(x+1) + \frac{\partial}{\partial h}\underline{V}_h(x) - 1\right], & x + \frac{rh}{2\beta} < -\tilde{x} + 1. \end{cases}
$$

I let ℓ be the number of integers in the interval I_h and ϖ be the vector

$$
\varpi = (-1/2, 0, \ldots, 0, 1/2)^\top.
$$

The linear system can be written as $A\, \frac{\partial}{\partial h}\underline{V}_h = \varpi$, where A is the matrix (E.1) in Online Appendix E of size $\ell \times \ell$.

For every $-\beta/r \le h \le \beta/r$, then $\underline{V}_h(0)$ is the middle entry of $\underline{V}_h$. Letting $s = \lfloor (\ell+1)/2 \rfloor$, it then follows from property *(vi)* of Lemma E.1 in Online Appendix E that

$$
0 \le \frac{\partial}{\partial h}\underline{V}_h(0) = \frac{1}{2}\left(-A^{-1}_{\ell+1-s,1} + A^{-1}_{\ell+1-s,\ell}\right) = \frac{1}{2}\left(-A^{-1}_{\ell+1-s,1} + A^{-1}_{s,1}\right) \le 1.
$$

$$
\Longrightarrow \underline{\psi}_h(0) - \underline{\psi}_0(0) = \underline{V}_h(0) - \underline{V}_0(0) \le h.
$$

For any $h \in \mathbb{R}$, I let $x \in \mathbb{Z}$ be such that $-\beta/r \leq h + 2\beta x/r \leq \beta/r$, then $x \in I_h$ and

$$\underline{\psi}_h(x) - \underline{\psi}_0(0) = \underline{\psi}_{h+2\beta x/r}(0) - \underline{\psi}_0(0) \leq h.$$

This establishes the desired inequality (D.14), which implies $m \leq m^*$.

D.4 Proof of Corollary 1

In network G, the sum of outdegrees equals to the sum of indegrees. Thus,

$$\sum_{j \in J} k_j = m^*(n - |J|).$$

Since $k_j \geq k(m^*)$ for every $j \in J$ (Theorem 2), it then follows from that

$$(k(m^*) + m^*)|J| \leq m^* n, \quad \Longrightarrow \quad |J| \leq \frac{m^* n}{k(m^*) + m^*}.$$

D.5 Proof of Proposition 4

Since a dealer's value $V_\vartheta(0)$ is increasing and strictly convex in its rate of customer order flow ϑ (Lemma 6), then by Jensen's inequality, it is strictly more efficient to concentrate the order flow on a smaller set of dealers. Thus, $U(\sigma) < U(\sigma^*_m)$.

D.6 Proof of Proposition 5

Step 1: I first show that the supporting equilibrium $\sigma^*(m^*)$ for $G(m^*)$ constructed in Appendix D.3 is coalition-proof. In any joint deviation by a coalition involving at least one buyside firm, at least one agent i does not receive any incoming link, thus is on the buyside in the deviation. If i was also on the buyside in the underlying equilibrium $\sigma^*(m^*)$, then its expected payoff in the proposed deviation cannot be higher than its equilibrium expected payoff $\Phi(m, p^*(m))/r$ in $\sigma^*(m^*)$. Otherwise, at least one agent j in the coalition would need to offer i a mid-to-bid spread tighter than $p^*(m)$ as part of its deviation. Then j would be strictly better off gouging i, deviating away from its deviation. Thus, the joint deviation is not coalition proof. If i was a dealer without the deviation, then i is worse off in the deviation because it no longer captures any intermediation profits.

In any joint deviation by a coalition of dealers, if some of the deviating dealers become a buyside firm in the deviation, then that dealer is worse off in the deviation. If all dealers in the coalition remain dealers in the deviation, then they cannot be better off. This is because all the buyside firms, not part of the deviating coalition, still employ the same grim trigger as their account maintenance strategy. In response, any dealer would optimally stick to its equilibrium pricing strategy.

Step 2: Given any equilibrium network G with a maximum outdegree $d(G) < m^*$, the buyside firms can jointly deviate to get rid of all dealers and trade in their own concentrated core-periphery network $G_I(m^*)$ with m^* dealers, in the same way as they would in the supporting equilibrium $\sigma_I^*(m^*)$ for $G_I(m^*)$. Such a deviation is coalition proof because $\sigma_I^*(m^*)$ is, and makes every buyside firm strictly better off. Hence, G is not coalition-proof.

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E A Symmetric Tri-Diagonal Matrix

This appendix establishes some properties for the inverse of a symmetric tri-diagonal matrix. I let n be a strictly positive integer. A vector ψ of length n is said to be *U-shaped* if

$$\psi_i = \psi_{n+1-i}, \qquad \forall\, 1 \le i \le n, \qquad \text{and} \quad \psi_1 > \psi_2 > \cdots > \psi_m, \text{ where } m = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

The vector ψ is *weakly U-shaped* if the inequalities above become weak inequalities while at least one remains strict. Given two vectors ψ and φ, I write $\psi < \varphi$ if ψ is strictly less than φ entry-wise. Given a constant $\zeta > 1$, I let A be the following $n \times n$ tri-diagonal matrix:

$$A = \begin{pmatrix} \zeta - \frac{1}{2} & -\frac{1}{2} & & & & \\ -\frac{1}{2} & \zeta & -\frac{1}{2} & & & \\ & -\frac{1}{2} & \zeta & -\frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{2} & \zeta & -\frac{1}{2} \\ & & & & -\frac{1}{2} & \zeta - \frac{1}{2} \end{pmatrix} \tag{E.1}$$

Lemma E.1. *The matrix A is invertible. Its inverse $M \equiv A^{-1}$ satisfies the following properties:*

(i) The matrix M is symmetric, with strictly positive entries.

(ii) For every $1 \le i \le n$, $M_{i,1} < M_{i,2} < \cdots < M_{i,i}$ and $M_{i,i} > M_{i,i+1} > \cdots > M_{i,n}$.

(iii) For every $i \neq j$, $M_{i,j-1} + M_{i,j+1} > 2M_{i,j}$.

(iv) For every i, j, $M_{i,j} = M_{n+1-i,n+1-j}$.

(v) If a vector ψ is weakly U-shaped, then $M\psi$ is U-shaped and

$$\frac{\psi_m}{\zeta - 1} < (M\psi)_n < \frac{\psi_n}{\zeta - 1}.$$

(vi) Letting $m = \lfloor (n+1)/2 \rfloor$, then $M_{m,1} - M_{n+1-m,1} < 2$.

(vii) If $n \to \infty$ and $\zeta \to 1$ with $(\zeta - 1)n \to 0$, letting $\operatorname{range} M = \max_{(i,j)} M_{ij} - \min_{(i,j)} M_{ij}$,

$$(\zeta - 1)\operatorname{range} M \sim (\zeta - 1)n.$$

Proof. Properties (i). Since $\zeta > 1$, the matrix A is diagonally dominant thus invertible. Its inverse M is symmetric since A is. The matrix A can be written as $A = \zeta I - D/2$, where

$$D = \begin{pmatrix} 1 & 1 & & & \\ 1 & & 1 & & \\ & \ddots & & \ddots & \\ & & 1 & & 1 \\ & & & 1 & 1 \end{pmatrix}$$

The sup-norm of the matrix D is $||D||_\infty = 2$. All entries of M are strictly positive, since

$$M = A^{-1} = \zeta^{-1}\left(I - \frac{D}{2\zeta}\right)^{-1} = \zeta^{-1}\left[I + \frac{D}{2\zeta} + \left(\frac{D}{2\zeta}\right)^2 + \dots\right].$$

Properties (ii) and (iii). One has $MB/2 = \zeta M - I$. Then for every $i > 1$,

$$\frac{M_{i,1} + M_{i,2}}{2} = \zeta M_{i,1} > M_{i,1} \implies M_{i,1} < M_{i,2}.$$

I suppose $M_{i,j-1} < M_{i,j}$ for some $j \in (1, i)$, then

$$\frac{M_{i,j-1} + M_{i,j+1}}{2} = \zeta M_{i,j} > M_{i,j} \implies M_{i,j} < M_{i,j+1}.$$

By induction, one has $M_{i,j} < M_{i,j+1}$ if $j < i$. Similarly, one has $M_{i,j} < M_{i,j-1}$ if $j > i$.

Property (iv). It is clear that $D_{i,j} = D_{n+1-i,n+1-j}$ for every i, j. If $D^\ell_{i,j} = D^\ell_{n+1-i,n+1-j}$, then

$$D^{\ell+1}_{i,j} = \sum_k D_{i,k}\left(D^\ell\right)_{k,j} = \sum_k D_{n+1-i,n+1-k}\left(D^\ell\right)_{n+1-k,n+1-j} = D^{\ell+1}_{n+1-i,n+1-j}.$$

Therefore, $M_{i,j} = M_{n+1-i,n+1-j}$ for every i, j.

Property (v). Given a weakly U-shaped vector ψ, then for every i,

$$(M\psi)_i = \sum_j M_{i,j}\psi_j = \sum_j M_{n+1-i,n+1-j}\psi_{n+1-j} = (M\psi)_{n+1-i}.$$

I let $e = (1, \ldots, 1)^\top$ and for every $0 < k \le m$, I let

$$w(k) = \left(\underbrace{1, \ldots, 1}_{k\ 1's}, \underbrace{0, \ldots, 0}_{(n-2k)\ 0's}, \underbrace{1, \ldots, 1}_{k\ 1's} \right)^\top .$$

Any weakly U-shaped vector ψ can be written as a linear combination of the vectors $w(k)$ and e. Thus, to show that $M\psi$ is U-shaped for any weakly U-shaped vectors ψ, it is sufficient to show that $Mw(k)$ is U-shaped for every $0 < k \le m$. For every $i \in [k, m)$,

$$\begin{aligned}[Mw(k)]_{i+1} &= \sum_{j \le k} M_{i+1,j} + \sum_{j > n-k} M_{i+1,j} \\ &= \sum_{j \le k} (M_{j,i+1} + M_{j,n-i}) < \sum_{j \le k} (M_{j,i} + M_{j,n-i+1}) = [Mw(k)]_i.\end{aligned}$$

Since $Ae = (\zeta - 1)e$, thus $Me = e/(\zeta - 1)$. Then for every $i < k$,

$$[Mw(k)]_{i+1} = \frac{1}{\zeta - 1} - \sum_{k < j \le n-k} M_{j,i+1} < \frac{1}{\zeta - 1} - \sum_{k < j \le n-k} M_{j,i} = [Mw(k)]_i.$$

Therefore, $Mw(k)$ is U-shaped. Given a weakly U-shaped vector ψ and for every i,

$$\frac{1}{\zeta - 1}\,\psi_m = \sum_j M_{n,j}\,\psi_m < (M\psi)_n = \sum_j M_{n,j}\,\psi_j < \sum_j M_{n,j}\,\psi_n = \frac{1}{\zeta - 1}\,\psi_n.$$

Property (vi). I let $H = (-2A)^{-1}$. Then property *(vi)* is equivalent to $H_{n+1-m,1} - H_{m,1} < 1$. If $n = 2$, $H_{n+1-m,1} - H_{m,1} = 1/(2\zeta) < 1$. If $n > 2$, I define the second-order linear recurrences

$$z_k = -2\zeta z_{k-1} - z_{k-2}, \qquad k = 2, 3, \ldots, n-1$$

where $z_0 = 1$, $z_1 = 1 - 2\zeta$. I let $\zeta = \cosh\gamma$ where $\gamma > 0$. It follows from induction that

$$z_k = (-1)^k \frac{\cosh\left(\left(k + \frac{1}{2}\right)\gamma\right)}{\cosh\frac{\gamma}{2}} \qquad k = 0, 1, \ldots, n-1.$$

Huang and McColl (1997) calculate the entries of H in closed form. In particular,

$$H_{1,1} = \frac{1}{1 - 2\zeta - \cfrac{z_{n-2}}{z_{n-1}}} = -\frac{\cosh\left(\left(n - \frac{1}{2}\right)\gamma\right)}{2\sinh\left(n\gamma\right)\,\sinh(\gamma/2)}, \tag{E.2}$$

$$H_{i,1} = (-1)^{i-1}\,\frac{z_{n-i}}{z_{n-1}}\,H_{1,1} \qquad \forall i > 1.$$

It then follows that for every $1 \leq i \leq n$,

$$\begin{aligned} H_{n+1-i,1} - H_{i,1} &= \left(\left|\frac{z_{i-1}}{z_{n-1}}\right| - \left|\frac{z_{n-i}}{z_{n-1}}\right|\right) H_{1,1} \\ &= \frac{\cosh\left(\left(n - i + \frac{1}{2}\right)\gamma\right) - \cosh\left(\left(i - \frac{1}{2}\right)\gamma\right)}{2\sinh(n\gamma)\,\sinh\frac{\gamma}{2}} \\ &= \frac{2\sinh(n\gamma/2)\,\sinh((n-2i+1)\gamma/2)}{2\sinh(n\gamma)\,\sinh(\gamma/2)} \end{aligned} \tag{E.3}$$

When $i = m$, one has

$$H_{n+1-m,1} - H_{m,1} \leq \frac{\sinh(n\gamma/2)\,\sinh\gamma}{\sinh\left(n\gamma\right)\,\sinh(\gamma/2)} = \frac{\cosh(\gamma/2)}{\cosh\left(n\gamma/2\right)} < 1.$$

Property (vii): Since $n\gamma$ goes to 0, it follows from (E.2) that $H_{1,1} \sim -1/(n\gamma^2)$. Letting $i = j = 1$ in (E.3), one has $H_{1,1} - H_{n,1} \sim -n/2$. This implies that

$$(\zeta - 1)\left(\max_{(i,j)} M_{ij} - \min_{(i,j)} M_{ij}\right) = (\zeta - 1)(M_{1,1} - M_{n,1}) \sim (\zeta - 1)n. \qquad \square$$

F Proofs for Section 4

F.1 Proof of Proposition 6

Part (i): I fix $m \geq 1$ and $p > 0$, and suppress η_m and p from the subscripts to simplify notation. Since $\frac{\partial}{\partial\vartheta}\,\underline{V}_\vartheta$ is U-shaped (Lemma 4), then for every $x \geq 0$,

$$\begin{aligned} &[\underline{V}_{n+1-m}(x) - \underline{V}_{n+1-m}(x+1)] - [\underline{V}_{n-m}(x) - \underline{V}_{n-m}(x+1)] \\ &= \int_{(n-m)\eta_m}^{(n-m+1)\eta_m} \left[\frac{\partial}{\partial\vartheta}\,\underline{V}_\vartheta(x) - \frac{\partial}{\partial\vartheta}\,\underline{V}_\vartheta(x+1)\right] d\vartheta < 0. \end{aligned}$$

Since $\underline{V}_{n-m}(x) - \underline{V}_{n-m}(x+1)$ is strictly decreasing in n, it admits some limit $\Delta_\infty(x)$.

It will be shown in the proof of Proposition 7 that $\bar{x}_n \to \infty$ as $n \to \infty$. Then given any $x \geq 0$, $x < \bar{x}_n$ for n sufficiently large, and it follows from (5) that

$$\begin{aligned} & r\underline{V}_{n-m}(x) \\ = \quad & -\beta x^2 + \frac{(n-m)\eta_m}{2}\left[\underline{V}_{n-m}(x+1) + \underline{V}_{n-m}(x-1) - 2\underline{V}_{n-m}(x) + 2p\right] \\ \sim \quad & \frac{n\eta_m}{2}\left[\Delta_\infty(x-1) - \Delta_\infty(x) + 2p\right] \end{aligned} \tag{F.1}$$

Where the symbol $\sim$ indicates asymptotic equivalence as $n \to \infty$. Letting $x = 0$, one has

$$r\underline{V}_{n-m}(0) \sim n\eta_m\left[-\Delta_\infty(0) + p\right]. \tag{F.2}$$

Given any $x \geq 0$, for n sufficiently large,

$$r[\underline{V}_{n-m}(0) - xp] \leq r\underline{V}_{n-m}(x) \leq r\underline{V}_{n-m}(0) \quad \Longrightarrow \quad r\underline{V}_{n-m}(x) \sim n\eta_m\left[-\Delta_\infty(0) + p\right]. \tag{F.3}$$

By comparing the asymptotic equivalences in (F.1) and (F.3), one obtains

$$\Delta_\infty(x) - \Delta_\infty(x-1) = 2\Delta_\infty(0),$$

for every $x \in \mathbb{Z}^+$. Thus

$$\Delta_\infty(x) = (2x+1)\Delta_\infty(0).$$

If $\Delta_\infty(0) > 0$, then $\Delta_\infty(x) > p$ for $x > [p/\Delta_\infty(0) - 1]/2$, which implies $\bar{x}_n \leq [p/\Delta_\infty(0) + 1]/2$ for n sufficiently large. This contradicts the fact that $\bar{x}_n$ goes to infinity as $n \to \infty$. Therefore, $\Delta_\infty(0) = 0$. It then follows from (F.2) that

$$r\underline{V}_{n-m}(0) \sim n\eta_m\, p. \tag{F.4}$$

Since $\underline{V}_{n-m}(0) - \underline{V}_{n-m-1}(0)$ is strictly increasing in $n \geq m$ (Lemma 6), it has a (possibly infinite) limit as $n \to \infty$. It then follows Cesàro's Theorem that

$$\frac{\underline{V}_{n-m}(0)}{n-m} = \frac{\sum_{k=1}^{n-m} \underline{V}_k(0) - \underline{V}_{k-1}(0)}{n-m} \xrightarrow{n\to\infty} \lim_{n\to\infty}[\underline{V}_{n-m}(0) - \underline{V}_{n-m-1}(0)].$$

The asymptotic equivalence in (F.4) then implies that

$$\lim_{n\to\infty}[\underline{V}_{n-m}(0) - \underline{V}_{n-m-1}(0)] = \frac{\eta_m p}{r}.$$

Given any $p > 0$ such that $\eta_m p/r \le \pi - p$, for every $n > m$,

$$\mathcal{L}(n-m, \eta_m, p) \le L_{n-m,\eta_m,p}(0) \le \underline{V}_{n-m}(0) - \underline{V}_{n-m-1}(0) < \frac{\eta_m p}{r} \le \pi - p.$$

Thus $\underline{p}(n-m, \eta_m) > p$ for every $n > m$, implying that

$$\liminf_{n} \underline{p}(n-m, \eta_m) \ge \frac{r\pi}{\eta_m + r}.$$

Given any p such that $\eta_m p/r > \pi - p$, I let $\varepsilon = \eta_m p/r - (\pi - p)$. For n sufficiently large, $\underline{V}_{n-m+1}(0) - \underline{V}_{n-m}(0) > \pi - p + \varepsilon/2$. If $\mathcal{L}(n-m, \eta_m, p) \le \pi - p$, it follows from (16) that

$$\begin{aligned}\mathcal{L}(n-m+1, \eta_m, p) &\ge \underline{V}_{n-m+1}(0) - \underline{V}_{n-m}(0) - (\pi - p) + \mathcal{L}(n-m, \eta_m, p)\\ &> \frac{\varepsilon}{2} + \mathcal{L}(n-m, \eta_m, p).\end{aligned}$$

Then it must be that $\mathcal{L}(n-m, \eta_m, p) > \pi - p$ for n sufficiently large, thus $\underline{p}(n-m, \eta_m) < p$ for n sufficiently large, implying that

$$\limsup_{n} \underline{p}(n-m, \eta_m) \le \frac{r\pi}{\eta_m + r}.$$

Thus, the limiting number m^*_∞ of dealers is the largest integer m such that

$$\frac{mr\pi}{\lambda\theta_m + mr} = \frac{r\pi}{\eta_m + r} = \lim_{n\to\infty} \underline{p}(n-m, \eta_m) < p^*(m).$$

Part (ii) (dependence of m^* on λ): Since the dealer-sustainable spread $\underline{p}(m)$ is strictly decreasing in λ (Proposition 3), and the equilibrium spread $p^*(m)$ is strictly increasing in λ (equation (3)), the core size m^* is thus weakly increasing in λ.

Part (ii) (dependence of m^* on c): As c decreases, $p^*(m)$ increases, while $\underline{p}(m)$ is not affected. The core size m^* thus weakly increases.

Part (iii): When the number n of agents increases, the equilibrium number m^* of dealers weakly increases (Part *(i)*). Competitive pressure pushes dealers to lower their equilibrium spread offer. This can be seen directly from expression (3) of the equilibrium spread $p^*(m^*)$.

F.2 Proof of Proposition 7

I let $\Delta_\vartheta(x) = \underline{V}_\vartheta(x) - \underline{V}_\vartheta(x+1)$ for every $x \in \mathbb{Z}$. Since $\frac{\partial}{\partial \vartheta} \underline{V}_\vartheta$ is U-shaped (Lemma 4), then

$$\Delta_{\vartheta_2}(x) - \Delta_{\vartheta_1}(x) = \int_{\vartheta_1}^{\vartheta_2} \frac{\partial}{\partial \vartheta} \underline{V}_\vartheta(x)\, d\vartheta - \int_{\vartheta_1}^{\vartheta_2} \frac{\partial}{\partial \vartheta} \underline{V}_\vartheta(x+1)\, d\vartheta < 0,$$

for every $x \in \mathbb{Z}^+$ and $\vartheta_1 < \vartheta_2$. Proposition 2 implies that $\bar{x}_{\vartheta_1,p,\beta} \leq \bar{x}_{\vartheta_2,p,\beta}$. The same technique can be applied to show that $\bar{x}_{\vartheta,p,\beta}$ is weakly decreasing in $\beta > 0$.

To drive the desired asymptotic, I fix some $m \geq 1$ and $p > 0$, and let $\vartheta = (n-m)\eta_m$. It is sufficient to show that $\bar{x}_\vartheta = \Theta\left(\vartheta^{1/3}\right)$ as ϑ goes to infinity. It follows from (5) that

$$\underline{V}_\vartheta(x) = T_1(\underline{V}_\vartheta)(x), \qquad -\bar{x}_\vartheta < x < \bar{x}_\vartheta. \tag{F.5}$$

$$\underline{V}_\vartheta(x) = T_2(\underline{V}_\vartheta)(x), \qquad x \geq \bar{x}_\vartheta. \tag{F.6}$$

where for every function $V : \mathbb{Z} \to \mathbb{R}$,

$$T_1(V)(x) = \frac{1}{\vartheta + r}\left(-\beta x^2 + \frac{\vartheta}{2}\left[V(x-1) + V(x+1) + 2p\right]\right)$$

$$T_2(V)(x) = \frac{1}{\vartheta + r}\left(-\beta x^2 + \frac{\vartheta}{2}\left[V(x-1) + V(x) + p\right]\right).$$

A quadratic solution U_ϑ^0 of (F.5) is given by

$$U_\vartheta^0(x) = -\frac{\beta}{r}x^2 + \frac{\vartheta}{r}\left(p - \frac{\beta}{r}\right).$$

To obtain all solutions of (F.5), I consider its homogeneous version:

$$rV(x) = \frac{\vartheta}{2}\left[V(x-1) + V(x+1) - 2V(x)\right]. \tag{F.7}$$

The set of solutions to the difference equation above forms a 2-dimensional vector space

$$\{ae^{d_\vartheta x} + \tilde{a}e^{-d_\vartheta x} : a, \tilde{a} \in \mathbb{R}\}, \qquad \text{where } d_\vartheta = \sqrt{\frac{2r}{\vartheta}} + O\left(\vartheta^{-\frac{3}{2}}\right).$$

Therefore, the solutions to (F.5) are

$$\mathbb{Z} \ni x \mapsto -\frac{\beta}{r}x^2 + \frac{\vartheta}{r}\left(p - \frac{\beta}{r}\right) + ae^{d_\vartheta x} + \tilde{a}e^{-d_\vartheta x},$$

where $a, \tilde{a} \in \mathbb{R}$. The value function $\underline{V}_\vartheta$ must be equal to one of the solutions U_ϑ in the region $-\bar{x}_\vartheta \leq x \leq \bar{x}_\vartheta$, for some $a = a_\vartheta$ and $\tilde{a} = \tilde{a}_\vartheta$. Since the function $\underline{V}_\vartheta$ is even, one must have $a_\vartheta = \tilde{a}_\vartheta$. Hence, for every integer $x \in [-\bar{x}_\vartheta, \bar{x}_\vartheta]$,

$$\underline{V}_\vartheta(x) = U_\vartheta(x) \equiv -\frac{\beta}{r}x^2 + \frac{\vartheta}{r}\left(p - \frac{\beta}{r}\right) + a_\vartheta \cosh(d_\vartheta x). \tag{F.8}$$

Solving equation (F.6), one obtains, for every integer $x \geq \bar{x}_\vartheta - 1$,

$$\begin{aligned} \underline{V}_\vartheta(x) = W_\vartheta(x) &\equiv W^0_\vartheta(x) + b_\vartheta e^{c_\vartheta x} \\ &\equiv -\frac{\beta}{r}x^2 + \frac{\vartheta}{r}\frac{\beta}{r}x - \left(\frac{\vartheta}{r}\right)^2\frac{\beta}{2r} + \frac{\vartheta}{2r}\left(p - \frac{\beta}{r}\right) + b_\vartheta e^{c_\vartheta x}, \end{aligned} \tag{F.9}$$

for some $b_\vartheta \in \mathbb{R}$, where $\quad c_\vartheta = -\frac{2r}{\vartheta} + 2\left(\frac{r}{\vartheta}\right)^2 + O\left(\vartheta^{-3}\right).$

I show that the undetermined coefficients a_ϑ and b_ϑ are non-negative. For this purpose, I define $\underline{V}^0_\vartheta$ as an even function from $\mathbb{Z}$ to $\mathbb{R}$ such that for every $x \in \mathbb{Z}^+$,

$$\underline{V}^0_\vartheta(x) = \max\left\{U^0_\vartheta(x), W^0_\vartheta(x)\right\}.$$

I let $\underline{B}_\vartheta$ be the Bellman operator defined in (8). Then one has

$$\begin{cases} \underline{B}_\vartheta\left(\underline{V}^0_\vartheta\right) \geq T_1\left(\underline{V}^0_\vartheta\right) \geq T_1\left(U^0_\vartheta\right) = U^0_\vartheta, \\ \underline{B}_\vartheta\left(\underline{V}^0_\vartheta\right) \geq T_2\left(\underline{V}^0_\vartheta\right) \geq T_2\left(W^0_\vartheta\right) = W^0_\vartheta, \end{cases} \implies \underline{B}_\vartheta\left(\underline{V}^0_\vartheta\right) \geq \max\left\{U^0_\vartheta, W^0_\vartheta\right\} = \underline{V}^0_\vartheta.$$

By iterating the Bellman operator $\underline{B}_\vartheta$, one obtains $\underline{V}_\vartheta \geq \underline{V}^0_\vartheta$, which implies $a_\vartheta \geq 0, b_\vartheta \geq 0$.

It follows from (F.8) and (F.9) that U_ϑ and W_ϑ have same values at $x = \bar{x}_\vartheta - 1$ and $\bar{x}_\vartheta$:

$$U_\vartheta(\bar{x}_\vartheta - 1) = W_\vartheta(\bar{x}_\vartheta - 1), \qquad U_\vartheta(\bar{x}_\vartheta) = W_\vartheta(\bar{x}_\vartheta), \tag{F.10}$$

which is the smooth pasting condition for the difference equations (F.5) and (F.6). Also,

$$T_1(U_\vartheta)(\bar{x}_\vartheta) = U_\vartheta(\bar{x}_\vartheta) = W_\vartheta(\bar{x}_\vartheta) = T_2(W_\vartheta)(\bar{x}_\vartheta) = T_2(U_\vartheta)(\bar{x}_\vartheta),$$

$$U_\vartheta(\bar{x}_\vartheta) - U_\vartheta(\bar{x}_\vartheta + 1) = p.$$

By an abuse of notation, I use U_ϑ and W_ϑ to denote the functions given by (F.8) and (F.9) respectively on the entire real line $\mathbb{R}$. There exists some $\tilde{x}_\vartheta \in (\bar{x}_\vartheta, \bar{x}_\vartheta + 1)$ such that

$$U'_\vartheta(\tilde{x}_\vartheta) = -p. \tag{F.11}$$

Similarly, there exists some $\hat{x}_\vartheta \in (\bar{x}_\vartheta - 1, \bar{x}_\vartheta + 1)$ such that

$$W'_\vartheta(\hat{x}_\vartheta) = -p, \tag{F.12}$$

Plugging the expressions of U_ϑ and W_ϑ into (F.10) to (F.12), one obtains

$$-\frac{2\beta}{r}\tilde{x}_\vartheta + a_\vartheta d_\vartheta \sinh(d_\vartheta \tilde{x}_\vartheta) = -p, \tag{F.13}$$

$$-\frac{2\beta}{r}\hat{x}_\vartheta + \frac{\vartheta}{r}\frac{\beta}{r} + b_\vartheta c_\vartheta e^{c_\vartheta \hat{x}_\vartheta} = -p. \tag{F.14}$$

$$\frac{\vartheta}{2r}\left(p - \frac{\beta}{r}\right) + a_\vartheta \cosh(d_\vartheta \bar{x}_\vartheta) = \frac{\vartheta}{r}\frac{\beta}{r}\bar{x}_\vartheta - \left(\frac{\vartheta}{r}\right)^2 \frac{\beta}{2r} + b_\vartheta e^{c_\vartheta \bar{x}_\vartheta}, \tag{F.15}$$

Equation (F.14) and $b_\vartheta \geq 0$ imply that

$$0 \leq -\frac{r}{\vartheta}\, b_\vartheta c_\vartheta e^{c_\vartheta \hat{x}_\vartheta} = \frac{\beta}{r}\left(1 - \frac{2r}{\vartheta}\hat{x}_\vartheta\right) + \frac{r}{\vartheta}\, p.$$

Thus, $b_\vartheta e^{c_\vartheta \bar{x}_\vartheta} = O(\vartheta^2)$ and $\hat{x}_\vartheta = O(\vartheta)$. I multiply (F.14) by $\vartheta/(2r)$ and subtract by (F.15),

$$a_\vartheta \cosh(d_\vartheta \bar{x}_\vartheta) = b_\vartheta O\left(\vartheta^{-1}\right) e^{c_\vartheta \bar{x}_\vartheta} + O(\vartheta) = O(\vartheta). \tag{F.16}$$

I show that $\hat{x}_\vartheta = o(\vartheta)$. If this is not the case, then there exists a sequence $(\vartheta_\ell)_{\ell \geq 0}$ going to infinity such that $\hat{x}_{\vartheta_\ell} = \Theta(\vartheta_\ell)$ as ℓ goes to infinity. It then follows from (F.13) that

$$a_{\vartheta_\ell} \sinh(d_{\vartheta_\ell} \tilde{x}_{\vartheta_\ell}) = \Theta\left(\vartheta_\ell^{3/2}\right), \qquad \text{thus} \quad a_{\vartheta_\ell} \cosh(d_{\vartheta_\ell} \tilde{x}_{\vartheta_\ell}) = \Theta\left(\vartheta_\ell^{3/2}\right).$$

This contradicts (F.16). Therefore, $\hat{x} = o(\vartheta)$, and thus $\bar{x}_\vartheta = o(\vartheta)$.

I derive a higher order Taylor expansion in (F.16):

$$a_\vartheta \cosh(d_\vartheta \bar{x}_\vartheta) \sim \frac{\beta}{r}\frac{\vartheta}{r}.$$
$$\Longrightarrow a_\vartheta d_\vartheta \sinh(d_\vartheta \bar{x}_\vartheta) \sim \frac{2\beta}{r}\sqrt{\frac{\vartheta}{2r}}\tanh(d_\vartheta \bar{x}_\vartheta).$$

It then follows from (F.13) that $d_\vartheta \bar{x}_\vartheta \sim \tanh(d_\vartheta \bar{x}_\vartheta)$. Since $y = \tanh y$ does not have non-zero solution. Thus, $\lim_\vartheta d_\vartheta \bar{x}_\vartheta = 0$. A higher-order Taylor expansion applied to (F.16) gives

$$a_\vartheta = \frac{\beta\vartheta}{r^2} - \frac{\beta\vartheta}{2r^2}\, d_\vartheta^2 \bar{x}_\vartheta^2 + O(\bar{x}_\vartheta). \tag{F.17}$$

Using (F.17), another Taylor expansion applied to (F.13) leads to

$$-\frac{\beta\vartheta}{3r^2}\, d_\vartheta^4\, \bar{x}_\vartheta^3 = -p,$$

which implies $\quad \tilde{x}_\vartheta = \Theta\left(\vartheta^{1/3}\right), \quad \bar{x}_\vartheta = \Theta\left(\vartheta^{1/3}\right), \quad V_\vartheta(0) - V_\vartheta(\bar{x}_\vartheta) = O\left(\vartheta^{1/3}\right). \qquad$ (F.18)

I show that the mixing time goes to 0 at the rate of $(n\lambda)^{-1/3}$ in Lemma G.2 which generalizes the mixing time result with a potential interdealer market.

F.3 Proof of Proposition 8

Part (i): Fixing some $m \geq 1$ and $p > 0$, I let $\vartheta = (n-m)\eta_m$. With reparametrization, I write $C(\vartheta)$ for $C(n, \lambda, m)$. The dealer value $V_\vartheta(0) = \underline{V}_\vartheta(0)$ increases superlinearly with ϑ (Lemma 6). Thus, the individual dealer inventory cost $C(\vartheta)$ is strictly concave in ϑ. Since $V_\vartheta(0) \leq \vartheta p/r$, then $C(\vartheta) \geq 0$. Hence, it must be that $C(\vartheta)$ is strictly increasing in $\vartheta \in \mathbb{R}^+$.

Equation (F.17) implies that $a_\vartheta = \beta\vartheta/r^2 + O\left(\vartheta^{2/3}\right)$. Letting $x = 0$ in (F.8), one has

$$V_\vartheta(0) = \frac{\vartheta p}{r} + O\left(\vartheta^{\frac{2}{3}}\right) \quad \Longrightarrow \quad C(\vartheta) = O\left(\vartheta^{2/3}\right).$$

Part (ii): One has $m\vartheta_m < (m+1)\vartheta_{m+1}$. Since the individual dealer inventory cost $C(\vartheta)$ is strictly increasing and strictly concave in ϑ, Jensen's inequality implies that

$$mC(\vartheta_m) = mC(\vartheta_m) + C(0) < (m+1)\, C\left(\frac{m\vartheta_m}{m+1}\right) < (m+1)\, C(\vartheta_{m+1}).$$

F.4 Proof of Proposition 9

With an instantaneous rate of benefit $\Phi_{m,p^*(m)}$, a buyside firm's equilibrium utility is

$$\frac{\Phi_{m,p^*(m)}}{r} = \frac{\lambda\theta_m(\pi - p^*(m)) - mc}{r}.$$

As n goes to infinity, it follows from (F.4) that

$$\underline{V}_{n-m,\eta_m,p^*(m)}(0) \sim n\lambda \frac{\theta_m}{rm} p^*(m). \tag{F.19}$$

Thus, $\quad U_m = (n-m)\Phi_{m,p^*(m)} + m\underline{V}_{n-m,\eta_m,p^*(m)}(0)$

$$\sim \left(\frac{\lambda\theta_m\pi - mc}{r}\right) n = \left[\sum_{1\leq m'\leq m} (\theta_{m'} - \theta_{m'-1})p^*(m')\right] \frac{\lambda n}{r} \equiv g(m)\frac{\lambda n}{r}.$$

Since $p^*(m') > 0$ for every $1 \leq m' \leq \overline{m}$, then $g(m)$ is strictly increasing in m for $0 \leq m \leq \overline{m}$. The asymptotic equivalence above implies that there exists some integer $n_0 > 0$, if the total number of agent $n > n_0$, the welfare U_m is strictly increasing in m for $0 \leq m \leq \overline{m}$.

G Proofs for Section 5

G.1 Proof of Theorem 3

With the presence of an interdealer market, the buyside firm's problem is not affected, thus the equilibrium spread remains to be $p^*(m)$ which solves the buyside firm's indifference condition. The dealer's value function $\widehat{V}_k$ solves a different HJB equation

$$\begin{aligned} r\widehat{V}_k(x) = &-\beta x^2 + k\,\frac{\eta}{2}\,\max\left\{\widehat{V}_k(x-1) - \widehat{V}_k(x) + p + h,\ 0,\ \widehat{V}_{k-1}(x-1) - \widehat{V}_k(x) + \pi\right\} \\ &+ k\,\frac{\eta}{2}\,\max\left\{\widehat{V}_k(x+1) - \widehat{V}_k(x) + p - h,\ 0,\ \widehat{V}_{k-1}(x+1) - \widehat{V}_k(x) + \pi\right\} \\ &+ (m-1)\xi\left[\widehat{V}_k(0) - \widehat{V}_k(x)\right]. \end{aligned}$$

Compared to the HJB equation (4) without the interdealer market, the HJB equation above includes an additional term $(m-1)\xi\left[\widehat{V}_k(0) - \widehat{V}_k(x)\right]$ that reflects the instantaneous rate of benefit from interdealer trading. On the equilibrium path, the dealer has no incentive to gouge any buyside firm. Thus $\widehat{V}_k = \underline{\widehat{V}}_k$ where $\underline{\widehat{V}}_k$ solves

$$r\underline{\widehat{V}}_k(x) = -\beta x^2 + (m-1)\xi \left[\underline{\widehat{V}}_k(0) - \underline{\widehat{V}}_k(x)\right] + k\,\frac{\eta}{2}\,\max\left\{\underline{\widehat{V}}_k(x-1) - \underline{\widehat{V}}_k(x) + p + h,\ 0\right\}$$
$$+ k\,\frac{\eta}{2}\,\max\left\{\underline{\widehat{V}}_k(x+1) - \underline{\widehat{V}}_k(x) + p - h,\ 0\right\}.$$

Similar to Lemma 2, one can show that the value function $\underline{\widehat{V}}_k$ is even and strictly concave. Thus the dealer optimally controls its inventory within some range $[\underline{x}_k, \bar{x}_k]$. I let $\xi_m = (m-1)\xi$, then

for $\underline{x}_k < x < \bar{x}_k$,

$$r\underline{\widehat{V}}_k(x) = -\beta x^2 + \xi_m \left[\underline{\widehat{V}}_k(0) - \underline{\widehat{V}}_k(x)\right] + \frac{k\eta}{2}\left[\underline{\widehat{V}}_k(x-1) + \underline{\widehat{V}}_k(x+1) - 2\underline{\widehat{V}}_k(x) + 2p\right] \tag{G.1}$$

for $x \geq \bar{x}_k$,

$$r\underline{\widehat{V}}_k(x) = -\beta x^2 + \xi_m \left[\underline{\widehat{V}}_k(0) - \underline{\widehat{V}}_k(x)\right] + \frac{k\eta}{2}\left[\underline{\widehat{V}}_k(x-1) - \underline{\widehat{V}}_k(x) + p + h\right], \tag{G.2}$$

for $x_k \leq \underline{x}_k$.

$$r\underline{\widehat{V}}_k(x) = -\beta x^2 + \xi_m \left[\underline{\widehat{V}}_k(0) - \underline{\widehat{V}}_k(x)\right] + \frac{k\eta}{2}\left[\underline{\widehat{V}}_k(x+1) - \underline{\widehat{V}}_k(x) + p - h\right], \tag{G.3}$$

I follow the same steps as in the proof of Proposition 7 to derive $\underline{\widehat{V}}_k$ as $k \to \infty$. First, one can solve the difference equations (G.1), (G.2) and (G.3) to obtain

$$\underline{\widehat{V}}_k(x) = \begin{cases} \widehat{U}_k(x), & \underline{x}_k \leq x \leq \bar{x}_k, \\ \widehat{W}_k^R(x), & x \geq \bar{x}_k - 1, \\ \widehat{W}_k^L(x), & x \leq \underline{x}_k + 1, \end{cases}$$

where for every $x \in \mathbb{Z}$,

$$\widehat{U}_k(x) = -\frac{\beta}{r+\xi_m}\,x^2 + \frac{k\eta}{r}\left(p - \frac{\beta}{r+\xi_m}\right) + \frac{a}{2}\left(e^{dx} + \frac{\xi_m}{r}\right) + \frac{\tilde{a}}{2}\left(e^{-dx} + \frac{\xi_m}{r}\right),$$

$$\widehat{W}_k^R(x) = -\frac{\beta}{r+\xi_m}\,x^2 + \frac{k\eta\beta}{(r+\xi_m)^2}\,x + \frac{k\eta}{2r}\left(p + h - \frac{\beta}{r+\xi_m} - \frac{k\eta\beta}{(r+\xi_m)^2}\right) + b\left(e^{cx} + \frac{\xi_m}{r}\right),$$

$$\widehat{W}_k^L(x) = -\frac{\beta}{r+\xi_m}\,x^2 - \frac{k\eta\beta}{(r+\xi_m)^2}\,x + \frac{k\eta}{2r}\left(p - h - \frac{\beta}{r+\xi_m} - \frac{k\eta\beta}{(r+\xi_m)^2}\right) + \tilde{b}\left(e^{-cx} + \frac{\xi_m}{r}\right),$$

for some non-negative constants $a, \tilde{a}, b, \tilde{b}$ to be determined, and as $k \to \infty$,

$$d = \sqrt{\frac{2(r+\xi_m)}{k\eta}} + O\left(k^{-\frac{3}{2}}\right), \quad c = -2\,\frac{r+\xi_m}{k\eta} + 2\left(\frac{r+\xi_m}{k\eta}\right)^2 + O\left(k^{-3}\right).$$

One can obtain the smooth pasting condition at the boundaries $\bar{x}_k$ and $\underline{x}_k$:

$$\begin{aligned}
&\widehat{U}_k(\bar{x}_k+1) - \widehat{U}_k(\bar{x}_k) = -(p-h), && \widehat{U}_k(\bar{x}_k) = \widehat{W}_k^R(\bar{x}_k), && \widehat{U}_k(\bar{x}_k-1) = \widehat{W}_k^R(\bar{x}_k-1),\\
&\widehat{U}_k(\underline{x}_k) - \widehat{U}_k(\underline{x}_k-1) = p+h, && \widehat{U}_k(\underline{x}_k) = \widehat{W}_k^L(\underline{x}_k), && \widehat{U}_k(\underline{x}_k+1) = \widehat{W}_k^L(\underline{x}_k+1),
\end{aligned}$$

As $k \to \infty$, one can solve for the smooth pasting conditions above to obtain

$$\bar{x}_k \sim \; -\underline{x}_k \sim \left(\frac{3p}{4}\right)^{\frac{1}{3}}(k\eta)^{\frac{1}{3}} + f_1(\beta, r, p, \xi_m, h) + f_2(\beta, r, p, \xi_m, h)(k\eta)^{-\frac{1}{3}} \tag{G.4}$$

$$\begin{aligned}
\widehat{U}_k(x) = &\left(\frac{k\eta p}{r} - \frac{\beta}{r}\left(\frac{3p}{4}\right)^{\frac{2}{3}}(k\eta)^{\frac{2}{3}} + f_6(\beta, r, p, \xi_m, h)(k\eta)^{\frac{1}{3}} + f_7(\beta, r, p, \xi_m, h)\right)\\
&+ f_3(\beta, r, p, \xi_m, h)x + f_4(\beta, r, p, \xi_m, h)(k\eta)^{-\frac{1}{3}}x^2 + f_5(\beta, r, p, \xi_m, h)(k\eta)^{-1}x^4 + g_k(x)
\end{aligned} \tag{G.5}$$

for some functions $f_1, \ldots, f_7$ independent of k, and g_k satisfying $\sup_{x\in[\underline{x}_k-1,\bar{x}_k+1]} |g_k(x)| \to 0$. Then for h in any compact set, in particular, for any mid-quote h such that the corresponding bid and ask quotes are both less than π, that is, $|h| + p \leq \pi$,

$$\widehat{U}_k(0) - \widehat{U}_{k-1}(0) < \eta p/r \text{ for } k \text{ sufficiently large,} \tag{G.6}$$

$$\text{and} \min_{x\in[\underline{x}_k-1,\bar{x}_k+1]} \left(\widehat{U}_k(x) - \widehat{U}_{k-1}(x)\right) \to \eta p/r \text{ as } k \to \infty. \tag{G.7}$$

Next, I show that for k sufficiently large, the dealer has no incentive to gouge for some h

if and only if $\eta_m p^*(m)/r > \pi - p^*(m)$, which is equivalent to $m \leq m^*$. No-gouging implies

$$\min_{x \in (\underline{x}_k, \bar{x}_k)} \left(\widehat{U}_k(x) - \widehat{V}_{k-1}(x) \right) \geq \pi - p + |h| \geq \pi - p.$$

Since $\widehat{U}_k - \widehat{V}_{k-1} \leq \widehat{U}_k - \underline{\widehat{V}}_{k-1}$, then the above inequality together with (G.6) imply $\pi - p < \eta_m p/r$. Conversely, a sufficient condition for no-gouging is

$$\widehat{\mathcal{L}}(k, \eta_m, p) := \min_{x \in [\underline{x}_k - 1, \bar{x}_k + 1]} \left(\widehat{U}_k(x) - \widehat{V}_{k-1}(x) \right) \geq \pi - p + |h|.$$

I let $h = 0$ and $W = \underline{\widehat{V}}_{k-1} + \left[\pi - p - \widehat{\mathcal{L}}(k-1, \eta_m, p) \right]^+$. Then $\widehat{V}_{k-1} \leq W$. If $\eta_m p^*(m)/r > \pi - p^*(m)$, I let $\varepsilon = \eta_m p/r - (\pi - p)$. For k sufficiently large,

$$\min_{x \in [\underline{x}_k - 1, \bar{x}_k + 1]} \left[\widehat{U}_k(x) - \widehat{U}_{k-1}(x) \right] > \pi - p + \varepsilon/2 \qquad \text{(following from (G.7))}.$$

On the other hand, one can show, by two separate inductions over x, that $\widehat{U}_{k-1}(x) \geq \underline{\widehat{V}}_{k-1}(x)$ for every $x > \bar{x}_{k-1}$ and $x < \underline{x}_{k-1}$, then $\widehat{U}_{k-1} \geq \underline{\widehat{V}}_{k-1}$. If $\widehat{\mathcal{L}}(k-1, \eta_m, p) \leq \pi - p$, then

$$\begin{aligned}
\widehat{\mathcal{L}}(k, \eta_m, p) &\geq \min_{x \in [\underline{x}_k - 1, \bar{x}_k + 1]} \left[\widehat{U}_k(x) - W(x) \right] \\
&= \min_{x \in [\underline{x}_k - 1, \bar{x}_k + 1]} \left[\widehat{U}_k(x) - \underline{\widehat{V}}_{k-1}(x) \right] - (\pi - p) + \widehat{\mathcal{L}}(k-1, \eta_m, p) \\
&\geq \min_{x \in [\underline{x}_k - 1, \bar{x}_k + 1]} \left[\widehat{U}_k(x) - \widehat{U}_{k-1}(x) \right] - (\pi - p) + \widehat{\mathcal{L}}(k-1, \eta_m, p) \\
&> \frac{\varepsilon}{2} + \widehat{\mathcal{L}}(k-1, \eta_m, p).
\end{aligned}$$

Then it must be that $\widehat{\mathcal{L}}(k, \eta_m, p) > \pi - p$ for k sufficiently large, ensuring that the dealer has no incentive to gouge.

G.2 Proof of Proposition 10

I define the long-run averages of interdealer volume and total trade volume by

$$\mathrm{Vol}_{\mathrm{ID}} = \lim_{T \to \infty} \frac{1}{T} \sum_{j \in J} \mathrm{Vol}_j(T), \qquad \mathrm{Vol} = \lim_{T \to \infty} \frac{1}{T} \sum_{i \in I, j \in J} \mathrm{Vol}_{i,j}(T),$$

where $\mathrm{Vol}_j(T)$ is the volume traded by dealer j in the interdealer market in the time interval $[0,T]$, and $\mathrm{Vol}_{i,j}(T)$ is the volume traded between buyside firm i and dealer j. The Ergodic Theorem implies that

$$\mathrm{Vol}_{\mathrm{ID}} = m(m-1)\xi\, \mathrm{E}(|X_j|)\,,$$

where the expectation E is taken with respect to the stationary distribution of the inventory position X_j of dealer j. Since the dealer controls its inventory within some range $[\underline{x}_n, \bar{x}_n]$ where both boundaries $\underline{x}_n$ and $\bar{x}_n$ are on the order of $(n\lambda)^{\frac{1}{3}}$ as $n\lambda \to \infty$ ((G.4)), then

$$\mathrm{E}(|X_j|) = \Theta\left((n\lambda)^{\frac{1}{3}}\right), \qquad \text{hence,} \quad \mathrm{Vol}_{\mathrm{ID}} = \Theta\left((n\lambda)^{\frac{1}{3}}\right).$$

On the other hand, $\mathrm{Vol} = \Theta(n\lambda)$ as $n \to \infty$. Therefore, the fraction $\mathrm{Vol}_{\mathrm{ID}}/\mathrm{Vol}$ of volume traded in the interdealer market is on the order of $\left((n\lambda)^{-2/3}\right)$.

G.3 Proof of Proposition 11

The proof works roughly as follows: Under the strategy profile $\hat{\sigma}^*_{\text{No DP}}(m)$, a buyside firm i receives a trading gain of either $\pi - p^*(m)$ or 0 each time it requests a quote from a dealer counterparty. This payoff is 0 only when the dealer's inventory is on one of the two boundaries $\pm\hat{x}_\vartheta$ (in which case the dealer rejects the trade request by offering either an ask price ∞ or a bid price $-\infty$). The probability of this event is arbitrarily close to 0 when the rate $\vartheta = (n-m)\eta_m$ of RFQ received by the dealer is large. Therefore, the continuation utility of i is arbitrarily close to its upper bound $\Phi_{m,p^*(m)}/r$, which is attained if all RFQ from i were served with the spread $p^*(m)$. Therefore, maintaining m or $m-1$ dealer accounts is an ε-optimal strategy for the buyside firm.

Formally, I first complement $\hat{\sigma}^*_{\text{No DP}}(m)$ with a consistent belief system. The belief system is identical to the one for $\sigma^*(m)$ constructed in the proof of Theorem 1 with one restriction on off-the-equilibrium-path beliefs: If a buyside firm i receives any off-the-equilibrium-path price offer from a dealer j, then i believes that j hits one of its inventory boundaries $\pm\hat{x}_\vartheta$. This restriction on off-the-equilibrium-path beliefs does not violate the no-signaling-what-you-don't-know requirement. Therefore, the belief system is consistent with respect to the strategy profile $\hat{\sigma}^*_{\text{No DP}}(m)$.

I let $\mathcal{G}\left(\widehat{G}(m), i\right)$ be the set of networks in which all agents other than i are connected to each other as in $\widehat{G}(m)$, and $\widehat{X}_t = (\widehat{X}_{jt})_{j\in J}$ be the dealers' inventory positions at time t. I fix a buyside firm i throughout this proof. At any information set of i (on or off the equilibrium path), the belief of i about the current state $(G_{t^-}, \widehat{X}_{t^-})$ assigns probability 1 to the set $\mathcal{G}\left(\widehat{G}(m), i\right)$ of networks and to the set $[-\hat{x}_\vartheta, \hat{x}_\vartheta]^m$ of dealer inventories. I let H_{t^-} include the support of all such inference distributions. That is,

$$H_{t^-} = \left\{G_{t^-} \in \mathcal{G}\left(\widehat{G}(m), i\right),\ \ \widehat{X}_{t^-} \in [-\hat{x}_\vartheta, \hat{x}_\vartheta]^m\right\}$$

For every game history $h_{t^-} \in H_{t^-}$, I let $U_i(h_{t^-}; \hat{\sigma}^*_{\text{No DP}}(m))$ be the continuation utility of i following the history h_{t^-} under the strategy profile $\sigma^*_{\text{No DP}}(m)$: it is the continuation utility of i if h_{t^-} has been realized and all agents follow the strategy profile $\sigma^*_{\text{No DP}}(m)$ in the continuation game. It suffices to show that this continuation utility is ε-close to its upper-bound $\Phi_{m,p^*(m)}/r$ for every history $h_{t^-} \in H_{t^-}$.

I let $x_{t^-} = (x_{jt^-})_{j\in J}$ be dealers' current inventories under a given history $h_{t^-} \in H_{t^-}$. Following h_{t^-}, a dealer's inventory process $(\widehat{X}_{js})_{s\geq t}$ is a Markov process starting at x_{jt^-} and moving in the state space $[-\hat{x}_\vartheta, \hat{x}_\vartheta]$. For every dealer j, I let τ_{jk} be the k'th time at which either *(i)* dealer j receives an RFQ, or *(ii)* dealer j makes an interdealer trade. If i currently has accounts with $m-1$ dealers under history h_{t^-}, then for every dealer j with whom i is connected, $(\tau_{jk})_{k\geq 1}$ are the event times of a Poisson process with intensity $\chi = (n-m-1)\eta_m + \eta_{m-1} + (m-1)\xi$. Otherwise, i would open and maintain accounts with every dealer j at and after time t, thus $(\tau_{jk})_{k\geq 1}$ are the event times of a Poisson process with intensity $\chi' = (n-m)\eta_m + (m-1)\xi$. I let $\text{RFQ}_{jk} \in \{0,1\}$ be the binary variable indicating whether buyside firm i submits an RFQ to dealer j at τ_{jk}. Then

$$
\begin{aligned}
&\Phi_{m,p^*(m)}/r - U_i(h_{t^-}; \hat{\sigma}^*_{\text{No DP}}(m)) \\
&\leq \mathrm{E}\left(\sum_{j \in J, \tau_{jk} \geq t} e^{-r(\tau_{jk}-t)} [\pi - p^*(m)] \, \mathbb{1}\left\{ \mathrm{RFQ}_{jk} = 1 \right\} \, \mathbb{1}\left\{ |\widehat{X}_{j\tau^-_{jk}}| = \hat{x}_\vartheta \right\} \;\middle|\; \widehat{X}_{jt^-} = x_{jt^-} \right) \\
&\leq [\pi - p^*(m)] \, \frac{\eta_{m-1}}{\chi} \sum_{j \in J, k \geq 1} \mathrm{E}\big(e^{-r\tilde{\tau}_{jk}}\big) \; \mathrm{P}\Big(|\widehat{Y}_{j(k-1)}| = \hat{x}_\vartheta \;\Big|\; \widehat{Y}_{j0} = x_{jt^-} \Big) \\
&\leq [\pi - p^*(m)] \, \frac{\eta_{m-1}}{\chi} \sum_{j \in J, k \geq 1} \left(\frac{\chi}{\chi + r} \right)^k \mathrm{P}\Big(|\widehat{Y}_{j(k-1)}| = \hat{x}_\vartheta \;\Big|\; \widehat{Y}_{j0} = x_{jt} \Big),
\end{aligned}
$$

where $(\tilde{\tau}_{jk})_{k \geq 1}$ are the event times of a Poisson process with the same intensity as that for $(\tau_{jk})_{k\geq 1}$, and $(\widehat{Y}_{jk})_{k \geq 0}$ is the embedded discrete-time Markov Chain of $(\widehat{X}_{js})_{s \geq t}$. The second equality above uses the independence between $(\tau_{jk})_{k \geq 1}$ and $(\widehat{X}_{j\tau_{jk}})_{k\geq 1}$, between RFQ_{jk} and $(\tau_{jk}, \widehat{X}_{j_k \tau_k})_{k \geq 1}$, and $\mathrm{P}\big(\mathrm{RFQ}_{jk} = 1\big) \leq \max\{\eta_{m-1}/\chi,\ \eta_m/\chi'\} = \eta_{m-1}/\chi$. Lemmas G.1 and G.2 imply that for every ε, there exists n_0 and $k(n) = o(n)$ such that for every $n > n_0$, $k > k(n)$ and $x_{jt^-} \in [-\hat{x}_\vartheta, \hat{x}_\vartheta]$,

$$
\mathrm{P}\Big(|\widehat{Y}_{j(k-1)}| = \hat{x}_\vartheta \;\Big|\; \widehat{Y}_{j0} = x_{jt^{-1}} \Big) < \varepsilon.
$$

Hence, there exists some n_1 such that for every $n > n_1$, t and $h_{t^-} \in H_{t^-}$,

$$
\Phi^*_{m,p^*(m)}/r - U_i(h_{t^-}; \hat{\sigma}^*_{\text{No DP}}(m)) < m\eta_{m-1}[\pi - p^*(m)] \left(\frac{k(n)}{\chi} + \frac{\varepsilon}{r} \right) < 2m\eta_{m-1}[\pi - p^*(m)] \, \frac{\varepsilon}{r}.
$$

Lemma G.1. *I let $\hat{\mu}$ be the stationary distribution of a dealer's inventory process $\hat{X}_{jt}$. Then*

$$
\hat{\mu}(\{-\hat{x}_\vartheta, \hat{x}_\vartheta\}) \leq \frac{2}{2\hat{x}_\vartheta + 1}.
$$

Proof. I show that if $\widetilde{X}_j \sim \mathrm{Unif}(-\hat{x}_\vartheta, \hat{x}_\vartheta)$ and $\widehat{X}_j \sim \hat{\mu}$, then

$$
|\widetilde{X}_j| \overset{d}{\geq} |\widehat{X}_j|, \tag{G.8}
$$

where $\overset{d}{\geq}$ denotes stochastic dominance. It would then follow that

$$
\hat{\mu}(\{-\hat{x}_\vartheta, \hat{x}_\vartheta\}) = \mathrm{P}\Big(|\widehat{X}_j| = \hat{x}_\vartheta \Big) \leq \mathrm{P}\Big(|\widetilde{X}_j| = \hat{x}_\vartheta \Big) = \frac{2}{2\hat{x}_\vartheta + 1}.
$$

I let $(\widetilde{Y}_k)$ be a bounded random walk that loops at the end points $\pm\hat{x}_\vartheta$, and $(\hat{y}_k, \tilde{y}_k, \hat{y}_{k+1})$ be integers within $[0, \hat{x}_\vartheta]$ such that $\hat{y}_k \geq \tilde{y}_k$. Since interdealer trading can only reduce the dealer's inventory, one has

$$P\Big(|\widetilde{Y}_{k+1}| \geq \widehat{y}_{k+1} \;\Big|\; |\widetilde{Y}_k| = \tilde{y}_k\Big) \geq P\Big(|\widehat{Y}_{j(k+1)}| \geq \hat{y}_{k+1} \;\Big|\; |\widehat{Y}_{jk}| = \hat{y}_k\Big).$$

Hence, the desired stochastic dominance (G.8) follows from an induction over k.

□

Lemma G.2. *As $n \to \infty$, the mixing time of a dealer's inventory process $(\widehat{X}_{jt})_{t\geq 0}$ is asymptotically bounded by $n^{-1/3}$.*

Proof. I use the *coupling* technique.[31] I consider a lazy version of $\widehat{Y}_{jk}$, which remains in its current position with probability $1/2$ and otherwise moves with the same transition probabilities as $\widehat{Y}_{jk}$. Further, I construct a coupling $(Y_k, Z_k)_{k\geq 0}$ of two such lazy chains on $[-\hat{x}_\vartheta, \hat{x}_\vartheta]$, starting from $Y_0 = y$ and $Z_0 = z$ respectively. At each event time τ_{jk}, dealer j either makes an interdealer trade or receives an RFQ from a buyside firm. In the first case, a fair coin is tossed to determine which of the two chains (Y_k) or (Z_k) makes the interdealer trade. In the second case, if $Y_{k-1} \neq Z_{k-1}$, then a fair coin is tossed to determine which of the two chains (Y_k) or (Z_k) receives the RFQ. If $Y_{k-1} = Z_{k-1}$, then a fair coin is tossed to determine whether both (Y_k) and (Z_k) receive the RFQ, or none does. Once the two chains (Y_k) and (Z_k) collide, they make identical moves thereafter. I let $D_k = Y_k - Z_k$ and $\hat{\mu}_{k,y}$ be the distribution of Y_k. Proposition 4.7 and Lemma 4.11 of Levin et al. (2009) imply that

$$\max_{y\in[-\hat{x}_\vartheta,\hat{x}_\vartheta]} ||\hat{\mu}_{k,y} - \hat{\mu}||_{\mathrm{TV}} \leq \max_{y,z\in[-\hat{x}_\vartheta,\hat{x}_\vartheta]} P_{y,z}(D_k \neq 0). \tag{G.9}$$

However, the process (D_k) is not Markovian. To bound the RHS of the above inequality, I construct a conditional Markov chain $(\bar{D}_k)$ such that $|\bar{D}_k| \geq |D_k|$ almost surely for every $k \geq 0$. The increment $\bar{D}_k - \bar{D}_{k-1}$ is set to be equal to $D_k - D_{k-1}$ unless at the k'th move, *(i)* either (Y_k) or (Z_k) (but not both) receives an RFQ, *(ii)* in the case where $(Y_k)_{k\geq 0}$ ($(Z_k)_{k\geq 0}$) receives the RFQ, $Y_{k-1} = \pm\hat{x}_\vartheta$ ($Z_{k-1} = \pm\hat{x}_\vartheta$) and the requested trade would further expand Y_k (Z_k). When these two conditions hold, I set $\bar{D}_k = (\bar{D}_{k-1} + \operatorname{sgn} Y_{k-1} - \operatorname{sgn} Z_{k-1}) \wedge (2\hat{x}_\vartheta) \vee (-2\hat{x}_\vartheta)$.

[31] Chapter 5 of Levin, Peres, and Wilmer (2009) provides relevant background for the coupling technique.

In other words, when (Y_k) or (Z_k) receives a trade request that would make it move beyond the boundaries $\pm\hat{x}_\vartheta$, $\bar{D}_k$ is determined if (Y_k) or (Z_k) were to move beyond the boundaries as long as $\bar{D}_k$ remains in the range $[-2\hat{x}_\vartheta, 2\hat{x}_\vartheta]$. One can verify that almost surely, $|\bar{D}_k| \geq |D_k|$ for every k. I let E_k be the event that no interdealer trade occurs up to period k. Conditional on the event E_k, the process $(\bar{D}_k)_{k\geq 0}$ is a random walk bounded in $[-2\hat{x}_\vartheta, 2\hat{x}_\vartheta]$ and loops at the end points $\pm 2\hat{x}_\vartheta$, up to period k and before being absorbed by 0. Therefore,

$$\mathrm{P}\big(\bar{D}_k = 0\big) > \mathrm{P}\big(\bar{D}_k = 0 \,|\, E_k\big)\, \mathrm{P}(E_k) = \mathrm{P}\left(\bar{D}_k = 0 \,|\, E_k\right) \left(\frac{(n-m)\eta_m}{\chi}\right)^k \tag{G.10}$$

To calculate $\mathrm{P}\big(\bar{D}_k = 0 \,|\, E_k\big)$, I consider a random walk $(\widetilde{D}_k)$ bounded in $[-2\hat{x}_\vartheta, 2\hat{x}_\vartheta]$ that starts from the same state $y - z$ as $(\bar{D}_k)$ but is not absorbed by 0. Then

$$\mathrm{P}\big(\bar{D}_k = 0 \,|\, E_k\big) = \mathrm{P}\Big(\widetilde{D}_k = 0 \;\Big|\; \widetilde{D}_0 = y - z\Big).$$

I let $\tau = \min\{k \,:\, \widetilde{D}_k = 0\}$ and $f_\ell = \mathrm{E}\Big(\tau \;\Big|\; \widetilde{D}_0 = \ell\Big)$, then $f_0 = 0$ and

$$\begin{aligned} f_\ell &= \frac{1}{2}\,(1 + f_{\ell-1}) + \frac{1}{2}\,(1 + f_{\ell+1}), \qquad 0 < |\ell| < 2\hat{x}_\vartheta, \\ f_{2\hat{x}_\vartheta} &= \frac{1}{2}\,(1 + f_{2\hat{x}_\vartheta - 1}) + \frac{1}{2}\,(1 + f_{2\hat{x}_\vartheta}). \end{aligned}$$

One can solve the system above to obtain $f_\ell = |\ell|\,(4\hat{x}_\vartheta - |\ell| + 1) \leq 2\hat{x}_\vartheta(2\hat{x}_\vartheta + 1)$. Hence,

$$\mathrm{P}\Big(\widetilde{D}_k \neq 0 \;\Big|\; \widetilde{D}_0 = y - z\Big) = \mathrm{P}\Big(\tau > k \;\Big|\; \widetilde{D}_0 = y - z\Big) < \frac{\mathrm{E}\Big(\tau \;\Big|\; \widetilde{D}_0 = y - z\Big)}{k} \leq \frac{2\hat{x}_\vartheta(2\hat{x}_\vartheta + 1)}{k}$$

It then follows from (G.9) and (G.10) that

$$\max_{y \in [-\hat{x}_\vartheta, \hat{x}_\vartheta]} ||\hat{\mu}_{k,y} - \hat{\mu}||_{\mathrm{TV}} \leq \mathrm{P}\big(\bar{D}_k \neq 0\big) < 1 - \left(1 - \frac{2\hat{x}_\vartheta(2\hat{x}_\vartheta + 1)}{k}\right) \left(\frac{(n-m)\eta_m}{\chi}\right)^k$$

As $n \to \infty$, the right hand side is arbitrarily close to 0 if $k = n^\alpha$ for any $\alpha \in (2/3, 1)$. Hence, the mixing time of $(\widehat{X}_{jt})$ is asymptotically bounded by $n^{-1/3}$. $\square$

G.4 Proof of Proposition 12

It follows from the asymptotic of $\widehat{\underline{V}}_k$ derived in the proof of Theorem 3 ((G.5)) that as $n \to \infty$,

$$\widehat{\underline{V}}_{n-m,m,p^*(m)}(0) - \underline{V}_{n-m,m,p^*(m)}(0) = O\left(n^{\frac{1}{3}}\right).$$

Since $\underline{V}_{n-m,m,p^*(m)}(0) = O(n)$ ((F.19)), then $\widehat{\underline{V}}_{n-m,m,p^*(m)}(0) \sim \underline{V}_{n-m,m,p^*(m)}(0)$ as $n \to \infty$. That is, for a given dealer in a large market, the surplus generated by trading with other dealers is negligible vis-à-vis the surplus generated by trading with buyside firms. Hence, for every $m \leq \overline{m}$, $\widehat{U}_m \sim U_m$ as $n\lambda \to \infty$. Proposition 12 thus follows as a corollary of Proposition 9.

References

Huang, Y. and W. F. McColl (1997): "Analytical Inversion of General Tridiagonal Matrices," *Journal of Physics A: Mathematical and General*, 30, 7919–7933.